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The Dirichlet-to-Neumann Map in Nonlinear Diffusion Problems

– Habilitation Thesis –

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Für meine Familie: Hanna, Nathaniel und Noémie.

Preface

This thesis is dedicated to the so-called Dirichlet-to-Neumann map associated with the weighted p-Laplace operator. In Chapter 1, we begin by deriving the Dirichlet-to-Neumann map by using classical modelling and outline why it is interesting to study this boundary operator. In the reminding part of Chapter 1, we dedicate each section an overview about the content of one chapter and summarize the main results. Chapter 2 is dedicated to the Poisson problem and the inverse of the Dirichlet-to-Neumann map. Chapter 3 provides the first main application of the Dirichlet-to-Neumann map, namely, it generates a strongly continuous semigroup of contractions on the Lebesgue space L^2 and this contraction can be extrapolated to a contraction on L^q for all $1 \leq q \leq \infty$. In Chapter 4, we develop an abstract theory to establish global L^q-L^∞ regularization estimates satisfied by the semigroup generated by the negative Dirichlet-to-Neumann map. Chapter 5 is concerned with L^1 and pointwise estimates on the time-derivative of the semigroup generated by the neagtive Dirichlet-to-Neumann map, which are known in the literatur as *Aronson-Bénilan type estimates*. In Chapter 6, we outline the theory of j-functional and its application to evolution problems. This theory allows us to study the Dirichlet problem on general open sets Ω, and to realize the Dirichlet-to-Neumann map as an operator in $L^2(\partial\Omega)$. In Chapter 7, we consider the limit case $p = 1$, which corresponds to the Dirichlet-to-Neumann map associated with the (unweighted) 1-Laplace operator. Each chapter covers parts of the authors papers mentioned in the references.

Sydney, 19 April 2024 *Daniel Hauer*

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Contents

Chapter 1
Introduction

Abstract In this introductory chapter, we begin by deriving a second-order diffusion equation, modeling various nonlinear diffusion phenomena where the flux obeys the non-Ohmic power law. If the underlying body conducts electricity, then the Dirichlet-to-Neumann map is the crucial mapping in Calderon's inverse problem, which assigns boundary data induced by voltages on the surface to the current flux through the boundary of the potential. In Section 1.2, we provide a mathematical rigorous derivation of the Dirichlet-to-Neumann map on domains with at least a Lipschitz continuous boundary. This mapping acts non-locally and requires the information of the data on the whole boundary (see Section 1.5). By using the functional analytical notion of j-elliptic functionals, we construct in Section 1.6, a realization of the Dirichlet-to-Neumann map in L^2 for domains, whose topological boundary does not need to be continuous. In Section 1.7, we present well-posedness of the elliptic Poisson problem for the Dirichlet-to-Neumann map and Hölder regularity of weak solutions of this problem. Section 1.8 is concerned with the well-posedness of the homogeneous Cauchy problem governed by the Dirichlet-to-Neumann map and the smoothing effect that mild solutions are strong ones. We show in Section 1.10 that this smoothing effect of mild solutions remains stable if the evolution problem is perturbed by a Lipschitz continuous lower-order term. Furthermore, mild solutions of the homogeneous evolution problem satisfy a global L^q-L^r-regularization estimates for $1 \le q < r \le \infty$, which also provide an immediate smoothing effect for arbitrarily small times $t > 0$ and long-time decay estimates (see Section 1.9). Section 1.11 is dedicated to the limit case $p = 1$; where we derive the Dirichlet-to-Neumann map associated with the 1-Laplace operator and present well-posedness of the Cauchy problem governed by this map. Except of the first two ones, each section introduces the topic and presents the main results. In the subsequent chapters, we introduce the necessary functional analytical framework and give the proofs of these results in full detail.

1.1 Motivation - physical background

Let $\sigma : \overline{\Omega} \to [0, \infty]$ be a positive proper function that models the *electrical conductivity* of a connected body Ω in the Euclidean space $\mathbb{R}^d$ with smooth boundary $\partial\Omega$. If $u_h : \overline{\Omega} \to \mathbb{R}$ is the electric *potential* corresponding to a given *voltage* h on the boundary $\partial\Omega$ of Ω, then by Ohm's law, the negative *current flux* $-\mathbf{J}$ is proportional to the *electric field* $\sigma\nabla u_h$ with respect to the conductivity σ; that is, one has that

$$-\mathbf{J} = \sigma\,\nabla u_h. \tag{1.1}$$

If we assume that there are no sources or sinks of electricity in Ω, then the law of conservation of energy says that for every open smooth sub-region O of Ω, the net flux $\mathbf{J}$ through the boundary ∂O of O is zero: that is,

$$\int_{\partial O} \mathbf{J} \cdot \nu \, \mathrm{d}\mathcal{H}^2 = 0, \tag{1.2}$$

where $\boldsymbol{\nu}$ denotes the outward pointing unit normal vector at the boundary ∂O. By the Gauss-Green theorem, (1.2) is equivalent to

$$\int_O \operatorname{div}(\mathbf{J})\,\mathrm{d}x = 0$$

and since this last equation holds for all open smooth sub-region O of Ω, we conclude that

$$\operatorname{div}(\mathbf{J}) = 0 \qquad \text{in } \Omega. \tag{1.3}$$

Inserting $\mathbf{J}$ given by (1.1) into (1.3) shows that the potential u_h is a solution of the *Dirichlet problem*

$$\begin{cases} -\nabla \cdot \Big(\sigma\,\nabla u_h\Big) = 0 & \text{in } \Omega, \\ \qquad\qquad u_h = h & \text{on } \partial\Omega. \end{cases}$$

It is worth noting that under sufficient regularity assumptions on the conductivity σ and the voltage h, one knows that the solution u_h of this boundary-value problem is unique.

However, there are many materials of the body Ω that are *non-Ohmic* and hence, the current flux $\mathbf{J}$ does not obey the *linear* law given by (1.1). One realistic alternative is to assume that the current flux $\mathbf{J}$ satisfies the following law.

The nonlinear power law (see, e.g., [20])

$$-\mathbf{J} = \sigma(x, \nabla u_h)\nabla u_h \qquad \text{with } \sigma(x, \nabla u_h) = \sigma_0 |\nabla u_h|^{p-2} \tag{1.4}$$

for some exponent $p \in \mathbb{R}$.

Then, the power law (1.4) leads to the *nonlinear diffusion equation*

$$-\nabla \cdot \left(\sigma_0 \, |\nabla u_h|^{p-2} \nabla u_h\right) = 0. \tag{1.5}$$

In electricity, the power law (1.4) is used, for example, to model conductivity properties in certain poly-crystalline materials near the superconducting-normal transition (see, e.g., [44, 71]).

But the quasi-linear second-order partial differential equation (1.5) appears in many other diffusion phenomena including nonlinear dielectrics [44, 74, 75] and plastic moulding [19]. In particular, (1.5) occurs to model electro-rheological [130] and thermorheological fluids [7], non-Newtonian fluids such as plasma or glue, viscous flows in glaciology [78], some plasticity phenomena [21], in image processing [95] and conformal geometry [102]. We refer, for example, to the PhD thesis [38] by Brander for further references.

In the case of normalized conductivity $\sigma_0 \equiv 1$, equation (1.5) reduces to the celebrated *p-Laplace equation*

$$-\Delta_p u := -\nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right) = 0, \tag{1.6}$$

which for $p = 2$ becomes the famous linear *Laplace equation*

$$-\Delta_2 u = -\Delta u := -\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = 0.$$

The equation (1.5) for $1 < p < \infty$ and its solution u_p may be used to model and approximate more complicated non-Ohmic problems; for example, by sending $p \to 1+$, one finds the stationary equation of the *total variation flow* [6, 88]

$$-\nabla \cdot \left(\sigma_0 \frac{\nabla u}{|\nabla u|}\right) = 0,$$

(see Section 1.11 and Chapter 7 for more details). Another limit equation can be derived by first rewriting the equation (1.6) in *non-divergence form*

$$-|\nabla u_p|^{p-4} \left[|\nabla u_p|^2 \Delta u_p + (p-2) \sum_{i,j=1}^{d} \frac{\partial u_p}{\partial x_i} \frac{\partial u_p}{\partial x_j} \frac{\partial^2 u_p}{\partial x_i \partial x_j}\right] = 0$$

and subsequently dividing by $|\nabla u_p|^{p-2}(p-2)$ and then sending $p \to \infty$. In this way, one arrives at the famous ∞*-Laplace equation* [58]

$$-\sum_{i,j=1}^{d} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0.$$

In medical imaging, *Electrical Impedance Tomography* refers to the imaging technique of reconstructing images of a specific region in the human body Ω by applying lower voltages h on the surface $\partial\Omega$ of the body Ω, and to measure the *current flux*

$$\Lambda_{2,\sigma_0} h := \sigma_0 \nabla u_h \cdot \boldsymbol{\nu}$$

at the boundary $\partial\Omega$ of the induced potential u_h. This is possible since biological tissue consists of multiple cells with conductive fluid surrounded by an insulating membrane providing naturally an electrical resistance (see, for instance, the survey paper [107]). The *inverse problem* of determining the conductivity function σ_0 from the measurement of the current flux $\Lambda_{2,\sigma_0} h$ was first formulated by Pedro Alberto Calderón [47] for providing a method for detecting oil wells by electronic measurements. We refer to the wonderful survey paper [147] by Uhlmann for further details. It is natural to study nonlinear generalizations of this inverse problem (cf [133, 37]).

Problem 1.1 (A nonlinear inverse problem) Determine the (conductivity/diffusivity) function σ_0 from the knowledge of the measurements of the flux (in normal direction) $\sigma_0 |\nabla u_h|^{p-2} \nabla u_h \cdot \boldsymbol{\nu}$ a cross the boundary $\partial\Omega$ induced by the boundary values h on $\partial\Omega$. In other words, recover σ_0 from the knowledge of the *Dirichlet-to-Neumann map*

$$\Lambda_{p,\sigma_0} : h \mapsto \sigma_0 |\nabla u_h|^{p-2} \nabla u_h \cdot \boldsymbol{\nu}_{|\partial\Omega}, \tag{1.7}$$

where u_h is a solution of the *Dirichlet problem*

$$\begin{cases} -\nabla \cdot \Big(\sigma_0 |\nabla u_h|^{p-2} \nabla u_h\Big) = 0 & \text{in } \Omega, \\ \qquad\qquad\qquad\quad u_h = h & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

In order to be able to attack this nonlinear inverse problem, it is crucial to know the analytic properties of the Dirichlet-to-Neumann map Λ_{p,σ_0}.

The aim of this thesis is to highlight some of the many properties of the Dirichlet-to-Neumann map Λ_{p,σ_0} associated with the *weighted p-Laplace operator*

$$\Delta_{p,\sigma_0} u := \nabla \cdot \Big(\sigma_0 |\nabla u|^{p-2} \nabla u\Big)$$

and mention some of its applications.

1.2 The Dirichlet-to-Neumann map - an analyst's perspective

Besides the physically motivated inverse problem (Problem 1.1), the Dirichlet-to-Neumann map Λ_{p,σ_0} also appears in the mathematical notion of p-capacity and therefore is also named *interior capacity operator* [64, Chapt. II.5.1] or, *Neumann operator* (see, for example, [141, p. 41]).

Throughout this section, we assume that Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 1$, which in dimension $d \geq 2$ has at least a Lipschitz continuous boundary $\partial\Omega$, and let $1 < p < \infty$. Further, we assume that the conductivity coefficient σ_0 is a *p-admissible* weight function. We refer to the Appendix A.1 for the precise definition and more details to such weights.

Notation 1.1 For every p-admissible weight function σ_0 on $\mathbb{R}^d$, we define the corresponding Radon measure μ by $\mathrm{d}\mu = \sigma_0\,\mathrm{d}x$.

Throughout this preliminary chapter, we make use of the following function spaces and corresponding norms. For $1 \leq p < \infty$, $\Omega \subseteq \mathbb{R}^d$ an open subset, and μ the Radon measure associated with a p-admissible σ_0 on $\mathbb{R}^d$, we denote by $L^p_\mu(\Omega)$ the classical *Lebesgue space* of all *μ-a.e.-equivalent* classes of measurable functions u with finite L^p_μ-norm

$$\|u\|_{p,\mu} = \left[\int_\Omega |u|^p \mathrm{d}\mu\right]^{\frac{1}{p}},$$

and we write $L^p(\Omega)$ equipped with its usual norm $\|\cdot\|_q$ provided the weight function $\sigma \equiv 1$. Further, we define the first *Sobolev space* $H^{1,p}_\mu(\Omega)$ by the completion of the set

$$\left\{u \in C^\infty(\Omega) \,\middle|\, \|u\|_{H^{1,p}_\mu} := \sqrt[p]{\|u\|^p_{p,\mu} + \|\nabla u\|^p_{p,\mu}} < \infty\right\}$$

with respect to $\|\cdot\|_{H^{1,p}_\mu}$, and $H^{1,p}_{\mu,0}(\Omega)$ as the closure of $C^\infty_c(\Omega)$ in $H^{1,p}_\mu(\Omega)$. We write $(H^{1,p}_\mu(\Omega))^*$ and $H^{-1,p'}_\mu(\Omega)$ to denote the corresponding *dual space* of $H^{1,p}_\mu(\Omega)$ and $H^{1,p}_{\mu,0}(\Omega)$.

The Dirichlet-to-Neumann map Λ_{p,σ_0} is constructed in two steps, which we intend to discuss now in more details.

1.2.1 Step 1. The Dirichlet problem

The main task of this step is to establish well-posedness of the elliptic boundary-value problem

$$\begin{cases} -\Delta_{p,\sigma_0} u_h = 0 & \text{in } \Omega, \\ u_h = h & \text{on } \partial\Omega \end{cases} \tag{1.9}$$

governed by the weighted p-Laplace operator Δ_{p,σ_0} for given Dirichlet boundary data h. Concerning the correct notion of solutions u_h of (1.9), it is worth noting that even for smooth σ_0 satisfying $\sigma_0 \geq c > 0$ on $\mathbb{R}^d$ for some $c > 0$, one cannot expect classical solutions u of the weighted p-Laplace equation (1.5) for powers $p \neq 2$. The reason for this comes from the fact that *near* regions $O \subseteq \Omega$ where $\nabla u = 0$, the conductivity coefficient $\sigma_0 |\nabla u|^{p-2}$ can become arbitrarily large if $p < 2$ and vanishes if $p > 2$. In other words, the nonlinear operator Δ_{p,σ_0} is a singular/degenerate second-order differential operator (see, for example, [77] or [97]) and hence, classical elliptic regularity theory (as treated, for example, in [97, 77]) cannot be applied. Therefore, it is natural to relax the notion of solutions, and to employ either the notion of *weak solutions* (see Definition 1.1) or *viscosity/Perron solutions* for the Dirichlet problem (1.9) (cf, [89]).

We note that even for normalized conductivity coefficient $\sigma_0 \equiv 1$, the *weak* solutions u of the p-Laplace equation (1.6) are merely differentiable with Hölder-continuous partial derivatives $\frac{\partial u}{\partial x_i}$ (cf, [144]).

For our purposes, it is sufficient to employ the notion of *weak* solutions u of (1.9) belonging to the Sobolev space $H^{1,p}_{\mu}(\Omega)$. For given $h \in H^{1,p}_{\mu}(\Omega)$, this notion of solutions is one-to-one related to the Euler equation

$$\varphi'(u_h) = 0 \tag{1.10}$$

of the corresponding minimization problem

$$\min\left\{ \varphi(u) \,\middle|\, u \in H^{1,p}_{\mu}(\Omega) \text{ with } u - h \in H^{1,p}_{\mu,0}(\Omega) \right\}, \tag{1.11}$$

where the energy functional $\varphi : H^{1,p}_{\mu}(\Omega) \to [0,\infty)$ is given by

$$\varphi(u) := \tfrac{1}{p} \int_{\Omega} |\nabla u|^p \, \mathrm{d}\mu. \tag{1.12}$$

Note, the functional φ is continuously differentiable and the (Fréchet-)derivative $\varphi' : H^{1,p}_{\mu}(\Omega) \to (H^{1,p}_{\mu}(\Omega))^*$ is given by

$$\langle \varphi'(u), \xi \rangle_{(H^{1,p}_{\mu}(\Omega))^*, H^{1,p}_{\mu}(\Omega)} = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \xi \, \mathrm{d}\mu \tag{1.13}$$

for every $u, \xi \in H^{1,p}_{\mu}(\Omega)$. Now, for a given boundary value $h \in H^{1,p}_{\mu}(\Omega)$, let $\mathcal{K}_h$ be the affine space

$$\mathcal{K}_h := h + H^{1,p}_{\mu,0}(\Omega).$$

Then set $\mathcal{K}_h$ is a closed convex subset of $H^{1,p}_{\mu}(\Omega)$, and due to Poincaré's inequality, the restriction $\varphi_{|\mathcal{K}_h}$ of φ on $\mathcal{K}_h$ is *coercive*; that is,

$$\lim_{\substack{\|u\|_{H_\mu^{1,p}(\Omega)}\to\infty \\ u\in\mathcal{K}_H}} \varphi(u) = \infty.$$

Thus, by the classical theory of convex minimization (see, for instance, [155, Theorem 2.E]), there is a minimizer $u_h \in H_\mu^{1,p}(\Omega)$ to problem (1.11). Moreover, this minimizer u_h is unique since the energy functional φ is strictly convex.

Now, since the functional φ is convex, a function $u_h \in H_\mu^{1,p}(\Omega)$ is a minimizer of (1.11) if and only if $u_h - h \in H_{\mu,0}^{1,p}(\Omega)$ and u_h satisfies $\varphi'(u_h) = 0$ (see [155, Theorem 2.E]), which by (1.13) means that

$$\int_\Omega |\nabla u_h|^{p-2}\nabla u_h \nabla\xi \,\mathrm{d}\mu = 0 \tag{1.14}$$

for every $\xi \in H_{\mu,0}^{1,p}(\Omega)$. From this, the following definition of a *weak* solution to the Dirichlet problem (1.9) is natural.

Definition 1.1 For given boundary value $h \in H_\mu^{1,p}(\Omega)$, we call a function $u_h \in H_\mu^{1,p}(\Omega)$ a *weak* solution of Dirichlet problem (1.9) on Ω if $u_h - h \in H_{\mu,0}^{1,p}(\Omega)$ and u_h satisfies (1.14).

Notation 1.2 For given boundary value $h \in H_\mu^{1,p}(\Omega)$, we set

$$P(h) = u_h, \tag{1.15}$$

where u_h is the unique weak solution of Dirichlet problem (1.9).

The mapping P admits several important properties, which we want to summarize in our next proposition. For normalized σ_0, this proposition has been proved in [85] by the author. We refer to Chapter 2.3 of this thesis for more details.

Proposition 1.1 *The following statements hold.*

1. *P defined by (1.15) is a well-defined, continuous, homogeneous (of order 1), and injective mapping*

$$P : H_\mu^{1,p}(\Omega) \to H_\mu^{1,p}(\Omega).$$

2. *Let $h_1,\ h_2 \in H_\mu^{1,p}(\Omega)$, $H \in H_\mu^{1,p}(\Omega)$ satisfying $h_2 - H \in H_{\mu,0}^{1,p}(\Omega)$ and $\lambda \in \mathbb{R}$. Then,*

$$\frac{1}{p}\int_\Omega |\nabla P(\lambda h_1 + h_2)|^p \,\mathrm{d}\mu \le \frac{1}{p}\int_\Omega |\lambda\nabla P(h_1) + \nabla H|^p \,\mathrm{d}\mu. \tag{1.16}$$

3. *Let $h \in H_\mu^{1,p}(\Omega)$. Then there exists a unique $\xi \in H_{\mu,0}^{1,p}(\Omega)$ such that $P(h) = \xi + h$.*

Since the proof of Proposition 1.1 remains the same as the one of Lemma 2.1 (see Chapter 2.3), we omit it here.

1.2.2 Step 2. The Neumann boundary operator

Now, if the boundary $\partial\Omega$ is smooth enough (for example, $C^{1,\alpha}$, see [101]), the conductivity coefficient σ_0 is smooth and satisfying $\sigma_0 \ge c$ on $\overline{\Omega}$ for some $c > 0$, and if the boundary data $h \in C^{1,\alpha}(\overline{\Omega})$, then the weak solution $P(h)$ of Dirichlet problem (1.9) is of the class $C^{1,\alpha}(\overline{\Omega})$ and hence the *co-normal derivative*

$$[\Lambda_{p,\sigma_0} h](x) := \sigma_0 |\nabla P(h)(x)|^{p-2} \nabla P(h)(x) \cdot \boldsymbol{\nu}(x) \tag{1.17}$$

exists at every $x \in \partial\Omega$.

Now, it is natural that one would like to define the Dirichlet-to-Neumann map Λ_{p,σ_0} for conductivity coefficients σ_0 and boundary data h with less regularity. To see how this can be achieved, we first multiply (1.17) by $P(\xi)$ for some $\xi \in C^{1,\alpha}(\overline{\Omega})$ and subsequently integrate over $\partial\Omega$ with respect to the Hausdorff measure $\mathrm{d}\mathcal{H}^{d-1}$. Then by Green's formula, one obtains that

$$\int_{\partial\Omega} \Lambda_{p,\sigma_0} h\, \xi \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla P(\xi)\,\mathrm{d}\mu. \tag{1.18}$$

Even if $P(h)$ and $P(\xi)$ merely belong to $H^{1,p}_\mu(\Omega)$, the integral on the right-hand side of (1.18) exists. Thus, the mapping

$$\xi \mapsto \int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla P(\xi)\,\mathrm{d}\mu \tag{1.19}$$

would be a suitable candidate to define the Dirichlet-to-Neumann map Λ_{p,σ_0} in a *weak sense* as a mapping from $H^{1,p}_\mu(\Omega)$ into the dual space $(H^{1,p}_\mu(\Omega))^*$. But, in general, P is not linear. Thus, it is *a priori* not clear whether the functional in (1.19) is a bounded linear functional on $H^{1,p}_\mu(\Omega)$. However, according to (3) of Proposition 1.1, for every $\xi \in H^{1,p}_\mu(\Omega)$, there is a unique $\xi_0 \in H^{1,p}_{\mu,0}(\Omega)$ such that $P(\xi) = \xi_0 + \xi$ and so,

$$\begin{aligned}
&\int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla P(\xi)\,\mathrm{d}\mu \\
&\qquad = \int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla \xi_0\,\mathrm{d}\mu + \int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla \xi\,\mathrm{d}\mu \\
&\qquad = \int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla \xi\,\mathrm{d}\mu
\end{aligned}$$

for every $h, \xi \in H^{1,p}_\mu(\Omega)$. This shows that the functional (1.19) is linear. Moreover, by Hölder's inequality, one sees that

$$|\langle \Lambda_{p,\sigma_0} h, \xi \rangle_{(H^{1,p}_\mu(\Omega))^*, H^{1,p}_\mu(\Omega)}| \le \|\nabla P(h)\|^{p-1}_{p,\mu} \|\nabla \xi\|_{p,\mu},$$

from where we can deduce that $\Lambda_{p,\sigma_0} \in (H^{1,p}_\mu(\Omega))^*$. This outline justifies our next definition and shows its consistency with the case of smooth functions.

Definition 1.2 The mapping $\Lambda_{p,\sigma_0} : H^{1,p}_\mu(\Omega) \to (H^{1,p}_\mu(\Omega))^*$ defined by

$$\langle \Lambda_{p,\sigma_0} h, \xi \rangle_{(H^{1,p}_\mu(\Omega))^*, H^{1,p}_\mu(\Omega)} = \int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla \xi \, \mathrm{d}\mu \tag{1.20}$$

for every $h, \xi \in H^{1,p}_\mu(\Omega)$ is called the *Dirichlet-to-Neumann map* from $H^{1,p}_\mu(\Omega)$ to $(H^{1,p}_\mu(\Omega))^*$ associated with the weighted p-Laplace operator Δ_{p,σ_0}.

The preceding definition provides a first rigorous mathematical realization of the Dirichlet-to-Neumann map Λ_{p,σ_0} for weak solutions of Dirichlet problem (1.9). Next, we consider the Dirichlet-to-Neumann map Λ_{p,σ_0} under the following assumption on σ_0.

Assumption 1.1 Assume that the conductivity function σ_0 is bounded from below and above; in other words, there is a constant $c > 0$ such that

$$c \leq \sigma_0(x) \leq c^{-1} \qquad \text{a.e. on } \overline{\Omega}.$$

The Assumption 1.1 has the advantage that one can use the classical Sobolev spaces $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$. Moreover, since by assumption, the boundary $\partial\Omega$ is Lipschitz continuous, there is a trace operator (see, e.g., [115])

$$\mathcal{Tr} : W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega), \tag{1.21}$$

where $W^{1-1/p,p}(\partial\Omega)$, denotes the *fractional* Sobolev-Slobodečki space. Then, one obtains that the Dirichlet-to-Neumann map Λ_{p,σ_0} is, in fact, the *boundary operator* $\Lambda_{p,\sigma_0} : W^{1-1/p,p}(\partial\Omega) \to (W^{1-1/p,p}(\partial\Omega))^*$ given by

$$\langle \Lambda_{p,\sigma_0} h, \xi \rangle_{(W^{1-1/p,p}(\partial\Omega))^*, W^{1-1/p,p}(\partial\Omega)} = \int_\Omega \sigma_0 |\nabla P(h)|^{p-2} \nabla P(h) \nabla \mathcal{Z} \xi \, \mathrm{d}x$$

for every $h, \xi \in W^{1-1/p,p}(\partial\Omega)$ (cf. Definition 2.4 in Chapter 2.4), where $\mathcal{Z}$ denotes the right-inverse of the trace operator $\mathcal{Tr}$.

1.3 The Dirichlet-to-Neumann map on L^2

In view of the nonlinear inverse problem (Problem 1.1), one disadvantage of Definition 1.2 of the Dirichlet-to-Neumann map Λ_{p,σ_0} is that one does not know whether $\Lambda_{p,\sigma_0} h$ is a regular distribution in the sense of L^1_{loc}-functions. Thus, the realization of the mapping Λ_{p,σ_0} as an operator from $H^{1,p}_\mu(\Omega)$ to $(H^{1,p}_\mu(\Omega))^*$ might not provide sufficient regularity.

Under Assumption 1.1, an alternative realization of the *Dirichlet-to-Neumann map* Λ_{p,σ_0} can be obtained by using the functional analytical approach of *subgradient operators* of convex functionals as follows. For every $h \in L^2(\partial\Omega)$, let

$$\varphi(h) := \begin{cases} \frac{1}{p}\int_\Omega \sigma_0|\nabla P(h)|^p \,\mathrm{d}x, & \text{if } h \in W^{1-1/p,p}(\partial\Omega)\cap L^2(\partial\Omega),\\ +\infty, & \text{otherwise.}\end{cases} \tag{1.22}$$

Then, by Proposition 1.1, φ is convex and proper. Moreover, φ is lower semicontinuous and densely defined in $L^2(\partial\Omega)$ (see [85, Lemma 3.13]). Since the restriction $\varphi_{|W^{1-1/p,p}(\partial\Omega)\cap L^2(\partial\Omega)}$ is Gâteaux-differentiable, the subgradient operator $\partial_{L^2}\varphi$ of φ in $L^2(\partial\Omega)$ is a single-valued mapping on $L^2(\partial\Omega)$ and can be characterized by

$$\begin{aligned} D(\partial_{L^2}\varphi) &= \left\{h \in D(\varphi) \,\middle|\, \exists\, g_h \in L^2(\partial\Omega) \text{ s.t. } g_h = \Lambda_{p,\sigma_0}h\right\}\\ \partial_{L^2}\varphi(h) &= g_h, \end{aligned} \tag{1.23}$$

where $D(\varphi) = W^{1-1/p,p}(\partial\Omega)\cap L^2(\partial\Omega)$ and $g_h = \Lambda_{p,\sigma_0}h$ means that

$$\int_{\partial\Omega} g_h\,\xi\,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega \sigma_0|\nabla P(h)|^{p-2}\nabla P(h)\nabla P(\xi)\,\mathrm{d}x \tag{1.24}$$

for every $\xi \in W^{1-1/p,p}(\partial\Omega)\cap L^2(\partial\Omega)$. We provide more details to this in Chapter 3.3.

We emphasize that the integral condition (1.24) is quite close to (1.18) from where we started. Thus, the subgradient operator $\partial_{L^2}\varphi$ is a good realization of the Dirichlet-to-Neumann map Λ_{p,σ_0} in $L^2(\partial\Omega)$. Moreover, due to (3) of Proposition 1.1, one has that

$$\int_{\partial\Omega} \partial_{L^2}\varphi(h)\,v\,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega \sigma_0|\nabla P(h)|^{p-2}\nabla P(h)\nabla \mathcal{Z}v\,\mathrm{d}x$$

for every $v \in W^{1-1/p,p}(\partial\Omega)\cap L^2(\partial\Omega)$ and $h \in D(\partial_{L^2}\varphi)$.

It is worth noting that the realization (1.23) of the Dirichlet-to-Neumann map Λ_{p,σ_0} on $L^2(\partial\Omega)$ has been first used in [68] (and see also [93]) and later in [85].

The above realization of the Dirichlet-to-Neumann map Λ_{p,σ_0} on $L^2(\partial\Omega)$ is based on the existence of the trace operator $\mathcal{T}r$ on $W^{1,p}(\Omega)$ and hence, on the assumption that the domain Ω has at least a Lipschitz continuous boundary $\partial\Omega$. In [51], this assumption could be relaxed by introducing the notion of a *weak trace* and the functional analytical notion of *j-elliptic functionals*. We give more details to this in Section 1.6.

1.4 The Dirichlet-to-Neumann map and Leray-Lions operators

Instead of working with the specific flux $\sigma_0|\nabla u|^{p-2}\nabla u$ involving a given p-admissible conductivity function σ_0, one can consider more general flux functions $\boldsymbol{\sigma}(x,\nabla u)$.

Assumption 1.2 Assume that the flux function $\boldsymbol{\sigma}(x,\nabla u)$ is a vector-valued Carathéodory function $\boldsymbol{\sigma} : \overline{\Omega}\times\mathbb{R}^d \to \mathbb{R}^d$; that is, for every $\xi \in \mathbb{R}^d$, $\boldsymbol{\sigma}(\cdot,\xi) : \overline{\Omega} \to \mathbb{R}^d$ is measurable, and for a.e. $x \in \Omega$, $\boldsymbol{\sigma}(x,\cdot) : R^d \to \mathbb{R}^d$ is continuous. Moreover, suppose

(Hi) $\boldsymbol{\sigma}$ satisfies the *growth condition*

$$|\boldsymbol{\sigma}(x,\xi)| \le \sigma_0(x)\,|\xi|^{p-1} + g(x) \qquad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^d,$$

for some p-admissible weight function σ_0 with corresponding Radon measure μ and $g \in L^{p'}(\Omega)$,

(Hii) $\boldsymbol{\sigma}$ is *(strict) monotone*; that is, for a.e. $x \in \Omega$, one has that

$$\big(\boldsymbol{\sigma}(x,\xi_1) - \boldsymbol{\sigma}(x,\xi_2)\big)\cdot(\xi_1-\xi_2) > 0$$

for every $\xi_1, \xi_2 \in \mathbb{R}^d$ satisfying $\xi_1 \neq \xi_2$, and $\boldsymbol{\sigma}(x,0) = 0$,

(Hiii) $\boldsymbol{\sigma}$ is *(p)-coercive*, meaning that there is an $\eta > 0$ such that

$$\boldsymbol{\sigma}(x,\xi)\xi \ge \eta\,\sigma_0(x)\,|\xi|^p$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^d$.

Under the Assumption 1.2, the classical theory of *monotone operators* (see, for instance, [103, Théorème 2.1, Chap. 2]) applies, yielding that for every $h \in H^{1,p}_{\mu}(\Omega)$, the Dirichlet problem

$$\begin{cases} -\operatorname{div}\big(\boldsymbol{\sigma}(x,\nabla u_h)\big) = 0 & \text{in } \Omega,\\ u_h = h & \text{on } \partial\Omega \end{cases}$$

admits a unique weak solution $u_h \in H^{1,p}_{\mu}(\Omega)$ in the sense of Definition 1.1, but where the integral equation (1.14) is replaced by

$$\int_{\Omega} \boldsymbol{\sigma}(x,\nabla u_h)\nabla\xi\,\mathrm{d}x = 0$$

for every $\xi \in H^{1,p}_{\mu,0}(\Omega)$.

Remark 1.1 Second-order differential operators $\mathcal{A}$ of the form

$$\mathcal{A}(u) := \operatorname{div}\big(\boldsymbol{\sigma}(x,\nabla u_h)\big), \tag{1.25}$$

where $\boldsymbol{\sigma}$ is a given Carathéodory vector-field satisfying the Assumption 1.2, were first introduced in [100] by Leray and Lions. Hence, such operators $\mathcal{A}$ are known in the literature under the name *Leray-Lions operators*.

The Dirichlet-to-Neumann map Λ_{σ} associated with the Leray-Lions operator $\mathcal{A}$ is defined as follows.

Definition 1.3 The mapping $\Lambda_{\sigma} : H^{1,p}_{\mu}(\Omega) \to (H^{1,p}_{\mu}(\Omega))^*$ defined by

$$\langle \Lambda_{\sigma} h, \xi \rangle_{(H^{1,p}_{\mu}(\Omega))^*, H^{1,p}_{\mu}(\Omega)} = \int_{\Omega} \boldsymbol{\sigma}(x, \nabla P(h)) \nabla \xi \, \mathrm{d}\mu \tag{1.26}$$

for every $h, \xi \in H^{1,p}_{\mu}(\Omega)$ is called the *Dirichlet-to-Neumann map* from $H^{1,p}_{\mu}(\Omega)$ to $(H^{1,p}_{\mu}(\Omega))^*$ associated with the Leray-Lions operator $\mathcal{A}$ given by (1.25).

Under both assumptions, Assumption 1.1 and Assumption 1.2, the Dirichlet-to-Neumann map Λ_{σ} associated with $\mathcal{A}$ is respectively the *boundary operator*

$$\Lambda_{\sigma} : W^{1-1/p,p}(\partial\Omega) \to (W^{1-1/p,p}(\partial\Omega))^*$$

given by

$$\langle \Lambda_{\sigma} h, \xi \rangle_{(W^{1-1/p,p}(\partial\Omega))^*, W^{1-1/p,p}(\partial\Omega)} = \int_{\Omega} \boldsymbol{\sigma}(x, \nabla P(h)) \nabla \mathcal{Z}\xi \, \mathrm{d}x$$

for every $h, \xi \in W^{1-1/p,p}(\partial\Omega)$. To the best of our knowledge, the boundary operator Λ_{σ} has been first introduced in [103, Chapter 4.2] as an application of the well-posedness theory of the nonlinear evolution problem

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) = f(t) & \text{on } (0,T), \\ u(0) = u_0 \end{cases}$$

governed by monotone operators $A : V \to V^*$, where V is a reflexive Banach space. The Dirichlet-to-Neumann map Λ_{σ} associated with the Leray-Lions operator $\mathcal{A}$ has been revisited in [2, 3] by Ammar, and later in [4] and [85]. We provide more details to these papers in Chapter 2.4.

1.5 The Dirichlet-to-Neumann map is a nonlocal operator

The Dirichlet-to-Neumann map Λ_{p,σ_0} is a *nonlocal* operator in the sense that for a given boundary function h, the values $[\Lambda_{p,\sigma_0} h](x)$ for x ranging in a relatively open neighborhood Γ_{x_0} of a given boundary point $x_0 \in \partial\Omega$ do not only depend on the values $h(x)$ for $x \in \Gamma_{x_0}$, but on the values of h along the whole boundary $\partial\Omega$. To be more precise, suppose the boundary $\partial\Omega$ is of class $C^{2,\beta}$ for a $\beta \in (0,1)$, the conductivity coefficient σ_0 is smooth and satisfies Assumption 1.1 and let $\partial\Omega = \Gamma_1 \dot{\cup} \Gamma_2$ with Γ_1 relatively open and nonempty. Further, suppose the boundary data $h \in C^{1,\alpha}(\partial\Omega)$ satisfies $h > 0$ on Γ_2 and $h = 0$ on Γ_1. By the weak maximum principle, $P(h)$ is positive in $\overline{\Omega}$ and attains its minimum on the boundary $\partial\Omega$. But since the boundary $\partial\Omega \in C^{2,\beta}$, at every boundary point $x \in \partial\Omega$ the inner ball condition holds. Thus Hopf's boundary-point lemma [148] yields that

$$[\Lambda_{p,\sigma_0} h](x) = \sigma_0 |\nabla P(h)(x)|^{p-2} \nabla P(h) \cdot \boldsymbol{\nu}(x) > 0$$

at every $x \in \Gamma_1$, which is in contrast to the condition $h = 0$ on Γ_1 satisfied by the boundary data h.

1.6 The Dirichlet-to-Neumann map on open sets

Under the assumption that Ω is a bounded domain in $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$ and σ_0 satisfies Assumption 1.1, we could realize the Dirichlet-to-Neumann map Λ_{p,σ_0} on $L^2(\partial\Omega)$ as the subgradient $\partial_{L^2}\varphi$ of the functional φ defined by (1.22) (see (1.23)–(1.24) in Section 1.2.2, and also Theorem 3.6 in Section 3.3). This construction requires the existence of a trace operator $\mathcal{Tr}$ on $W^{1,p}(\Omega)$ as given in (1.21). However, if Ω has a boundary $\partial\Omega$ with much weaker regularity, then a trace operator $\mathcal{Tr}$ on $W^{1,p}(\Omega)$ might not exist and the Dirichlet problem (1.9) is, in general, ill-posed (see [16] for $p = 2$). Hence, the subgradient $\partial_{L^2}\varphi$ given by (1.22)–(1.24) of the Dirichlet-to-Neumann map Λ_{p,σ_0} would not be available anymore.

An alternative approach to the realization of the Dirichlet-to-Neumann map Λ_{p,σ_0} on $L^2(\partial\Omega)$ for *general* open subsets Ω of $\mathbb{R}^d$ with finite Lebesgue measure was established in [51] by Chill, Kennedy and the author using the nonlinear functional analytical framework of *j-elliptic functionals*. In order to outline this construction, we briefly recall some basic definitions and results from [51].

1.6.1 j-elliptic functionals and their j-subgradient

Let V be a real locally convex topological vector space, H a real Hilbert space, and $j : V \to H$ be a linear operator which is merely weak-to-weak continuous (and, in general, not injective). In this setting, we introduce our first definition.

Definition 1.4 A functional $\varphi : V \to (-\infty, +\infty]$ is called *j-elliptic* if there exists $\omega \geq 0$ such that the "shifted" function $\varphi_\omega : V \to (-\infty, +\infty]$ given by

$$\varphi(\hat{u}) + \frac{\omega}{2} \|j(\hat{u})\|_H^2 \qquad \text{for every } \hat{u} \in V, \tag{1.27}$$

is convex and for every $c \in \mathbb{R}$, the sublevel sets $\{\hat{u} \in V \,|\, \varphi_\omega(u) \leq c\}$ are relatively weakly compact.

Further, we introduce the following notion of operators.

Definition 1.5 Given a functional $\varphi : V \to (-\infty, +\infty]$, then the *$j$-subgradient* is the operator

$$\partial_j \varphi := \left\{ (u, f) \in H \times H \;\middle|\; \begin{array}{l} \exists \hat{u} \in D(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{v} \in V, \\ \liminf\limits_{t \searrow 0} \dfrac{\varphi(\hat{u} + t\hat{v}) - \varphi(\hat{u})}{t} \geq (f, j(\hat{v}))_H \end{array} \right\}.$$

Next, we require the following type of convexity.

Definition 1.6 A functional $\varphi : V \to (-\infty, +\infty]$ is called *j-semiconvex* if there exists $\omega \in \mathbb{R}$ such that the shifted functional φ_ω given by (1.27) is convex.

If $V = H$ and $j = I_H$, then j-semiconvex functionals φ are the *semiconvex* ones (see Definition 3.6 in Chapter 3.2.2).

Finally, in order to complete this variational construction, we require the following type of closeness of the sublevel sets of the functional φ.

Definition 1.7 We call a functional $\varphi : V \to (-\infty, +\infty]$ *lower semicontinuous* if the sublevel sets $\{\varphi \leq c\}$ are closed in the topology of V for every $c \in \mathbb{R}$.

The main result in [51] is that the j-subgradient $\partial_j \varphi$ of a j-elliptic functional φ is already a classical subgradient. More precisely, the following holds.

Theorem 1.1 *Let $\varphi : V \to (-\infty, +\infty]$ be proper, lower semicontinuous, and j-elliptic. Then there is a proper, lower semicontinuous, semiconvex function $\varphi^H : H \to (-\infty, +\infty]$ such that $\partial_j \varphi = \partial \varphi^H$. The functional φ^H is unique up to an additive constant.*

We prove this theorem in Corollary 6.2 in Chapter 6.2.1. The next result provides a description of φ^H in the convex case and will be important for our intentions in this section.

Theorem 1.2 *Assume that $\varphi : V \to (-\infty, +\infty]$ is convex, proper, lower semicontinuous and j-elliptic, and let $\varphi^H : H \to (-\infty, +\infty]$ be the functional from Theorem 1.1. Then, there is a constant $c \in \mathbb{R}$ such that*

$$\varphi^H(u) = c + \inf_{\hat{u} \in j^{-1}(\{u\})} \varphi(\hat{u}) \qquad \textit{for every } u \in H$$

with effective domain $D(\varphi^H) = j(D(\varphi))$.

The statement of this theorem is proved in Theorem 6.5 in Chapter 6.2.1.

1.6.2 The construction of a weak trace on open sets

Here, we outline how to construct a trace in a *weak sense* on the boundary $\partial\Omega$ on general open sets. This theory is made possible by the following inequality (see [108] and [109, Cor. 2, Sec. 4.11.1, p.258]), which states that if Ω has finite Lebesgue measure, and if the parameters $1 \le p,\, q,\, r < \infty$ satisfy

$$(d-p)r \le p(d-1) \;\text{ and }\; q \le rd/(d-1), \tag{1.28}$$

then there is a constant $C = C(d,p,q,r,|\Omega|) > 0$ such that

$$\|u\|_{L^q(\Omega)} \le C\left(\|\nabla u\|_{L^p(\Omega)^d} + \|u_{|\partial\Omega}\|_{L^r(\partial\Omega)}\right) \tag{1.29}$$

for all $u \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$. Here $C_c(\overline{\Omega})$ is the set of all functions $u \in C(\overline{\Omega})$ with compact support in $\overline{\Omega}$, and $W^{1,p}(\Omega)$ is the classical Sobolev space.

We shall refer to inequality (1.29) as *Maz'ya's inequality*. This inequality motivates the introduction of the following Sobolev-type spaces.

Definition 1.8 For $1 \le p,\, q \le \infty$ let $W^1_{p,q}(\Omega)$ be the Banach space of all $u \in L^q(\Omega)$ having all distributional partial derivatives $\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d} \in L^p(\Omega)$. We equip $W^1_{p,q}(\Omega)$ with the natural norm $\|u\|_{W^1_{p,q}} := \|u\|_{L^q(\Omega)} + \|\nabla u\|_{L^p(\Omega)^d}$.

Further, we define the space $V_{p,r}(\Omega)$ to be the abstract completion of

$$V_0 := \left\{ u \in W^{1,p}(\Omega) \cap C_c(\overline{\Omega}) \,\middle|\, \|u\|_{V_{p,r}} < \infty \right\} \tag{1.30}$$

with respect to the norm

$$\|u\|_{V_{p,r}} := \|\nabla u\|_{L^p(\Omega)^d} + \|u_{|\partial\Omega}\|_{L^r(\partial\Omega)},$$

where $L^r(\partial\Omega) := L^r(\partial\Omega, \mathcal{H}^{d-1})$, and $\mathcal{H} = \mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure on the boundary $\partial\Omega$.

Remark 1.2 Note that in [109], the function space $V_{p,r}(\Omega)$ is denoted by $W^1_{p,r}(\Omega, \partial\Omega)$.

Maz'ya's inequality (1.29) says that if $1 \le p,\, q,\, r < \infty$ satisfy (1.28), then the natural embedding

$$j_0 : V_0 \to W^1_{p,q}(\Omega), \quad u \mapsto u \tag{1.31}$$

is well defined and bounded. Moreover, by definition of V_0, the operator

$$\iota_0 : V_0 \to W^1_{p,q}(\Omega) \times L^r(\partial\Omega), \quad u \mapsto (u, u_{|\partial\Omega}),$$

is well defined and bounded, too, and it is an isomorphism from V_0 onto its image. The operator ι_0 then has a unique extension to a bounded linear operator

$$\iota : V_{p,r}(\Omega) \to W^1_{p,q}(\Omega) \times L^r(\partial\Omega)$$

which is again an isomorphism from $V_{p,r}(\Omega)$ onto its image. This means we may identify $V_{p,r}(\Omega)$ with a closed linear subspace of $W^1_{p,q}(\Omega)\times L^r(\partial\Omega)$. Let $p_1 : W^1_{p,q}(\Omega)\times L^r(\partial\Omega) \to W^1_{p,q}(\Omega)$ and $p_2 : W^1_{p,q}(\Omega)\times L^r(\partial\Omega) \to L^r(\partial\Omega)$ be the canonical coordinate projections. We then define the bounded linear operators

$$j := p_1 \circ \iota : V_{p,r}(\Omega) \to W^1_{p,q}(\Omega), \tag{1.32}$$

and

$$\mathcal{T}r := p_2 \circ \iota : V_{p,r}(\Omega) \to L^r(\partial\Omega). \tag{1.33}$$

For example, j may be regarded as the embedding of $V_{p,r}(\Omega)$ into $W^1_{p,q}(\Omega)$ induced by Maz'ya's inequality. Or, in other words, j is the bounded linear extension of the natural embedding j_0 from (1.31).

Notation 1.3 In an abuse of notation, we will also use j to denote the map $V_{p,r}(\Omega) \to L^q(\Omega)$ given by $i \circ p_1 \circ \iota$, where $i : W^1_{p,q}(\Omega) \to L^q(\Omega)$ is the natural embedding, if there is no danger of confusion.

Definition 1.9 The operator $\mathcal{T}r$ is a natural extension of the trace operator $u \mapsto u|_{\partial\Omega}$ defined on V_0, and we therefore call $\mathcal{T}r(u)$ the *weak trace* of an element $u \in V_{p,r}(\Omega)$.

Notation 1.4 We write $\mathrm{Rg}(\mathcal{T}r)$ for the set of all $h \in L^r(\partial\Omega)$ such that there exists $u \in V_{p,r}(\Omega)$ satisfying $\mathcal{T}r(u) = h$. The set $\mathrm{Rg}(\mathcal{T}r)$ is called the *range* of the weak trace operator $\mathcal{T}r$.

Remark 1.3 There is a potential complication with the map j which Maz'ya did not explore in [108] or [109], but which has subsequently received a certain amount of attention: j is not necessarily injective. Since an element u belongs to $\ker j$ if and only if there is a sequence (u_n) in $W^{1,p}(\Omega)\cap C_c(\overline{\Omega})$ such that

$$\begin{gathered}\lim_{n\to\infty} \nabla u_n = 0 \text{ in } L^p(\Omega)^d, \qquad \lim_{n\to\infty} u_n = 0 \text{ in } L^q(\Omega),\\ \text{and } \lim_{n\to\infty} u_{n|\partial\Omega} = h \text{ in } L^r(\partial\Omega)\end{gathered} \tag{1.34}$$

for some $h \in L^r(\partial\Omega)$, the map j being injective is equivalent to $h = 0$ whenever (1.36) holds. This is certainly true if, for example, Ω is a bounded Lipschitz domain, since in that case we have a trace inequality (see, for instance, [115]); but of course such an inequality does not hold on arbitrary open sets. This important point was first raised by Daners in [61]; soon afterwards an example of an Ω for which j is not injective was constructed by Warma [152]. This issue reemerged some time later when Arendt and ter Elst [12, 13] introduced a generalization of the notion of trace valid on an arbitrary open set, based in large part on Maz'ya's inequality.

There is another possible definition of trace, which is a further generalization (to p, q, $r \neq 2$) of the generalization of trace in [12]. In particular, the following definition agrees with [12, Section 1] when $p = q = r = 2$.

Definition 1.10 For $1 \le p,\ q,\ r \le \infty$, following [12], we say that $h \in L^r(\partial\Omega)$ is a *weak trace* of $u \in W^1_{p,q}(\Omega)$ if there is a sequence $(u_n)_{n\ge 1}$ in $W^1_{p,q}(\Omega) \cap C_c(\overline{\Omega})$ such that $u_n \to u$ in $W^1_{p,q}(\Omega)$ and $u_{n|\partial\Omega} \to h$ in $L^r(\partial\Omega)$.

In other words, $\varphi \in L^r(\partial\Omega)$ is a weak trace of $u \in W^1_{p,q}(\Omega)$ if and only if the pair $(u, \varphi) \in \iota(V_{p,r}(\Omega))$.

Remark 1.4 (a) It is known that there are domains on which functions may have multiple weak traces in the sense of Definition 1.10; this is immediately seen to be the case exactly when the map j is not injective, which in particular is a property of the *domain* Ω and not the function(s) in question. This can happen if $\partial\Omega$ becomes too "disconnected" from Ω in a sense which can be made precise using the notion of relative capacity; we refer to [12] for more details in the case $p = q = r = 2$. Of course, functions in $V_{p,r}(\Omega)$ always have unique traces in the sense of (1.33), since the map $\mathcal{T}r$ is well defined.

(b) Weak traces in the sense of Definition 1.10 depend intrinsically on all three parameters $p,\ q,\ r$. We expect it is possible that a given function in $W^1_{p,q}(\Omega)$ may have multiple traces for some r and only one (or even none) for other r, although we do not explore this here.

(c) If $p, r \ge 1$ satisfy the first inequality in (1.28), then one can always find a $q \ge 1$ such that $V_{p,r}(\Omega)$ maps into $W^1_{p,q}(\Omega)$ (take $q = rd/(d-1)$) so that the maps j and $\mathcal{T}r$ from (1.32) and (1.33), respectively, are well defined. Moreover, if $u \in C_c^\infty(\mathbb{R}^d)$, then, approximating u by itself wherever necessary, we may identify u canonically with an element of $V_{p,r}(\Omega)$ and $W^1_{p,q}(\Omega)$, and $u_{|\partial\Omega}$ is both a trace and a weak trace of u.

(d) The definition (1.33) of the trace of a function in $V_{p,r}(\Omega)$ can be easily extended to any pair $p,\ r \ge 1$, even if they do not satisfy the first inequality in (1.28), since one can always identify $V_{p,r}(\Omega)$ canonically with a closed subset of $L^p(\Omega)^d \times L^r(\partial\Omega)$; the trace is simply the composition of this embedding and the projection onto $L^r(\partial\Omega)$. In the sequel, however, we will always assume that (1.28) holds, and so we will tend not to distinguish between the various possible notions of trace.

With these preliminaries, we can now begin to construct the Dirichlet-to-Neumann map.

1.6.3 Construction of the Dirichlet-to-Neumann map

We impose the following regularity on the boundary $\partial\Omega$ of the open set Ω in $\mathbb{R}^d$.

Assumption 1.3 Assume that $\Omega \subseteq \mathbb{R}^d$ is an open set of finite Lebesgue measure $|\Omega|$ for which the topological boundary $\partial\Omega$ has locally finite $(d-1)$-dimensional Hausdorff measure, that is, $\mathcal{H}^{d-1}(K)$ is finite for every compact $K \subseteq \partial\Omega$.

Further, for $1 \le p, r < \infty$, we consider the space $V_{p,r}(\Omega)$ and it will be convenient to write $\hat{u}$ for elements in $V_{p,r}(\Omega)$, $u = \mathcal{T}r(\hat{u})$ for their traces, and $j(\hat{u})$ for their embeddings into $L^q(\Omega)$ given by Maz'ya's inequality (1.29) for p, q satisfying (1.28). However, by definition, elements $\hat{u} \in V_{p,r}(\Omega)$ still admit both a natural gradient $\nabla\hat{u} \in L^p(\Omega)$ and a trace $u = \mathcal{T}r(\hat{u}) \in L^r(\partial\Omega)$ since $V_{p,r}(\Omega)$ may be identified with a closed subset of $L^p(\Omega)^d \times L^r(\partial\Omega)$ in a natural way; see Remark 1.4 (d).

Now, we can introduce the notion of *weak* solutions of Dirichlet problem (1.9) which is appropriate for this general framework.

Definition 1.11 For given $h \in L^r(\partial\Omega)$, we call a function $\hat{u} \in V_{p,r}(\Omega)$ a *weak* solution of Dirichlet problem (1.9) if $\mathcal{T}r\hat{u} = h$ $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$, and

$$\int_\Omega \sigma_0 |\nabla\hat{u}|^{p-2} \nabla\hat{u} \nabla\hat{v} \, \mathrm{d}x = 0$$

for every $\hat{v} \in \ker(\mathcal{T}r)$.

Since $V_{p,r}(\Omega)$ can be identified as a closed subset of $L^p(\Omega)^d \times L^r(\partial\Omega)$, one knows that $V_{p,r}(\Omega)$ is reflexive for $1 < p, r < \infty$. Thus, for given $h \in L^r(\partial\Omega)$ and $H \in V_{p,r}(\Omega)$ satisfying $\mathcal{T}r(H) = h$ $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$, classical theory of convex minimization yields that the variational problem

$$\min \left\{ \varphi(\hat{u}) \,\middle|\, \hat{u} \in V_{p,r}(\Omega), \text{ with } \hat{u} - H \in \ker(\mathcal{T}r) \right\}$$

admits a solution $\hat{u}_h$ for the functional φ defined by

$$\varphi(\hat{u}) = \tfrac{1}{p} \int_\Omega \sigma_0 |\nabla\hat{u}|^p \, \mathrm{d}x \tag{1.35}$$

for every $\hat{u} \in V_{p,2}(\Omega)$. This solution $\hat{u}_h$ satisfies the Euler equation (1.10), which again is equivalent to $\hat{u}_h$ being a weak solution of Dirichlet problem (1.9) in the sense of Definition 1.11. It is worth noting that even though a function $\hat{u} \in V_{p,r}(\Omega)$ can have multiple *weak traces* (cf. Remark 1.4), a weak solution $\hat{u}$ of the Dirichlet problem (1.9) with boundary value $h \in \mathrm{Rg}(\mathcal{T}r)$ is unique.

Remark 1.5 Note, the well-posedness of the Dirichlet problem (1.9) obtained in this general framework is not in contradiction to the counter-example constructed in [16] since they look for solutions u of Dirichlet problem (1.9) (in the case $p = 2$), which are minimizers of (1.35) over the affine space $H + W^{1,2}_0(\Omega)$ for a given $H \in W^{1,2}(\Omega)$. But, in fact, there are open sets Ω satisfying Assumption 1.3 such that for the weak trace operator $\mathcal{T}r$, the space $W^{1,2}_0(\Omega) \subsetneq \ker(\mathcal{T}r)$.

By using the notion of the *weak trace*, we can now define the Dirichlet-to-Neumann map Λ_{σ_0} in $L^2(\partial\Omega)$ as follows.

Definition 1.12 We define the Dirichlet-to-Neumann map Λ_{p,σ_0} in $L^2(\partial\Omega)$ by setting

$$\Lambda_{p,\sigma_0} = \Big\{(h,g) \in L^2(\partial\Omega)\times L^2(\partial\Omega)\Big|\, \exists\, \hat{u} \in V_{p,2}(\Omega) \text{ s.t. } \mathcal{T}r(\hat{u}) = u \text{ and}$$
$$\int_\Omega \sigma_0 |\nabla\hat{u}|^{p-2}\nabla\hat{u}\nabla\hat{v}\,\mathrm{d}x = \int_{\partial\Omega} g\mathcal{T}r(\hat{v})\,\mathrm{d}\mathcal{H}^{d-1} \quad \forall\, \hat{v}\in V_{p,2}(\Omega)\Big\}$$

The next result states that the Dirichlet-to-Neumann map Λ_{p,σ_0} in $L^2(\partial\Omega)$ as given in Definition 1.12 can be realized as the $\mathcal{T}r$-subgradient of an appropriate functional φ.

Theorem 1.3 *The functional $\varphi : V_{p,2}(\Omega) \to \mathbb{R}$ defined by (1.35) is convex, continuously differentiable, and $\mathcal{T}r$-elliptic. Moreover, the $\mathcal{T}r$-subgradient $\partial_{\mathcal{T}r}\varphi$ of φ is densely defined and coincides with the Dirichlet-to-Neumann map Λ_{p,σ_0}.*

Remark 1.6 If Ω has Lipschitz boundary and $g = 0$, then our construction coincides with the variational definition of the Dirichlet-to-Neumann map Λ_{p,σ_0} as introduced in Section 1.3 (see also Section 3.3 or [85], or, for example,[12, 13] in the linear case $p = 2$). In this case, the trace inequality together with Maz'ya's inequality (1.29) (from Section 1.6) imply that $V_{p,2}(\Omega)$ coincides with the Sobolev space $W^1_{p,2}(\Omega)$, up to an equivalent norm. Moreover, $\ker(\mathcal{T}r)$ coincides exactly with $W^{1,p}_0(\Omega)$, the closure of $C^\infty_c(\Omega)$ in the $W^{1,p}$-norm.

We outline the proof of Theorem 1.3 in Chapter 6.

1.7 The Poisson problem and the Neumann-to-Dirichlet map

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, which in dimension $d \geq 2$ has at least a Lipschitz continuous boundary $\partial\Omega$, and Λ_{p,σ_0} the Dirichlet-to-Neumann map associated with the weighted p-Laplace operator Δ_{p,σ_0} for $1 < p < \infty$ and conductivity σ_0 satisfying Assumption 1.1.

Ammar showed in [2, 3] (see also [4]) that the *Poisson problem*

$$\Lambda_{p,\sigma_0} h = f \qquad \text{on } \partial\Omega \tag{1.36}$$

is well-posedness in the sense of *entropy solutions* for every $f \in L^1(\partial\Omega)$, and provided h and f have *zero mean* on the boundary $\partial\Omega$; that is,

$$\int_{\partial\Omega} h\,\mathrm{d}\mathcal{H}^{d-1} = 0 \quad \text{and} \quad \int_{\partial\Omega} f\,\mathrm{d}\mathcal{H}^{d-1} = 0.$$

More than 10 years later, it was shown in [85] by the author that well-posedness of problem (1.36) also holds in the sense of *weak solutions* for given $f \in W_m^{-(1-1/p),p'}(\partial\Omega)$ defined as follows.

Here, $W_m^{1-1/p,p}(\partial\Omega)$ denotes the subspace of all functions $h \in W^{1-1/p,p}(\partial\Omega)$ satisfying $\int_{\partial\Omega} h\,\mathrm{d}\mathcal{H}^{d-1} = 0$, and we write $W_m^{-(1-1/p),p'}(\partial\Omega)$ for its dual space.

Definition 1.13 For given $f \in W_m^{-(1-1/p),p'}(\partial\Omega)$, we call a boundary function h a *weak solution* of the elliptic equation (1.36) if $h \in W^{1-1/p,p}(\partial\Omega)$ and satisfies

$$\int_\Omega \sigma_0 |\nabla P(h)|^{p-2} \nabla \mathcal{Z}(\xi)\,\mathrm{d}x = \langle f, \xi \rangle_{W^{-(1-1/p),p'}(\partial\Omega), W^{1-1/p,p}(\partial\Omega)}$$

for every $\xi \in W^{1-1/p,p}(\partial\Omega)$.

The following characterization of weak solutions h of Poisson problem (1.36) is worth noting. Since the proof is obvious, we omit it.

Proposition 1.2 *For given $f \in W_m^{-(1-1/p),p'}(\partial\Omega)$, a function $h \in W_m^{1-1/p,p}(\partial\Omega)$ is a* weak solution *of the Poisson problem* (1.36) *if and only if $P(h) \in W^{1,p}(\Omega)$ and $P(h)$ is a weak solution of the inhomogeneous* Neumann problem

$$\begin{cases} -\Delta_{p,\sigma_0} P(h) = 0 & \text{in } \Omega, \\ \sigma_0 |\nabla P(h)|^{p-2} \nabla P(h) \cdot \nu = f & \text{on } \partial\Omega. \end{cases}$$

On the other hand, for given $h \in W_m^{1-1/p,p}(\partial\Omega)$ satisfying the Poisson problem (1.36), the p*-harmonic extension* $P(h)$ in Proposition 1.2 is the unique solution of the Dirichlet problem (1.9). Thus, the resolution map $\mathcal{R} : W_m^{-(1-1/p),p'}(\partial\Omega) \to W_m^{1-1/p,p}(\partial\Omega)$ assigning $f \mapsto \mathcal{R}(f) := h$, where h is the unique solution of the , is nothing less than the *Neumann-to-Dirichlet map* $\Lambda_{p,\sigma_0}^{-1}$.

Further, it turns out that weak solutions h of the Poisson problem (1.36) are Hölder continuous provided f belongs to $L^q(\partial\Omega)$ for $q \geq 1$ large enough and has zero mean $\int_{\partial\Omega} f\,\mathrm{d}\mathcal{H}^{d-1} = 0$. We summarize these results in the next theorem.

Theorem 1.4 *Assume that the conductivity coefficient σ_0 satisfies Assumption* 1.1*. Then the following assertions hold.*

1. *For every $f \in W_m^{-(1-1/p),p'}(\partial\Omega)$, there is a unique weak solution*

$$h \in W_m^{1-1/p,p}(\partial\Omega) \qquad \text{of Poisson problem (1.36).}$$

   Moreover, the Neumann-to-Dirichlet map $\Lambda_{p,\sigma_0}^{-1}$ is continuous a mapping from $W_m^{-(1-1/p),p'}(\partial\Omega)$ to $W_m^{1-1/p,p}(\partial\Omega)$.
2. *Let $q = \frac{d-1}{p-1-\varepsilon}$ for some $\varepsilon \in (0,1)$ if $p \leq d$ and $q = 1$ if $p > d$. Further, let $f \in L^q(\partial\Omega)$ have zero mean $\int_{\partial\Omega} f\,\mathrm{d}\mathcal{H}^{d-1} = 0$. Then there are $\alpha \in (0,1)$ and $c_\alpha \geq 0$ such that every weak solution $h \in W^{1-1/p,p}(\partial\Omega)$ of* (1.36) *belongs to $C^{0,\alpha}(\partial\Omega)$ and satisfies*

$$\|h\|_{C^{0,\alpha}(\partial\Omega)} \leq c_\alpha \left(\|f\|_{L^q(\partial\Omega)}^{\frac{1}{p-1}} + \|P(h)\|_{L^p(\Omega)} \right) + c_\alpha. \tag{1.37}$$

The proof of this theorem follows from Theorem 2.1 in Chapter 2.1, which considers the Dirichlet-to-Neumann operator Λ_σ associated with the Leray-Lions operator $\mathcal{A}$ given by (1.25).

1.8 Evolution problems governed by the Dirichlet-to-Neumann map

In the past much research has been done and is still evolving to investigate evolution problems governed by the Dirichlet-to-Neumann map $\Lambda_{2,1}$ associated with the *linear* Laplace operator $\Delta_2 = \Delta$ with normalized conductivity coefficient $\sigma_0 \equiv 1$. To mention a few (but from being a complete list), see, for instance, [64, Proposition 1 in Chapter II.5.1], [12, 72, 142, 143, 14, 25]). But before the author's paper [85], only little was know about the evolution problem governed by the Dirichlet-to-Neumann map Λ_{p,σ_0} associated with the *nonlinear* weighted p-Laplace operator Δ_{p,σ_0} for general $1 \le p < \infty$ (see, for example, [103, 68, 93, 3, 4]).

Throughout this section, assume that Ω is a bounded domain in $\mathbb{R}^d$, $d \ge 1$, which in dimension $d \ge 2$ has at least a Lipschitz continuous boundary $\partial\Omega$. Further, suppose that the conductivity function σ_0 satisfies Assumption 1.1, and for $1 < p < \infty$, let Λ_{p,σ_0} be the Dirichlet-to-Neumann map associated with the weighted p-Laplace operator Δ_{p,σ_0}. Then, the Cauchy problem governed by Λ_{p,σ_0} reads as follows

$$\begin{cases} \partial_t h + \Lambda_{p,\sigma_0} h = 0 & \text{on } \partial\Omega \times (0,\infty), \\ h(0) = h_0 & \text{on } \partial\Omega. \end{cases} \tag{1.38}$$

It is worth mentioning that Cauchy problem (1.38) is equivalent to the *elliptic-parabolic* initial boundary-value problem

$$\begin{cases} -\Delta_{p,\sigma_0} P(h) = 0 & \text{in } \Omega \times (0,\infty), \\ \partial_t h + \sigma_0 |\nabla P(h)|^{p-2} \nabla P(h) \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ h(0) = h_0 & \text{on } \partial\Omega. \end{cases}$$

In order to study well-posedness and regularity properties of solutions of Cauchy problem (1.38), it is standard to employ a functional analytical framework (see, for example, [103, 23]). To do this, one realizes the Dirichlet-to-Neumann map Λ_{p,σ_0} as an (abstract) operator

$$\Lambda_{|X} \qquad \text{on a Banach space } X,$$

and the Banach space X should be contained in $L^1_{loc}(\partial\Omega)$. For example, the subgradient $\partial_{L^2}\varphi$ given by (1.23) (in Section 1.2.2) provides a realization of the Dirichlet-to-Neumann map Λ_{p,σ_0} in the Hilbert space $X = L^2(\partial\Omega)$ (see also Section 3.3). We refer to Definition 3.21 in Section 3.4 for a more detailed definition of the Dirichlet-to-Neumann map $\Lambda_{|X}$ in X.

For the Dirichlet-to-Neumann map $\Lambda_{|X}$ in the Banach space X, one aims to establish well-posedness of the first-order Cauchy problem (in X)

$$\begin{cases} \frac{dh}{dt}(t) + \Lambda_{|X} h(t) \ni 0 & \text{for } t \in (0,T), \\ h(0) = h_0, \end{cases} \tag{1.39}$$

for a specific notion of solutions, as well as regularity properties and the long-time asymptotic behavior of solutions. Here, we use the following notion of solutions.

Definition 1.14 Let $X \subseteq L^1_{loc}(\partial\Omega)$ be a Banach space. Then, for given $h_0 \in \overline{D(\Lambda_{|X})}^X$, a function $h \in C([0,\infty);X)$ is called a *mild (in t)* solution of Cauchy problem (1.39) with initial value h_0 if $h(0) = h_0$ and for every $\varepsilon > 0$, there is a *partition* $\tau_\varepsilon : 0 = t_0 < t_1 < \cdots < t_N = T$ satisfying $t_i - t_{i-1} < \varepsilon$ for every $i = 1, \dots, N$, and a *step function*

$$h_{\varepsilon,N}(t) = h_0 \mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^{N} h_i \mathbb{1}_{(t_{i-1},t_i]}(t) \qquad \text{for every } t \in [0,T]$$

with coefficients h_i, which are iteratively solutions of the discrete in time (elliptic) inclusion

$$\frac{h_i - h_{i-1}}{t_i - t_{i-1}} + \Lambda_{|X} h_i \ni 0 \qquad \text{for every } i = 1, \dots, N,$$

such that

$$\sup_{t\in[0,T]} \|h(t) - h_{\varepsilon,N}(t)\|_X < \varepsilon.$$

Once, one can show that the Dirichlet-to-Neumann map $\Lambda_{|X}$ is *m-accretive* in the Banach space X (see Definition 3.3 in Section 3.2.2.1), then it follows from the celebrated Crandall-Liggett theorem (cf [59]) that the Cauchy problem (1.39) admits a mild solution.

Since mild solutions are limits of step functions, it is natural to ask whether those functions are differentiable in time with values in X. This leads to the following notion of solutions of Cauchy problem (1.39).

Definition 1.15 Let $X \subseteq L^1_{loc}(\partial\Omega)$ be a Banach space. For given $h_0 \in \overline{D(\Lambda_{|X})}^X$, a function $h \in C([0,T];X)$ is called a *strong (in t)* solution of Cauchy problem (1.39) if $h(0) = h_0$, for a.e. $t \in (0,\infty)$, one has that $h(t)$ is differentiable with values in X at t,

$$h(t) \in D(\Lambda_{|X}), \quad \text{and} \quad -\tfrac{dh}{dt}(t) \in \Lambda_{|X} h(t). \tag{1.40}$$

We need to discuss further notions of solutions.

Remark 1.7 (Strong variational solutions in $L^2(\partial\Omega)$) As outlined in Section 1.3 but also in Theorem 1.3 (see Section 1.6.3), the Dirichlet-to-Neumann operator Λ_{p,σ_0} can be realized as a subgradient $\Lambda_{|L^2} = \partial_{L^2}\varphi$ in the Hilbert space $L^2(\partial\Omega)$ with dense domain. This has the advantage that due to a result by Brezis [39] (cf [23, 40]), for every initial value $h_0 \in L^2(\partial\Omega)$, there is a unique mild (in t) solution h of Cauchy problem (1.39) and this solution is *strong* (in t). Moreover, since the operator $\Lambda_{|L^2}$ has a *variational structure*, it is known explicitely (see, e.g. (1.23)). Thus, the property (1.40) implies that the strong solution h satisfies

$$h \in C([0,\infty);L^2(\partial\Omega)) \cap L^p_{loc}((0,\infty);W^{1-1/p,p}(\partial\Omega))$$

and for every $t_2 > t_1 > 0$,

$$\int_{\partial\Omega} h(t)\,\xi(t)\,\mathrm{d}\mathcal{H}^{d-1}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\partial\Omega} h(t)\partial_t\xi(t)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\int_{\Omega}\sigma_0|\nabla P(h(t))|^{p-2}\nabla P(h(t))\nabla\mathcal{Z}(\xi(t))\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{1.41}$$

for every $\xi \in C([0,\infty);L^2(\partial\Omega))\cap L^p_{\mathrm{loc}}([0,\infty);W^{1-1/p,p}(\partial\Omega))$. Measurable functions h satisfying an integral identity similar to (1.41) are also known as *weak* solutions in the literature (cf [69, 135]), and important tfor prove further properties of solutions as, for instance, maximum principles or Hölder-regularity.

Remark 1.8 (Entropy solutions) If $L^2(\partial\Omega)$ is continuously embedded into the Banach space $X \subseteq L^1_{\mathrm{loc}}(\partial\Omega)$, then the graph of the Dirichlet-to-Neumann operator $\Lambda_{|X}$ in X is merely the restriction on $X\times X$ of the closure of $\Lambda_{|L^2}$ with respect to the norm topology of $L^{1+\infty}(\partial\Omega)$, where $L^{1+\infty}(\partial\Omega)$ is an abbreviation of the sum space $L^1(\partial\Omega)+L^\infty(\partial\Omega)$ (see Definition 3.21 and Proposition 3.7 in Chapter 3); in symbols, this means that

$$\Lambda_{|X} = \overline{\Lambda_{|L^2}}^{L^{1+\infty}(\partial\Omega)} \cap (X\times X). \tag{1.42}$$

Then by Definition 1.14, every mild solution $h : [0,\infty) \to X$ of the Cauchy problem (1.39) in X with an initial value $h_0 \in X\setminus L^2(\partial\Omega)$ is merely a limit of step functions. But until 1995, it was not clear, in which sense these limit solutions satisfy the parabolic partial differential equation equation

$$\partial_t h(t) + \Lambda_{p,\sigma_0} h(t) = 0 \quad \text{on } \partial\Omega\times(0,\infty). \tag{1.43}$$

Thus, it was a big achievement in the field of nonlinear semigroup theory when in [27] the six authors Bénilan, Boccardo, Gallouët, Gariepy, Pierre, and Vázquez introduced the notion of *entropy* solutions for elliptic equations in L^1. In the same paper, the authors also gave hints about this notion of solutions of nonlinear parabolic equations. Later, Ammar [3] precised this notion of solutions concerning the Cauchy problem (1.43) in $X = L^1(\partial\Omega)$. Here, $T_k(s) := s/|s|$ if $|s| \le k$ and $T_k(s) := \mathrm{sign}(s)k$ if $|s| \ge k$ for every $s \in \mathbb{R}$ and $k > 0$.

Definition 1.16 For given $h_0 \in L^1(\partial\Omega)$, a function $h \in C([0,\infty);L^1(\partial\Omega))$ is called an *entropy solution* of Cauchy problem (1.43) if for all $k > 0$ and $T > 0$, $T_k(h) \in L^p(0,T;W^{1-1/p,p}(\partial\Omega))$, there exists a measurable function $u : (0,T) \to \mathbb{R}$ such that $T_k(u) \in W^{1-1/p,p}(\partial\Omega)$ and is an extension of $T_k(h(t))$ satisfying

$$\int_0^t\int_{\Omega}\sigma_0|\nabla u(r)|^{p-2}\nabla T_k(u(r)-v(r))\,\mathrm{d}x\mathrm{d}r + \int_{\partial\Omega}\int_0^{h(t)-g(t)} T_k(s)\,ds\,\mathrm{d}\mathcal{H}^{d-1}$$
$$\le \int_0^t\int_{\partial\Omega}\tfrac{\mathrm{d}g}{\mathrm{d}s}(s)T_k(h(s)-g(s))\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s + \int_{\partial\Omega}\int_0^{h(0)-g(0)} T_k(s)\,ds\,\mathrm{d}\mathcal{H}^{d-1}$$

for every $g \in L^\infty(\Omega\times(0,T))\cap L^\infty(0,T;W^{1-1/p,p}(\partial\Omega))\cap W^{1,1}(0,T;L^1(\partial\Omega))$.

In [3, Théorème 4.1.1], Ammar proved that mild solutions h of the Cauchy problem (1.39) in $X = L^1(\partial\Omega)$ and *entropy* solutions h of (1.43) satisfying the same initial condition $h(0) = h_0$ are, indeed, the same.

Remark 1.9 (Strong variational solutions in X) Assume again that $L^2(\partial\Omega)$ is continuously embedded into the Banach space $X \subseteq L^1_{\mathrm{loc}}(\partial\Omega)$. Even if $h : [0,\infty) \to X$ is a strong solution of the Cauchy problem (1.39) in X, then it is *a priori* not clear for the closed operator $\Lambda_{|X}$ given by (1.42), (1.40) implies that h satisfies the integral equation (1.41). We show in Theorem 1.7 (in the subsequent Section 1.9) that if the Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ in $L^2(\partial\Omega)$ satisfies a Sobolev inequality, then strong solutions h of (1.39) satisfy (1.41). To highlight, when strong solutions h of the Cauchy problem (1.39) in X satisfy (1.41), we introduce the following notion of solutions.

Definition 1.17 Let $X \subseteq L^1_{\mathrm{loc}}(\partial\Omega)$ be a Banach space, and X^* its dual space. Then we call a measurable function $h : (0,\infty) \to X$ a *variational* solution of equation (1.43), if

$$h \in C((0,\infty); X) \cap L^p_{\mathrm{loc}}((0,\infty); W^{1-1/p,p}(\partial\Omega))$$

and for every $t_2 > t_1 > 0$, h satisfies (1.41) for every test function ξ belonging to $C((0,\infty); X^*) \cap L^p_{\mathrm{loc}}((0,\infty); W^{1-1/p,p}(\partial\Omega))$. Furthermore, we call a measurable function $h : (0,\infty) \to X$ a *strong variational* solution of (1.43) if h is a.e. differentiable on $(0,\infty)$ with values in X, satisfies (1.40), and is a variational solution of (1.43).

In [68], Díaz and Jiménez studied the problem (1.38) with normalized conductivity coefficient σ_0 and established existence of *strong solutions* in $L^2(\partial\Omega)$ (see also [93]). In the case σ_0 satisfies Assumption 1.1 (or, more specifically, for a Carathéodory function $\boldsymbol{\sigma}$ satisfying (2.2), (2.3), (2.4), (2.5) from Chapter 2.1), first results on existence and uniqueness in the sense of *mild solutions* in $L^1(\partial\Omega)$ of Cauchy problem (1.39) for initial data $h_0 \in L^1(\partial\Omega)$ were announced in [2] and later established in the PhD thesis [3] by Ammar.

In [85], we could complement the existing literature by proving well-posedness of *mild* solutions of (1.39) in $L^q(\partial\Omega)$ for every all $1 \le q < \infty$ and in $C(\partial\Omega)$, and by establishing the smoothing effect that mild solutions in $L^q(\partial\Omega)$ are, in fact, *strong* in $L^q(\partial\Omega)$. In particular, we established that every strong solution h of (1.39) is Hölder-continuous on $\partial\Omega$ provided $h_0 \in L^q(\partial\Omega)$ for $q \ge 1$ large enough (which depends on p). Furthermore, we established in [85] the long time asymptotic behavior of mild solutions h to (1.39).

In fact, we present here a slightly improved version of the results obtained in [85]. In particular, we establish existence of mild solutions of the Cauchy problem (1.39) in X, when the Banach space $X = L^\psi(\partial\Omega)$ is an *Orlicz space* with an N-function satisfying the so-called Δ_2*-condition* (see Definition 3.1 and Definition 3.2 in Chapter 3.2.2.2).

We note that all Banach spaces X employed here are linear subspaces of $L^1(\partial\Omega)$. Thus, it makes sense to introduce the *mean-value* $\overline{h} := \int_{\partial\Omega} h \,\mathrm{d}\mathcal{H}^{d-1}/\mathcal{H}^{d-1}(\partial\Omega)$ for a given function $h \in L^1(\partial\Omega)$. Further, we write $[h]^+$ for the *positive* part of $h \in X$ and we sometimes write $[h]^1$ for h.

Now, we summarize our results in the next theorem.

Theorem 1.5 *Assume that the conductivity coefficient σ_0 satisfies Assumption 1.1. Further, let X be either $L^1(\partial\Omega)$, $L^\psi(\partial\Omega)$ for any N-function satisfying the Δ_2-condition, or $X = C(\partial\Omega)$. Then the following statements hold.*

1. *For every $h_0 \in X$, there exists a unique mild solution h of Cauchy problem (1.39) in X. Moreover, for every h_0, $\hat{h}_0 \in X$, the corresponding mild solutions h and $\hat{h}$ of Cauchy problem (1.39) satisfy*
$$\|[h(t) - \hat{h}(t)]^\nu\|_X \le \|[h_0 - \hat{h}_0]^\nu\|_X$$
for every $t \ge 0$ and $\nu \in \{1, -\}$, and
$$\|h(t) - c\mathbb{1}_{\partial\Omega}\|_X \le \|h_0 - c\mathbb{1}_{\partial\Omega}\|_X$$
for every $t \ge 0$ and $c \in \mathbb{R}$.
2. *If X is continuously embedded into $L^2(\partial\Omega)$, then for every h_0, the mild solution h of Cauchy problem (1.39) in X coincides with the unique strong variational solution of Cauchy problem (1.39) in $L^2(\partial\Omega)$ and has the regularity*
$$h \in C((0,\infty); W^{1-1/p,p}(\partial\Omega)) \cap W^{1,\infty}([\delta,\infty); L^2(\partial\Omega))$$
for every $\delta > 0$, $h(t)$ is right-hand side differentiable with values in $L^2(\partial\Omega)$ at every $t > 0$, and for every $t > 0$, one has that
$$\int_{\partial\Omega} \frac{\mathrm{d}h}{\mathrm{d}t}_+(t)\,\xi\,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega \boldsymbol{\sigma}(x, \nabla P(h(t)))\nabla\mathcal{Z}(\xi)\,\mathrm{d}x = 0$$
for every $\xi \in W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega)$. In particular, h conserves mass, meaning that
$$\int_{\partial\Omega} h(t)\,\mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\Omega} h_0\,\mathrm{d}\mathcal{H}^{d-1}$$
for every $t \ge 0$. The function
$$t \mapsto \varphi(h(t)) := \int_\Omega \mathcal{A}(x, \nabla P(h(t)))\,\mathrm{d}x$$
is convex, decreasing, Lipschitz continuous on $[\delta,\infty)$ for every $\delta > 0$, and
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi(h(t)) = -\|\tfrac{\mathrm{d}h}{\mathrm{d}t}(t)\|^2_{L^2(\partial\Omega)} \qquad \text{for a.e. } t > 0.$$
3. *Suppose that $p \ne 2$ and $\xi \mapsto \boldsymbol{\sigma}(\cdot, \xi)$ satisfies the homogeneity condition (2.17). Then for every $h_0 \in X$, the mild solution h of Cauchy problem (1.39) in X is a*

strong solution with the regularity property that

$$h \in W^{1,\infty}([\delta,\infty);X)\cap C([0,\infty);X) \tag{1.44}$$

for every $\delta > 0$, $h(t)$ *is differentiable with values in* X *from the right hand-side at every* $t > 0$ *and satisfies*

$$\begin{aligned} |(\Lambda_{|X})^{\circ}h(t)| &\le \tfrac{2}{|p-2|}\frac{|h_0|}{t} && \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega,\\ \left\|\frac{\mathrm{d}h}{\mathrm{d}t_+}(t)\right\|_X &\le C_p\frac{\|h_0\|_X}{t} && \text{for every } t>0, \end{aligned} \tag{1.45}$$

where $C_p = 2/|p-2|$, *and*

$$(p-2)(\Lambda_{|X})^{\circ}h(t) \le \frac{h(t)}{t} \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega$$

for every $t > 0$ *provided* $h_0 \ge 0$. *In particular, if* X *is continuously embedded into* $L^2(\partial\Omega)$, *then for every* $h_0 \in X$, *the mild solution* h *of Cauchy problem* (1.39) *in* X *is strong variational.*

4. *Let* $p = 2$ *and* $X = L^q(\partial\Omega)$ *for* $1 < q < \infty$. *Then, for every* $h_0 \in X$, *the mild solution* h *of Cauchy problem* (1.39) *in* X *is strong satisfying* (1.44) *and* (1.45) *for some constant* $C_2 = C_2(q) > 0$. *In particular, if* X *is continuously embedded into* $L^2(\partial\Omega)$ *then for every* $h_0 \in X$, *the mild solution* h *of Cauchy problem* (1.39) *in* X *is strong variational.*
5. *For* $(2 \vee \frac{d-1}{p-1-\varepsilon}) \le q \le \infty$ *with some* $\varepsilon \in (0,1)$ *if* $p \le d$ *and for* $2 \le q \le \infty$ *if* $p > d$, *there are* $\alpha \in (0,1)$ *and* $c_\alpha > 0$ *such that*

$$\|h(t)\|_{C^{0,\alpha}(\partial\Omega)} \le c_\alpha\left[\left(\frac{\|h(t)\|_{L^q(\partial\Omega)}}{t}\right)^{\frac{1}{p-1}} + \Phi(h(t))\right] + c_\alpha$$

for every $t > 0$, *where the function*

$$t \mapsto \Phi(h(t)) := \varphi(h(t))^{1/p} + \|h(t)\|_{L^2(\partial\Omega)}$$

is continuous and decreasing on $(0,\infty)$.

6. *For every* $h_0 \in X$, *one has that*

$$\lim_{t\to+\infty} h(t) = \overline{h_0} := \tfrac{1}{\mathcal{H}(\partial\Omega)}\int_{\partial\Omega} h_0\,\mathrm{d}\mathcal{H}^{d-1} \qquad \text{in } X. \tag{1.46}$$

In particular, if $(2 \vee \frac{d-1}{p-1-\varepsilon}) \le q \le \infty$ *for some* $\varepsilon \in (0,1)$ *provided* $p \le d$, *or if* $2 \le q \le \infty$ *provided* $p > d$, *then for every* $h_0 \in L^q(\partial\Omega)$, *one has that* (1.46) *holds in* $C(\partial\Omega)$.

For the proof of Theorem 1.5, we employ various techniques from functional analysis and elliptic regularity theory. In particular, claim (1) is established by using nonlinear semigroup theory ([30, 23, 28]), claim (2) employs the theory of functional inequalities and regularizing properties of nonlinear semigroups from [56], and for the claim (3), we employ the elliptic regularity estimate (1.37) from Theorem 1.4. We give the details of the proof in Chapter 3.

We conclude this section with the following conjecture.

Conjecture. There are thresholds $1 < p_0 < 2$ and $1 \le q_0 < \infty$ such that for every $h_0 \in L^{q_0}(\partial\Omega)$, the mild solution h of the Cauchy problem (1.39) becomes constant in finite time, or equivalently, the trajectory $t \mapsto h(t) - \overline{h_0}$ extincts in finite time.

1.9 L^q-L^∞ regularization and decay estimates

Let X be either $L^1(\partial\Omega)$, $L^\psi(\partial\Omega)$ for any N-function satisfying the Δ_2-condition, or $X = C(\partial\Omega)$. Assume that the conductivity coefficient σ_0 satisfies Assumption 1.1, and $\Lambda_{|X}$ denotes the realization in the Banach space X of the Dirichlet-to-Neumann map Λ_{σ_0} associated with the weighted negative p-Laplace operator $A = -\mathrm{div}(\sigma_0|\nabla\cdot|^{p-2}\nabla\cdot)$.

Due to the well-posedness of the Cauchy problem (1.39) in the sense of mild/strong solutions h in $L^2(\partial\Omega)$ (Theorem 1.5), setting

$$e^{-t\Lambda_{|L^2}} h_0 := h(t),\ t \ge 0,\ h_0 \in L^2(\partial\Omega), \tag{1.47}$$

defines a C_0- semigroup $\{e^{-t\Lambda_{|L^2}}\}_{t\ge 0}$ on $L^2(\partial\Omega)$ of contractions $e^{-t\Lambda_{|L^2}}$ on $L^2(\partial\Omega)$. Since the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ is *completely accretive* (see Definition 3.16 and Theorem 3.6 in Chapter 3), this semigroup $\{e^{-t\Lambda_{\sigma_0}}\}_{t\ge 0}$ can be extended to a C_0-semigroups $\{e^{-t\Lambda_{|X}}\}_{t\ge 0}$ of contractions $e^{-t\Lambda_{|X}}$ on X for each of the above given Banach spaces X. Then, by construction, one has the *consistency equation*

$$e^{-t\Lambda_{|X}} h_0 = e^{-t\Lambda_{|L^2}} h_0 \tag{1.48}$$

for every $t \ge 0$ and $h_0 \in L^2(\partial\Omega) \cap X$.

On the other hand, for given initial value $h_0 \in X$, setting

$$h(t) := e^{-t\Lambda_{|X}} h_0 \qquad \text{for ever } t \ge 0$$

defines the unique mild solution of Cauchy problem (in X)

$$\begin{cases} \frac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda_{|X} h(t) \ni 0 & \text{for } t \in (0,T), \\ h(0) = h_0, \end{cases} \tag{1.39}$$

(cf Theorem 3.4 in Chapter 3.2.2.2). Thus, we fix the following convention.

Notation 1.5 For X being either $L^1(\partial\Omega)$, $L^\psi(\partial\Omega)$ for any N-function satisfying the Δ_2-condition, or $X = C(\partial\Omega)$, we write $\{e^{-t\Lambda}\}_{t\geq 0}$ also for the extension $\{e^{-t\Lambda_{|X}}\}_{t\geq 0}$ on X of the semigroup $\{e^{-t\Lambda_{|L^2}}\}_{t\geq 0}$ on $L^2(\partial\Omega)$. Moreover, we refer to $\{e^{-t\Lambda}\}_{t\geq 0}$ as the *semigroup generated by the negative Dirichlet-to-Neumann map* $-\Lambda$ (on X).

In the next theorem, we provide global L^q-L^∞ *regularization estimates* for $1 \leq q < \infty$, which also provide decay estimates in time. This result has been first announced in the monograph [56].

Theorem 1.6 *Assume that the conductivity coefficient σ_0 satisfies Assumption 1.1, and let $\{e^{-t\Lambda}\}_{t\geq 0}$ be the semigroup of the negative Dirichlet-to-Neumann map $-\Lambda$. For $1 < p < \infty$, choose $1 \leq q_0 \geq p$ minimal such that*

$$((d-1)/(d-p)-1)q_0 + p - 2 > 0, \tag{1.49}$$

and let $1 \leq q \leq \infty$. Then, the following statements hold.

1. *If $1 < p \leq 2d/(d+1)$, then for every $1 \leq q \leq (d-1)q_0/(d-p)$ satisfying $q > (2-p)(d-1)/(p-1)$, one has that*

$$\|e^{-t\Lambda}h_0 - \overline{h_0}\|_\infty \lesssim t^{-\alpha_q}\,\|h_0 - \overline{h_0}\|_q^{\gamma_q} \tag{1.50}$$

for every $t > 0$ and $h_0 \in L^q(\partial\Omega)$, where the exponents α_q and γ_q are given by

$$\begin{aligned} \alpha_q &= \frac{\alpha^*}{1-\gamma^*\left(1-\frac{q(d-p)}{(d-1)q_0}\right)}, \quad \gamma_q = \frac{\gamma^*\,\frac{q(d-p)}{(d-1)q_0}}{1-\gamma^*\left(1-\frac{q(d-p)}{(d-1)q_0}\right)}, \\ \alpha^* &= \frac{d-p}{(p-1)q_0+(d-p)(p-2)}, \quad \gamma^* = \frac{(p-1)q_0}{(p-1)q_0+(d-p)(p-2)}. \end{aligned} \tag{1.51}$$

2. *For every $\frac{2d}{d+1} < p < d$, condition (1.49) holds for $q_0 = p$. Thus, for every $1 \leq q \leq \frac{(d-1)p}{d-p}$ satisfying $q > \frac{(2-p)(d-1)}{p-1}$, the semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ satisfies the L^q-L^∞-regularizing estimate (1.50) with exponents α_q and γ_q given by (1.51) with $q_0 = p$.*
3. *If $\frac{2d-1}{d} < p < d$, then for every $1 \leq q \leq \frac{(d-1)p}{d-p}$, the semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ satisfies the L^q-L^∞-regularizing estimate (1.50) with exponents α_q and γ_q given by (1.51) with $q_0 = p$.*
4. *Let $p = d \geq 2$ and $1 - \frac{1}{p} < \theta < 1$. Then, for every $1 \leq q \leq \frac{1}{1-\theta}$, the semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ satisfies the L^q-L^∞-regularizing estimate (1.50) with exponents*

$$\begin{aligned} \alpha_q &= \frac{\alpha_\theta^*}{1-\gamma_\theta^*(1-q(1-\theta))}, \quad \gamma_q = \frac{\gamma_\theta^* q(1-\theta)}{1-\gamma_\theta^*(1-q(1-\theta))}, \\ \alpha_\theta^* &= \frac{1}{\frac{1}{1-\theta}-2}, \quad \gamma_\theta^* = \frac{\frac{1}{1-\theta}-p}{\frac{1}{1-\theta}-2}. \end{aligned} \tag{1.52}$$

5. *Let $d < p < \infty$. Then, for every $1 \le q \le 2$, the semigroup $\{e^{-t\Lambda}\}_{t\ge 0}$ satisfies the L^q-L^∞-regularizing estimate (1.50) with exponents*

$$\alpha_q = \frac{1}{p-2+q}, \quad \gamma_q = \frac{q}{p-2+q}. \tag{1.53}$$

We outline the proof of Theorem 1.6 in Chapter 4. Combining estimates (1.45) for $X = L^\infty(\partial\Omega)$ from Theorem 1.5 with (1.50) yields (1.55) (below) and due to the elliptic regularity inequality (1.37) (in Theorem 1.4), one obtains Hölder-continuity w.r.t. the space variable.

Corollary 1.1 *Assume that the conductivity coefficient σ_0 satisfies Assumption 1.1, let $1 < p < \infty$ but $p \neq 2$, $q_0 \ge p$ satisfy (1.49), and $1 \le q \le \infty$ be such that*

$$\begin{aligned} 1 \le q \le q_0\frac{d-1}{d-p} \text{ and } q > (2-p)\frac{d-1}{d-p} \qquad & \text{if } 1 < p \le \frac{2d}{d+1}, \\ 1 \le q \le p\frac{d-1}{d-p} \text{ and } q > (2-p)\frac{d-1}{d-p} \qquad & \text{if } \frac{2d}{d+1} < p \le \frac{2d-1}{d}, \\ 1 \le q \le p\frac{d-1}{d-p} \qquad & \text{if } \frac{2d-1}{d} < p < d, \\ 1 \le q \le \frac{1}{1-\theta} \qquad & \text{if } p = d \text{ and } 1-\frac{1}{p} < \theta < 1, \\ 1 \le q \le 2 \qquad & \text{if } p > d. \end{aligned} \tag{1.54}$$

Then for every $h_0 \in L^q(\partial\Omega)$, the mild solution h of the homogeneous Cauchy problem (1.39) satisfies

$$\left\| \frac{\mathrm{d}h}{\mathrm{d}t_+}(t) \right\|_\infty \lesssim \frac{\|h_0 - \overline{h_0}\|_q^{\gamma_q}}{t^{\delta_q+1}} + 2C_p \frac{|\overline{h_0}|}{t} \tag{1.55}$$

for every $t > 0$, where the exponents α_q, γ_q are the same as given in (1.51), (1.52), and (1.53). Moreover, the mild solution h of (1.39) is strong variational and Hölder-continuous (w.r.t. $x \in \partial\Omega$).

Now, by the *semigroup property*

$$e^{-(t+s)\Lambda}h_0 = e^{-t\Lambda}\left(e^{-s\Lambda}h_0\right)$$

t, $s \ge 0$, $h_0 \in X$, and due to the consistency equation (1.48), a mild solution h of Cauchy problem (1.39) in X is a strong variational solution of equation (1.43) on (t_0, ∞) provided there exists a $t_0 > 0$ such that

$$e^{-t_0\Lambda}h_0 \in L^2(\partial\Omega).$$

In other words, one requires an X-L^2 *regularization effect* of the semigroup $\{e^{-t\Lambda_{\sigma_0}}\}_{t\ge 0}$ generated by the negative Dirichlet-to-Neumann map Λ on X.

We conclude this section with the following result on *strong variational* solutions.

Theorem 1.7 *Assume that the conductivity coefficient σ_0 satisfies Assumption 1.1. Then the following statements hold.*

1. *Let $2d/(d+1) \le p < d$. Then, for every $2 - p/2 < q \le 2$, every mild solution of Cauchy problem (1.39) in $X = L^q(\partial\Omega)$ with initial value $h_0 \in L^q(\partial\Omega)$ is strong variational.*
2. *Let $p \ge d$. Then for every $h_0 \in L^1(\partial\Omega)$, the corresponding mild solution of Cauchy problem (1.39) in $X = L^1(\partial\Omega)$ is strong variational.*

1.10 Aronson-Bénilan type estimates

As in Section 1.9, let X be either $L^1(\partial\Omega)$, $L^\psi(\partial\Omega)$ for any N-function satisfying the Δ_2-condition, or $X = C(\partial\Omega)$. Let Λ_{σ_0} be the Dirichlet-to-Neumann map Λ_{σ_0} associated with the weighted negative p-Laplace operator $A = -\mathrm{div}(\sigma_0|\nabla\cdot|^{p-2}\nabla\cdot)$, where we assume that the conductivity coefficient σ_0 satisfies Assumption 1.1. Further, we follow the convention introduced in Notation 1.5 that Λ refers to the realization $\Lambda_{|X}$ in the Banach space X of the Dirichlet-to-Neumann map Λ_{σ_0}.

In (1.45) of Theorem 1.5(see Section 1.8, we outlined that every mild solution h of the homogeneous Cauchy problem (in X)

$$\begin{cases} \frac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda_{|X} h(t) \ni 0 & \text{for } t \in (0,T), \\ h(0) = h_0, \end{cases} \tag{1.39}$$

satisfies that immediate *smoothing effect* that it becomes a strong solution of (1.39) and satisfies the inequality

$$\left\| \frac{\mathrm{d}h}{\mathrm{d}t_+}(t) \right\|_X \le C_p \frac{\|h_0\|_X}{t} \qquad \text{for every } t > 0. \tag{1.56}$$

It is worth noting that Inequality (1.56) is also known under the name L^1-*Aronson-Bénilan type estimate* acknowledging the pioneering work [18] by Aronson and Bénilan. Mild solutions h of the Cauchy problem (1.39) governed by the Dirichlet-to-Neumann map Λ satisfy (1.56) since the operator Λ associated with either the *linear* weighted Laplace operator Δ_{2,σ_0} generates an analytic semigroup $\{e^{t\Lambda}\}_{t\ge 0}$ on $L^q(\partial\Omega)$, or for $p \ne 2$, from the property that Λ is *homogeneous of order* $\alpha := p-1 > 0$ (see Proposition 2.1 for more details).

In [17, p. 5], Aronson announced that positive (classic) solutions $u = u(x,t)$ of the porous medium equation

$$\partial_t u = \Delta u^m \qquad \text{in } \mathbb{R}^d \times (0,+\infty) \tag{1.57}$$

satisfy the *point-wise estimate*

$$\partial_t u(x,t) \geq -\frac{k}{t}\, u(x,t) \tag{1.58}$$

for a.e. $x \in \mathbb{R}^d$ and all $t > 0$, where $k = 1/(m-1+2/d)$ and $m > [d-2]^+/d$. Shortly after [17], Aronson & Bénilan [18] proved inequality (1.58). Therefore, one refers to (1.58) as the *point-wise Aronson-Bénilan type estimate*. In the same paper [18, Théorème 2], they also proved that every (strong) solutions u of the porous medium equation (1.60) satisfies the L^1*-estimate* (1.56).

Another big achievement is this direction has been obtained by Bénilan and Crandall [28]. They showed that if A is a (possibly nonlinear) m-accretive operators on a Banach space X, which is homogeneous of order $\alpha > 0$, $\alpha \neq 0$, then every mild solution u of the homogeneous Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) \ni 0 & \text{in } (0,\infty),\\ \qquad u(0) = u_0, \end{cases} \tag{1.59}$$

is Lipschitz continuous and satisfies

$$\limsup_{h\to 0+} \frac{\|u(t+h)-u(t)\|_X}{h} \leq \frac{2\,\|u_0\|_X}{|\alpha-1|\,t}$$

at every every $t > 0$. If the Banach space X has the Radon-Nikodým properties, then the last inequality implies that u satisfies the L^1-estimate (1.56). Moreover, they showed that if X is an ordered Banach space with an order relation $\geq$, then every positive solution u of the homogeneous Cauchy problem (1.59) also satisfies the point-wise Aronson-Bénilan type estimate (1.58). The class of m-accretive and homogeneous operators A includes the local p-Laplace operator Δ_p, the local doubly nonlinear operator $\Delta_p u^m$, $1 < p < \infty$, $m > 0$, as well as the nonlocal fractional p-Laplace operator $(-\Delta_p)^s$, respectively equipped with various boundary conditions (see, for instance, [56] for more details to the analytic properties of these quasi-linear 2nd-order differential operators).

In the papers [57] and [60] Crandall and Pierre showed that every mild solution u of the more general version of the porous medium equation

$$\partial_t u = \Delta\phi(u) \qquad \text{in } \Omega\times(0,\infty) \tag{1.60}$$

also satisfies the point-wise Aronson-Bénilan type estimate (1.58). In (1.60), $\phi \in C(\mathbb{R})\cap C^2(\mathbb{R}\setminus\{0\})$ is a monotonically increasing function and Ω and open subset of $\mathbb{R}^d$. These two results by Crandall and Pierre were slightly improved in the short paper [50] by Chasseigne for ϕ with less regularity. The point-wise Aronson-Bénilan type estimate (1.58) was revisited in various settings; on manifolds (see, e.g. [106, 48]), and with drift-term (see, e.g, [92]), or with a linear perturbation (see, e.g., [49]). One important reason among others, for the strong further development of the point-wise estimate (1.58) is that it can be used to study the regularity of the free-boundaries (see, for instance, [125] or [149]). We refer the interested reader to the book [150] by Vázquez (and more recently [33]) for a detailed exposition concerning the development of the point-wise Aronson-Bénilan estimate (1.58) satisfied by solutions to the porous media equation (1.60).

Recently, Mazón and the author showed in [87] that the two Aronson-Bénilan type estimates (1.56) and (1.58) are satisfied by the mild solutions h of the unperturbed Cauchy problem (1.61) for *homogeneous operators of order zero* (i.e., $\alpha = 0$). This class of operators includes, for example, the (negative) total variation flow operator $Au = -\mathrm{div}(\frac{Du}{|Du|})$, or the fractional 1-Laplacian $A = (-\Delta_1)^s$ for $s \in (0,1)$ respectively equipped with some boundary conditions. In [88], Mazón and the author applied this idea to the Dirichlet-to-Neumann map Λ_{σ_0} associated with the 1-Laplace operator. We refer either to the subsequent Section 1.11 or to Chapter **??** for more details.

Let's come back to the evolution problem governed by the Dirichlet-to-Neumann map Λ. In [86], the author could show that mild solutions h of the *perturbed* Cauchy problem (in X)

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t}(t) + \Lambda h(t) + F(h(t)) \ni 0 & \text{for } t \in (0,T), \\ h(0) = h_0, \end{cases} \tag{1.61}$$

also satisfy the smoothing effect that they are strong and satisfy a similar Aronson-Bénilan type estimate as the one given in (1.56), provided the perturbation F is Lipschitz continuous. More precisely, suppose that $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a *Lipschitz continuous Carathéodory function*; that is, f satisfies the following three properties:

- $f(\cdot, h) : \partial\Omega \to \mathbb{R}$ is measurable on $\partial\Omega$ for every $h \in \mathbb{R}$, (1.62)
- $f(x, 0) = 0$ for a.e. $x \in \partial\Omega$, and (1.63)
- there is a constant $\omega \geq 0$ such that

$$|f(x,h) - f(x,\hat{h})| \leq \omega\,|h - \hat{h}| \quad \text{for all } h, \hat{h} \in \mathbb{R}, \text{ a.e. } x \in \partial\Omega. \tag{1.64}$$

Further, let $F : X \to X$ be the *Nemytskii operator* of f on X given by

$$(Fh)(x) := f(x, h(x)) \qquad \text{for a.e. } x \in \partial\Omega,$$

for every $h \in X$. Then, the operator $\Lambda_{\sigma_0} + F$ is *ω-quasi m-completely accretive* on X (see Definition 3.16), implying that the perturbed Cauchy problem (1.61) is well-posed in the sense of mild solutions.

Remark 1.10 We note that by taking $\tilde{\Lambda} = \Lambda_{\sigma_0} + F$, the Definitions 1.14 and 1.15 of *mild* and *strong* solutions of the perturbed Cauchy problem (1.61) remain valid. Here, we call a function h *positive* if $h(x) \geq 0$ for a.e. $x \in \partial\Omega$.

Now, mild solutions h of the perturbed Cauchy problem (1.61) admit the following smoothing effect (cf [86]).

Theorem 1.8 *Assume that the conductivity coefficient σ_0 satisfies Assumption 1.1 and $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous Carathéodory function. Further, suppose, $1 < p < \infty$ but $p \neq 2$. Then every mild solution h of the perturbed Cauchy problem (1.61) admits the following additional regularity.*

1. *(*L^1 Aronson-Bénilan type estimates*) For every* $h_0 \in X$*, the mild solution* h *of* (1.61) *is a* strong solution *of* (1.61) *in* X*, and satisfies*

$$\left\|\frac{\mathrm{d}h}{\mathrm{d}t_+}(t)\right\|_X \le \frac{[2+\omega t]\, e^{\omega t}}{|p-2|\,t}\|h_0\|_X \quad \textit{for every } t>0.$$

2. *(*Point-wise Aronson-Bénilan type estimates*) For every positive* $h_0 \in X$*, the strong solution* h *of the perturbed problem* (1.61) *in* X *satisfies*

$$(p-2)\frac{\mathrm{d}h}{\mathrm{d}t_+}(t) \ge -\frac{h(t)}{t} + (p-2)\,g_0(t),$$

for a.e. $t>0$*, where* $g_0 : (0,\infty) \to X$ *is a measurable function.*

We outline the auxiliary results and the proof of Theorem 1.8 in Chapter 5.

1.11 The Dirichlet-to-Neumann map associated with the 1-Laplacian

This section is dedicated to the limiting case $p = 1$; we discuss the realization of the Dirichlet-to-Neumann map $\Lambda : L^1(\partial\Omega) \to L^\infty(\partial\Omega)$ associated with the singular 1-*Laplace operator*

$$\Delta_1 u := \mathrm{div}\left(\frac{Du}{|Du|}\right)$$

and discuss properties, and the well-posedness of the inhomogeneous Cauchy problems governed by the Dirichlet-to-Neumann map Λ.

Throughout this section, we assume that $\Omega \subseteq \mathbb{R}^d$ for $d \ge 2$, is a bounded domain with a C^1-boundary $\partial\Omega$.

We begin by discussing the well-posedness of the singular *Dirichlet problem*

$$\begin{cases} -\Delta_1 u_h = 0 & \text{in } \Omega, \\ \quad\;\; u_h = h & \text{on } \partial\Omega, \end{cases} \tag{1.65}$$

for the 1-Laplace operator Δ_1 in the following sense of solutions.

Definition 1.18 For given $h \in L^1(\partial\Omega)$, we call a function $u_h \in BV(\Omega)$ a *weak* solution of Dirichlet problem (1.65) if there is a vector field $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ generalizing $Du/|Du|$ through the three conditions

$$\|\mathbf{z}_h\|_\infty \le 1, \tag{1.66}$$

$$-\mathrm{div}(\mathbf{z}_h) = 0 \qquad \text{in } \mathcal{D}'(\Omega), \text{ and} \tag{1.67}$$

$$(\mathbf{z}_h, Du_h) = |Du_h| \qquad \text{as Radon measures} \tag{1.68}$$

and the *weak trace* $[\mathbf{z}_h, \nu]$ on $\partial\Omega$ (see Definition 7.1 in Chapter 7.1.1) of the generalized co-normal derivative $\mathbf{z}_h \cdot \nu$ satisfies

$$[\mathbf{z}_h, \nu] \in \operatorname{sign}(h - \mathcal{T}r(u)) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{1.69}$$

It is well-known (see Section 7.2) that for every $h \in L^1(\partial\Omega)$, there exists at least one weak solution u_h of Dirichlet problem (1.65). But difficulties arise by deriving properties of the Dirichlet-to-Neumann map Λ from the fact that the notion of weak solutions u_h of (1.65) merely requires that the Dirichlet boundary data h on $\partial\Omega$ is satisfied in the *very weak* sense (1.69). Because of this, the Dirichlet problem (1.65) might have infinitely many weak solutions u_h (see Remark 7.1 in Section 7.2 for more details). Further, for each weak solution u_h of (1.65), there might be infinitely many vector fields $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (1.66)-(1.69) with u_h, and if $\hat{\mathbf{z}}_h$ is a second vector field satisfying (1.66)-(1.69) for another weak solution $\hat{u}_h$ of (1.65) then $\mathbf{z}_h$ also satisfies (1.68)-(1.69) with $\hat{u}_h$ and $\hat{\mathbf{z}}_h$ satisfies (1.68)-(1.69) with u_h. We prove this in Theorem 7.4 in Section 7.2. Thus, for given Dirichlet boundary data $h \in L^1(\partial\Omega)$, we introduce the set

$$\mathcal{Z}_h := \left\{ \mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d) \;\middle|\; \begin{array}{l} \mathbf{z}_h \text{ satisfying (1.66)-(1.69) for a weak} \\ \text{solution } u_h \text{ of Dirichlet problem (1.65)} \end{array} \right\} \tag{1.70}$$

In Section 7.2, we provide a brief review of the current state of the literature about existence and (non)-uniqueness of weak solutions to Dirichlet problem (1.65). But we emphasize that due to the existence theory of weak solutions of Dirichlet problem (1.65), the set $\mathcal{Z}_h$ is non-empty for every $h \in L^1(\partial\Omega)$.

Thus the following realization of the Dirichlet-to-Neumann operator Λ_1 in $L^1(\partial\Omega)$ defines a possibly multi-valued operator.

Definition 1.19 Let $\overline{B}_{L^\infty(\partial\Omega)}$ denote the closed unit ball of $L^\infty(\partial\Omega)$ centered at zero. Then, we call the operator Λ_1 defined by

$$\Lambda_1 = \left\{ (h,g) \in L^1(\partial\Omega) \times L^1(\partial\Omega) \;\middle|\; \begin{array}{c} g \in \overline{B}_{L^\infty(\partial\Omega)} \text{ and } \exists\, \mathbf{z}_h \in \mathcal{Z}_h \text{ satisfying} \\ [\mathbf{z}_h, \nu] = g \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \end{array} \right\}$$

the *Dirichlet-to-Neumann map in* $L^1(\partial\Omega)$ *associated with the* 1*-Laplace operator* Δ_1.

Now, our first main result reads as follows. Here, we write $L^\infty_\sigma(\partial\Omega)$ for the space $L^\infty(\partial\Omega)$ equipped with the weak*-topology $\sigma(L^\infty(\partial\Omega), L^1(\partial\Omega))$.

Theorem 1.9 *For given $h \in L^1(\partial\Omega)$, let $\mathcal{Z}_h$ be the set of divergence-free vector fields defined by* (1.70)*. Then the Dirichlet-to-Neumann map Λ_1 in $L^1(\partial\Omega)$ associated with the* 1*-Laplace operator Δ_1 admits the following properties.*

1. *Λ_1 is m-completely accretive in $L^1(\partial\Omega)$ and has the effective domain*

$$D(\Lambda_1) = L^1(\partial\Omega);$$

2. *Λ_1 is homogeneous of order zero;*
3. *Λ_1 is closed in $L^1(\partial\Omega)\times L^\infty_\sigma(\partial\Omega)$;*
4. *Λ_1 can be characterized by*
$$\Lambda_1 = \partial\varphi, \tag{1.71}$$
$\partial\varphi$ is the subdifferential operator in $L^1\times L^\infty(\partial\Omega)$ of the convex, even, homogeneous (of order one), and continuous functional $\varphi : L^1(\partial\Omega)\to[0,\infty)$ defined by
$$\varphi(h) = \int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h \,\mathrm{d}\mathcal{H}^{d-1} \tag{1.72}$$
for every $h \in L^1(\partial\Omega)$ and $\mathbf{z}_h \in \mathcal{Z}_h$. In particular, the value of the integral on the right-hand side in (1.72) does not change among all vector fields $\mathbf{z}_h \in \mathcal{Z}_h$.

Both statements (1) and (2) in Theorem 1.9 follow from a careful study of the Dirichlet problem (1.65) (see Proposition 7.12 and Proposition 7.13). In Proposition 7.14, we show that φ is even, continuous, homogeneous of order one, and convex. The fact that the value of the integral on the right-hand side in (1.72) does not change among all vector fields $\mathbf{z}_h \in \mathcal{Z}_h$ follows from Theorem 7.5 stated in Section 7.2. To establish the characterization (1.71) for the Dirichlet-to-Neumann operator Λ_1, we first show in Proposition 7.15 that the closure $\overline{\Lambda_1}^{L^1\times L^\infty_\sigma}$ of Λ_1 in $L^1(\partial\Omega)\times L^\infty_\sigma(\partial\Omega)$ is contained in the subdifferential operator $\partial\varphi$. Since $\partial\varphi$ is completely accretive in $L^1(\Omega)$, the characterization (1.71) follows from a classical result by Bénilan and Crandall (see Theorem 3.4 in Chapter 3.2.2.2), once we have shown that Λ_1 is m-completely accretive in $L^1(\Omega)$. The property that the Dirichlet-to-Neumann operator Λ_1 is closed in $L^1(\partial\Omega)\times L^\infty_\sigma(\partial\Omega)$ (statement (3)) is proved in Proposition 7.16, and provides the following surprising stability/compactness result related to the Dirichlet problem (1.65). In Section 7.4, we give the details of the proof of this theorem.

Corollary 1.2 (stability/compactness) *For every sequence $(h_n)_{n\geq 1}$ in $L^1(\partial\Omega)$ converging to some h in $L^1(\partial\Omega)$, there is a weak solution u_h of Dirichlet problem (1.65) satisfying boundary data $u_h = h$ and a sub-sequence $(h_{k_n})_{n\geq1}$ of $(h_n)_{n\geq1}$ such that the generalized co-normal derivative $[\mathbf{z}_{h_{k_n}}, \nu]$ corresponding to h_{k_n} converges weakly* to $[\mathbf{z}_h, \nu]$ in $L^\infty(\partial\Omega)$ and*
$$\lim_{n\to\infty} \varphi(h_n) = \varphi(h). \tag{1.73}$$

Of course, the limit (1.73) follows from the continuity property of φ, but the surprising fact here is that the two divergence free vector fields $\tilde{\mathbf{z}}_h$ in $\varphi(h)$ and $\mathbf{z}_h$ in the limit $[\mathbf{z}_h, \nu]$ do not have to be the same. In fact, we show in Theorem 7.4 that any two divergence free vector fields $\mathbf{z}_h$ and $\hat{\mathbf{z}}_h$ are interchangeable among the set of weak solutions u_h and $\hat{u}_h$ of Dirichlet problem (1.65). But according to Theorem 7.5, the value of $\varphi(h)$ is invariant among all divergence free vector fields $\mathbf{z}_h$, for which there is a weak solution u_h of Dirichlet problem (1.65). Hence, φ given by (1.72) is a well-defined mapping on $L^1(\partial\Omega)$. The fact that φ is continuous on $L^1(\partial\Omega)$ can easily be deduced from its convexity property and that φ is upper bounded on any bounded subset of its effective domain $D(\varphi) = L^1(\partial\Omega)$ (see Proposition 7.14 for more details).

Further, we establish well-posedness and comparison principles in the sense of *mild* solutions (see Definition 3.7 in Chapter 3.2.2), sufficient conditions implying improved regularity properties of mild solutions, and the long-time stability (in the case $F \equiv 0$ and $g \equiv 0$) of the inhomogeneous Cauchy problem (in $L^1(\partial\Omega)$)

$$\begin{cases} \dfrac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda_1 h(t) + F(h(t)) \ni g(t) & \text{for } t \in (0,T), \\ h(0) = h_0. & \text{on } \partial\Omega, \end{cases} \tag{1.74}$$

for every given $g \in L^1(0,T;L^1(\partial\Omega))$ and $h_0 \in L^1(\partial\Omega)$. In the differential inclusion (1.74), the lower order term $F : L^1(\partial\Omega) \to L^1(\partial\Omega)$ denotes the Nemytskii operator

$$F(h)(x) := f(x,h(x)) \quad \text{for a.e. } x \in \partial\Omega, \tag{1.75}$$

and every $h \in L^1(\partial\Omega)$, of a *Lipschitz Carathéodory function* $f : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ satisfying $f(x,0) = 0$, $(x \in \partial\Omega)$; that is,

$$\begin{cases} \text{there is an } \omega > 0 \text{ such that } |f(x,h) - f(x,\hat{h})| \le \omega\,|h-\hat{h}| \\ \text{for all } h,\, \hat{h} \in \mathbb{R}, \text{ uniformly for a.e. } x \in \partial\Omega, \text{ and} \\ \text{for all } h \in \mathbb{R},\, x \mapsto f(x,h) \text{ is measurable on } \partial\Omega. \end{cases} \tag{1.76}$$

It is worth noting that well-posedness, regularity and stability analysis of Cauchy problem (1.74) are equivalent topics of the following singular *elliptic-parabolic boundary value problem*

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du_h}{|Du_h|}\right) = 0 & \text{in } \Omega\times(0,T), \\ u_h = h & \text{on } \partial\Omega\times(0,T), \\ \partial_t h + \dfrac{Du_h}{|Du_h|}\cdot\nu + f(\cdot,h) \ni g & \text{on } \partial\Omega\times(0,T), \\ h = h_0 & \text{on } \partial\Omega\times\{t=0\}. \end{cases} \tag{1.77}$$

Recently, existence and uniqueness to the elliptic-parabolic boundary value problem

$$\begin{cases} \lambda h - \operatorname{div}\left(\dfrac{Du_h}{|Du_h|}\right) = 0 & \text{in } \Omega\times(0,T), \\ u_h = h & \text{on } \partial\Omega\times(0,T), \\ \partial_t h + \dfrac{Du_h}{|Du_h|}\cdot\nu \ni g & \text{on } \partial\Omega\times(0,T), \\ h = h_0 & \text{on } \partial\Omega\times\{t=0\} \end{cases}$$

for parameter $\lambda > 0$ and with initial data $h_0 \in L^2(\partial\Omega)$ was obtained in [98]. We emphasize that for $\lambda > 0$, the associated Dirichlet problem

$$\begin{cases} \lambda u_h - \mathrm{div}\left(\dfrac{Du_h}{|Du_h|}\right) = 0 & \text{in } \Omega, \\ u_h = h & \text{on } \partial\Omega. \end{cases}$$

is uniquely solvable, but which is not true for the case $\lambda = 0$ (that is, Dirichlet problem (1.65)). This makes this singular elliptic-parabolic boundary value problem (1.77) more appealing, but also complements the research in [98]. In addition, problem (1.77) models an evolution problem with dynamic boundary conditions. Related problems were studied, for instance, in [76].

To study further regularity properties of mild solutions to Cauchy problem (1.74), we introduce the following operators.

Notation 1.6 For every $1 \le q \le \infty$, we write $\Lambda_{1,|L^q}$ for the restriction of Λ_1 on $L^q(\partial\Omega) \times L^q(\partial\Omega)$. In other words,

$$\Lambda_{1|L^q} = \Lambda_1 \cap (L^q(\partial\Omega) \times L^q(\partial\Omega)),$$

and call the operator $\Lambda_{1|L^q}$ the *Dirichlet-to-Neumann map on* $L^q(\partial\Omega)$.

Thanks to the continuous embedding from $L^q(\partial\Omega)$ into $L^1(\partial\Omega)$, the first three statements in the next corollary follow immediately from (1) of Theorem 1.9 and Corollary 1.2. Statement (4) with the characterization (1.78) are concluded from Proposition 7.19 (in Section 7.4.2).

Corollary 1.3 *Let $1 \le q \le \infty$. Then the following statements on the Dirichlet-to-Neumann operator $\Lambda_{1|L^q}$ in $L^q(\partial\Omega)$ hold.*

1. *$\Lambda_{1|L^q}$ is m-completely accretive in $L^q(\partial\Omega)$ with effective domain*

$$D(\Lambda_{1|L^q}) = L^q(\partial\Omega);$$

2. *$\Lambda_{1|L^q}$ is homogeneous of order zero;*
3. *$\Lambda_{1|L^q}$ is closed in $L^q(\partial\Omega) \times L^\infty_\sigma(\partial\Omega)$;*
4. *$\Lambda_{1|L^2}$ can be characterized by*

$$\Lambda_{1|L^2} = \partial_{L^2}\varphi_{|L^2} \tag{1.78}$$

where $\partial_{L^2}\varphi_{|L^2}$ denotes the subgradient on $L^2(\partial\Omega)$ of the restriction $\varphi_{1|L^2}$ on $L^2(\partial\Omega)$ of the functional φ given by (1.72).

The property that the operator $\Lambda_{1|L^q}$ is m-accretive in $L^q(\partial\Omega)$ yields the well-posedness of Cauchy problem (1.74) for initial values u_0 in $L^q(\partial\Omega)$ and forcing term $g \in L^1(0,T;L^q(\partial\Omega))$ in the sense of *mild solutions* in $L^q(\partial\Omega)$.

Corollary 1.4 (Existence and uniqueness in $L^q(\partial\Omega)$) *Let $1 \le q \le \infty$ and suppose F is given by (4.71) with f satisfying (4.68). Then, for every $u_0 \in L^q(\partial\Omega)$ and*

$g \in L^1(0,T;L^q(\partial\Omega))$, there is a unique mild solution of Cauchy problem (5.34) in $L^q(\partial\Omega)$.

Due to the fact that $\Lambda_{1|L^q}$ is completely accretive and by [30] (see also [56]), the following comparison principle is available. Here, we write $[u]^\nu$ with $\nu \in \{+,1\}$ for either denoting the *positive part* $[u]^+ = \max\{0,u\}$ of u or $[u]^1 := u$ itself.

Corollary 1.5 (Comparison principle and well-posedness) *Let $1 \le q \le \infty$ and suppose F is given by (1.75) with f satisfying (1.76). Then, for every h_0 and $\hat{h}_0 \in L^q(\partial\Omega)$, g, $\hat{g} \in L^1(0,T;L^q(\partial\Omega))$, and corresponding two mild solutions h and v of Cauchy problem (1.74), one has that*

$$\|[h(t)-\hat{h}(t)]^\nu\|_q \le e^{\omega t}\|[h(s)-\hat{h}(s)]^\nu\|_q + \int_s^t e^{\omega(t-s)}\|[g(r)-\hat{g}(r)]^\nu\|_q \,\mathrm{d}r$$

for every $0 \le s < t \le T$, and $\nu \in \{+,1\}$.

Our next theorem is concerned with the *regularizing effect* that a mild solution of Cauchy problem (1.74) is, indeed, a *strong solution* of (1.74). The regularizing effect described in the first two statements of Theorem 1.10 is due to the fact that the Dirichlet-to-Neumann map $\Lambda_{1|L^2}$ in $L^2(\partial\Omega)$ can be realized as a subgradient (see (1.71) or (1.78)) and hence, follows from a classic result due to Brezis [40] (see, also [11]). The regularizing effect stated in Theorem 1.10 (3) results from the property that the Dirichlet-to-Neumann map $\Lambda_{1|L^q}$ is homogeneous of order zero and so, follows from an application of [86] (see also [28] and [87]). We give the details of the proof of this theorem in Chapter 7.4.3.

Theorem 1.10 (Regularizing effect) *Let F be given by (1.75) with f satisfying (1.76), and $\varphi_f : L^2(\partial\Omega) \to \mathbb{R}$ denote the functional given by*

$$\varphi_f(h) := \varphi(h) + \int_{\partial\Omega}\int_0^{h(x)} f(x,r)\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^{d-1} \qquad \text{for every } h \in L^2(\partial\Omega), \tag{1.79}$$

where φ is the functional defined by (1.72). Then the following statements hold.

1. *(Max. L^2-regularity) If there is a $b \in L^\infty(\partial\Omega)$ such that*

$$|f(x,h)| \le b(x) \qquad \text{for all } h \in \mathbb{R} \text{ and } \mathcal{H}^{d-1}\text{-a.e. } x \in \partial\Omega, \tag{1.80}$$

then the functional φ_f defined by (1.79) can be extended continuously to a functional on $L^1(\partial\Omega)$. Moreover, then for every $h_0 \in L^1(\partial\Omega)$ and $g \in L^2(0,T;L^2(\partial\Omega))$, the mild solution h of Cauchy problem (1.74) in $L^1(\partial\Omega)$ is strong variational with time-derivative

$$\frac{\mathrm{d}h}{\mathrm{d}t} \in L^2(0,T;L^2(\partial\Omega))$$

and global estimate

$$\frac{1}{2}\int_0^t \left\| \frac{\mathrm{d}h}{\mathrm{d}s}(s) \right\|_2^2 \mathrm{d}s + \varphi_f(h(t)) \le \varphi_f(h_0) + \frac{1}{2}\int_0^t \|g(s)\|_2^2 \mathrm{d}s \tag{1.81}$$

for every $0 \le t \le T$.

2. *For every* $h_0 \in L^2(\partial\Omega)$ *and* $g \in L^2(0,T;L^2(\partial\Omega))$, *the mild solution* u *of Cauchy problem* (1.74) *in* $L^2(\partial\Omega)$ *is a strong variational solution in* $L^2(\partial\Omega)$ *with*

$$h \text{ and } \frac{\mathrm{d}h}{\mathrm{d}t} \in L^2(0,T;L^2(\partial\Omega)).$$

3. *(L^1 Aronson-Bénilan type estimates) Let* $1 \le q \le \infty$. *Then for every* $h_0 \in L^q(\partial\Omega)$ *and* $g \in W^{1,1}(0,T;L^q(\partial\Omega))$, *the mild solution* h *of Cauchy problem* (1.74) *in* $L^q(\partial\Omega)$ *is a strong solution in* $L^q(\partial\Omega)$ *satisfying*

$$\left\| \frac{\mathrm{d}h}{\mathrm{d}t_+}(t) \right\|_q \le \frac{1}{t}\left[a_\omega(t) + \omega \int_0^t a_\omega(s) e^{\omega(t-s)} \mathrm{d}s \right] \quad \text{for a.e. } t \in (0,T), \tag{1.82}$$

where

$$a_\omega(t) := \int_0^t \|g'(s)\|_q \, s \, \mathrm{d}s + \left[\left(1 + e^{\omega t}\right) \|h_0\|_q \right.$$
$$\left. + \int_0^t \|g(s)\|_q \, \mathrm{d}s + \omega \int_0^t \int_0^s e^{-\omega r} \|g(r)\|_q \mathrm{d}r \, \mathrm{d}s \right].$$

According to Corollary 1.3, for every $1 \le q \le \infty$, the operator $-(\Lambda_{1|L^q} + F)$ generates a strongly continuous semigroup $\{e^{-t(\Lambda_{1|L^q}+F)}\}_{t\ge 0}$ of quasi-contractions on $L^q(\partial\Omega)$ (see Section 3.2.2.1 for a concise review of nonlinear semigroup theory). But since $\partial\Omega$ is assumed to be compact, the semigroup $\{e^{-t(\Lambda_{1|L^q}+F)}\}_{t\ge 0}$ generated by $-(\Lambda_{1|L^q} + F)$ on $L^q(\partial\Omega)$ coincides with the semigroup $\{e^{-t(\Lambda_1+F)}\}_{t\ge 0}$ generated by $-(\Lambda_1 + F)$ on $L^1(\partial\Omega)$. For this reason, it is sufficient to consider only the semigroup $\{e^{-t(\Lambda_1+F)}\}_{t\ge 0}$ on $L^q(\partial\Omega)$, which is quasi-contractive on $L^q(\partial\Omega)$ for all $1 \le q \le \infty$.

The next corollary summarizes the regularity properties of the semigroup $\{e^{-t(\Lambda_1+F)}\}_{t\ge 0}$. Here, Λ_1° denotes the *minimal selection* of Λ defined by (3.28) in Chapter 3.2.2.

Corollary 1.6 *Let* F *be given by* (1.75) *with* f *satisfying* (1.76) *and* $1 \le q \le \infty$. *Then the operator* $-(\Lambda_1 + F)$ *generates a strongly continuous semigroup* $\{e^{-t(\Lambda_1+F)}\}_{t\ge 0}$ *on* $L^1(\partial\Omega)$, *which is* ω*-quasi complete contractive on* $L^q(\partial\Omega)$ *for every* q. *Moreover,* $\{e^{-t(\Lambda_1+F)}\}_{t\ge 0}$ *has the following regularity properties:*

1. *(L^1 Aronson-Bénilan type estimates) For every* $h_0 \in L^q(\partial\Omega)$, *the mapping* $t \mapsto e^{-t(\Lambda_1+F)}h_0$ *is differentiable in* $L^q(\partial\Omega)$ *at a.e.* $t \in (0,\infty)$ *and*

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t_+} e^{-t(\Lambda_1+F)} h_0 \right\|_q \le \frac{2+\omega t}{t} e^{\omega t} \|h_0\|_q \qquad \text{for every } t > 0;$$

2. *If $F \equiv 0$, then for every $h_0 \in L^1(\partial\Omega)$ and $t > 0$, $\frac{\mathrm{d}}{\mathrm{d}t_+} e^{-t\Lambda_1} h_0$ exists in $L^1(\partial\Omega)$ and*
$$|\Lambda^\circ e^{-t\Lambda_1} h_0| \le 2\frac{|h_0|}{t} \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega; \tag{1.83}$$
3. *(Order preservation of the semigroup) For every h_0, $\hat{h}_0 \in L^q(\partial\Omega)$, one has that $h_0 \le \hat{h}_0$ yields that*
$$e^{-t(\Lambda_1+F)} h_0 \le e^{-t(\Lambda_1+F)} \hat{h}_0 \qquad \text{for all } t \ge 0.$$
4. *(Point-wise Aronson-Bénilan type estimates) For every positive $h_0 \in L^q(\partial\Omega)$, one has that*
$$\frac{\mathrm{d}}{\mathrm{d}t_+} e^{-t(\Lambda_1+F)} h_0 \le \frac{1}{t} e^{-t(\Lambda_1+F)} h_0 + g_0(t) \quad \text{for a.e. } t > 0, \tag{1.84}$$
where $g_0 : (0, \infty) \to L^q(\partial\Omega)$ is a measurable function satisfying
$$\|g_0(t)\|_q \le \frac{\omega}{t} \int_0^t e^{\omega(t-r)} \left\| \frac{\mathrm{d}}{\mathrm{d}t_+} e^{-r(\Lambda_1+F)} h_0 \right\|_q \mathrm{d}r \quad \text{for a.e. } t > 0.$$

The statements (1) and (3) of Corollary 1.6 follow from Theorem 1.10 and Corollary 1.5. Statements (2) and (4) of Corollary 1.6 are direct applications of Theorem 5.6 and Theorem 5.5 in Section 5.2.3 (alterantively, see [86, Theorem 2.9 and Theorem 4.14] and [87] for inequality (1.83)).

Remark 1.11 For $1 < p < \infty$, we outlined in Theorem 1.6 (in Section 1.9) the *L^q-L^r regularization effect* of the semigroup $\{e^{-\Lambda}\}_{t\ge 0}$ generated by the negative Dirichlet-to-Neumann map $-\Lambda$ associated with the weighted p-Laplace operator Δ_{p,σ_0} under the assumption that the conductivity coefficient σ_0 satisfies Assumption 1.1. We stress that one cannot expect a similar regularization effect in the case $p = 1$; that is, for the semigroup $\{e^{-\Lambda_1)}\}_{t\ge 0}$ generated by the negative Dirichlet-to-Neumann operator $-\Lambda_1$ associated with the 1-Laplace operator Δ_1 since the *trace-Sobolev* inequality on $BV(\Omega)$, merely maps into $L^1(\partial\Omega)$. We refer the interested reader to the monograph [56] for further discussion on this topic.

Our last theorem is dedicated to the long-time stability of the semigroup $\{e^{-t(\Lambda_{1|L^q}+F)}\}_{t\ge 0}$ generated by $-(\Lambda_{1|L^q(\partial\Omega)} + F)$ on $L^q(\partial\Omega)$.

Theorem 1.11 *Let $1 \le q \le \infty$, F be given by (1.75) with f satisfying (1.76), and φ_f the functional given by (1.79). Then the following statements hold.*

1. *(Energy decreasing) For every $h_0 \in L^1(\partial\Omega)$, the function $t \mapsto \varphi_f(e^{-t(\Lambda_1+F)} h_0)$ is monotonically decreasing along $(0, \infty)$.*
2. *(Conservation of mass) If $F \equiv 0$, then one has that*
$$\int_{\partial\Omega} e^{-t\Lambda_1} h_0 \,\mathrm{d}\mathcal{H}^{d-1} = \overline{h}_0 := \frac{1}{\mathcal{H}^{d-1}(\partial\Omega)} \int_{\partial\Omega} h_0 \,\mathrm{d}\mathcal{H}^{d-1} \qquad \text{for all } t \ge 0$$
and all $h_0 \in L^1(\partial\Omega)$.

3. *(Long-time stability in $L^q(\partial\Omega)$) If $F \equiv 0$, then for every $h_0 \in L^q(\partial\Omega)$ and $q < \infty$, one has that*
$$\lim_{t\to\infty} e^{-t\Lambda_1} h_0 = \overline{h}_0 \qquad \text{in } L^q(\partial\Omega)$$
and $\varphi(\overline{h}_0) = 0$.
4. *(Entropy-transport inequality) If $F \equiv 0$, then there is a $C > 0$ such that*
$$\|e^{-t\Lambda_1} h_0 - \overline{h_0}\|_1 \le C\,\varphi(e^{-t\Lambda_1} h_0) \qquad \text{for all } t > 0;$$
5. *For every $h_0 \in L^2(\partial\Omega)$, one has that*
$$\varphi(e^{-t\Lambda_1} h_0) \le 2\frac{\|h_0\|_2^2}{t} \qquad \text{for all } t > 0.$$

The statements of Theorem 1.11 are established in Proposition 7.20, Proposition 7.21, and Proposition 7.22 in Chapter 7.4.4.

It is worth noting that the semigroup $\{e^{t\Lambda_1}\}_{t\ge 0}$ does not admit an L^q-L^∞-regularization effect since the trace operator $\mathcal{T}r$ only maps $BV(\Omega)$ into $L^1(\partial\Omega)$. Moreover, we believe that the following conjecture holds.

Conjecture. For every $h_0 \in L^1(\partial\Omega)$, the trajectory $t \mapsto e^{-t\Lambda_1} h_0 - \overline{h_0}$ extincts in finite time.

We conclude this subsection with some important remarks and historical development on the 1-Laplace operator and the Dirichlet-to-Neumann operator.

Remark 1.12 1. As outlined in Section 7.2, for every $h \in L^2(\partial\Omega)$, there is a vector field $\mathbf{z}_h \in \mathcal{Z}_h$ and by the definition of $\Lambda_{1|L^2}$, one has that $(h, [\mathbf{z}_h, \nu]) \in \Lambda_{1|L^2}$. Now, on the one hand, an integration by parts (see Proposition **??**) yields that
$$\int_{\partial\Omega} [\mathbf{z}_h, \nu]\, u_h \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega |Du_h|.$$
But on the other hand, the integral equality
$$\int_{\partial\Omega} [\mathbf{z}, \nu]\, h \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega |Du_h| \tag{1.85}$$
is, in general, not true since for the notion of *weak solutions* u_h of Dirichlet problem (1.65), it is not required that the Dirichlet boundary condition $u_h = h$ on $\partial\Omega$ is satisfied in the *trace sense*: that is, there is a $H \in BV(\Omega)$ such that $\mathcal{T}r(H) = h$ and $H - u_h \in BV_0(\Omega)$. Here, we denote by $BV_0(\Omega)$ the closure $\overline{C_c^\infty(\Omega)}^{BV(\Omega)}$ of the set of test functions $C_c^\infty(\Omega)$ in $BV(\Omega)$.

2. Our comment in (1) of this remark provides a strong reasoning, but not an explicit proof, for why the recently developed theory [51] of *j-elliptic functionals* cannot be applied to the functional

$$\hat{\varphi}(u) = \int_{\Omega} |Du|, \qquad (u \in V_2(\Omega)), \tag{1.86}$$

where $V_2(\Omega) := \{u \in BV(\Omega) \mid \mathcal{T}r(u) \in L^2(\partial\Omega)\}$, in order to obtain well-posedness of the Cauchy problem (1.74) in $L^2(\partial\Omega)$. To be more precise, we briefly recall from [51] that for $\hat{\varphi}$ given by (1.86) and $j = \mathcal{T}r_{|V_2}$ the standard trace operator $\mathcal{T}r : BV(\Omega) \to L^1(\partial\Omega)$ restricted on V_2, the j-subgradient $\partial_j\hat{\varphi}$ on $L^2(\partial\Omega)$ can be expressed by

$$\partial_j\hat{\varphi} = \left\{ (h,g) \in L^2(\partial\Omega)^2 \;\middle|\; \begin{array}{l} \exists u_h \in V_2 \text{ s.t. } \mathcal{T}r(u_h) = h \;\&\; \exists \mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d) \\ \text{satisfying (1.66)-(1.69) with } u_h \;\&\; g = [\mathbf{z}_h, \nu] \end{array} \right\}.$$

Thus, one has that $\partial_{\mathcal{T}r_{|V_2}}\hat{\varphi} \subseteq \Lambda_{1|L^2}$. But we claim that the equation

$$\Lambda_{1|L^2} = \partial_{\mathcal{T}r_{|V_2}}\hat{\varphi} \tag{1.87}$$

cannot be true in general. To see this, D_1 be the unit disc in the plane $\mathbb{R}^2$ and $S_1 = \partial D_1$ the unit circle. Then, recall that Spradlin and Tamasan [137] (see also [70]) constructed a boundary function $h_0 \in L^\infty(\mathcal{S}_1)$, for which there is no solution $u_{h_0} \in BV(D_1)$ of the minimization problem

$$\inf\left\{ \int_{D_1} |Dv| \;\middle|\; v \in BV(D_1),\ v = h_0 \text{ in the weak } \textit{sense of traces} \right\}. \tag{1.88}$$

Hence, one has that $h_0 \notin D(\partial_{\mathcal{T}r_{|V_2}}\hat{\varphi})$. But on the other hand, since the effective domain $D(\Lambda_{1|L^2})$ of $\Lambda_{1|L^2}$ coincides with $L^2(S_1)$, and since $\mathcal{S}_1$ is compact, we have that $h_0 \in D(\Lambda_{1|L^2})$, contradicting (1.87).

3. Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ whose boundary $\partial\Omega$ satisfies the following two conditions:

   a. For every $x \in \partial\Omega$ there exists a $r_0 > 0$ such that for every set $A \subset\subset B(x,r_0)$ of finite perimeter (that is, $P(A,\Omega) := |D\mathbb{1}_A|(\Omega)$ is finite), one has that
   $$P(\Omega,\mathbb{R}^d) \le P(\Omega \cup A,\mathbb{R}^d);$$

   b. For every $x \in \partial\Omega$ and every $r > 0$ there is a set $A \subset\subset B(x,r)$ of finite perimeter such that
   $$P(\Omega, B(x,r)) > P(\Omega \setminus A, B(x,r)).$$

   Then by [138, Theorem 3.7 & Corollary 4.2] and by the characterization [110, Theorem 1.1] of functions of least gradients and weak solutions to Dirichlet problem (1.65), for every boundary data $h \in C(\partial\Omega)$ there is a unique weak solution $u \in BV(\Omega)$ of (1.65) satisfying $u = h$ a.e. on $\partial\Omega$. Due to this existence and uniqueness result, we know that at least in this situation, the integral equation (1.85) holds for every boundary data $h \in C(\partial\Omega)$.

The 1-Laplace operator Δ_1 is not only interesting from its geometrical perspectives and its applications to engineering sciences, but also by his mathematical challenges. For a given $u \in BV(\Omega)$, $\Delta_1 u$ is the *scalar mean curvature* of the level sets of u. Thus, every level surface $\{u = t\}$ of a function u of least gradient has mean curvature zero; a necessary condition for functions u whose super-level sets $\{u \geq t\}$ are area-minimizing. Functions of least gradient do not have much regularity, in the sense that even though u might be essentially bounded, necessarily, u need not admit a continuous representative on $\overline{\Omega}$. In fact, in some applications, this property of functions of least gradient is strongly desired, for example, in image processing (see [5] and the references therein); if the nonlinear diffusion process associated with Δ_1 is used to recover a blurred picture $u_0 : \Omega \to [0,1]$, $(\Omega \subseteq \mathbb{R}^2)$, then the contours in u_0 are maintained and not smoothened as compared to diffusion processes involving linear or degenerate differential operators. But the operator Δ_1 also appears in other engineering fields. For example in free material design (see [82]), or conductivity imaging (see [91]).

Chapter 2
The Poisson problem and the Neumann-to-Dirichlet map

Abstract In this chapter, we intend to establish Theorem 1.4 for the Dirichlet-to-Neumann map Λ_{σ} associated with Leray-Lions operators. The weighted p-Laplace operator Δ_{p,σ_0} for σ_0 satisfying Assumption 1.1 is an important prototype of Leray-Lions operators. The content of this chapter covers parts of the paper [85].

2.1 The Poisson problem

This chapter is devoted to establish well-posedness of the Poisson problem

$$\Lambda_{\sigma} h = f \qquad \text{on } \partial\Omega. \tag{2.1}$$

and Hölder-regularity of weak solutions h satisfying the compatibility condition (2.31), where the leading operator $\Lambda_{\sigma} : W^{-(1-1/p),p'}(\partial\Omega) \to W^{1-1/p,p}(\partial\Omega)$ is the Dirichlet-to-Neumann map associated with the second order quasi-linear operator $A = -\operatorname{div}(\boldsymbol{\sigma}(x,\nabla\cdot))$ (cf. (2.20) below).

Throughout this chapter, we assume that Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 2$ with at least a Lipschitz-continuous boundary $\partial\Omega$ in the sense of [115, Sect. 1.3]. Further, let $1 < p < \infty$.

If nothing else is said, then let $\boldsymbol{\sigma} : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}^d$ and $\mathcal{A} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be two Carathéodory functions satisfying

$$\boldsymbol{\sigma}(x,\xi)\xi \geq \eta|\xi|^p \tag{2.2}$$

$$|\boldsymbol{\sigma}(x,\xi)| \leq c|\xi|^{p-1} \tag{2.3}$$

$$(\boldsymbol{\sigma}(x,\xi_1) - \boldsymbol{\sigma}(x,\xi_1))(\xi_1 - \xi_2) > 0 \tag{2.4}$$

$$\nabla_{\xi}\mathcal{A}(\cdot,\xi) = \boldsymbol{\sigma}(\cdot,\xi) \tag{2.5}$$

for a.e. $x \in \Omega$ and all $\xi, \xi_1, \xi_2 \in \mathbb{R}^d$ with $\xi_1 \neq \xi_2$.

We begin this section with the following definition.

Definition 2.1 Let $f \in W^{-(1-1/p),p'}(\partial\Omega)$ satisfy the compatibility condition (2.31). Then, we call a function h a *weak solution* of the Poisson problem (2.1) if $h \in W^{1-1/p,p}(\partial\Omega)$ and satisfies

$$\int_\Omega \sigma(x, \nabla P(h)) \nabla \mathcal{Z}\xi \, dx = \langle f, \xi \rangle \tag{2.6}$$

for all $\xi \in W^{1-1/p,p}(\partial\Omega)$.

Now, we are in a position to formulate our main result of this chapter, from where the statement of Theorem 1.4 follows as a special case.

Theorem 2.1 *Suppose that $\sigma : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (2.2)-(2.4). Then the following assertions hold.*

1. *For every $f \in W_m^{-(1-1/p),p'}(\partial\Omega)$ there is a unique weak solution $h \in W_m^{1-1/p,p}(\partial\Omega)$ of the Poisson problem (2.1). Moreover, the Neumann-to-Dirichlet map Λ^{-1} assigning Neumann data f to the unique weak solution h of (2.1) is continuous from $W_m^{-(1-1/p),p'}(\partial\Omega)$ to $W_m^{1-1/p,p}(\partial\Omega)$.*
2. *Let $q = \frac{d-1}{p-1-\varepsilon}$ for some $\varepsilon \in (0,1)$ if $p \le d$ and $q = 1$ if $p > d$. Further, let $\psi \in L^q(\partial\Omega)$ satisfying (2.31). Then there are $\alpha \in (0,1)$ and $c_\alpha \ge 0$ such that every weak solution $h \in W^{1-1/p,p}(\partial\Omega)$ of Poisson problem (2.1) belongs to $C^{0,\alpha}(\partial\Omega)$ and satisfies*

$$\|h\|_{C^{0,\alpha}(\partial\Omega)} \le c_\alpha \left(\|f\|_{L^q(\partial\Omega)}^{\frac{1}{p-1}} + \|P(h)\|_{L^p(\Omega)} \right) + c_\alpha. \tag{2.7}$$

We outline the proof of this theorem in Section 2.5 after introducing some notions, definitions, and results, which we will require later on.

2.2 Preliminaries

For $1 \le q \le \infty$, we denote by $L^q(\Omega)$ and $W^{1,q}(\Omega)$ the usual Lebesgue and first Sobolev spaces. We denote by $C^{0,1}(\overline{\Omega})$ the space of all Lipschitz-continuous functions on the closure $\overline{\Omega}$ of Ω. The boundary $\partial\Omega$ of Ω is equipped with the $(d-1)$-dimensional Hausdorff measure when we work with the Lebesgue space $L^q(\partial\Omega)$. Moreover, $C(\partial\Omega)$ denotes the set of all real-valued continuous functions on $\partial\Omega$.

Since Ω is a Lipschitz domain, the mapping $u \mapsto u_{|\partial\Omega}$ from $C^{0,1}(\overline{\Omega})$ to $C^{0,1}(\partial\Omega)$ has a unique continuous extension mapping

$$\mathcal{T}r : W^{1,p}(\Omega) \to L^{p^*}(\partial\Omega)$$

called *trace operator* with $p^* = \frac{p(d-1)}{d-p}$ if $1 \le p < d$, $p^* \ge 1$ if $p = d$ and $p^* = \infty$ if $p > d$ (cf. [115, Théorème 4.2, 4.6, and 3.8]). For convenience, we write $u_{|\partial\Omega} := Tr(u)$ for $u \in W^{1,p}(\Omega)$ even if u does not belong to $C(\overline{\Omega})$ and call $u_{|\partial\Omega}$ the *trace* of u. The following properties of the trace operator $\mathcal{T}r$ will be used frequently throughout this article:

1. the kernel $ker(\mathcal{T}r) := \{u \in W^{1,p}(\Omega) \,|\, Tr(u) = 0\}$ of $\mathcal{T}r$ coincides with the Sobolev space $W_0^{1,p}(\Omega)$,
2. the range $Rg(\mathcal{T}r) := \{Tr(u) \,|\, u \in W^{1,p}(\Omega)\}$ of $\mathcal{T}r$ coincides with the Sobolev-Slobodečki space $W^{1-1/p,p}(\partial\Omega)$, which is defined to be the linear subspace of all $h \in L^p(\partial\Omega)$ with finite semi-norm
$$[h]_p^p := \int_{\partial\Omega}\int_{\partial\Omega} \tfrac{|h(x)-h(y)|^p}{|x-y|^{d-2+p}}\,\mathrm{d}x\mathrm{d}y$$
and is equipped with the norm $\|h\|_{W^{1-1/p,p}(\partial\Omega)} := \|h\|_{L^p(\partial\Omega)} + [h]_p$ for every $h \in W^{1-1/p,p}(\partial\Omega)$ (cf. [115, Section 3.8]),
3. the trace operator $\mathcal{T}r$ has a linear and bounded right inverse (cf. [115, Théorème 5.7])
$$\mathcal{Z} : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega). \tag{2.8}$$

Here, we denote by $W^{-(1-1/p),p'}(\partial\Omega)$ the dual space of $W^{1-1/p,p}(\partial\Omega)$ and by $\langle\chi,\psi\rangle$ the value of $\chi \in W^{-(1-1/p),p'}(\partial\Omega)$ at $\psi \in W^{1-1/p,p}(\partial\Omega)$.

Another crucial property of a Lipschitz domain Ω is that $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ for $p \in [1,\infty)$. We state this standard result explicitly for later use.

Lemma 2.1 *The space $C^\infty(\overline{\Omega})$ lies dense in $W^{1,p}(\Omega)$ and the set $\{\xi_{|\partial\Omega} \,|\, \xi \in C^\infty(\overline{\Omega})\}$ is dense in $W^{1-1/p,p}(\partial\Omega)$ and $L^2(\partial\Omega)$. In particular, $C^\infty(\overline{\Omega})$ is dense in the space*
$$V_{p,2}(\Omega) := \left\{\xi \in W^{1,p}(\Omega) \,\middle|\, \mathcal{T}r(\xi) \in L^2(\partial\Omega)\right\}$$
equipped with the sum norm.

Proof First, by the Stone-Weierstraß Theorem, the set $\{\xi_{|\partial\Omega} \,|\, \xi \in C^\infty(\mathbb{R}^d)\}$, which may be identified with a subset of $\{\xi_{|\partial\Omega} \,|\, \xi \in C^\infty(\overline{\Omega})\}$, is dense in $C(\partial\Omega)$. Since the $(d-1)$-dimensional Hausdorff measure is Borel regular (cf. [73, Theorem 1, Sec. 2.1]), the latter set is dense in $L^2(\partial\Omega)$. Note that this also means that $W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega)$ lies dense in $L^2(\partial\Omega)$. Since the set $\{\xi_{|\overline{\Omega}} \,|\, \xi \in C^\infty(\mathbb{R}^d)\}$ is a subset of $C^\infty(\overline{\Omega})$, the last claims follows from [115, Théorème 3.1]. □

Besides Lemma 2.1, we need the following p-variant of Maz'ya's remarkable inequality
$$\|u\|_{L^{pd/(d-1)}(\Omega)} \le C\left(\|\nabla u\|_{L^p(\Omega)^d} + \|\mathcal{T}r(u)\|_{L^p(\partial\Omega)}\right) \tag{2.9}$$
holding for all $u \in W^{1,p}(\Omega)$ provided $1 \le p < \infty$. Here the constant $C > 0$ depends on p, the volume $|\Omega|$, and the isoperimetric constant $C(d)$ (cf. [109, Cor. 3.6.3] and see also [62]). Using inequality (2.9) and Poincaré's inequality on $W^{1,p}(\Omega)$, yields the following useful inequality
$$\|u\|_{L^p(\Omega)} \le \tilde{C}\left(\|\nabla u\|_{L^p(\Omega)^d} + \|\mathcal{T}r(u)\|_{L^2(\partial\Omega)}\right) \tag{2.10}$$

for every $u \in W^{1,p}(\Omega)$ with trace $\mathcal{T}r(u) \in L^2(\partial\Omega)$, where $\tilde{C} > 0$ is some constant independent of u. To see this, set $u_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}x$ for given $u \in L^1(\Omega)$. Then Poincaré's inequality says that there is a $C_1 > 0$ such that

$$\|u - u_\Omega\|_{L^p(\Omega)} \le C_1 \|\nabla u\|_{L^p(\Omega)^d}$$

for every $u \in W^{1,p}(\Omega)$. Rearranging this inequality yields

$$\|u\|_{L^p(\Omega)} \le C_2\Big(\|\nabla u\|_{L^p(\Omega)^d} + \|u\|_{L^1(\Omega)}\Big)$$

for every $u \in W^{1,p}(\Omega)$. Applying to this Maz'ya's inequality (2.9) for $p = 1$ and the two inequalities $\|\nabla u\|_{L^1(\Omega)^d} \le C\,\|\nabla u\|_{L^p(\Omega)^d}$ for $u \in W^{1,p}(\Omega)$ and $\|u\|_{L^1(\partial\Omega)} \le C\,\|u\|_{L^2(\partial\Omega)}$ for $u \in L^2(\partial\Omega)$ leads to (2.10).

Concluding this section, we emphasise that we follow here the convention that constants denoted by C or c_α may vary from line to line.

2.3 The Dirichlet problem

In this section we review some basic facts about weak solutions of the nonlinear Dirichlet problem

$$\begin{cases} -\operatorname{div}(\boldsymbol{\sigma}(x,\nabla u)) = 0 & \text{in } \Omega,\\ \qquad\qquad\qquad u = h & \text{on } \partial\Omega \end{cases} \tag{2.11}$$

for given boundary data $h \in W^{1-1/p,p}(\partial\Omega)$. We start by recalling the following definition.

Definition 2.2 We call $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ a *weak solution* of equation

$$-\operatorname{div}(\boldsymbol{\sigma}(x,\nabla u)) = 0 \qquad \text{in } \Omega \tag{2.12}$$

if u satisfies the integral equation

$$\int_\Omega \boldsymbol{\sigma}(x,\nabla u)\nabla\xi\,\mathrm{d}x = 0 \tag{2.13}$$

for every $\xi \in W^{1,p}_0(\Omega)$.

For later use, we need the following compactness result concerning weak solutions of (2.12), which is an immediate consequence of [34, Theorem 2.1 & Remark 2.1]. We leave the details of the proof of this lemma to the reader.

Lemma 2.2 *Let $(u_n)_{n\ge1}$ be a bounded sequence in $W^{1,p}(\Omega)$ of weak solution of (2.12), then there is subsequence $(u_{k_n})_{n\ge1}$ of $(u_n)_{n\ge1}$ and a weak solution $u \in W^{1,p}(\Omega)$ of (2.12) such that u_{k_n} converges to u weakly in $W^{1,p}(\Omega)$, strongly* *in* $L^p(\Omega)$*,* ∇u_{k_n} *converges to* ∇u *a.e in* Ω *and* $\boldsymbol{\sigma}(x, \nabla u_{k_n})$ *converges to* $\boldsymbol{\sigma}(x, \nabla u)$ *weakly in* $L^{p'}(\Omega)$ *and a.e. on* Ω*.*

Now, for any boundary value $h \in W^{1-1/p,p}(\partial\Omega)$, let $\Phi \in W^{1,p}(\Omega)$ be such that $\Phi_{|\partial\Omega} = h$ and set $\mathcal{K}_\Phi := \Phi + W_0^{1,p}(\Omega)$. We consider the energy functional $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) = \int_\Omega \mathcal{A}(x, \nabla u)\,\mathrm{d}x \tag{2.14}$$

for every $u \in W^{1,p}(\Omega)$. The functional φ is continuously differentiable by growth condition (2.3), the derivative $\varphi' : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ is given by

$$\langle \varphi'(u), v \rangle = \int_\Omega \boldsymbol{\sigma}(x, \nabla u) \nabla v \, dx \tag{2.15}$$

for every $u, v \in W^{1,p}(\Omega)$ and φ is convex by the monotonicity condition (2.4) (cf. [96, Theorem 6.2.1]). The restriction $\varphi_{|\mathcal{K}_\Phi}$ of φ on $\mathcal{K}_\Phi$ is coercive by condition (2.2) and by Poincaré's inequality. Furthermore, by (2.4), the Fréchet-derivative φ' is strictly monotone on the set $\mathcal{K}_\Phi$. Thus the convex minimization principle (cf. [155, Theorem 2.E]) implies that there is a unique solution $u \in W^{1,p}(\Omega)$ of the minimization problem:

$$\min \left\{ \int_\Omega \mathcal{A}(x, \nabla u)\,\mathrm{d}x \,\middle|\, u \in W^{1,p}(\Omega) \text{ with } u - \Phi \in W_0^{1,p}(\Omega) \right\}. \tag{2.16}$$

Note that the uniqueness of the minimizer of (2.16) is independent of the choice of Φ. Hence we can take $\Phi = \mathcal{Z}(h)$ for every boundary value $h \in W^{1-1/p,p}(\partial\Omega)$, where $\mathcal{Z}$ denotes the right inverse of the trace operator $\mathcal{T}r$ given in (2.8). Moreover, $u \in W^{1,p}(\Omega)$ is the minimizer of (2.16) if and only if $u - \Phi \in W_0^{1,p}(\Omega)$ and u is a weak solution of (2.12) on Ω (cf. [155, Theorem 2.E]). Because of the regularity and the boundary value of u, we arrive to the following definition.

Definition 2.3 For given boundary value $h \in W^{1-1/p,p}(\partial\Omega)$, we call a function $u \in W^{1,p}(\Omega)$ a $W^{1,p}$*-solution* of Dirichlet problem (2.11) on Ω if $u - \mathcal{Z}(h) \in W_0^{1,p}(\Omega)$ and u is a weak solution of equation (2.12).

In the next proposition, we collect some well-known properties about $W^{1,p}$-solutions of (2.11) for later use.

Proposition 2.1 *Consider the mapping* $P : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ *defined by assigning to each* $h \in W^{1-1/p,p}(\partial\Omega)$ *the unique* $W^{1,p}$*-solution* u *of* (2.11) *with boundary value* h*. Then the following statements are true.*

1. *P is well-defined and injective mapping.*
2. *P is continuous.*
3. *Let* $h, \psi \in W^{1-1/p,p}(\partial\Omega)$*,* $\Psi \in W^{1,p}(\Omega)$ *such that* $\Psi_{|\partial\Omega} = \psi$*, and* $\lambda \in \mathbb{R}$*. Then we have that*

$$\int_\Omega \mathcal{A}(x, \nabla P(\lambda h+\psi))\,\mathrm{d}x \le \int_\Omega \mathcal{A}(x, \lambda\nabla P(h)+\nabla\Psi)\,\mathrm{d}x.$$

4. *Let $h\in W^{1-1/p,p}(\partial\Omega)$. Then for every $\Phi\in W^{1,p}(\Omega)$ with $\Phi_{|\partial\Omega}=h$, there is a unique $u_\Phi\in W^{1,p}_0(\Omega)$ such that $P(h)=u_\Phi+\Phi$.*
5. *If the function $\boldsymbol{\sigma}:\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ satisfies* homogeneity condition

$$\boldsymbol{\sigma}(x,\lambda\xi)=|\lambda|^{p-2}\lambda\,\boldsymbol{\sigma}(x,\xi) \tag{2.17}$$

for every $\lambda\in\mathbb{R}$, $\xi\in\mathbb{R}^d$, and a.e. $x\in\Omega$, then P satisfies

$$P(\lambda h)=\lambda P(h)$$

for every $h\in W^{1-1/p,p}(\partial\Omega)$ and $\lambda\in\mathbb{R}$.

Proof Claim (1), (4) and (5) follow from the existence and uniqueness of boundary value problem (2.11). Since $W^{1,p}$-solutions of Dirichlet problem (2.11) can be characterized as the unique minimizer of problem (2.16), it follows that claim (3) holds. In particular, claim (2) is well-known, but for the sake of completeness, we give here the proof. Let $(h_n)_{n\ge1}$ be a sequence in $W^{1-1/p,p}(\partial\Omega)$ and $h\in W^{1-1/p,p}(\partial\Omega)$ such that h_n converges to h in $W^{1-1/p,p}(\partial\Omega)$. Since $P(h_n)$ is the unique minimizer of problem (2.16),

$$\int_\Omega \mathcal{A}(x,\nabla P(h_n))\mathrm{d}x \le \int_\Omega \mathcal{A}(x,\nabla\mathcal{Z}(h_n))\mathrm{d}x.$$

Combining this with the two conditions (2.2) and (2.3) gives

$$\frac{\eta}{p}\int_\Omega |\nabla P(h_n)|^p\mathrm{d}x \le c\int_\Omega|\nabla\mathcal{Z}(h_n)|^p\mathrm{d}x \tag{2.18}$$

for every n. Since $\mathcal{Z}:W^{1-1/p,p}(\partial\Omega)\to W^{1,p}(\Omega)$ is bounded, the sequence $(\mathcal{Z}(h_n))_{n\ge1}$ is bounded in $W^{1,p}(\Omega)$. Thus, by the boundedness of $(\nabla\mathcal{Z}(h_n))_{n\ge1}$ in $L^p(\Omega)^d$ and by using the estimate (2.18), we see that that $(\nabla P(h_n))_{n\ge1}$ is bounded in $L^p(\Omega)^d$. Now, Maz'ya's inequality (2.9) implies that $(P(h_n))_{n\ge1}$ is bounded in $W^{1,p}(\Omega)$. By Lemma 2.2, there is a weak solution $u\in W^{1,p}(\Omega)$ of (2.12) on Ω and there is a subsequence $(h_{k_n})_{n\ge1}$ of $(h_n)_{n\ge1}$ such that $P(h_{k_n})$ converges to u weakly in $W^{1,p}(\Omega)$, $P(h_{k_n})$ converges to $P(h)$ in $L^p(\Omega)$, $\nabla P(h_{k_n})$ converges to ∇u a.e. on Ω and $\boldsymbol{\sigma}(x,\nabla P(h_{k_n}))$ converges to $\boldsymbol{\sigma}(x,\nabla P(h))$ weakly in $L^{p'}(\Omega)^d$. By the compactness of the trace operator $\mathcal{T}r:W^{1,p}(\Omega)\to L^p(\partial\Omega)$ and since h_{k_n} converges to h in $L^p(\partial\Omega)$, we can conclude that $u=P(h)$. Since $P(h_{k_n})$ converges to $P(h)$ in $L^p(\Omega)$, it remains to show that $\nabla P(h_{k_n})$ converges to $\nabla P(h)$ in $L^p(\Omega)^d$. To see this, note that by monotonicity condition (2.4),

$$\chi_{k_n}(x):=(\boldsymbol{\sigma}(x,\nabla P(h_{k_n})(x))-\boldsymbol{\sigma}(x,\nabla P(h)(x)))(\nabla P(h_{k_n})(x)-\nabla P(h)(x)) \tag{2.19}$$

is non-negative for a.e. $x\in\Omega$. Since $P(h_{k_n})$ is a weak solution of (2.12),

$$\int_{\Omega} \chi_{k_n} \mathrm{d}x = \int_{\Omega} \boldsymbol{\sigma}(x, \nabla P(h_{k_n})) \nabla \mathcal{Z}(h_{k_n}) \mathrm{d}x - \int_{\Omega} \boldsymbol{\sigma}(x, \nabla P(h_{k_n})) \nabla P(h) \mathrm{d}x \\ - \int_{\Omega} \boldsymbol{\sigma}(x, \nabla P(h)) (\nabla P(h_{k_n}) - \nabla P(h))\, \mathrm{d}x.$$

Therefore and by the weak convergence of $\boldsymbol{\sigma}(x, \nabla P(h_{k_n}))$ to $\boldsymbol{\sigma}(x, \nabla P(h))$ in $L^{p'}(\Omega)^d$, the strong convergence of $\mathcal{Z}(h_{k_n})$ to $\mathcal{Z}h$ in $W^{1,p}(\Omega)$, and since $P(h)$ is a weak solutions of (2.12), it follows that χ_{k_n} converges to 0 in $L^1(\Omega)$. By using the definition of χ_{k_n}, coercivity condition (2.2), and Hölder's inequality, we see

$$\begin{aligned} \eta \int_E |\nabla P(h_{k_n})|^p \mathrm{d}x &\le \int_E \boldsymbol{\sigma}(x, \nabla P(h_{k_n})) \nabla P(h_{k_n}) \mathrm{d}x \\ &= \int_E \chi_{k_n} \mathrm{d}x + \int_E \boldsymbol{\sigma}(x, \nabla P(h_{k_n})) \nabla P(h) \mathrm{d}x \\ &\quad + \int_E \boldsymbol{\sigma}(x, \nabla P(h)) (\nabla P(h_{k_n}) - \nabla P(h)) \mathrm{d}x \\ &\le \int_E \chi_{k_n} \mathrm{d}x + \|\boldsymbol{\sigma}(\cdot, \nabla P(h_{k_n}))\|_{L^{p'}(\Omega)} \left(\int_E |\nabla P(h)|^p \mathrm{d}x \right)^{1/p} \\ &\quad + \left(\int_E \boldsymbol{\sigma}(x, \nabla P(h)) \mathrm{d}x \right)^{1/p'} \|\nabla P(h_{k_n}) - \nabla P(h)\|_{L^p(\Omega)} \end{aligned}$$

for every measurable subset $E \subseteq \Omega$. Since the measurable set E was arbitrary, and since χ_{k_n} is equi-integrable in $L^1(\Omega)$, $(\boldsymbol{\sigma}(\cdot, \nabla P(h_{k_n})))_{n \ge 1}$ is bounded in $L^{p'}(\Omega)$ and $(\nabla P(h_{k_n}))_{n \ge 1}$ is bounded in $L^p(\Omega)$, our last estimates show that $(|\nabla P(h_{k_n})|^p)_{n \ge 1}$ is equi-integrable in $L^1(\Omega)$. Thus and since $\nabla P(h_{k_n})$ converges to $\nabla P(h)$ a.e. on Ω, it follows (see [35, Corollary 4.5.5]) that $\nabla P(h_{k_n})$ converges to $\nabla P(h)$ in $L^p(\Omega)^d$. Since the same arguments hold, for each subsequence of a convergent sequence $(h_n)_{n \ge 1}$ in $W^{1-1/p,p}(\partial\Omega)$, we have thereby shown that the operator P is continuous from $W^{1-1/p,p}(\partial\Omega)$ to $W^{1,p}(\Omega)$. □

We conclude this section with the following remark concerning the well-posedness of Dirichlet problem (2.11).

Remark 2.1 In order to establish well-posedness of Dirichlet problem (2.11) one does not need to assume the gradient condition (2.5). Indeed, the well-posedness of (2.11) is well-known (see [103, Chapitre 2. Sect. 2.3.2] and use Lemma 2.2).

2.4 The Dirichlet-to-Neumann map

This section is devoted to introduce the Dirichlet-to-Neumann map Λ_{σ} associated with the second order quasi-linear operator (also called a *Leray-Lions operator*)

$$Au := -\operatorname{div}(\boldsymbol{\sigma}(x, \nabla u)) \tag{2.20}$$

in the sense of distributions for $u \in W^{1,p}(\Omega)$ and to discuss some basic properties.

For given boundary value $h \in W^{1-1/p,p}(\partial\Omega)$, let $P(h)$ be the unique $W^{1,p}$-solution of Dirichlet-problem (2.11) for boundary value h as introduced in Proposition 2.1. If $P(h)$ is smooth enough up to the boundary, then the Dirichlet-to-Neumann map Λ_σ associated with the Leray-Lions operator (2.20) is *formally* defined by

$$\Lambda_\sigma h := \boldsymbol{\sigma}(x, \nabla P(h)) \cdot \boldsymbol{\nu}$$

on $\partial\Omega$. Let $\psi \in W^{1-1/p,p}(\partial\Omega)$ and suppose that $P(\psi)$ is smooth. Then multiplying this equation by $P(\psi)$ with respect to the inner product on $L^2(\partial\Omega)$ and applying Green's formula yields

$$\int_{\partial\Omega} \Lambda_\sigma h \psi \, \mathrm{d}\mathcal{H}^{d-1} = \int_\Omega \boldsymbol{\sigma}(x, \nabla P(h)) \nabla P(\psi) \, \mathrm{d}x. \tag{2.21}$$

Even if $P(h)$ and $P(\psi)$ merely belong to $W^{1,p}(\Omega)$, the integral on the right-hand side of this equation exists. Thus, we could use this integral to define the Dirichlet-to-Neumann map Λ_σ as a mapping from $W^{1-1/p,p}(\partial\Omega)$ into $W^{-(1-1/p),p'}(\partial\Omega)$. But, since the mapping P, in general, is not linear, it is *a priori* not clear whether the functional

$$\psi \mapsto \int_\Omega \boldsymbol{\sigma}(x, \nabla P(h)) \nabla P(\psi) \, \mathrm{d}x \tag{2.22}$$

is linear on $W^{1-1/p,p}(\partial\Omega)$. However, for every $\psi \in W^{1-1/p,p}(\partial\Omega)$, there is a unique $u_{\mathcal{Z}\psi} \in W_0^{1,p}(\Omega)$ such that $P(\psi) = u_{\mathcal{Z}\psi} + \mathcal{Z}(\psi)$, according to claim (4) of Proposition 2.1. Thus,

$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma}(x, \nabla P(h)) \nabla P(\psi) \, \mathrm{d}x \\ &\qquad = \int_\Omega \boldsymbol{\sigma}(x, \nabla P(h)) \nabla u_{\mathcal{Z}\psi} \, \mathrm{d}x + \int_\Omega \boldsymbol{\sigma}(x, \nabla P(h)) \nabla \mathcal{Z}(\psi) \, \mathrm{d}x. \end{aligned}$$

Since $P(h)$ is a weak solution of (2.12), the first integral on the right-hand side equals zero. Therefore,

$$\int_\Omega \boldsymbol{\sigma}(x, \nabla P(h)) \nabla P(\psi) \mathrm{d}x = \int_\Omega \boldsymbol{\sigma}(x, \nabla P(h)) \nabla \mathcal{Z}(\psi) \, \mathrm{d}x \tag{2.23}$$

for every h, $\psi \in W^{1-1/p,p}(\partial\Omega)$. This shows that the functional in (2.22) can be rewritten as

$$\psi \mapsto \int_\Omega \boldsymbol{\sigma}(x, \nabla P(h)) \nabla \mathcal{Z}(\psi) \, \mathrm{d}x \tag{2.24}$$

and is linear by the linearity of $\mathcal{Z}$. Using Hölder's inequality and by growth condition (2.3), one easily shows that the functional (2.24) belongs to the dual space $W^{-(1-1/p),p'}(\partial\Omega)$. This justifies why our following definition makes sense and is consistent to the case of smooth functions.

Definition 2.4 We call $\Lambda_\sigma : W^{1-1/p,p}(\partial\Omega) \to W^{-(1-1/p),p'}(\partial\Omega)$ defined by

$$\langle \Lambda_\sigma h, \psi \rangle = \int_\Omega \sigma(x, \nabla P(h)) \nabla \mathcal{Z}(\psi)\, \mathrm{d}x \tag{2.25}$$

for every h, $\psi \in W^{1-1/p,p}(\partial\Omega)$ the *Dirichlet-to-Neumann map* associated with the Leray-Lions operator A given by (2.20).

The next proposition contains the key properties to establish well-posedness of the elliptic problems associated with the Dirichlet-to-Neumann map Λ_σ. Some results stated in our proposition are already known but they also complement and improve the known literature (cf. [68, Lema in Section 2] and [3, Lemme 2.1.1]).

Proposition 2.2 *Suppose that $\sigma : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying* (2.2)-(2.4)*. Then the mapping $\Lambda_\sigma : W^{1-1/p,p}(\partial\Omega) \to W^{-(1-1/p),p'}(\partial\Omega)$ defined by* (2.25) *is continuous and monotone, that is,*

$$\langle \Lambda_\sigma h_1 - \Lambda_\sigma h_2, h_1 - h_2 \rangle \geq 0.$$

Moreover, there are constants C_1, $C_2 > 0$ such that

$$\|\Lambda_\sigma h\|_{W^{-(1-1/p),p'}(\partial\Omega)} \leq C_1 \|h\|^{p-1}_{W^{1-1/p,p}(\partial\Omega)} \tag{2.26}$$

for every $h \in W^{1-1/p,p}(\partial\Omega)$ and

$$\langle \Lambda_\sigma h, h \rangle \geq C_2 \|h\|^p_{W^{1-1/p,p}(\partial\Omega)} \tag{2.27}$$

for every $h \in W^{1-1/p,p}_m(\partial\Omega)$.

Proof First, we show that Λ_σ is continuous. Let $(h_n)_{n\geq 1} \subseteq W^{1-1/p,p}(\partial\Omega)$ and $h \in W^{1-1/p,p}(\partial\Omega)$ such that h_n converges to h in $W^{1-1/p,p}(\partial\Omega)$. Then by claim (2) of Proposition 2.1, $P(h_n)$ converges to $P(h)$ in $W^{1,p}(\Omega)$. Since the flux function $\sigma : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is Carathéodory and satisfies growth condition (2.3), it follows that

$$\lim_{n\to\infty} \sigma(x, \nabla P(h_n)) = \sigma(x, \nabla P(h)) \qquad \text{in } L^{p'}(\Omega)^d. \tag{2.28}$$

We fix a $\psi \in W^{1-1/p,p}(\partial\Omega)$. By using Hölder's inequality, we see that

$$\begin{aligned} &|\langle \Lambda_\sigma h_n - \Lambda h, \psi \rangle| \\ &\qquad \leq \|\sigma(x, \nabla P(h_n)) - \sigma(x, \nabla P(h))\|_{L^{p'}(\Omega)^d} \|\nabla \mathcal{Z}\psi\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Therefore, and since $\mathcal{Z} : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ is linear and bounded,

$$\begin{aligned} &\|\Lambda_\sigma h_n - \Lambda_\sigma h\|_{W^{-(1-1/p),p'}(\partial\Omega)} \\ &\qquad \leq C \|\sigma(x, \nabla P(h_n)) - \sigma(x, \nabla P(h))\|_{L^{p'}(\Omega)^d} \end{aligned}$$

for some constant $C > 0$. Due to this estimate, limit (2.28) implies that $\Lambda_\sigma h_n$ converges to $\Lambda_\sigma h$ in $W^{-(1-1/p),p'}(\partial\Omega)$. This proves the continuity of Λ_σ.

To see that Λ_σ is monotone, let $h_1, h_2 \in W^{1-1/p,p}(\partial\Omega)$. Then by the linearity of the operator $\mathcal{Z}$ and by (2.23), we see that

$$\langle \Lambda_\sigma h_1, h_1 - h_2 \rangle = \int_\Omega \boldsymbol{\sigma}(x, \nabla P h_1) \nabla (P h_1 - P h_2)\, \mathrm{d}x$$

and similarly,

$$\langle \Lambda_\sigma h_2, h_1 - h_2 \rangle = \int_\Omega \boldsymbol{\sigma}(x, \nabla P h_2) \nabla (P h_1 - P h_2)\, \mathrm{d}x.$$

Therefore,

$$\begin{aligned}
&\langle \Lambda_\sigma h_1 - \Lambda_\sigma h_2, h_1 - h_2 \rangle \\
&\quad = \langle \Lambda_\sigma h_1, h_1 - h_2 \rangle - \langle \Lambda_\sigma h_2, h_1 - h_2 \rangle \\
&\quad = \int_\Omega (\boldsymbol{\sigma}(x, \nabla P h_1) - \boldsymbol{\sigma}(x, \nabla P h_2))(\nabla P h_1 - \nabla P h_2)\, \mathrm{d}x
\end{aligned}$$

and so monotonicity condition (2.4) implies that Λ_σ is monotone.

Now, let $h, \psi \in W^{1-1/p,p}(\partial\Omega)$. Then, by Hölder's inequality, growth condition (2.3), claim (3) of Proposition 2.1, and the boundedness of the operator $\mathcal{Z} : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$, we see that

$$|\langle \Lambda_\sigma h, h \rangle| \le \|\nabla P(h)\|_{L^p(\Omega)^d}^{p-1} \|\nabla P\psi\|_{L^p(\Omega)^d} \le C\, \|h\|_{W^{1-1/p,p}(\partial\Omega)}^{p-1} \|\psi\|_{W^{1-1/p,p}(\partial\Omega)},$$

which leads to inequality (2.26).

To see that Λ_σ satisfies (2.27), let $h \in W_m^{1-1/p,p}(\partial\Omega)$. Then, by (2.23) and since $\boldsymbol{\sigma}$ is coercive (see (2.2)), one has that

$$\langle \Lambda_\sigma h, \psi \rangle \ge \eta\, \|\nabla P(h)\|_{L^p(\Omega)^d}^p. \tag{2.29}$$

On the other hand, the boundedness of the trace operator $\mathcal{T}r$ from $W^{1,p}(\Omega)$ to $W^{1-1/p,p}(\partial\Omega)$ and Maz'ya's inequality (2.9) yields

$$\begin{aligned}
\|h\|_{W^{1-1/p,p}(\partial\Omega)}^p &= \|\mathcal{T}r P(h)\|_{W^{1-1/p,p}(\partial\Omega)}^p \\
&\le C \left(\|\nabla P(h)\|_{L^p(\Omega)^d} + \|P(h)\|_{L^p(\Omega)} \right)^p \\
&\le C \left(\|\nabla P(h)\|_{L^p(\Omega)^d} + \|h\|_{L^p(\partial\Omega)} \right)^p \\
&\le C \left(\|\nabla P(h)\|_{L^p(\Omega)^d}^p + \|h\|_{L^p(\partial\Omega)}^p \right)
\end{aligned}$$

Applying Poincaré's inequality (2.30) stated in Lemma 2.3 below to the term $\|h\|_{L^p(\partial\Omega)}^p$ in the latter estimate shows that

$$\|h\|_{W^{1-1/p,p}(\partial\Omega)}^p \le C \|\nabla P(h)\|_{L^p(\Omega)^d}^p.$$

Hence by (2.29), Λ satisfies inequality (2.27). This completes the proof of this proposition. □

Lemma 2.3 *For a function $u \in L^1(\partial\Omega)$, we set $\overline{u} = \frac{1}{\mathcal{H}(\partial\Omega)} \int_{\partial\Omega} u \,\mathrm{d}\mathcal{H}^{d-1}$. Then, there is a constant $C > 0$ such that*

$$\int_{\partial\Omega} |u - \overline{u}|^p \,\mathrm{d}\mathcal{H}^{d-1} \le C \int_{\Omega} |\nabla u|^p \,\mathrm{d}x \tag{2.30}$$

for every $u \in W^{1,p}(\Omega)$.

The proof of Lemma (2.3) is standard in the literature. Hence we omit it.

Remark 2.2 We underline that the operator Λ_σ is not *strictly monotone* without an additional condition on $h \in W^{1-1/p,p}(\partial\Omega)$. In other words, for given h_1, $h_2 \in W^{1-1/p,p}(\partial\Omega)$, the implication

$$\langle \Lambda_\sigma h_1 - \Lambda_\sigma h_2, h_1 - h_2 \rangle = 0 \quad \Longrightarrow \quad h_1 = h_2$$

does not hold in general as the following counter-example shows. Let $h_1 \equiv c_1$ and $h_2 \equiv c_2$ on $\partial\Omega$ for some $c_1, c_2 \in \mathbb{R}$ such that $c_1 \neq c_2$. Then $Ph_i \equiv c_i$ on $\overline{\Omega}$ and hence, $\Lambda_\sigma h_i \equiv 0$ on $\partial\Omega$ for $i = 1, 2$. Thus,

$$\langle \Lambda_\sigma h_1 - \Lambda_\sigma h_2, h_1 - h_2 \rangle = 0 \qquad \text{but} \qquad h_1 \neq h_2.$$

A natural condition on $h \in W^{1-1/p,p}(\partial\Omega)$ would be *compatibility condition*

$$\int_{\partial\Omega} h \,\mathrm{d}\mathcal{H}^{d-1} = 0. \tag{2.31}$$

As a result, the space $W_m^{1-1/p,p}(\partial\Omega)$ has been introduced by us in Section 1. In fact, Λ_σ restricted on $W_m^{1-1/p,p}(\partial\Omega)$ is strictly monotone. To see this, let h_1, $h_2 \in W_m^{1-1/p,p}(\partial\Omega)$ such that

$$\langle \Lambda_\sigma h_1 - \Lambda_\sigma h_2, h_1 - h_2 \rangle = 0.$$

Then by (2.23) and by the linearity of the operator $\mathcal{Z}$, this equality can be rewritten as

$$\int_{\Omega} \sigma(x, \nabla Ph_1) \nabla (Ph_1 - Ph_2) \,\mathrm{d}x = 0.$$

By the strict monotonicity condition (2.4), this implies that $\nabla Ph_1 = \nabla Ph_2$ a.e. on Ω. By [156, Corollary 2.1.9] and since Ω is connected, $Ph_1 = Ph_2 + c$ on Ω for some $c \in \mathbb{R}$. This means that $P(h_1)_{|\partial\Omega} = h_1 = h_2 + c$ on $\partial\Omega$. However, h_1 and h_2 satisfy condition (2.31). This implies that $c = 0$ and thereby we have shown that $h_1 = h_2$ on $\partial\Omega$, proving that Λ_σ is strictly monotone on the space $W_m^{1-1/p,p}(\partial\Omega)$.

With this remark we can conclude the following statement.

Corollary 2.1 *Suppose that* $\boldsymbol{\sigma} : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}^d$ *is a Carathéodory function satisfying* (2.2)-(2.4)*. The restriction of the operator* Λ_σ *defined by* (2.25) *on the space* $W_m^{1-1/p,p}(\partial\Omega)$*, denoted again by* Λ_σ*, is a well-defined, continuous, strictly monotone operator from* $W_m^{1-1/p,p}(\partial\Omega)$ *to the dual space* $W_m^{-(1-1/p),p'}(\partial\Omega)$ *satisfying the inequalities* (2.26) *and* (2.27)*.*

Proof Due to Proposition 2.2, it remains to show that the restriction of Λ_σ on $W_m^{1-1/p,p}(\partial\Omega)$ is a well-defined mapping from $W_m^{1-1/p,p}(\partial\Omega)$ into the dual space $W_m^{-(1-1/p),p'}(\partial\Omega)$. To see this, we first note that $W_m^{-(1-1/p),p'}(\partial\Omega)$ can be identified with the closed linear subspace

$$\left\{\psi \in W^{-(1-1/p),p'}(\partial\Omega) \;\middle|\; \langle \psi, c\,\mathbb{1}_{\partial\Omega}\rangle = 0 \text{ for every } c \in \mathbb{R}\right\}$$

of $W^{-(1-1/p),p'}(\partial\Omega)$. Moreover, since $P(c \cdot \mathbb{1}_{\partial\Omega}) \equiv c$ on $\overline{\Omega}$ for any $c \in \mathbb{R}$,

$$\langle \Lambda_\sigma h, c \cdot \mathbb{1}_{\partial\Omega}\rangle = \int_\Omega \boldsymbol{\sigma}(x, \nabla P(h)) \nabla P(c \cdot \mathbb{1}_{\partial\Omega})\,\mathrm{d}x = 0$$

for every $c \in \mathbb{R}$ and any $h \in W^{1-1/p,p}(\partial\Omega)$. Therefore, $\Lambda_\sigma h$ belongs to the space $W_m^{-(1-1/p),p'}(\partial\Omega)$ for every $h \in W^{1-1/p,p}(\partial\Omega)$. □

After discussing these preliminary results, we can now focus on the proof of Theorem 2.1.

2.5 Proof of Theorem 2.1

The proof Theorem 2.1 is devided into two parts. We begin by proving claim (1) of this theorem.

2.5.1 Proof of claim (1) of Theorem 2.1

By Corollary 2.1, the operator $\Lambda_\sigma : W_m^{1-1/p,p}(\partial\Omega) \to W_m^{-(1-1/p),p'}(\partial\Omega)$ is a continuous, strictly monotone, bounded and coercive. The Banach space $W_m^{1-1/p,p}(\partial\Omega)$ is reflexive and separable as a closed subspace of the reflexive and separable Banach space $W^{1-1/p,p}(\partial\Omega)$. Therefore, we can apply [103, Théorème 2.1 & Sect. 2.2] and obtain that for every $\psi \in W_m^{-(1-1/p),p'}(\partial\Omega)$ there is a unique weak solution $h \in W_m^{1-1/p,p}(\partial\Omega)$ of Poisson problem (2.1).

Next, let $(\psi_n) \subseteq W_m^{-(1-1/p),p'}(\partial\Omega)$ and $\psi \in W_m^{-(1-1/p),p'}(\partial\Omega)$ such that ψ_n converges to ψ in $W_m^{-(1-1/p),p'}(\partial\Omega)$. By the preceding part of this proof, there are unique weak solutions h_n and $h \in W_m^{1-1/p,p}(\partial\Omega)$ of Poisson problem (2.1) with right-hand sides ψ_n and ψ, respectively. By coercivity inequality (2.27) and by Young's inequality, we obtain

$$\tfrac{C_2}{2}\,\|h_n\|^p_{W^{1-1/p,p}(\partial\Omega)} \le C_3\,\|\psi_n\|_{W^{-(1-1/p),p'}(\partial\Omega)}, \tag{2.32}$$

where $C_2 > 0$ comes from (2.27) and $C_3 > 0$ is a constant independent of n. Since $(\psi_n)_{n\ge 1}$ is bounded in $W_m^{-(1-1/p),p'}(\partial\Omega)$, inequality (2.32) implies that $(h_n)_{n\ge 1}$ is bounded in $W_m^{1-1/p,p}(\partial\Omega)$. Since $W_m^{1-1/p,p}(\partial\Omega)$ is reflexive there is a $\tilde{h} \in W_m^{1-1/p,p}(\partial\Omega)$ and there is a subsequence $(\psi_{k_n})_{n\ge 1}$ of $(\psi_n)_{n\ge 1}$ such that

$$\lim_{n\to\infty} h_{k_n} = \tilde{h} \qquad \text{weakly in } W^{1-1/p,p}(\partial\Omega). \tag{2.33}$$

Since the operator $Z : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ is bounded, $(\mathcal{Z}(h_{k_n}))_{n\ge 1}$ is bounded in $W^{1,p}(\Omega)$ and so, inequality (2.18) implies that $(\nabla P(h_{k_n}))_{n\ge 1}$ is bounded in $L^p(\Omega)^d$. Moreover, $(h_{k_n})_{n\ge 1}$ is bounded in $L^p(\partial\Omega)$ and so by Maz'ya's inequality (2.9), we obtain that $(P(h_{k_n}))_{n\ge 1}$ is bounded in $W^{1,p}(\Omega)$.

Note, in order to prove the boundedness of $(P(h_{k_n}))$ in $W^{1,p}(\Omega)$, the flux function $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ does not need to satisfy gradient condition (2.5). Instead one uses that $P(h_n) = u_{0n} + \mathcal{Z}(h_n)$, where $u_{0n} \in W_0^{1,p}(\Omega)$ is the unique solution of $-\operatorname{div}(\boldsymbol{\sigma}(x, \nabla u_{0n} + \mathcal{Z}(h_n))) = 0$ in $W^{-1,p'}(\Omega)$, the conditions (2.2) and (2.3) and the boundedness of $(\mathcal{Z}(h_n))_{n\ge 1}$ in $W^{1,p}(\Omega)$ (cf. Remark 2.1).

By Lemma 2.2, there is a weak solution $u \in W^{1,p}(\Omega)$ of (2.12) on Ω and there is a subsequence of $(\psi_{k_n})_{n\ge 1}$ denoted again by $(\psi_{k_n})_{n\ge 1}$ such that

$$\lim_{n\to\infty} P(h_{k_n}) = u \qquad \text{weakly in } W^{1,p}(\Omega). \tag{2.34}$$

Since the trace operator $\mathcal{T}r : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is compact, by the convergence of $Ph_{k_n|\partial\Omega} = h_{k_n}$ to $u_{|\partial\Omega}$ in $L^p(\partial\Omega)$ and by (2.33), we can conclude that $u = P\tilde{h}$ and h_{k_n} converges to $\tilde{h}$ in $L^p(\partial\Omega)$. Thus and since every h_{k_n} satisfies the compatibility condition (2.31), we find $\tilde{h} \in W_m^{1-1/p,p}(\partial\Omega)$. It remains to show that $\tilde{h} = h$ and h_{k_n} converges to h in $W_m^{1-1/p,p}(\partial\Omega)$ with respect to the norm topology. To see that $\tilde{h} = h$, note by Lemma 2.2, $\boldsymbol{\sigma}(x, \nabla P(h_{k_n}))$ converges to $\boldsymbol{\sigma}(x, \nabla P\tilde{h})$ weakly in $L^{p'}(\Omega)^d$. Furthermore, every h_{k_n} is the unique weak solution of (2.1) with right-hand side ψ_{k_n}. Thus, one has that

$$\langle \psi_{k_n}, \xi\rangle = \langle \Lambda h_{k_n}, \xi\rangle = \int_\Omega \boldsymbol{\sigma}(x, \nabla P(h_{k_n}))\nabla\mathcal{Z}(\xi)\,\mathrm{d}x$$

and so,

$$\lim_{n\to\infty} \langle \psi_{k_n}, \xi\rangle = \int_\Omega \boldsymbol{\sigma}(x, \nabla P\tilde{h})\nabla\mathcal{Z}(\xi)\,\mathrm{d}x$$

for every $\xi \in W_m^{1-1/p,p}(\partial\Omega)$. On the other hand, $\langle \psi_{k_n}, \xi\rangle$ converges to $\langle \psi, \xi\rangle$ for every $\xi \in W_m^{1-1/p,p}(\partial\Omega)$. Therefore, $\tilde{h}$ is a weak solution of (2.1) with right-hand side ψ and so by uniqueness, $\tilde{h} = h$. To see that h_{k_n} converges to h in $W^{1-1/p,p}(\partial\Omega)$, recall that by Lemma 2.2, $P(h_{k_n})$ converges to $P\tilde{h}$ in $L^p(\Omega)$, and $\nabla P(h_{k_n})$ converges to $\nabla P(\tilde{h})$ a.e. on Ω. Thus it remains to show that $(|\nabla P(h_{k_n})|^p)$ is equi-integrable in $L^1(\Omega)$. Following the same idea as given in the proof of Proposition 2.1, it suffices to show that the non-negative function χ_{k_n} defined by (2.19) converges to 0 in $L^1(\Omega)$. Since h_{k_n} and h are the unique weak solutions of (2.1) with right-hand side ψ_{k_n} and ψ, respectively, we have that

$$\int_\Omega \chi_{k_n}(x), \mathrm{d}x = \langle \psi_{k_n}, h_{k_n} - h\rangle - \langle \psi_{k_n}, h_{k_n} - h\rangle.$$

Therefore and by the convergence of ψ_{k_n} to ψ in $W_m^{-(1-1/p),p'}(\partial\Omega)$ and by the weak limit (2.33) with $\tilde{h} = h$, we see that χ_{k_n} defined by (2.19) converges to 0 in $L^1(\Omega)$. Thereby we have shown that for every sequence $(\psi_n) \subseteq W_m^{-(1-1/p),p'}(\partial\Omega)$ converging to some $\psi \in W_m^{-(1-1/p),p'}(\partial\Omega)$, there is a subsequence $(\psi_{k_n})_{n\geq 1}$ of $(\psi_n)_{n\geq 1}$ such that h_{k_n} converges to h in $W_m^{1-1/p,p}(\partial\Omega)$. This proves that the mapping $\psi \mapsto h$ is continuous from $W_m^{-(1-1/p),p'}(\partial\Omega)$ to $W_m^{1-1/p,p}(\partial\Omega)$, completing the proof of claim (1) of Theorem 2.1.

2.5.2 Preliminaries for the proof of claim (2) of Theorem 2.1

For $1 \leq q \leq \infty$, let $L_m^q(\partial\Omega)$ denote the set of all $\psi \in L^q(\partial\Omega)$ satisfying (2.31) and let q' be the Hölder-conjugate of q given by $\frac{1}{q} + \frac{1}{q'} = 1$. The next result seems to be well-known. But for the sake of completeness, we supply the proof here.

Lemma 2.4 *Let $q = \frac{d-1}{p-1}$ if $p \leq d$ and $q = 1$ if $p > d$. Then $W_m^{1-1/p,p}(\partial\Omega)$ is continuously embedded into $L_m^{q'}(\partial\Omega)$ by a continuous injection with a dense image. Moreover, $L_m^q(\partial\Omega)$ is continuously embedded into $W_m^{-(1-1/p),p'}(\partial\Omega)$.*

Proof We prove the claim of this lemma only for $p < d$ since the case $p \geq d$ is similar. Observe that the Lebesgue space $L_m^{q'}(\partial\Omega)$ is a closed linear subspace of $L^{q'}(\partial\Omega)$ and can be identified with the quotient space $L^{q'}(\partial\Omega)/\mathbb{R}$. Here we identify the set of constant functions on $\partial\Omega$ with $\mathbb{R}$. Thus, the dual space $(L_m^{q'}(\partial\Omega))'$ can be identified with $L_m^q(\partial\Omega)$.

Now, we show that $W_m^{1-1/p,p}(\partial\Omega) \hookrightarrow L_m^{q'}(\partial\Omega)$ by a continuous injection for $q = (d-1)/(p-1)$. For this, we note first that $q = (d-1)/(p-1)$ if and only if $q' = (d-1)/(d-p)$ and recall that the trace operator $\mathcal{T}r : W^{1,p}(\Omega) \to L^{p^*}(\partial\Omega)$ with $p^* = (d-1)p/(d-p)$ (see [115, Théorème 4.2, p. 84]) and that its right-inverse $\mathcal{Z} : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ are bounded. By using this together with the fact that $\partial\Omega$ has finite $\mathcal{H}$-measure, one sees that

$$\|h\|_{L^{q'}(\partial\Omega)} \leq C\|\mathcal{T}r\mathcal{Z}(h)\|_{L^{p^*}(\partial\Omega)} \leq C\|\mathcal{Z}(h)\|_{W^{1,p}(\Omega)} \leq C\|h\|_{W^{1-1/p,p}(\partial\Omega)}$$

for every $h \in W^{1-1/p,p}(\partial\Omega)$. This shows that $W^{1-1/p,p}(\partial\Omega)$ is continuously injected into $L^{q'}(\partial\Omega)$ and hence, in particular, there is a continuous injection

$i : W_m^{1-1/p,p}(\partial\Omega) \hookrightarrow L_m^{q'}(\partial\Omega)$. If we can show that $W_m^{1-1/p,p}(\partial\Omega)$ lies dense in $L_m^{q'}(\partial\Omega)$, then the adjoint operator $i' : L_m^q(\partial\Omega) \to W_m^{-(1-1/p),p'}(\partial\Omega)$ is also a continuous injection. For this, we set $e_1 = \mathcal{H}(\partial\Omega)^{-1/2}\mathbb{1}_{\partial\Omega}$. Then $\omega_h := h - (h, e_1)_{L^2(\partial\Omega)}\, e_1 \in W_m^{1-1/p,p}(\partial\Omega)$ for every $h \in W^{1-1/p,p}(\partial\Omega)$. Now, let $g \in L_m^q(\partial\Omega)$ satisfy

$$(g, h)_{L^2(\partial\Omega)} = 0 \qquad \text{for every } h \in W_m^{1-1/p,p}(\partial\Omega). \tag{2.35}$$

Then, $(g, \omega_h)_{L^2(\partial\Omega)} = 0$ and $(g, e_1)_{L^2(\partial\Omega)} = 0$. Hence,

$$(g, h)_{L^2(\partial\Omega)} = (g, \omega_h)_{L^2(\partial\Omega)} + (h, e_1)_{L^2(\partial\Omega)}\,(g, e_1)_{L^2(\partial\Omega)} = 0$$

for every $h \in W^{1-1/p,p}(\partial\Omega)$. Since the space $W^{1-1/p,p}(\partial\Omega)$ lies dense in $L^{q'}(\partial\Omega)$ (see Lemma 2.1), it follows that $g = 0$. As $g \in L_m^q(\partial\Omega)$ satisfying (2.35) was arbitrary, we have thereby proved that $W_m^{1-1/p,p}(\partial\Omega)$ lies dense in $L_m^{q'}(\partial\Omega)$, concluding the proof of this Lemma. □

Next, let $q = \frac{d-1}{p-1}$ if $p \le d$ and $q = 1$ if $p > d$. In order to prove that weak solutions $h \in W^{1-1/p,p}(\partial\Omega)$ of Poisson problem (2.1) with right-hand side $\psi \in L_m^q(\partial\Omega)$ are Hölder-continuous, it is crucial to know that $u := P(h)$ solves the elliptic Neumann boundary value problem

$$\begin{cases} -\operatorname{div}\big(\boldsymbol{\sigma}(x, \nabla u)\big) = 0 & \text{in } \Omega, \\ \boldsymbol{\sigma}(x, \nabla u)\cdot \boldsymbol{\nu} = \psi & \text{on } \partial\Omega \end{cases} \tag{2.36}$$

in a weak sense. We recall the definition of a weak solution of problem (2.36).

Definition 2.5 For $\psi \in L_m^q(\partial\Omega)$, we call a function $u \in W^{1,p}(\Omega)$ a *weak solution* of the elliptic Neumann boundary-value problem (2.36) if u satisfies

$$\int_\Omega \boldsymbol{\sigma}(x, \nabla u)\nabla\xi \,\mathrm{d}x = \int_{\partial\Omega} \psi\,\xi\,\mathrm{d}\mathcal{H}^{d-1} \tag{2.37}$$

for all $\xi \in W^{1,p}(\Omega)$.

We have the following characterisation of weak solutions to Poisson problem (2.1).

Lemma 2.5 *For $\psi \in L_m^q(\partial\Omega)$, the function $h \in W^{1-1/p,p}(\partial\Omega)$ is a weak solution of problem (2.1) if and only if $P(h)$ is a weak solution of (2.36).*

Proof Let $\psi \in L_m^q(\partial\Omega)$ arbitrary but fixed and suppose $h \in W^{1-1/p,p}(\partial\Omega)$ is a weak solution of Problem (2.1). Then, by Definition 2.1, $P(h)$ satisfies

$$\int_\Omega \boldsymbol{\sigma}(x, \nabla P(h))\nabla\mathcal{Z}(\xi)\,\mathrm{d}x = \int_{\partial\Omega} \psi\,\xi\,\mathrm{d}\mathcal{H}^{d-1} \tag{2.38}$$

for every $\xi \in W^{1-1/p,p}(\partial\Omega)$. For every $\zeta \in W^{1,p}(\Omega)$, the function $\xi := \mathcal{Tr}(\zeta)$ belongs to $W^{1-1/p,p}(\partial\Omega)$ and satisfies $\zeta - \mathcal{Z}(\xi) \in W_0^{1,p}(\Omega)$. Thus and since $P(h)$ is a weak solution of equation (2.12), equation (2.38) is equivalent to

$$\int_\Omega \boldsymbol{\sigma}(x, \nabla P(h))\nabla\zeta \,\mathrm{d}x = \int_{\partial\Omega} \psi\, \zeta \,\mathrm{d}\mathcal{H}^{d-1}$$

for every $v \in W^{1,p}(\Omega)$. Therefore, $h \in W^{1-1/p,p}(\partial\Omega)$ is a weak solution of Poisson problem (2.1) if and only if $P(h)$ satisfies equation (2.37), meaning $P(h)$ is a weak solution of Neumann problem (2.36). □

2.5.3 Proof of claim (2) of Theorem 2.1

Suppose that $q_\varepsilon := \frac{d-1}{p-1-\varepsilon}$ if $p \le d$ and $q_\varepsilon := 1$ if $p > d$ for $\varepsilon \in [0,1)$. Since $\partial\Omega$ has finite measure, and since $q_0 \le q_\varepsilon$, Lemma 2.4 yields that

$$L^{q_\varepsilon}_m(\partial\Omega) \hookrightarrow L^{q_0}_m(\partial\Omega) \hookrightarrow W^{-(1-1/p),p'}_m(\partial\Omega)$$

respectively by continuous injections. Thus, claim (1) of Theorem 2.1 ensures that for every $\psi \in L^{q_\varepsilon}_m(\partial\Omega)$, there is a unique weak solution $h \in W^{1-1/p,p}_m(\partial\Omega)$ of the Poisson problem (2.1).

Now, let $\varepsilon \in (0,1)$ and suppose that $h \in W^{1-1/p,p}(\partial\Omega)$ is a weak solution of (2.1) for some right-hand side $\psi \in L^{q_\varepsilon}(\partial\Omega)$. By Lemma 2.5, $P(h)$ is a weak solution of Neumann problem (2.36). Thus, [116, Theorem 3.7] implies that $P(h) \in C^{0,\alpha}(\overline{\Omega})$ and satisfies

$$\|P(h)\|_{C^{0,\alpha}(\overline{\Omega})} \le c_\alpha \left(\|\psi\|^{\frac{1}{p-1}}_{L^{q_\varepsilon}(\partial\Omega)} + \|P(h)\|_{L^p(\Omega)} \right) + c_\alpha$$

for some $\alpha \in (0,1)$ and $C_\alpha \ge 0$ independent of ψ and h. Therefore, $h \in C^{0,\alpha}(\partial\Omega)$ and since $\|h\|_{C^{0,\alpha}(\partial\Omega)} \le \|P(h)\|_{C^{0,\alpha}(\overline{\Omega})}$, the last estimate shows that h satisfies the desired inequality (2.7). Thus, claim (2) of Theorem 2.1 holds, completing the proof of this theorem.

Chapter 3
Nonlinear elliptic-parabolic evolution problems

Abstract This chapter, is dedicated to establish the well-posedness in the sense of mild solutions of nonlinear evolution problems driven by the Dirichlet-to-Neumann map, smoothing effects of mild solutions, and the long-time asymptotic behavior. In particular, we outline the proof of Theorem 1.5. The content of this chapter covers parts of the paper [85] and the monograph [56].

3.1 Main result

In this chapter, we establish the well-posedness of the Cauchy problem (in X)

$$\begin{cases} \frac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda_{|X} h(t) \ni 0 & \text{for } t \in (0,T), \\ h(0) = h_0, \end{cases} \tag{3.1}$$

in the sense of *mild solutions*, where in the Banach space X is either $L^1(\partial\Omega)$, the Orlicz space $L^\psi(\partial\Omega)$ for every N-function satisfying the Δ_2-condition (see Definition 3.1 and Definition 3.2 in Section 3.2.2.2 below), or $C(\partial\Omega)$. Here, $\Lambda_{|X}$ denotes the realization in X of the Dirichlet-to-Neumann map Λ_{σ} associated with the Leray-Lions operator $A = -\operatorname{div}(\boldsymbol{\sigma}(x, \nabla\cdot))$.

As in Chapter 2, we assume throughout this chapter that Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a Lipschitz-continuous boundary $\partial\Omega$. Further, for $1 < p < \infty$, suppose that the flux function $\boldsymbol{\sigma}$ is a Carathéodory function satisfying (2.2), (2.3), (2.4), (2.5) from Chapter 2.1.

The following theorem is the main result of this chapter. Note, Theorem 1.5 is a special case of this result.

Theorem 3.1 *Suppose that* $\boldsymbol{\sigma} : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}^d$ *and* $\mathcal{A} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ *are two Carathéodory functions satisfying* (2.2)-(2.5). *Further, let* X *be either* $L^1(\partial\Omega)$, $L^\psi(\partial\Omega)$ *for any* N*-function* ψ *satisfying the* Δ_2*-condition, or* $X = C(\partial\Omega)$. *Then the following statements hold.*

1. *For every $h_0 \in X$, there exists a unique mild solution h of Cauchy problem (3.1). Moreover, for every h_0, $\hat{h}_0 \in X$, the corresponding mild solutions h and $\hat{h}$ of Cauchy problem (3.1) satisfy*
$$\|[h(t) - \hat{h}(t)]^{\nu}\|_X \le \|[h_0 - \hat{h}_0]^{\nu}\|_X \tag{3.2}$$
for every $t \ge 0$ and $\nu \in \{+, 1\}$, and
$$\|h(t) - c\mathbb{1}_{\partial\Omega}\|_X \le \|h_0 - c\mathbb{1}_{\partial\Omega}\|_X \tag{3.3}$$
for every $t \ge 0$ and $c \in \mathbb{R}$.
2. *If X satisfies the continuous embedding*
$$X \hookrightarrow L^2(\partial\Omega),$$
then the mild solution h of Cauchy problem (3.1) in X coincides with the unique strong solution of (3.1) in $L^2(\partial\Omega)$ and has the regularity
$$h \in C((0,\infty); W^{1-1/p,p}(\partial\Omega)) \cap W^{1,\infty}([\delta,\infty); L^2(\partial\Omega))$$
for every $\delta > 0$. Moreover, h is right-hand side differentiable with values in $L^2(\partial\Omega)$ at every $t > 0$ satisfying
$$\int_{\partial\Omega} \frac{\mathrm{d}h}{\mathrm{d}t}_{+}(t)\,\xi \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Omega} \boldsymbol{\sigma}(x, \nabla P(h(t)))\nabla \mathcal{Z}(\xi)\,\mathrm{d}x = 0 \tag{3.4}$$
for every $\xi \in W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega)$. In particular, h satisfies the conservation of mass equation
$$\int_{\partial\Omega} h(t)\,\mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\Omega} h_0\,\mathrm{d}\mathcal{H}^{d-1} \tag{3.5}$$
for every $t \ge 0$. The function
$$t \mapsto \varphi(h(t)) := \int_{\Omega} \mathcal{A}(x, \nabla P(h(t)))\,\mathrm{d}x$$
is convex, decreasing, Lipschitz continuous on $[\delta,\infty)$ for every $\delta > 0$, and
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi(h(t)) = -\|\tfrac{\mathrm{d}h}{\mathrm{d}t}(t)\|^2_{L^2(\partial\Omega)} \tag{3.6}$$
for a.e. $t > 0$.
3. *Suppose that $p \neq 2$ and $\xi \mapsto \boldsymbol{\sigma}(\cdot, \xi)$ satisfies the homogeneity condition (2.17). Then for every $h_0 \in X$, the mild solution h of the Cauchy problem (3.1) in X is a strong solution and has the regularity*
$$h \in W^{1,\infty}([\delta,\infty); X) \cap C([0,\infty); X) \tag{3.7}$$

for every $\delta > 0$, is differentiable with values in X from the right at every $t > 0$, and satisfies

$$|(\Lambda_{|X})^{\circ} h(t)| \le \tfrac{2}{|p-2|}\frac{|h_0|}{t} \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega, \tag{3.8}$$

$$\left\|\frac{\mathrm{d}h}{\mathrm{d}t_+}(t)\right\|_X \le C_p \frac{\|h_0\|_X}{t} \qquad \text{for every } t > 0, \tag{3.9}$$

where $C_p = 2/|p-2|$. Further, for every positive $h_0 \in X$, one has that

$$(p-2)(\Lambda_{|X})^{\circ} h(t) \le \frac{h(t)}{t} \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{3.10}$$

In particular, if X is continuously embedded into $L^2(\partial\Omega)$, then for every $h_0 \in X$, the corresponding strong solution h of Cauchy problem (3.1) in X is weak.

4. *Let $p = 2$ and suppose that the flux function $\xi \mapsto \boldsymbol{\sigma}(\cdot,\xi)$ is linear. Further, let $X = L^q(\partial\Omega)$ for $1 < q < \infty$. Then, for every $h_0 \in X$, the mild solution h of Cauchy problem (3.1) in X is a strong solution of (3.1) satisfying (3.7) and (3.9) for some constant $C_2 = C_2(q) > 0$. In particular, if X is continuously embedded into $L^2(\partial\Omega)$ then for every $h_0 \in X$, the corresponding strong solution h of Cauchy problem (3.1) in X is also weak.*
5. *For $(2 \vee \frac{d-1}{p-1-\varepsilon}) \le q \le \infty$ with some $\varepsilon \in (0,1)$ if $p \le d$ and for $2 \le q \le \infty$ if $p > d$, there are $\alpha \in (0,1)$ and $c_\alpha > 0$ such that*

$$\|h(t)\|_{C^{0,\alpha}(\partial\Omega)} \le c_\alpha \left[\left(\frac{\|h(t)\|_{L^q(\partial\Omega)}}{t}\right)^{\frac{1}{p-1}} + \Phi(h(t))\right] + c_\alpha \tag{3.11}$$

for every $t > 0$, where the function

$$t \mapsto \Phi(h(t)) := \varphi(h(t))^{1/p} + \|h(t)\|_{L^2(e^{-t\Lambda}h)} \tag{3.12}$$

is continuous and decreasing on $(0,\infty)$.

6. *For every $h_0 \in X$, one has that*

$$\lim_{t\to+\infty} h(t) = \overline{h_0} \qquad \text{in } X. \tag{3.13}$$

In particular, if $(2 \vee \frac{d-1}{p-1-\varepsilon}) \le q \le \infty$ for some $\varepsilon \in (0,1)$ provided $p \le d$, or if $2 \le q \le \infty$ provided $p > d$, then for every $h_0 \in L^q(\partial\Omega)$, one has that (3.13) holds in $C(\partial\Omega)$.

For the proof of Theorem 3.1, we use nonlinear semigroup theory. We provide a brief review of this theory in Section 3.2.2 summarizing the most relevant results for this thesis. In Section 3.3, we first realize the Dirichlet-to-Neumann map $\Lambda_{\boldsymbol{\sigma}}$ as an operator $\Lambda_{|L^2}$ acting on $L^2(\partial\Omega)$. Built on the operator $\Lambda_{|L^2}$, we introduce in Definition 3.21 (see Section 3.4 below) the Dirichlet-to-Neumann map as an operator $\Lambda_{|X}$ on the Banach space X, where X is either $L^1(\partial\Omega)$, $L^\psi(\partial\Omega)$ for every N-function, and $X = C(\partial\Omega)$.

3.2 Preliminaries

Here, we recall various tools from nonlinear functional analysis. We begin by introducing some function spaces, which we use throughout this thesis.

3.2.1 Some function spaces

Throughout this preliminary section, let $(\Sigma, \mathcal{B}, \mu)$ be a σ-finite measure space, and $M(\Sigma,\mu)$ the space of μ-a.e. equivalent classes of measurable functions $u : \Sigma \to \mathbb{R}$. For $u \in M(\Sigma,\mu)$, we write $[u]^+$ to denote $\max\{u, 0\}$ and set $[u]^- = -\min\{u, 0\}$. Further, we call a function $u \in M(\Sigma,\mu)$ *positive* if $u \geq 0$ μ-a.e. on Σ. We denote by L^q, $1 \leq q \leq \infty$, the standard Lebesgue space $L^q(\Sigma,\mu)$ with norm

$$\|u\|_q = \begin{cases} \left(\displaystyle\int_\Sigma |u|^q \,\mathrm{d}\mu\right)^{1/q} & \text{if } 1 \leq q < \infty, \\ \inf\left\{k \in [0,+\infty] \,\middle|\, |u| \leq k \ \mu\text{-a.e. on } \Sigma\right\} & \text{if } q = \infty. \end{cases}$$

For $1 \leq q < \infty$, we identify the dual space $(L^q)^*$ with $L^{q'}(\Sigma,\mu)$, where q' is the conjugate exponent of q given by $1 = \frac{1}{q} + \frac{1}{q'}$.

Next, we briefly recalling the notion of *Orlicz spaces* (cf. [126, Chapter 3]).

Definition 3.1 A continuous function $\psi : [0,+\infty) \to [0,+\infty)$ is an *N-function* if it is convex, $\psi(s) = 0$ if and only if $s = 0$, $\lim_{s\to 0+} \psi(s)/s = 0$, and $\lim_{s\to\infty} \psi(s)/s = \infty$. Given an N-function ψ, the *Orlicz space* L^ψ is defined as follows

$$L^\psi := L^\psi(\Sigma,\mu) := \left\{u \in M(\Sigma,\mu) \,\middle|\, \int_\Sigma \psi\left(\frac{|u|}{\alpha}\right) \mathrm{d}\mu < \infty \text{ for some } \alpha > 0\right\}$$

and equipped with the *Orlicz-Minkowski norm*

$$\|u\|_\psi := \inf\left\{\alpha > 0 \,\middle|\, \int_\Sigma \psi\left(\frac{|u|}{\alpha}\right) \mathrm{d}\mu \leq 1\right\}.$$

Definition 3.2 Further, an N-function is said to satisfy a Δ_2*-condition* (globally) if there is a $K \geq 1$ such that $\psi(2x) \leq K\psi(x)$ for all $x \geq 0$.

We require the following result from [126, Corollary 5 in Chapter 3.4].

Proposition 3.1 *If $\psi : [0,+\infty) \to [0,+\infty)$ is an N-function satisfying the Δ_2-condition, then the simple functions are dense in L^ψ.*

Further, we require the following function spaces. Let

$$L^{1\cap\infty} := L^{1\cap\infty}(\Sigma,\mu) := L^1(\Sigma,\mu) \cap L^\infty(\Sigma,\mu)$$

be the *intersection* space of L^1 and L^∞, and

$$L^{1+\infty} := L^{1+\infty}(\Sigma,\mu) := L^1(\Sigma,\mu) + L^\infty(\Sigma,\mu)$$

the *sum* space of L^1 and L^∞, which respectively equipped with the norms

$$\|u\|_{1\cap\infty} := \max\left\{\|u\|_1, \|u\|_\infty\right\},$$
$$\|u\|_{1+\infty} := \inf\left\{\|u_1\|_1 + \|u_2\|_\infty \,\middle|\, u = u_1 + u_2,\ u_1 \in L^1, u_2 \in L^\infty\right\}$$

are Banach spaces. In fact, $L^{1\cap\infty}$ and $L^{1+\infty}$ are respectively the smallest and the largest of the rearrangement-invariant Banach function spaces (cf [32, Chapter 3.1]). If $\mu(\Sigma)$ is finite, then $L^{1+\infty} = L^1$ with equivalent norms, but if $\mu(\Sigma) = \infty$ then

$$\bigcup_{1\le q\le\infty} L^q \subset L^{1+\infty}.$$

Further, we will employ the space

$$L_0 := L_0(\Sigma,\mu) := \left\{u \in M(\Sigma,\mu) \,\middle|\, \int_\Sigma \left[|u| - k\right]^+ \mathrm{d}\mu < \infty \text{ for all } k > 0\right\},$$

equipped with the $L^{1+\infty}$-norm. This space is a closed subspace of $L^{1+\infty}$. In fact, one has that $L_0 = \overline{L^1 \cap L^\infty}^{1+\infty}$ (see [29]). Since for every $k > 0$, the function $T_k(s) := [|s| - k]^+$ is a Lipschitz mapping $T_k : \mathbb{R} \to \mathbb{R}$ satisfying $T_k(0) = 0$, and by using Chebyshev's inequality, it is not difficult to see that $L^q \hookrightarrow L_0$ for every $1 \le q < \infty$ (and $q = \infty$ if the measure $\mu(\Sigma)$ is finite), and $L^\psi \hookrightarrow L_0$ for every N-function ψ if $\mu(\Sigma)$ is finite.

3.2.2 Nonlinear semigroup theory - Part I

Suppose that X is a Banach space with norm $\|\cdot\|_X$, X^* its dual space, $\langle\cdot,\cdot\rangle_{X^*,X}$ the duality brackets on $X^* \times X$, and let I denote the *identity* on X.

In this framework, we deal with usually nonlinear and possibly multivalued mappings $A : X \to 2^X$. It is standard to identify such a mapping A with its *graph*

$$A := \left\{(u,v) \in X \times X \,\middle|\, v \in Au\right\} \quad \text{in } X \times X$$

and so, one can see A as a subset of $X\times X$. Thus, one also calls a possibly multivalued mapping $A : X \longrightarrow 2^X$ an *operator on* X with *domain* $D(A) := \{u \in X \,|\, Au \neq \emptyset\}$ and *range* $\mathrm{Rg}(A) := \bigcup_{u\in D(A)} Au \subseteq X$.

3.2.2.1 Accretive operators and nonlinear semigroups

The following class of operators is crucial.

Definition 3.3 An operator A on a Banach space X is called *accretive* if for every (u, v), $(\hat{u}, \hat{v}) \in A$ and every $\lambda > 0$,

$$\|u - \hat{u}\|_X \leq \|u - \hat{u} + \lambda(v - \hat{v})\|_X.$$

Further, A is called *m-accretive* if A is accretive and A satisfies for some (or equiv., all) $\lambda > 0$ the *range condition*

$$\mathrm{Rg}(I + \lambda A) = X. \tag{3.14}$$

For $\omega \in \mathbb{R}$, an operator A on X is called *ω-quasi m-accretive operator* on X if $A+\omega I$ is *accretive*, and if for some (or equiv., all) $\lambda > 0$ satisfying $\lambda\omega < 1$, (3.14) holds.

There are several characterizations of A being accretive, which we will use from time to time throughout this thesis. The first one states that an operator A is accretive on X if and only if

$$\begin{cases} \text{for every } (u, v),\ (\hat{u}, \hat{v}) \in A, \text{ there exists } \psi \in J(u - \hat{u}) \\ \text{satisfying } \langle \psi, v - \hat{v}\rangle \geq 0, \end{cases} \tag{3.15}$$

where $J : X \longrightarrow 2^{X^*}$ denotes the *duality mapping* of X, which is given by

$$J(u) = \left\{\psi \in X^* \,\middle|\, \langle \psi, u\rangle = \|u\|_X \text{ and } \|\psi\|_{X^*} \leq 1\right\}$$

for every $u \in X$ (cf. [30, Theorem (2.15)] or [23, Proposition 3.1]).

It is not difficult to verify (cf. [30, Example (2.11)]) that for $X = L^1$, the duality mapping J on L^1 is given by

$$J(u) = \left\{\psi \in L^\infty \,\middle|\, \psi(x) \in \mathrm{sign}(u(x)) \text{ for a.e. } x \in \Sigma\right\} \tag{3.16}$$

for every $u \in L^1$, where the multi-valued *signum* function is defined by

$$\mathrm{sign}(s) := \begin{cases} 1 & \text{if } s > 0, \\ [-1, 1] & \text{if } s = 0, \\ -1 & \text{if } s < 0 \end{cases}$$

for every $s \in \mathbb{R}$. Note, $J(u)$ is multi-valued exactly when the set $\{u = 0\}$ has strictly positive μ-measure. Further, for $1 < q < \infty$, the duality mapping $J : L^q \to L^{q'}$ is given by

$$J(u) = u_q \, \|u\|_q^{1-q} \tag{3.17}$$

for every $u \in L^q$, where $u_q := |u|^{q-2}u$.

In the case $X = H$ is a Hilbert space with inner product $(\cdot,\cdot)_H$, the notion of A being m-accretive is equivalent to A being *maximal monotone* (cf [40, Proposition 2.2]).

Definition 3.4 An operator A on a Hilbert space H is called *monotone* if

$$(\hat{v} - v, \hat{u} - u)_H \geq 0 \qquad \text{for all } (u,v),\ (\hat{u},\hat{v}) \in A.$$

If A is a monotone operator on H then A is called *maximal monotone* if A is monotone and A is not properly contained in any other monotone subset of $H \times H$.

An important class of possibly multivalued operators is given by the class of *subdifferentials*.

Definition 3.5 For a given functional $\varphi : X \to (-\infty, +\infty]$ on Banach space X with *effective domain* $D(\varphi) := \{u \in X \,|\, \varphi(u) < \infty\}$, the *subdifferential* $\partial\varphi : X \to 2^{X^*}$ is defined by

$$\partial\varphi := \left\{ (u,x') \in X \times X^* \,\middle|\, \langle x', \xi - u\rangle_{X^*,X} \leq \liminf_{t \downarrow 0} \frac{\varphi(u + t\xi) - \varphi(u)}{t} \ \forall\, \xi \in D(\varphi) \right\}.$$

Further, we say that φ is *lower semicontinuous* if for every $c \in \mathbb{R}$, the sublevel set

$$E_c := \left\{ u \in D(\varphi) \,\middle|\, \varphi(u) \leq c \right\}$$

is closed in X.

Notation 3.1 If $X = H$ is a Hilbert space, then after identifying the dual space H^* with H, the subdifferential operator $\partial_{H \times H'}\varphi$ reduces to a (possibly multi-valued) operator on H. In this setting, we simply write $\partial_H\varphi$ for the operator $\partial_{H \times H'}\varphi$ and call $\partial_H\varphi$ the *subgradient* of φ.

Definition 3.6 A function $\varphi : H \to (-\infty, +\infty]$ defined on a Hilbert space H is called *semiconvex* if there exists an $\omega \in \mathbb{R}$ such that the shifted function $\varphi_\omega : H \to (-\infty, +\infty]$ defined by

$$\varphi_\omega(u) := \varphi(u) + \frac{\omega}{2}\|u\|_H^2, \qquad (u \in H),$$

is convex.

Then, $\varphi_{\hat{\omega}}$ is convex for all $\hat{\omega} \geq \omega$, and φ_ω is lower semicontinuous if and only if φ is lower semicontinuous. If φ_ω is convex, then the subgradient $\partial_H\varphi$ of φ reduces to the set

$$\partial_H \varphi = \Big\{(u,h) \in H \times H \,\Big|\, \varphi_\omega(u+\xi) - \varphi_\omega(u) \geq (h+\omega u, \xi)_H \, \forall \xi \in D(\varphi)\Big\}$$

and if, in addition, φ is proper and lower semicontinuous, then the subgradient $\partial_H \varphi$ is ω-quasi m-accretive on H.

If A is quasi m-accretive in X, then by the classical existence theory (see, for example, [59], [30, Theorem 6.5] or [23, Corollary 4.2]), for every $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0,T;X)$, the first-order Cauchy problem (in X)

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t}(t) + A(u(t)) \ni g(t) & \text{for } t \in (0,T), \\ u(0) = u_0 \end{cases} \tag{3.18}$$

is well-posed in the following *mild sense*.

Definition 3.7 For given $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0,T;X)$, a function $u \in C([0,T];X)$ is called a *mild solution* of the inhomogeneous differential inclusion (3.18) with initial value u_0 if $u(0) = u_0$ and for every $\varepsilon > 0$, there is a *partition* $\tau_\varepsilon : 0 = t_0 < t_1 < \cdots < t_{N_\varepsilon} = T$ and a *step function*

$$u_\varepsilon(t) = u_0 \mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^{N_\varepsilon} u_i \mathbb{1}_{(t_{i-1},t_i]}(t) \qquad \text{for every } t \in [0,T]$$

satisfying

- $t_i - t_{i-1} < \varepsilon \qquad \text{for all } i = 1, \dots, N_\varepsilon,$
- $\displaystyle\sum_{i=1}^{N_\varepsilon} \int_{t_{i-1}}^{t_i} \|g(t) - \overline{g}_i\| \,\mathrm{d}t < \varepsilon \qquad \text{where } \overline{g}_i := \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} g(t)\,\mathrm{d}t,$
- $\displaystyle\frac{u_i - u_{i-1}}{t_i - t_{i-1}} + Au_i \ni \overline{g}_i \qquad \text{for all } i = 1, \dots, N,$

and

$$\sup_{t \in [0,T]} \|u(t) - u_\varepsilon(t)\|_X < \varepsilon.$$

Since a mild solution u is merely the locally uniform (in time t) limit of a sequence $(u_{\varepsilon_n})_{n\geq 1}$ of step functions u_{ε_n}, u is not necessarily differentiable in time. This leads to the following stronger notion of solutions to Cauchy problem (3.18) (cf [30] or [23]).

Definition 3.8 For given $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0,T;X)$, a function $u \in C([0,T];X)$ is called a *strong solution* of the Cauchy problem (3.18) if $u(0) = u_0$, and for a.e. $t \in (0,T)$, u is differentiable at t, $u(t) \in D(A)$, and $Au(t) \ni g(t) - \frac{\mathrm{d}u}{\mathrm{d}t}(t)$.

It is not difficult to show (cf. [30] or [23]) that every strong solution is a mild one. Thus, the next result provides a sufficient condition when mild solutions are strong.

Theorem 3.2 ([30, Theorem 7.1]) *Let A be quasi m-accretive in X and $g \in L^1(0,T;X)$. Then u is a strong solution of the differential inclusion (in X)*

$$\tfrac{\mathrm{d}u}{\mathrm{d}t}(t) + A(u(t)) \ni g(t) \qquad \text{in } (0,T)$$

if and only if u is a mild solution on $[0,T]$ and u is absolutely continuous and differentiable with values in X at a.e. $t \in (0,T)$.

If $X = H$ is a Hilbert space and A has a *subgradient structure*, that is, $A = \partial_H \varphi$ for some proper, semiconvex, and lower semicontinuous function $\varphi : H \to (-\infty, \infty]$, then it follows from a result by Brezis [39] that every mild solution u of Cauchy problem (3.18) admits the immediate *smoothing effect* that u is a strong solution of (3.18). For later use, we summarize these results in one theorem (cf. [11, Theorem 2.2]).

Theorem 3.3 (Brezis' L^2-maximal regularity for semiconvex φ) *Suppose, $\varphi : H \to (-\infty, \infty]$ is proper, semiconvex, and lower semicontinuous. Then, for every $u_0 \in \overline{D(\varphi)}^H$ and $g \in L^2(0,T;H)$, there exists a unique $u \in W^{1,2}_{loc}((0,T];H) \cap C([0,T];H)$ satisfying*

$$\begin{cases} \tfrac{\mathrm{d}u}{\mathrm{d}t}(t) + \partial_H \varphi(u(t)) \ni g(t) & \text{a.e. on } (0,T), \\ u(0) = u_0. \end{cases} \tag{3.19}$$

Moreover,

$$\varphi \circ u \in W^{1,1}_{loc}((0,T]) \cap L^1(0,T), \tag{3.20}$$

$$\|u(t)\|_H \le \left(\|u_0\|_H^2 + \int_0^T \|g(s)\|_H^2 \,\mathrm{d}s \right)^{\frac{1}{2}} e^{\frac{1+2\omega}{2} t} \tag{3.21}$$

for every $t \in (0,T]$,

$$\int_0^T \varphi(u(s))\,\mathrm{d}s \le \tfrac{1}{2}\|f\|^2_{L^2(0,T;H)} + \tfrac{1+\omega}{2}\|u\|^2_{L^2(0,T;H)} + \tfrac{1}{2}\|u_0\|_H^2, \tag{3.22}$$

$$t\varphi(u(t)) \le \int_0^T \varphi(u(s))\,\mathrm{d}s + \tfrac{1}{2}\|\sqrt{\cdot}\, g\|^2_{L^2(0,T;H)} \tag{3.23}$$

for every $t \in (0,T]$,

$$\|\sqrt{\cdot}\, \tfrac{\mathrm{d}u}{\mathrm{d}t}\|^2_{L^2(0,T;H)} \le 2\int_0^T \varphi(u(t))\,\mathrm{d}t + \|\sqrt{\cdot}\, g\|^2_{L^2(0,T;H)}. \tag{3.24}$$

Finally, if $u_0 \in D(\varphi)$, then $u \in W^{1,2}(0,T;H)$.

Next, let A be a quasi m-accretive operator on a Banach space X, and for every given $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0,T;X)$, let u be the unique mild solution of the Cauchy problem (3.18). Then the family $\{e^{-tA}\}_{t=0}^T$ of mappings $e^{-tA} : \overline{D(A)}^X \times L^1(0,T;X) \to \overline{D(A)}^X$ defined by

$$e^{-tA}(u_0, g) := u(t) \qquad \text{for every } t \in [0,T], \tag{3.25}$$

$u_0 \in \overline{D(A)}^X$, and $g \in L^1(0,T;X)$ forms a strongly continuous *semigroup* as defined as follows.

Definition 3.9 Given a subset C of X, a family $\{e^{-tA}\}_{t=0}^T$ of mapping $e^{-tA} : C \times L^1(0,T;X) \to C$ is called a *strongly continuous semigroup (or, for simplicity, C_0-semigroup) of quasi-contractive mappings e^{-tA}* if $\{e^{-tA}\}_{t=0}^T$ satisfies the following three properties:

- (*semigroup property*) for every $(u_0,g) \in \overline{D(A)}^X \times L^1(0,T;X)$,
$$e^{-(t+s)A}(u_0,T_s g) = e^{-tA}(e^{-sA}(u_0,g),T_s g)$$
for every t, $s \in [0,T]$ with $t+s \le T$. Here, T_t denotes the *shift operator* and is given by
$$(T_t g)(s) := g(t+s) \tag{3.26}$$
for every $s \in [0,T-t]$, $t \in [0,T)$, and $g \in L^1(0,T;X)$;
- (*strong continuity*) for every $(u_0,g) \in \overline{D(A)}^X \times L^1(0,T;X)$,
$$t \mapsto e^{-tA}(u_0,T_0 g) \text{ belongs to } C([0,T];X);$$
- (*ω-quasi contractivity*) e^{-tA} satisfies
$$\begin{aligned}\|e^{-tA}(u_0,T_0 g) - e^{-tA}(\hat{u}_0,T_0\hat{g})\|_X &\le e^{\omega t}\|u_0-\hat{u}_0\|_X \\ &\quad + \int_0^t e^{\omega(t-s)}\|g(s)-\hat{g}(s)\|_X\,\mathrm{d}s\end{aligned}$$

Notation 3.2 In the case $g \equiv 0$, we simply write $e^{-tA}u_0$ instead of $e^{-tA}(u_0,T_0 0)$. To a given semigroup $\{e^{-tA}\}_{t\ge 0}$ one assigns the following operator.

Definition 3.10 Given a C_0-semigroup $\{e^{-tA}\}_{t\ge 0}$ of ω-quasi contractions e^{-tA} on $\overline{D(A)}^X$, then the operator
$$A_0 := \left\{(u_0,w) \in X\times X \,\middle|\, \lim_{h\downarrow 0}\frac{e^{-hA}u_0-u_0}{h} = w \quad \text{in } X\right\} \tag{3.27}$$
is an ω-quasi accretive well-defined mapping $A_0 : D(A_0) \to X$ and called the *infinitesimal generator* of $\{e^{-tA}\}_{t\ge 0}$.

If the Banach space X and its dual space X^* are both uniformly convex (see [23, Proposition 4.3]), then one has that $-A_0 = A^\circ$, where A° is the minimal selection of A defined by
$$A^\circ := \left\{(u,w) \in A \,\middle|\, \|w\|_X = \inf_{\hat{w}\in Au}\|\hat{w}\|_X\right\}. \tag{3.28}$$
Moreover, for every initial value $u_0 \in D(A)$ and $g \in W^{1,1}(0,T;X)$, the mild solution u of the Cauchy problem (3.18) belongs to the space $W^{1,\infty}_{\text{loc}}([0,\infty);X)$, u is almost

everywhere differentiable on $(0,\infty)$, differentiable from the right at every $t \geq 0$, $\frac{du}{dt}_+(t)$ is right continuous on $[0,\infty)$, and for every $t \in [0,T)$,

$$u(t) \in D(A) \quad \text{and} \quad \tfrac{\mathrm{d}}{\mathrm{d}t}_+ u(t) + (Au(t) - g(t))^\circ = 0 \tag{3.29}$$

(cf. [23, Theorem 4.6] or [30, Corollary (7.11)]).

For simplicity, we ignore the additional geometric assumptions on the Banach space X, and fix the following convention.

Definition 3.11 Given a quasi m-accretive operator A on X, then we refer to the family $\{e^{tA}\}_{t\geq 0}$ defined by (3.25) for $g \equiv 0$ as the C_0*-semigroup generated by* $-A$.

3.2.2.2 Completely accretive operators

Here, we briefly recall the notion of completely accretive operators, which was introduced by Bénilan and Crandall [29] and further developed in [56].

Definition 3.12 Let $X \subseteq M(\Sigma,\mu)$ be a normed space. Then, a mapping $S : D(S) \to X$ with domain $D(S) \subseteq X$ is called a T*-contraction* if

$$\|[Su - S\hat{u}]^+\|_X \leq \|[u - \hat{u}]^+\|_X \qquad \text{for every } u,\, \hat{u} \in D(S).$$

Obviously every T-contraction S on X is *order preserving* as defined as follows.

Definition 3.13 We call a family $\{T_\lambda\}_{\lambda>0}$ of mappings $T_\lambda : D \to M(\Sigma,\mu)$ with (common) domain $D \subseteq M(\Sigma,\mu)$ to be *order preserving* if for every $\lambda > 0$ and u, $v \in D$ with $u \leq v$, one has that $T_\lambda u \leq T_\lambda v$.

Further, for $X = L^q$, $1 \leq q \leq \infty$, or $X = C(\Sigma)$, every T-contraction on X is a contraction on X. In general Banach spaces X this result is not true (cf the appendix of [6]).

Definition 3.14 Let $X \subseteq M(\Sigma,\mu)$ be a normed space. An operator A on X is called T*-accretive* on X if for every $\lambda > 0$, the resolvent operator J_λ of A is a T-contraction.

Next, let

$$\mathcal{J}_0 := \Big\{ j : \mathbb{R} \to [0,+\infty] \,\Big|\, j \text{ is convex, lower semicontinuous, } j(0) = 0 \Big\}. \tag{3.30}$$

Then, for every u, $v \in M(\Sigma,\mu)$, we write

$$u \ll v \quad \text{if and only if} \quad \int_\Sigma j(u)\,\mathrm{d}\mu \leq \int_\Sigma j(v)\,\mathrm{d}\mu \quad \text{for all } j \in \mathcal{J}_0.$$

Remark 3.1 Due to the interpolation result [29, Proposition 1.2], for given u, $v \in M(\Sigma,\mu)$, the relation $u \ll v$ is equivalent to the following two conditions

$$\begin{cases} \int_{\Sigma} [u-k]^{+}\,\mathrm{d}\mu & \le \int_{\Sigma} [v-k]^{+}\,\mathrm{d}\mu \qquad \text{for all } k>0 \text{ and} \\ \int_{\Sigma} [u+k]^{-}\,\mathrm{d}\mu & \le \int_{\Sigma} [v+k]^{-}\,\mathrm{d}\mu \qquad \text{for all } k>0. \end{cases}$$

By this characterization, it is clear that for every $u, v, w \in M(\Sigma,\mu)$,

$$\text{if } u \ll v \text{ and } 0 \le w \le u \text{ then } w \ll v. \tag{3.31}$$

Thus, the relation $\ll$ is closely related to the theory of rearrangement-invariant function spaces (cf [32]). Another, useful characterization of the relation $\ll$ is the following (cf [29, Remark 1.5]): for every $u, v \in M(\Sigma,\mu)$, one has that

$$u \ll v \text{ if and only if } u^{+} \ll v^{+} \text{ and } u^{-} \ll v^{-}.$$

Further, the relation $\ll$ on $M(\Sigma,\mu)$ has the following properties. We omit the proof of this proposition.

Proposition 3.2 *For every $u, v, w \in M(\Sigma,\mu)$, one has that*

1. $u^{+} \ll u$, $u^{-} \ll -u$;
2. $u \ll v$ *if and only if* $u^{+} \ll v^{+}$ *and* $u^{-} \ll v^{-}$;
3. (positive homogeneity) *if* $u \ll v$ *then* $\alpha u \ll \alpha v$ *for all* $\alpha > 0$;
4. (transitivity) *if* $u \ll v$ *and* $v \ll w$ *then* $u \ll w$;
5. *if* $u \ll v$ *then* $|u| \ll |v|$;
6. (convexity) *for every* $u \in M(\Sigma,\mu)$, *the set* $\{w \mid w \ll u\}$ *is convex.*

With this in mind, we can now state the following two definitions.

Definition 3.15 A mapping $S : D(S) \to M(\Sigma,\mu)$ with domain $D(S) \subseteq M(\Sigma,\mu)$ is called a *complete contraction* if

$$Su - S\hat{u} \ll u - \hat{u} \qquad \text{for every } u, \hat{u} \in D(S).$$

Now, we can introduce the class of completely accretive operators.

Definition 3.16 An operator A on $M(\Sigma,\mu)$ is called *completely accretive* if for every $\lambda > 0$, the resolvent operator J_{λ} of A is a complete contraction, or equivalently, if for every (u_1, v_1), $(u_2, v_2) \in A$ and $\lambda > 0$, one has that

$$u_1 - u_2 \ll u_1 - u_2 + \lambda(v_1 - v_2).$$

If X is a linear subspace of $M(\Sigma,\mu)$ and A an operator on X, then A is *m-completely accretive on X* if A is completely accretive and satisfies the range condition (3.14). Further, for $\omega \in \mathbb{R}$, an operator A on a linear subspace $X \subseteq M(\Sigma,\mu)$ is called *ω-quasi (m)-completely accretive* in X if $A + \omega I$ is (m)-completely accretive in X. Finally, an operator A on a linear subspace $X \subseteq M(\Sigma,\mu)$ is called *quasi m-completely accretive* if there is some $\omega \in \mathbb{R}$ such that $A + \omega I$ is m-completely accretive in X.

Remark 3.2 Note, for every $1 \le q < \infty$, one has that $j_q(\cdot) = |[\cdot]^+|^q \in \mathcal{J}_0$ and $j_\infty(\cdot) = [[\cdot]^+ - k]^+ \in \mathcal{J}_0$ for $k \ge 0$ large enough if $q = \infty$. Further, for every N-function ψ and $\alpha > 0$, one has that $j_{\psi,\alpha}(\cdot) = \psi(\frac{[\cdot]^+}{\alpha}) \in \mathcal{J}_0$. Thus, every complete contraction S is a T-contraction on L^q for every $1 \le q \le \infty$ and on the Orlicz space L^ψ for every N-function ψ.

The following characterization of completely accretive operators is turns out to be very useful in the application.

Proposition 3.3 ([29] for $\omega = 0$, [56] for general ω) *Let P_0 be the set of (smooth) truncator functions given by*

$$P_0 = \left\{ p \in C^\infty(\mathbb{R}) \,\middle|\, 0 \le p' \le 1,\ \text{supp}(p')\ \textit{compact},\ 0 \notin \text{supp}(p) \right\} \tag{3.32}$$

and $\omega \in \mathbb{R}$. Then, an operator $A \subseteq L_0 \times L_0$ is ω-quasi completely accretive if and only if

$$\int_\Sigma p(u - \hat{u})(v - \hat{v})\,\mathrm{d}\mu + \omega \int_\Sigma p(u - \hat{u})(u - \hat{u})\,\mathrm{d}\mu \ge 0$$

for every $p \in P_0$ and every (u, v), $(\hat{u}, \hat{v}) \in A$.

For our next theorem, we require a specific class of Banach spaces $X \subseteq M(\Sigma, \mu)$, which were introduced first in [29].

Definition 3.17 A Banach space $X \subseteq M(\Sigma, \mu)$ with norm $\|\cdot\|_X$ is called *normal* if the norm $\|\cdot\|_X$ has the following property:

$$\begin{cases} \text{for every } u \in X,\ v \in M(\Sigma, \mu) \text{ satisfying} \quad v \ll u, \\ \text{one has that} \quad v \in X \quad \text{and} \quad \|v\|_X \le \|u\|_X. \end{cases}$$

Remark 3.3 Typical examples of normal Banach spaces $X \subseteq M(\Sigma, \mu)$ are Orlicz-spaces L^ψ for every N-function ψ, L^q for every $1 \le q \le \infty$, $L^{1\cap\infty}$, L_0, and $L^{1+\infty}$.

Remark 3.4 It is important to point out that if X is a normal Banach space, then for every $u \in X$, one always has that u^+, u^- and $|u| \in X$. To see this, recall that by (1) Proposition 3.2, if $u \in X$, then $u^+ \ll u$ and $u^- \ll -u$. Thus, u^+ and $u^- \in X$ and since $|u| = u^+ + u^-$, one also has that $|u| \in X$.

Remark 3.5 The dual space $(L_0)^*$ of L_0 is isometrically isomorphic to the space $L^{1\cap\infty}$. Thus, a sequence $(u_n)_{n\ge1}$ in L_0 is said to be *weakly convergent* to u in L_0 if

$$\langle v, u_n \rangle := \int_\Sigma v\, u_n\,\mathrm{d}\mu \to \int_\Sigma v\, u\,\mathrm{d}\mu \qquad \text{for every } v \in L^{1\cap\infty}.$$

For the rest of this paper, we write $\sigma(L_0, L^{1\cap\infty})$ to denote the *weak topology* on L_0.

Concerning the weak topology $\sigma(L_0, L^{1\cap\infty})$ on L_0, we have the following compactness result.

Proposition 3.4 ([29, Proposition 2.11]) *Let $u \in L_0$. Then, the following statements hold.*

1. *The set $\left\{v \in M(\Sigma,\mu) \,\middle|\, v \ll u\right\}$ is $\sigma(L_0, L^{1\cap\infty})$-sequentially compact in L_0;*
2. *Let $X \subseteq M(\Sigma,\mu)$ be a normal Banach space satisfying $X \subseteq L_0$ and*

$$\begin{cases} \text{for every } u \in X,\ (u_n)_{n\ge 1} \subseteq M(\Sigma,\mu) \text{ with } u_n \ll u \text{ for all } n \ge 1 \\ \text{and } \lim_{n\to+\infty} u_n(x) = u(x)\ \mu\text{-a.e. on } \Sigma, \text{ yields } \quad \lim_{n\to+\infty} u_n = u \text{ in } X. \end{cases} \tag{3.33}$$

Then for every $u \in X$ and sequence $(u_n)_{n\ge 1} \subseteq M(\Sigma,\mu)$ satisfying

$$u_n \ll u \text{ for all } n \ge 1 \quad \text{and} \quad \lim_{n\to+\infty} u_n = u\ \sigma(L_0, L^{1\cap\infty})\text{-weakly in } X,$$

one has that

$$\lim_{n\to+\infty} u_n = u \qquad \text{in } X.$$

Remark 3.6 Note, examples of normal Banach spaces $X \subseteq L_0(\Sigma,\mu)$ satisfying (3.33) are $X = L^p(\Sigma,\mu)$ for $1 \le p < \infty$, $L^\psi(\Sigma,\mu)$ for every N-function satisfying the Δ_2-condition (see [126, Theorem 12 in Chap. 3.4]), and $L_0(\Sigma,\mu)$.

To conclude this preliminary section, we state the following theorem summarizing statements from [56], which we require in Section 3.4. For the class of completely accretive operators, this theorem is due to Bénilan and Crandall [29].

Theorem 3.4 *For $\omega \in \mathbb{R}$, let A be ω-quasi completely accretive in L_0.*

1. *If there is a $\lambda_0 > 0$ such that $Rg(I+\lambda_0 A)$ is dense in L_0, then for the closure $\overline{A}^{L_0}$ of A in L_0 and every normal Banach space $X \subseteq L_0$, the restriction*

$$\overline{A}_X^{L_0} := \overline{A}^{L_0} \cap (X \times X)$$

of A on X is the unique ω-quasi m-completely accretive extension of the part $A_X = A \cap (X\times X)$ of A in X.

2. *For a given normal Banach space $X \subseteq L_0$, suppose A is ω-quasi m-completely accretive in X for some $\omega \in \mathbb{R}$, and $\{e^{-tA}\}_{t\ge 0}$ be the semigroup generated by $-A$ on $\overline{D(A)}^X$. Further, let $\{e^{-t\overline{A}^{L_0}}\}_{t\ge 0}$ be the semigroup generated by $-\overline{A}^{L_0}$, where $\overline{A}^{L_0}$ denotes the closure of A in $\overline{X}^{L_0}$. Then, the following statements hold.*

   a. *The semigroup $\{e^{-t\overline{A}^{L_0}}\}_{t\ge 0}$ is ω-quasi completely contractive on $\overline{D(A)}^{L_0}$. Moreover, one has that*

$$e^{-tA}u = e^{-t\overline{A}^{L_0}}u \qquad \text{for every } u \in \overline{D(A)}^X,$$

where $e^{-t\overline{A}^{L_0}}$ is the closure of e^{-tA} in L_0. In particular, one has that

$$e^{-t\overline{A}^{L_0}}u = L_0 - \lim_{n\to+\infty}\left(I + \frac{t}{n}A\right)^{-n} u \tag{3.34}$$

for every $u \in \overline{D(A)}^{L_0} \cap X$.

b. *The restriction* $A_X := \overline{A}^{L_0} \cap (X \times X)$ *of* $\overline{A}^{L_0}$ *on* X *is the unique* ω*-quasi* m*-complete extension of* A *in* X*; that is,* $A = A_X$.

c. *The operator* A *is sequentially closed in* $X \times X$ *equipped with the relative* $(L_0 \times (X, \sigma(L_0, L^{1\cap\infty})))$*-topology.*

d. *The domain of* A *is characterized by*

$$D(A) = \left\{ u \in \overline{D(A)}^{L_0} \cap X \;\middle|\; \begin{array}{l} \textit{there exits } v \in X \textit{ such that} \\ e^{-\omega t}\, \frac{\overline{A}^{L_0} u - u}{t} \ll v \textit{ for all small } t > 0 \end{array} \right\};$$

e. *For every* $t \geq 0$*, one has* $S_t(D(A)) \subseteq D(A)$*;*

f. *For every* $u \in D(A)$*, one has that*

$$\lim_{t\to 0+} \frac{e^{-t\overline{A}^{L_0}} u - u}{t} = -A^{\circ} u \qquad \textit{strongly in } L_0. \tag{3.35}$$

3.2.3 Homogeneous operators - Part I

The aim of this part is to briefly summarize the smoothing effect of mild solutions u of the homogeneous Cauchy problem

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t}(t) + A(u(t)) \ni 0 & \text{for } t \in (0,T), \\ \qquad\qquad u(0) = u_0 \end{cases} \tag{3.36}$$

provided the m-accretive operator A admits the following homogeneity property.

Definition 3.18 An operator A on X is called *homogeneous of order* $\alpha \in \mathbb{R}$ if $(0,0) \in A$ and for every $u \in D(A)$ and $\lambda \geq 0$, one has that $\lambda u \in D(A)$ and

$$A(\lambda u) = \lambda^{\alpha} A u. \tag{3.37}$$

For the class of operators $A = \partial_H \varphi$ admitting a subgradient structure defined on a Hilbert space $X = H$, every mild solution u of the Cauchy problem (3.36) admits the regularizing effect that it is indeed a strong solution of(3.36) (see Theorem 3.3).

In the pioneering work [28], Bénilan and Crandall showed that for the class of *homogeneous operators* A *of order* $\alpha > 0$ *with* $\alpha \neq 1$ defined on reflexive Banach space $(X, \|\cdot\|_X)$, every mild solution of the Cauchy problem (3.36) admits the same regularization effect. In particular, for this class of operators defined on a general Banach space X, every mild solution u of the Cauchy problem (3.36) satisfies the follwoing global regularity estimate

$$\limsup_{h\to 0+} \frac{\|u(t+h)-u(t)\|_X}{h} \le 2\,\frac{\|u_0\|_X}{|\alpha-1|}\,\frac{1}{t} \tag{3.38}$$

for every $t > 0$. Moreover, they showed that if X is equipped with a partial ordering "$\le''$ such that $(X,\le)$ defines an ordered vector space, and if A is T m-accretive, then every positive mild solution u of the Cauchy problem (3.36) satisfies the point-wise estimate

$$(\alpha-1)\frac{u(t+h)-u(t)}{h} \ge (\alpha-1)\frac{(1+h/t)^{\frac{1}{1-\alpha}}-1}{h}\,u(t) \tag{3.39}$$

for every $t > 0$. In the second paper [29], Bénilan and Crandall shows that the limit in (3.38) and (3.39) (for $h \to 0+$) exist provided A is a homogeneous (or order $\alpha > 0$, $\alpha \neq 1$), m-completely accretive defined on a normal Banach space $X \subseteq L_0$ satisfying (3.33).

These results were recently extended to the class of homogeneous operators of *order zero* in [87] by Mazon and the author and in [86] for homogeneous evolution equations with perturbation.

For later use, we summarize these results in following theorem.

Theorem 3.5 *Let X be Banach space and A an m-accretive operator on X, which is homogeneous operators A of order $\alpha \in \mathbb{R}\setminus\{1\}$. Then the following statements hold.*

1. *For every $u_0 \in \overline{D(A)}^X$, the corresponding mild solution u of Cauchy problem (3.36) satisfies the the global regularity estimate (3.38).*
2. *If X is equipped with a partial ordering "$\le''$ such that $(X,\le)$ defines an ordered vector space, and if, in addition, A is T-accretive on X, then for every positive $u_0 \in \overline{D(A)}^X$, the corresponding mild solution u of (3.36) satisfies the point-wise estimate (3.39).*
3. *If $X \subseteq L_0(\Sigma,\mu)$ is normal Banach space satisfying (3.33), then for every $u_0 \in \overline{D(A)}^X$, the corresponding mild solution u of Cauchy problem (3.36) is locally absolutely continuous on $(0,\infty)$, differentiable at a.e. $t \in (0,T)$ satisfying*

$$|A^{\circ}h(t)| \le \frac{2}{|\alpha-1|}\,\frac{|h_0|}{t} \qquad \mu\text{-a.e. on } \Sigma, \tag{3.40}$$

and

$$\left\|\frac{\mathrm{d}h}{\mathrm{d}t_+}(t)\right\|_X \le \frac{2}{|\alpha-1|}\,\frac{\|h_0\|_X}{t} \tag{3.41}$$

for every $t > 0$, and

$$(\alpha-1)A^{\circ}h(t) \le \frac{h(t)}{t} \qquad \mu\text{-a.e. on } \Sigma \tag{3.42}$$

for every $t > 0$ provided $h_0 \ge 0$.

The statements of Theorem 3.5 follow from Theorem 5.4 and Theorem 5.5 in a slightly more general setting (see Section 5.2.2).

3.3 The Dirichlet-to-Neumann operator on L^2

Suppose that the flux function $\boldsymbol{\sigma}(x,\nabla u)$ is a vector-valued Carathéodory function $\boldsymbol{\sigma}:\overline{\Omega}\times\mathbb{R}^d\to\mathbb{R}^d$ satisfying (2.2)-(2.4). Further, we assume that $\boldsymbol{\sigma}(x,\boldsymbol{\xi})$ is a gradient field with respect to the second variable $\boldsymbol{\xi}$. More precisely, suppose there is a second Carathéodory function $\mathcal{A}:\Omega\times\mathbb{R}^d\to\mathbb{R}$ satisfying (2.5).

Under these assumptions, we begin this section by realizing the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ on $L^2(\partial\Omega)$ and then subsequently, introduce the Dirichlet-to-Neumann map $\Lambda_{|X}$ on the Banach spaces $X=L^1(\partial\Omega)$, $X=L^\psi(\partial\Omega)$ for any N-function ψ satisfying the Δ_2-condition, and $X=C(\partial\Omega)$.

We begin by introducing the following two definitions.

Definition 3.19 For a function $u\in W^{1,p}(\Omega)$ we say that the *generalized normal derivative* $\boldsymbol{\sigma}(\cdot,\nabla u)\cdot\boldsymbol{\nu}$ *of* u *exists in* $L^2(\partial\Omega)$ and write $\boldsymbol{\sigma}(\cdot,\nabla u)\cdot\boldsymbol{\nu}\in L^2(\partial\Omega)$ if there are $\psi\in L^2(\partial\Omega)$ and $F\in L^1(\Omega)$ satisfying

$$\int_\Omega \boldsymbol{\sigma}(\cdot,\nabla u)\nabla\xi\,\mathrm{d}x=\int_{\partial\Omega}\psi\,\xi\,\mathrm{d}\mathcal{H}^{d-1}-\int_\Omega F\,\xi\,\mathrm{d}x \tag{3.43}$$

for every $\xi\in C^\infty(\overline{\Omega})$. In order to justify this definition, recall that by Lemma 2.1, the set $\{\xi_{|\partial\Omega}\,|\,\xi\in C^\infty(\overline{\Omega})\}$ lies dense in $L^2(\partial\Omega)$. Thus, any $\psi\in L^2(\partial\Omega)$ satisfying equation (3.43) for the same $F\in L^1(\Omega)$ and all $\xi\in C^\infty(\overline{\Omega})$ is unique and hence, it makes sense to set $\psi=\boldsymbol{\sigma}(\cdot,\nabla u)\cdot\boldsymbol{\nu}$.

Now, we are in a position to define the Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ on $L^2(\partial\Omega)$.

Definition 3.20 Let $D(\Lambda_{|L^2})$ be the set of all $h\in L^2(\partial\Omega)\cap W^{1-1/p,p}(\partial\Omega)$ with the property that there is a $\psi\in L^2(\partial\Omega)$ satisfying

$$\int_\Omega \boldsymbol{\sigma}(x,\nabla P(h))\nabla\mathcal{Z}(\xi)\,\mathrm{d}x=\int_{\partial\Omega}\psi\,\xi\,\mathrm{d}\mathcal{H}^{d-1} \tag{3.44}$$

for all $\xi\in L^2(\partial\Omega)\cap W^{1-1/p,p}(\partial\Omega)$. Since $\psi\in L^2(\partial\Omega)$ is uniquely determined by (3.44), we can set

$$\Lambda_{|L^2}h=\psi \qquad \text{for every } h\in D(\Lambda_{|L^2}). \tag{3.45}$$

We call the operator $\Lambda_{|L^2}:D(\Lambda_{|L^2})\to L^2(\partial\Omega)$ defined by (3.45) the *Dirichlet-to-Neumann operator* on $L^2(\partial\Omega)$ associated with the quasi-linear operator A given by (2.20).

Now, let

$$V_0:=W^{1-1/p,p}(\partial\Omega)\cap L^2(\partial\Omega)$$

be equipped with the sum norm and V_0^* be the dual space of V_0. Then by construction, $V_0\hookrightarrow L^2(\partial\Omega)$ by a bounded linear injection i with a dense image. Hence, the adjoint operator $i^*: L^2(\partial\Omega)^* \hookrightarrow V_0^*$ is a continuous injection as well. We identify $L^2(\partial\Omega)$ with its dual space $L^2(\partial\Omega)^*$. Then, the space V_0 can be considered as a linear subspace of $L^2(\partial\Omega)$ and V_0^*, where $L^2(\partial\Omega)$ is an intermediate space between V_0 and V_0^*. To this end, we note that if Λ is the operator defined in (2.25) (see Definition 2.4), then the restriction $\Lambda_{|V_0}: V_0 \to V_0^*$ of Λ on V_0 remains a continuous and monotone mapping.

Our next proposition shows that the operator $\Lambda_{|L^2}$ is the *part of* Λ *in* $L^2(\partial\Omega)$.

Proposition 3.5 *Let $\Lambda_{|V_0}: V_0 \to V_0^*$ be the operator defined by*

$$\langle \Lambda_\sigma h, \xi\rangle = \int_\Omega \boldsymbol{\sigma}(x, \nabla P(h))\nabla \mathcal{Z}(\xi)\,\mathrm{d}x$$

for every h, $\xi \in V_0$, where $\boldsymbol{\sigma}: \overline{\Omega}\times\mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (2.2)-(2.4). Then the following statements hold true.

1. *For every $h \in D(\Lambda_{|L^2})$, we have that*

$$\boldsymbol{\sigma}(\cdot, \nabla P(h))\cdot \boldsymbol{\nu} \in L^2(\partial\Omega) \qquad \text{and} \qquad \Lambda_{|V_0} h = \boldsymbol{\sigma}(\cdot, \nabla P(h))\cdot\boldsymbol{\nu}.$$

2. *The operator $\Lambda_{|L^2}$ is the part of Λ in $L^2(\partial\Omega)$, that is, $\Lambda_{|L^2}$ has effective domain*

$$D(\Lambda_{|L^2}) = \left\{ h \in L^2(\partial\Omega)\cap W^{1-1/p,p}(\partial\Omega) \,\middle|\, \Lambda h \in L^2(\partial\Omega)\right\} \quad \text{and}$$
$$\Lambda_{|L^2} h = \Lambda h \quad \text{for every } h \in D(\Lambda_{|L^2}).$$

Proof Since claim (2) is a consequence of claim (1), we only show that claim (1) holds. For this, let $h \in D(\Lambda_{|L^2})$ and suppose $\psi \in L^2(\partial\Omega)$ satisfies (3.44). For every $\xi \in C^\infty(\overline{\Omega})$, $\mathcal{Z}(\xi_{|\partial\Omega}) = \xi$. Therefore,

$$\int_{\partial\Omega} \psi\,\xi\,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega \boldsymbol{\sigma}(x, \nabla P(h))\nabla\xi\,\mathrm{d}x$$

for every $\xi \in C^\infty(\overline{\Omega})$. Thus, ψ satisfies equation (3.43) for every $\xi \in C^\infty(\overline{\Omega})$. This shows that claim (1) of this proposition is true. □

Our next theorem contains the key property of $\Lambda_{|L^2}$, which ensures that $-\Lambda_{|L^2}$ generates a strongly continuous order preserving semigroup $\{e^{-t\Lambda_{|L^2}}\}$ of contractions $e^{-t\Lambda_{|L^2}}$ on $L^2(\partial\Omega)$. Moreover, each mapping $e^{-t\Lambda_{|L^2}}$ can be extrapolated to a contractive mapping on a normal Banach space $X \subseteq L_0(\partial\Omega, \mathcal{H}^{d-1})$ and on $X = C(\partial\Omega)$ (see Section 3.4).

Theorem 3.6 *Suppose $\boldsymbol{\sigma}: \overline{\Omega}\times\mathbb{R}^d \to \mathbb{R}^d$ is a vector-valued Carathéodory function satisfying (2.2)-(2.4), and there is a second Carathéodory function $\mathcal{A}: \Omega\times\mathbb{R}^d \to \mathbb{R}$ satisfying (2.5). Then, the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ on $L^2(\partial\Omega)$ can be realized as the subgradient $\partial_{L^2}\varphi$ in $L^2(\partial\Omega)$ of the proper, convex, densely defined and lower semicontinuous functional*

$$\varphi(h) := \begin{cases} \displaystyle\int_\Omega \mathcal{A}(x, \nabla P(h))\,\mathrm{d}x, & \text{if } h \in W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega),\\ +\infty, & \text{if otherwise} \end{cases} \tag{3.46}$$

for every $h \in L^2(\partial\Omega)$. In particular, $\Lambda_{|L^2}$ is m-completely accretive in $L^2(\partial\Omega)$.

The statement of Theorem 3.6 generalizes [68, part (A) of Theorema in Section 2.] and [93, Theorem 2]. It was first proven in [3] (see also [4]) that the realization $\Lambda_{|L^1}$ of the Dirichlet-to-Neumann map in $L^1(\partial\Omega)$ is m-completely accretive in $L^1(\partial\Omega)$.

As mentioned above, we can conclude from Theorem 3.6 and the classical theory of non-linear semigroups in Hilbert spaces (cf. [40, Théorème 3.2]) the following corollary.

Corollary 3.1 *The negative Dirichlet-to-Neumann operator $-\Lambda_{|L^2}$ in $L^2(\partial\Omega)$ generates an order preserving C_0-semigroup $\{e^{-t\Lambda_{|L^2}}\}_{t\ge 0}$ of contractions $e^{-t\Lambda_{|L^2}}$ on $L^2(\partial\Omega)$, and each mapping $e^{-t\Lambda_{|L^2}}$ can be extrapolated to a contractive mapping on a normal Banach space $X \subseteq L_0(\partial\Omega, \mathcal{H}^{d-1})$.*

The proof of Theorem 3.6 is given by the following four lemmas.

Lemma 3.1 *Let $(h_n)_{n\ge1}$ be a sequence in $L^2(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega)$ and suppose $(h_n)_{n\ge1}$ is bounded in $L^2(\partial\Omega)$ such that $(\nabla P(h)_n)_{n\ge1}$ is bounded in $L^p(\Omega)^d$. Then there is a $h \in L^2(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega)$ and a subsequence $(h_{k_n})_{n\ge1}$ of $(h_n)_{n\ge1}$ such that h_{k_n} converges weakly to h in $L^2(\partial\Omega)$, $P(h_{k_n})_{n\ge1}$ converges to $P(h)$ weakly in $W^{1,p}(\Omega)$ and $\mathcal{A}(x, \nabla P(h_{k_n}))$ converges to $\mathcal{A}(x, \nabla P(h))$ a.e. on Ω.*

Proof To see that the claim of this lemma holds, let $(h_n)_{n\ge1}$ be a sequence in $L^2(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega)$ such that $(h_n)_{n\ge1}$ is bounded in $L^2(\partial\Omega)$ and $(\nabla P(h_n)_{n\ge1})$ is bounded in $L^p(\Omega)^d$. By inequality (2.10), we can conclude that $(P(h_n)_{n\ge1})$ is bounded in $W^{1,p}(\Omega)$. Thus by Lemma 2.2 and since $L^2(\partial\Omega)$ is reflexive, there is a subsequence $(h_{k_n})_{n\ge1}$ of $(h_n)_{n\ge1}$, a weak solution $u \in W^{1,p}(\Omega)$ of (2.12) and some $h \in L^2(\partial\Omega)$ such that $P(h_{k_n})$ converges to u weakly in $W^{1,p}(\Omega)$, and h_{k_n} converges weakly to h in $L^2(\partial\Omega)$. Thus and since the trace operator $\mathcal{T}r$ from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$ is compact, it follows that $h \in L^2(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega)$ and $u = P(h)$. Since $\mathcal{A}$ and $\boldsymbol{\sigma}$ are Carathéodory functions satisfying (2.5) and since $\boldsymbol{\sigma}$ satisfies the growth condition (2.3), Lemma 2.2 implies that $\mathcal{A}(x, \nabla P(h_{k_n}))$ converges to $\mathcal{A}(x, \nabla P(h))$ a.e. on Ω, completing the proof of this lemma. □

Lemma 3.2 *The functional φ defined by (3.46) is proper, convex, densely defined, and lower semicontinuous in $L^2(\partial\Omega)$ with domain $D(\varphi) = W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega)$.*

Proof Since $\mathcal{A}$ is a Carathéodory function and by the growth condition (2.3), the functional φ given by (3.46) is well-defined and proper. Furthermore by Lemma 2.1, the domain $D(\varphi)$ lies dense in $L^2(\partial\Omega)$. To see that φ is convex, let h_1, $h_2 \in D(\varphi)$ and $\lambda \in [0,1]$. Then, $\lambda h_1 + (1-\lambda)h_2 \in D(\varphi)$. Thus, by claim (3) of Proposition 2.1 and since $\xi \mapsto \mathcal{A}(x,\xi)$ is convex on $\mathbb{R}^d$ for a.e. $x \in \Omega$, we have that

$$\begin{aligned}\varphi(\lambda h_1+(1-\lambda)h_2) &\le \int_\Omega \mathcal{A}(x,\lambda\nabla P(h_1)+(1-\lambda)\nabla P(h_2))\,\mathrm{d}x\\ &\le \lambda\,\varphi(h_1)+(1-\lambda)\,\varphi(h_2).\end{aligned}$$

It remains to show that φ is lower semicontinuous in $L^2(\partial\Omega)$. For this, let $\alpha \ge 0$, $(h_n)_{n\ge1}$ be a sequence in $D(\varphi)$, and $h \in L^2(\partial\Omega)$. Suppose $\varphi(h_n)_{n\ge1} \le \alpha$ for all n and h_n converges to h in $L^2(\partial\Omega)$. Then the sequence $(h_n)_{n\ge1}$ satisfies the hypotheses of Lemma 3.1 and hence, $h \in D(\varphi)$ and there is a subsequence $(h_{k_n})_{n\ge1}$ of $(h_n)_{n\ge1}$ such that $\mathcal{A}(x,\nabla P(h_{k_n})_{n\ge1})$ converges to $\mathcal{A}(x,\nabla P(h))$ a.e on Ω. Thus and since $\mathcal{A}(x,\nabla P(h_{k_n}))$ are non-negative measurable functions from Ω to $\mathbb{R}$, Fatou's lemma and the assumption $\varphi(h_{k_n}) \le \alpha$ for all $n \ge 1$ imply

$$\varphi(h) \le \liminf_{n\to\infty} \varphi(h_{k_n}) \le \alpha.$$

As $\alpha \ge 0$ and the convergent sequence $(h_n)_{n\ge1} \subseteq D(\varphi)$ were arbitrary, we have thereby shown that φ is lower semicontinuous in $L^2(\partial\Omega)$. □

Lemma 3.3 *Let φ be the functional given by (3.46). Then the subgradient operator $\partial_{L^2}\varphi$ in $L^2(\partial\Omega)$ is single-valued and coincides with $\Lambda_{|L^2}$ on $L^2(\partial\Omega)$. In particular, $\Lambda_{|L^2}$ is m-accretive in $L^2(\partial\Omega)$.*

Proof First, let $h \in D(\Lambda_{|L^2})$. Then by Proposition 3.5 and using (2.23), h belongs to $D(\varphi)$ such that $\Lambda_{|L^2}h \in L^2(\partial\Omega)$ and satisfies

$$\int_{\partial\Omega} \Lambda_{|L^2}h\,\xi\,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega \boldsymbol{\sigma}(x,\nabla P(h))\nabla P(\xi)\,\mathrm{d}x \tag{3.47}$$

for all $\xi \in L^2(\partial\Omega)\cap W^{1-1/p,p}(\partial\Omega)$. For any $\psi \in D(\varphi)$, $\Phi := P(\psi)-P(h) \in W^{1,p}(\Omega)$ and has trace $\Phi_{|\partial\Omega} = \psi - h \in L^2(\partial\Omega)$. By claim (4) of Proposition 2.1, there is a unique $u_\Phi \in W^{1,p}_0(\Omega)$ such that $P(\psi-h) = u_\Phi + (P(psi)-P(h))$. Hence taking $\xi = \psi - h$ in (3.47) and using that $P(h)$ is a weak solution of (2.12), we see

$$\int_{\partial\Omega} \Lambda_{|L^2}h\,(\psi-h)\,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega \boldsymbol{\sigma}(x,\nabla P(h))(\nabla P(\psi)-\nabla P(h))\,\mathrm{d}x$$

By the assumptions (2.4) and (2.5), $\xi \mapsto \mathcal{A}(x,\xi)$ is convex on $\mathbb{R}^d$ satisfying $\nabla_\xi \mathcal{A}(x,\xi) = \boldsymbol{\sigma}(x,\xi)$ for every $\xi \in \mathbb{R}^d$ and for a.e. $x \in \Omega$. Thus

$$\boldsymbol{\sigma}(x,\xi_0)(\xi-\xi_0) \le \mathcal{A}(x,\xi) - \mathcal{A}(x,\xi_0)$$

for all ξ, $\xi_0 \in \mathbb{R}^d$ and a.e. $x \in \Omega$. Applying this inequality with $\xi_0 = \nabla P(h)$ and $\xi = \nabla P(\psi)$ to the right-hand side of the last equation, we see that

$$\int_{\partial\Omega} \Lambda_{|L^2}h\,(\psi-h)\,\mathrm{d}\mathcal{H}^{d-1} \le \varphi(\psi) - \varphi(h).$$

As $\psi \in D(\varphi)$ was arbitrary, this shows that $h \in D(\partial_{L^2}\varphi)$ and $\Lambda_{|L^2}h \in \partial_{L^2}\varphi(h)$.

Next, take $h \in D(\partial_{L^2}\varphi)$ and $\chi \in \partial_{L^2}\varphi(h)$. Then $h \in D(\varphi)$, $\chi \in L^2(\partial\Omega)$, and

$$\int_{\partial\Omega} \chi\,(\psi - h)\,\mathrm{d}\mathcal{H}^{d-1} \leq \int_{\Omega} \mathcal{A}(x, \nabla P(\psi))\,\mathrm{d}x - \int_{\Omega} \mathcal{A}(x, \nabla P(h))\,\mathrm{d}x$$

for every $\psi \in D(\varphi)$. Now, for $\lambda > 0$ and $\zeta \in D(\varphi)$, we take $\psi = h + \lambda\,\zeta$ in the previous inequality and subsequently divide the resulting inequality by λ. Then,

$$\int_{\partial\Omega} \chi\,\zeta\,\mathrm{d}\mathcal{H}^{d-1} \leq \frac{1}{\lambda}\left(\int_{\Omega} \mathcal{A}(x, \nabla P(h + \lambda\,\zeta))\,\mathrm{d}x - \int_{\Omega} \mathcal{A}(x, \nabla P(h))\,\mathrm{d}x\right).$$

By claim (3) of Proposition 2.1,

$$\int_{\Omega} \mathcal{A}(x, \nabla P(h + \lambda\,\zeta))\,\mathrm{d}x \leq \int_{\Omega} \mathcal{A}(x, \nabla P(h) + \lambda\,P(\zeta))\,\mathrm{d}x$$

and hence

$$\int_{\partial\Omega} \chi\,\zeta\,\mathrm{d}\mathcal{H}^{d-1} \leq \frac{1}{\lambda}\left(\int_{\Omega} \mathcal{A}(x, \nabla P(h) + \lambda\,P(\zeta))\,\mathrm{d}x - \int_{\Omega} \mathcal{A}(x, \nabla P(h))\,\mathrm{d}x\right)$$

for all $\lambda > 0$. Recall that the functional $\mathcal{F}$ given by (2.14) is convex, Gâteaux-differentiable on $W^{1,p}(\Omega)$ and its derivative is given by (2.15). Thus taking the infimum over all $\lambda > 0$ in the last inequality yields

$$\int_{\partial\Omega} \chi\,\zeta\,\mathrm{d}\mathcal{H}^{d-1} \leq \int_{\Omega} \boldsymbol{\sigma}(x, \nabla P(h))\nabla P(\zeta)\,\mathrm{d}x$$

and so by using again (2.23),

$$\int_{\partial\Omega} \chi\,\zeta\,\mathrm{d}\mathcal{H}^{d-1} \leq \int_{\Omega} \boldsymbol{\sigma}(x, \nabla P(h))\nabla \mathcal{Z}(\zeta)\,\mathrm{d}x.$$

Note, this inequality holds for all $\zeta \in D(\varphi)$, which is a linear vector space. Thus replacing ζ by $-\zeta$ in the latter inequality shows that χ and h satisfy (3.44). Hence, $h \in D(\Lambda_{|L^2})$ with $\Lambda_{|L^2} h = \chi$. As $\chi \in \partial_{L^2}\varphi(h)$ was arbitrary and $\Lambda_{|L^2} h$ is uniquely determined by (3.44), we have thereby shown that $\partial_{L^2}\varphi(h) = \{\Lambda_{|L^2} h\}$ for every $h \in D(\partial_{L^2}\varphi)$. This shows that the claim of this lemma holds. □

It remains to prove that $\Lambda_{|L^2}$ is completely accretive.

Lemma 3.4 *The Dirichlet-to-Neumann map $\Lambda_{|L^2}$ is completely accretive in $L^2(\partial\Omega)$.*

Proof Let $T \in P_0$ and $h, \hat{h} \in D(\Lambda_{|L^2})$. Since $T : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous such that $T(0) = 0$, $\Phi := T(P(h) - P(\hat{h})) \in W^{1,p}(\Omega)$ and has trace $T(P(h) - P(\hat{h}))_{|\partial\Omega} = T(h - \hat{h})$. Therefore, by claim (4) of Proposition 2.1 there is a unique $u_\Phi \in W_0^{1,p}(\Omega)$ satisfying $PT(h - \hat{h}) = u_\Phi + \Phi$. Applying this to formula (2.21) and using that $P(h)$ and $P(\hat{h})$ are weak solutions of equation (2.12) yields

$$\begin{aligned}
&\int_{\partial\Omega}(\Lambda_{|L^2}h-\Lambda_{|L^2}\hat{h})T(h_1-h_2)\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad=\int_{\Omega}(\boldsymbol{\sigma}(x,\nabla P(h))-\boldsymbol{\sigma}(x,\nabla P(\hat{h})))\nabla P(T(h_1-h_2))\,\mathrm{d}x\\
&\quad=\int_{\Omega}(\boldsymbol{\sigma}(x,\nabla P(h))-\boldsymbol{\sigma}(x,\nabla P(\hat{h})))\nabla T(P(h)-P(\hat{h}))\,\mathrm{d}x\\
&\quad=\int_{\Omega}(\boldsymbol{\sigma}(x,\nabla P(h))-\boldsymbol{\sigma}(x,\nabla P(\hat{h})))(\nabla P(h)-\nabla P(\hat{h}))T'(P(h)-P(\hat{h}))\,\mathrm{d}x.
\end{aligned}$$

Since $T'(P(h)-P(\hat{h}))\geq 0$ and by monotonicity condition (2.4), the integrand in the last integral of this calculation is non-negative a.e. on Ω, proving that the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ is completely accretive in $L^2(\partial\Omega)$. □

The claim of Lemma 3.4 completes the proof of Theorem 3.6.

3.4 The Dirichlet-to-Neumann map on L^1, L^ψ and C

This section is dedicated to realize the Dirichlet-to-Neumann map $\Lambda_{|X}$ on normal Banach space $X\subseteq L_0(\partial\Omega,\mathcal{H}^{d-1})$, and on $C(\partial\Omega)$.

Since the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ on $L^2(\partial\Omega)$ is an m-completely accretive operator (see Theorem 3.6), and since $L^2(\partial\Omega)\hookrightarrow L_0(\partial\Omega,\mathcal{H}^{d-1})$ by a continuous embedding with dense image, $\Lambda_{|L^2}$ is also completely accretive in $L_0(\partial\Omega,\mathcal{H}^{d-1})$ and the range $\mathrm{Rg}(I+A)$ is dense in $L_0(\partial\Omega,\mathcal{H}^{d-1})$. Therefore, we can introduce the following definition of $\Lambda_{|X}$ by following the construction given in Theorem 3.4.

Definition 3.21 Let $\Lambda_{|L^2}$ be the Dirichlet-to-Neumann map on $L^2(\partial\Omega)$ and let $X\subseteq L_0(\partial\Omega,\mathcal{H}^{d-1})$ be a normal Banach space or $X=C(\partial\Omega)$. Then we define the *Dirichlet-to-Neumann map* $\Lambda_{|X}$ *on* X by

$$\Lambda_{|X}=\left\{(h,\psi)\in X\times X\;\middle|\;\begin{array}{l}\text{there exists }((h_n,\psi_n))_{n\geq 1}\subseteq\Lambda_{|L^2}\text{ s.t.}\\ L_0-\lim_{n\to\infty}h_n=h\text{ and }L_0-\lim_{n\to\infty}\psi_n=\psi\end{array}\right\}.$$

Since, $X=L^\psi(\partial\Omega)$ for every N-function ψ, $X=L^1(\partial\Omega)$, $X=L^\infty(\partial\Omega)$ are normal Banach spaces (cf. Remark 3.3), we have thereby defined the *Dirichlet-to-Neumann map* $\Lambda_{|X}$ on those spaces X.

Notation 3.3 In the following, we write $D(\Lambda_{|X})$ for the domain of the Dirichlet-to-Neumann map $\Lambda_{|X}$ on X.

Remark 3.7 We note that the Definition 3.21 of the Dirichlet-to-Neumann map $\Lambda_{|X}$ on X differs from the one given in [85]. In fact, the definition of the Dirichlet-to-Neumann map $\Lambda_{|X}$ on X uses the closure in $L_0(\partial\Omega,\mathcal{H}^{d-1})$.

Theorem 3.7 *Let $X \subseteq L_0(\partial\Omega, \mathcal{H}^{d-1})$ be either a normal Banach space, or $X = C(\partial\Omega)$. Then, the following statements hold.*

1. *The Dirichlet-to-Neumann map $\Lambda_{|X}$ on X is m-completely accretive in X.*
2. *If X is continuously embedded into $L^2(\partial\Omega)$, then one has that*

$$\Lambda_{|X} = \left\{ (h,\psi) \in X \times X \,\middle|\, (h,\psi) \in \Lambda_{|L^2} \right\}. \tag{3.48}$$

Proof Concerning claim (1), we note that for the operator $\Lambda_{|X}$ (as defined in Definition 3.21) can be expressed as

$$\Lambda_{|X} = \overline{\Lambda_{|L^2(\partial\Omega)}}^{L_0} \cap (X \times X).$$

Therefore, it follows from Theorem 3.4 that $\Lambda_{|X}$ is the unique m-completely accretive extension in X of the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ in $L^2(\partial\Omega)$.

Next, we note that by Theorem 3.4 and since $\Lambda_{|L^2}$ is m-completely accretive (see Theorem 3.6), one has for $X = L^2(\partial\Omega)$ that

$$\Lambda_{|L^2} = \overline{\Lambda_{|L^2}}^{L_0}_{L^2(\partial\Omega)} := \overline{\Lambda_{|L^2(\partial\Omega)}}^{L_0} \cap (X \times X).$$

Now, let X not be $L^2(\partial\Omega)$ but still a normal Banach space, which is continuously embedded into $L^2(\partial\Omega)$ and set

$$B_{|X} = \left\{ (h,\psi) \in X \times X \,\middle|\, (h,\psi) \in \Lambda_{|L^2} \right\}.$$

Since for every pair $(h,\psi) \in B_{|X}$, the constant sequence $((h_n,\psi_n))_{n\geq 1}$ given by $h_n = h$ and $\psi_n = \psi$ for all $n \geq 1$ lies in $\Lambda_{|L^2}$ and converges to (h,ψ) in $L_0(\partial\Omega, \mathcal{H}^{d-1}) \times L_0(\partial\Omega, \mathcal{H}^{d-1})$, one has that $B_{|X}$ is contained in the operator $\overline{\Lambda_{|L^2}}^{L_0}_X$. Further, since $\Lambda_{|L^2}$ is completely accretive, also $B_{|X}$ needs to be completely accretive. Next, let $\psi \in X$. Then, by the continuous embedding of X into $L^2(\partial\Omega)$, and since $\Lambda_{|L^2}$ is m-completely accretive in $L^2(\partial\Omega)$, there is a solution $h \in D(\Lambda_{|L^2})$ of equation

$$h + \Lambda_{|L^2} h = \psi$$

Further, since the resolvent operator $J_1^{\Lambda_{|L^2}} := (I + \Lambda_{|L^2})^{-1}$ of $\Lambda_{|L^2}$ is completely contractive and by $0 \in \Lambda_{|L^2} 0$, one also has that $h \in X$ provided $X \neq C(\partial\Omega)$. Therefore, one has that $\Lambda_{|L^2} h = \psi - h \in X$ and so, $h \in D(\Lambda_{|X})$ satisfying $h + \Lambda_{|X} h = \psi$ in X, proving that $\Lambda_{|X}$ satisfies the range condition in X. If $\psi \in C(\partial\Omega)$. Then, $\psi \in L^\infty(\partial\Omega)$, which is a normal Banach space and hence, $h \in D(\Lambda_{|L^\infty(\partial\Omega)})$ satisfying $h + \Lambda_{|L^\infty(\partial\Omega)} h = \psi$ in $L^\infty(\partial\Omega)$. Applying claim (2) of Theorem 2.1 yields that $h \in C^{0,\alpha}(\partial\Omega)$ for some $0 < \alpha < 1$. Thus, $h \in D(\Lambda_{|C(\partial\Omega)})$ with $h + \Lambda_{C(\partial\Omega)} h = \psi$ in $C(\partial\Omega)$, proving that $\Lambda_{|C(\partial\Omega)}$ is m-accretive in $C(\partial\Omega)$. This shows that the operator $B_{|X}$ is m-completely accretive and $B_{|X} \subseteq \overline{\Lambda_{|L^2}}^{L_0}_X$. Since according to Theorem 3.4,

$\overline{\Lambda_{|L^2}}_X^{L_0}$ is the unique m-completely accretive extension of $\Lambda_{|L^2}$ in X, we have thereby proven that the characterization (3.48) for $\Lambda_{|X}$ holds. This completes the proof of this theorem. □

We can conclude the following consequence from the previous theorem.

Corollary 3.2 *Let $X \subseteq L_0(\partial\Omega, \mathcal{H}^{d-1})$ be either a normal Banach space, or $X = C(\partial\Omega)$. Then, the following statements hold.*

1. *The Dirichlet-to-Neumann map $\Lambda_{|X}$ on X generates an order preserving C_0-semigroup $\{e^{-t\Lambda_{|X}}\}_{t\ge 0}$ of contractions $e^{-t\Lambda_{|X}}$ on $\overline{D(\Lambda_{|X})}^X$. Moreover, the restiction*
$$e^{-t\Lambda_{|X}}_{|L^2(\partial\Omega)} = e^{-t\Lambda_{|L^2}}$$
of the semigroup $\{e^{-t\Lambda_{|L^2}}\}_{t\ge 0}$ generated by $-\Lambda_{|L^2}$ on $L^2(\partial\Omega)$.
2. *For $X = L^\psi(\partial\Omega)$ and ψ being an N-function satisfying the Δ_2-condition, $X = L^1(\partial\Omega)$, or for $X = C(\partial\Omega)$, one has that $\overline{D(\Lambda_{|X})}^X = X$.*

Proof Claim (1) follows from Theorem 3.7, Theorem 3.6 and by applying Theorem 3.4 to either $A = \Lambda_{|L^{1\cap\infty}}$ or $A = \Lambda_{|L^2}$. Since by Theorem 3.4, the semigroup $\{e^{-t\Lambda_{|X}}\}_{t\ge 0}$ is defined on the closure

$$\overline{\overline{D(\Lambda_{|L^2})}^{L^2(\partial\Omega)} \cap L^{1\cap\infty}(\partial\Omega)}^X \quad \text{in } X$$

and since the domain $D(\Lambda_{|L^2})$ is dense in $L^2(\partial\Omega)$ (see Theorem 3.6), claim (2) for $X = L^1(\partial\Omega)$ or $X = L^\psi(\partial\Omega)$ follows from the fact that $L^{1\cap\infty}(\partial\Omega)$ is dense in $L^1(\partial\Omega)$ and in $L^\psi(\partial\Omega)$ provided ψ is an N-function satisfying the Δ_2-condition (see Proposition 3.1). To see that this density result als holds for $X = C(\partial\Omega)$, we use that due to the embedding $C(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$, the operator $\Lambda_{|C(\partial\Omega)}$ is contained in $\Lambda_{|L^2}$. Thus, for every $\lambda > 0$, the resolvent operator $J_\lambda^{\Lambda_{|C(\partial\Omega)}}$ of $\Lambda_{|C(\partial\Omega)}$ and the resolvent operator $J_\lambda^{\Lambda_{|L^2}}$ of $\Lambda_{|L^2}$ coincide on $C(\partial\Omega)$. Since the domain $D(\Lambda_{|L^2})$ is dense in $L^2(\partial\Omega)$, for very $\xi \in C(\partial\Omega)$, $J_\lambda^{\Lambda_{|C(\partial\Omega)}}\xi \to \xi$ in $L^2(\partial\Omega)$ as $\lambda \to 0+$. Moreover, one has that $J_\lambda^{\Lambda_{|C(\partial\Omega)}}\xi \in D(\Lambda_{|C(\partial\Omega)})$ and $\|J_\lambda^{\Lambda_{|C(\partial\Omega)}}\xi\|_\infty \le \|\xi\|_\infty$ for all $\lambda > 0$. Since $h_\lambda := J_\lambda^{\Lambda_{|C(\partial\Omega)}}\xi$ is a weak solution of $h_\lambda + \Lambda_{|L^2} h_\lambda = \xi$, it follows from claim (2) of Theorem 2.1 that $h_\lambda \in C^{0,\alpha}(\partial\Omega)$ for some $0 < \alpha < 1$. In fact, $(h_\lambda)_{\lambda>0}$ is bounded in $C^{0,\alpha}(\partial\Omega)$. Thus, for every zero sequence $(\lambda_n)_{n\ge 1}$ in $(0,1)$, there is a subsequence of $(\lambda_n)_{n\ge 1}$ such that $h_{\lambda_n} = J_{\lambda_n}^{\Lambda_{|C(\partial\Omega)}}\xi \to \xi$ in $C(\partial\Omega)$. This completes the proof of this corollary. □

3.5 Proof of Theorem 3.1

This section is dedicated to outline the proof of Theorem 3.1. This proof optimizes the one given in [85].

Proof Let X be either $L^1(\partial\Omega)$, $L^\psi(\partial\Omega)$ for any N-function satisfying the Δ_2-condition, or $X = C(\partial\Omega)$.

By Corollary 3.2, for every $h_0 \in X$, there is a unique mild solution h of the homogeneous Cauchy problem (3.1), and for every h_0, $\hat{h}_0 \in X$, the corresponding mild solutions h and $\hat{h}$ satisfy (3.2). Since for every $c \in \mathbb{R}$, $\Lambda_{|L^2}(c\mathbb{1}_{\partial\Omega}) = 0$, it follows that

$$\Lambda_{|X}(c\mathbb{1}_{\partial\Omega}) = 0 \qquad \text{for all } c \in \mathbb{R}, \tag{3.49}$$

for every space X introduced in the header of this proof. Due to (3.49), one has that the constant functions $c\mathbb{1}_{\partial\Omega}$ are *fixed points* of the semigroup $\{e^{-t\Lambda_{|X}}\}_{t\geq 0}$, meaning that

$$e^{-t\Lambda_{|X}}(c\mathbb{1}_{\partial\Omega}) = c\mathbb{1}_{\partial\Omega} \qquad \text{for all } t \geq 0.$$

Thus and by (3.2) with $v = 1$, it follows that (3.3) holds.

Next, suppose X is continuously embedded into $L^2(\partial\Omega)$. Since the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ can be realized as a subgradient $\partial_{L^2}\varphi$ (see Theorem 3.6), the classical generation theorem (see Theorem 3.3) implies that for every $h_0 \in X$, the mild solution $t \mapsto h(t) := e^{-\Lambda_{|L^2}} h_0$ of (3.1) in X is a strong solution of (3.1) in $L^2(\partial\Omega)$ satisfying $h \in W^{1,\infty}([\delta,\infty); L^2(\partial\Omega))$ for every $\delta > 0$, and h is differentiable with values in $L^2(\partial\Omega)$ from the right at every $t > 0$. Moreover, for every $t > 0$, h satisfies equation (3.4) and the function $t \mapsto \varphi(h(t))$ with φ given by (3.46) is convex, decreasing, Lipschitz continuous on $[\delta,\infty)$ for every $\delta > 0$, and satisfies (3.6).

Further, since every strong solution h of the homogeneous Cauchy problem (3.1) in $X = L^2(\partial\Omega)$ satisfies

$$h(t) \in D(\partial_{L^2}\varphi) \quad \text{with} \quad -\frac{\mathrm{d}h}{\mathrm{d}t}(t) \in \partial_{L^2}\varphi(h(t)) \quad \text{for a.e. } t > 0,$$

it follows from the definition of the subgradient $\partial_{L^2}\varphi$ that every strong solution of the Cauchy problem (3.1) in $L^2(\partial\Omega)$ is also a strong variational solution of the parabolic equation

$$\partial_t h(t) + \Lambda_\sigma h(t) = 0 \qquad \text{on } \partial\Omega \times (0,\infty).$$

Next, let $h_0 \in L^2(\partial\Omega)$. Then, we show that $P(h) \in C((0,\infty); W^{1,p}(\Omega))$. For this, suppose $(t_n)_{n\geq 1}$ is a sequence in $(0,\infty)$ converging to some $t_0 \in (0,\infty)$. Then, $h(t_n)$ converges to $h(t_0)$ in $L^2(\partial\Omega)$. Thus and by the continuity of the function $t \mapsto \varphi(h(t))$, inequality (2.10) implies that the sequence $(P(h(t_n)))_{n\geq 1}$ is bounded in $W^{1,p}(\Omega)$. Since $W^{1,p}(\partial\Omega)$ is reflexive, there is a subsequence $(t_{k_n})_{n\geq 1}$ of $(t_n)_{n\geq 1}$ such that $P(h(t_{k_n}))$ converges to u weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$, $\nabla Ph(t_{k_n})$ converges to ∇u a.e. on Ω and $\boldsymbol{\sigma}(x, \nabla Ph(t_{k_n}))$ converges to $\boldsymbol{\sigma}(x, \nabla u)$ weakly in $L^{p'}(\Omega)^d$ and a.e. on Ω. Moreover, thanks to Lemma 2.2, $u \in W^{1,p}(\Omega)$ is a weak solution of equation (2.12) on Ω. Thus and since the trace operator $\mathcal{T}r : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is compact, $Ph(t_{k_n})_{|\partial\Omega} = h(t_{k_n})$ converges to $u_{|\partial\Omega}$ in $L^p(\partial\Omega)$. On the other hand, $h(t_{k_n})$ converges to $h(t_0)$ in $L^2(\partial\Omega)$. Thus $u_{|\partial\Omega} = h(t_0)$ and since $\boldsymbol{\sigma}(x, \nabla Ph(t_{k_n}))$ converges to $\boldsymbol{\sigma}(x, \nabla u)$ weakly in $L^{p'}(\Omega)^d$, we can conclude that $u = P(h(t_0))$. It remains to show that $\nabla P(h(t_{k_n}))$ converges to $\nabla Ph(t_0)$ strongly in $L^p(\Omega)$. Since $\nabla P h(t_{k_n})$ converges to ∇u a.e. on Ω, it remains to show that $(|\nabla P(h_{k_n})_{n\geq 1}|^p)_{n\geq 1}$ is equi-integrable in $L^1(\Omega)$. To see this, consider χ_{k_n} defined by (2.19) with $h_{k_n} := h(t_{k_n})$ and $h := h(t_0)$. Since $\boldsymbol{\sigma}(x,\nabla u)$ weakly in $L^{p'}(\Omega)^d$ and a.e. on Ω, the non-negative function χ_{k_n} converges to 0 a.e. on Ω. Moreover, since $P(h(t_{k_n}))$ and $P(h(t_0))$ are weak solution of (2.12), by (2.23), and since $h(t_{k_n})$ and $h(t_0)$ are solutions of the parabolic equation in (3.1), it follows that

$$\begin{aligned}\int_\Omega \chi_{k_n}\,\mathrm{d}x &= (\partial_{t+}h(t_0) - \partial_{t+}h(t_{k_n}), h(t_{k_n}) - h(t_0))_{L^2(\partial\Omega)}\\ &\leq \|\partial_{t+}h(t_0) - \partial_{t+}h(t_{k_n})\|_{L^2(\partial\Omega)}\,\|h(t_{k_n}) - h(t_0)\|_{L^2(\partial\Omega)}.\end{aligned}$$

Note that $\partial_{t+}h \in L^\infty([\delta,\infty);L^2(\partial\Omega))$ for every $\delta > 0$. Therefore the convergence of $h(t_{k_n})$ to $h(t_0)$ in $L^2(\partial\Omega)$ yields χ_{k_n} converges to 0 in $L^1(\Omega)$. Now, we employ the same arguments as in the proof of Proposition 2.1 to conclude that $(|\nabla P(h_{k_n})|^p)_{n\geq 1}$ is equi-integrable in $L^1(\Omega)$. Since the convergent sequence $(t_n)_{n\geq 1}$ in $(0,\infty)$ was arbitrary, we have thereby shown that $P(h) \in C((0,\infty);W^{1,p}(\Omega))$ and hence by the continuity of the trace operator $\mathcal{T}r : W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega)$, we obtain that $h \in C((0,\infty);W^{1-1/p,p}(\partial\Omega))$.

Further, to see that for given $h_0 \in L^2(\partial\Omega)$, the corresponding strong solution h of (3.1) satisfies (3.5), one simply multiplies the equation in (3.1) by $\mathbb{1}_{\partial\Omega}$. Since $\nabla P(\mathbb{1}_{\partial\Omega}) = 0$ on Ω, the second integral in (3.4) vanishes. Now, integrating the resulting equation over $(0,t)$ for given $t > 0$, one immediately sees that h satisfies (3.5). Next, to see that (3.5) is satisfied by every mild solution h of (3.1) for general initial value $h_0 \in X$, one uses that the semigroup

In order to see that claim (3) holds, we consider first the case $p = 2$ and suppose that the Carathéodory function $\boldsymbol{\sigma} : \Omega\times\mathbb{R}^d \to \mathbb{R}^d$ is linear in the second component. Then it is well-known (cf. [12]) that $\Lambda_{|L^2}$ is a positive and self-adjoint operator on $L^2(\partial\Omega)$. Moreover, $-\Lambda_{|L^2}$ generates a strongly continuous semigroup $\{e^{-t\Lambda_{|L^2}}\}$ of self-adjoint contractions on $L^2(\partial\Omega)$, which is Markovian and has a holomorphic extension on some sector. Hence, by the classical theory of extrapolation of linear C_0-semigroups (cf. [9, Sect. 7.1 & Sect. 7.2]), the semigroup $\{e^{-t\Lambda_{|L^2}}\}_{t\geq 0}$ extends the positive holomorphic C_0-semigroup $\{e^{-t\Lambda_{|L^q}}\}_{t\geq 0}$ generated by $-\Lambda_{|L^q}$ of contractive linear operators $e^{-t\Lambda_{|L^q}}$ on $L^q(\partial\Omega)$ for $1 < q < \infty$. Thus, for $1 < q < \infty$, the infinitesimal generator $-\Lambda_{|L^q}$ satisfies inequality (3.9) with a constant $C_2 = C_2(q) > 0$ (for example, see [121]). In particular, the corresponding mild solution h of the Cauchy problem (3.1) belongs to $C^1((0,\infty);L^q(\partial\Omega))$ and satisfies, in particular, (3.7).

Now, suppose that $1 < p < \infty$, $p \neq 2$, and the flux function $\boldsymbol{\sigma} : \Omega\times\mathbb{R}^d \to \mathbb{R}^d$ merely satisfies homogeneity condition (2.17). Then, by Proposition 2.1, the Dirichlet-to-Neumann map $\Lambda_\sigma : W^{1-1/p,p}(\partial\Omega) \to W^{-(1-1/p),p'}(\partial\Omega)$ defined in (2.25) (see Definition 2.4 in Section 2.4) is homogeneous of order $\alpha := p-1 > 0$, that is,

$$\Lambda_\sigma(rh) = r^\alpha \Lambda_\sigma h \qquad \text{for every } r \geq 0 \text{ and } h \in W^{1-1/p,p}(\partial\Omega).$$

Owing to Proposition 3.5, also the realization $\Lambda_{|L^2}$ in $L^2(\partial\Omega)$ of the Dirichlet-to-Neumann map Λ_σ is homogeneous of order $p-1$, and by construction of the realization $\Lambda_{|X}$ in X of Λ_σ, also $\Lambda_{|X}$ is homogeneous of order $p-1$.

Further, according to Remark 3.6, the Banach spaces $X = L^1(\partial\Omega)$ and $X = L^\psi(\partial\Omega)$ for any N-function ψ satisfying the Δ_2-condition, also satisfy (3.33). Thus, it follows from Theorem 3.5 that every mild solution h of the the Cauchy problem (3.1) is differentiable with values in X at a.e. $t > 0$. Moreover, h is differentiable with values in X from the right at every $t > 0$ satisfying (3.7)–(3.10).

Further, if X is continuously embedded into $L^2(\partial\Omega)$, then by Theorem 3.7, the strong solution h of Cauchy problem (3.1) in X is also a strong variational solution of (3.1) in $L^2(\partial\Omega)$.

Next, we aim to show that claim 5. holds. To see this, let $h_0 \in L^q(\partial\Omega)$ for $(2 \vee \frac{d-1}{p-1-\varepsilon}) \le q \le \infty$ for some $\varepsilon \in (0,1)$ if $p \le d$ and $2 \le q \le \infty$ if $p > d$. Then $L^q(\partial\Omega)$ is continuously embedded into $L^2(\partial\Omega)$. Thus, the corresponding milds solution h of Cauchy problem (3.1) in $L^q(\partial\Omega)$ is strong and weak and for every $t > 0$, $h(t)$ is a weak solution of the stationary Poisson problem (2.1) (in Chapter 2.1) with right-hand side $f = -\frac{\mathrm{d}h}{\mathrm{d}t_+}(t)$. By estimate (3.9) with respect to the norm of $X = L^q(\partial\Omega)$, one can apply Theorem 2.1. Therefore, there are $\alpha \in (0,1)$ and $c_\alpha > 0$ such that $h(t) \in C^{0,\alpha}(\partial\Omega)$ satisfying

$$\|h(t)\|_{C^{0,\alpha}(\partial\Omega)} \le c_\alpha \left[\left(\frac{\|h_0\|_{L^q(\partial\Omega)}}{t} \right)^{\frac{1}{p-1}} + \|P(h(t))\|_{L^p(\Omega)} \right] + c_\alpha$$

for every $t > 0$. Next, applying (2.10) to the term $\|P(h(t)\|_{L^p(\Omega)}$ in the latter inequality, and by using the coercivity condition (2.2) shows that (3.11) holds for the function Φ defined in (3.12). Recall the function $t \mapsto \varphi(h(t))$ is continuous and decreasing on $(0,\infty)$. Since the semigroup $\{e^{-t\Lambda_{|L^q}}\}_{t\ge 0}$ is L^2-contractive on $L^q(\partial\Omega)$ and since $e^{-t\Lambda_{|L^q}}0 = 0$ for all $t \ge 0$, the function $t \mapsto \|h(t)\|_{L^2(\partial\Omega)}$ is decreasing and continuous on $[0,\infty)$. Therefore, the function $t \mapsto \Phi(e^{-t\Lambda_{|L^q}}h_0)$ is continuous and decreasing on $(0,\infty)$ and so, inequality (3.11) implies that the set $\{h(t)\,|\, t \ge \delta\}$ is bounded in $C^{0,\alpha}(\partial\Omega)$ for every $\delta > 0$, and thanks to the Arzelà-Ascoli theorem, we can conclude that $h \in C((0,\infty) \times \partial\Omega)$.

Finally, we show that for every $h_0 \in X$, the corresponding mild solution h of Cauchy problem (3.1) has the long-time stability as stated in claim (6).

First, we note that by (3.6), functional φ given by (3.46) is a strict Lyapunov function for the dynamical system $\{e^{-t\Lambda_{|L^2}}\}_{t\ge 0}$ on $L^2(\partial\Omega)$; (cf. [83] or [84, Section 2.2]). Moreover, the set $(\partial_{L^2}\varphi)^{-1}(0)$ of weak solutions $h \in W^{1-1/p,p}(\partial\Omega)$ of equation

$$\Lambda_\sigma h = 0 \qquad \text{on } \partial\Omega \tag{3.50}$$

coincides with the set $\mathbb{R}\mathbb{1}_{\partial\Omega} := \{c \cdot \mathbb{1}_{\partial\Omega}\,|\, c \in \mathbb{R}\}$ of constant functions on $\partial\Omega$. To see this, note that $P(c \cdot \mathbb{1}_{\partial\Omega}) \equiv c$ on $\overline{\Omega}$ for every $c \in \mathbb{R}$. Hence, the set $\mathbb{R}\mathbb{1}_{\partial\Omega}$ is contained in the set $(\partial_{L^2}\varphi)^{-1}(0)$. On the other hand, if $h \in W^{1-1/p,p}(\partial\Omega)$ is a weak solution of (3.50), then by (2.23), one finds that

$$\int_{\Omega} \boldsymbol{\sigma}(x, \nabla P(h)) \nabla P(\xi)\, \mathrm{d}x = 0$$

for every $\xi \in W^{1-1/p,p}(\partial\Omega)$. Taking $\xi = h$ in this equation and subsequently applying coercivity condition (2.2) yields that $\nabla P(h) = 0$ a.e. on Ω. Since Ω is connected, the function $P(h) \equiv c$ on $\overline{\Omega}$ for some $c \in \mathbb{R}$, according to [156, Corollary 2.1.9]).

Now, for given $h_0 \in L^2(\partial\Omega)$, let $\omega(h_0)$ be the ω-limit set equipped with the norm topology of $L^2(\partial\Omega)$of the corresponding mild solution h of Cauchy problem (3.1) with initial value h_0; that is, for $X = L^2(\partial\Omega)$, one has that

$$\omega(h_0) := \Big\{ \xi \in X \,\Big|\, \exists t_n \uparrow 0 \text{ s.t. } X - \lim_{n\to\infty} h(t_n) = \xi \Big\}. \tag{3.51}$$

Then, by LaSalle's invariant principle (see, for example [83]), one has that $\omega(h_0) \subseteq \mathbb{R}\mathbb{1}_{\partial\Omega}$. But since for every $c \in \omega(h_0)$, the conservation of mass equality (3.5) implies that $c = \overline{h_0} = ⨍_{\partial\Omega} h_0 \,\mathrm{d}\mathcal{H}^{d-1}$, we have thereby shown that $\omega(h_0) = \{\overline{h_0}\}$, implying that the limit (3.13) holds in $X = L^2(\partial\Omega)$.

Next, let X be either $L^1(\partial\Omega)$ or $L^{\psi}(\partial\Omega)$ for any N-function saitsfying the Δ_2-condition and such that $L^2(\partial\Omega) \hookrightarrow L^{\psi}(\partial\Omega)$ by a continuous embedding. Since for two the semigroup $\{e^{-t\Lambda_{|X}}\}_{t\geq 0}$ and $\{e^{-t\Lambda_{|L^2}}\}_{t\geq 0}$, one has that

$$h(t) = e^{-t\Lambda_{|X}} h_0 = e^{-t\Lambda_{|L^2}} h_0 \tag{3.52}$$

for every $h_0 \in L^2$, it follows from the continuous embedding $L^2(\partial\Omega) \hookrightarrow X$ that the mild solution h of Cauchy problem (3.1) satisfies

$$\|h(t) - \overline{h}\|_X \leq C\, \|h(t) - \overline{h_0}\|_{L^2(\partial\Omega)}$$

for every $t \geq 0$ and every initial data $h_0 \in L^2(\partial\Omega)$. Thus, and since the projection $h_0 \mapsto \overline{h_0}\,\mathbb{1}_{\partial\Omega}$ is contractive on X, and since $L^2(\partial\Omega)$ is dense in X, it follows that (3.13) holds in X.

Next, let X be such that $X \hookrightarrow L^2(\partial\Omega)$ by a continuous embedding. Then, for every $h_0 \in X$, we equip the ω-limit set $\omega(h_0)$ defined by (3.51) with the strong topology of X. By the continuous injection of $X \hookrightarrow L^2(\partial\Omega)$, one has that (3.52) holds for all $h_0 \in X$. Thus, and since the limit (3.13) holds in $X = L^2(\partial\Omega)$, it necessary follows that $\omega(h_0) = \{\overline{h_0}\}$ also with respect to the norm-topology of X. This proves that (3.13) holds in X provided $X \hookrightarrow L^2(\partial\Omega)$ by a continuous embedding.

Now, let $h_0 \in L^q(\partial\Omega)$ with $(2 \vee \frac{d-1}{p-1-\varepsilon}) \leq q \leq \infty$ for some $\varepsilon \in (0,1)$ if $p \leq d$ or $2 \leq q \leq \infty$ if $p > d$. Since for Φ defined in (3.12), the function $t \mapsto \Phi(e^{-t\Lambda_{L^q}} h_0)$ is continuous and decreasing along $(0,\infty)$, inequality (3.11) together with Arzelà-Ascoli's theorem imply that the set $\{h(t) \,|\, t \geq 1\}$ is relatively compact in $C(\partial\Omega)$. Thus, limit (3.13) holds in $C(\partial\Omega)$ for such initial values $h_0 \in L^q(\partial\Omega)$, and this completes the proof of claim (6). □

Chapter 4
L^q-L^∞ regularization and decay estimates

Abstract In this chapter, we show that the semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ generated by the negative Dirichlet-to-Neumann map $-\Lambda$ satisfies gobal L^q-L^∞ regularization and decay estimates for $1 \leq q < \infty$. Here the Dirichlet-to-Neumann map Λ is associated with a Leray-Lions operator, which includes the weighted p-Laplace operator as a special case. Thus, the main result of this chapter includes the statement of Theorem 1.6 as a special case. The content of this chapter covers parts of the monograph [56].

4.1 Main results

In this chapter, we establish global L^q-L^∞ regularization and decay estimates for $1 \leq q < \infty$ satisfied by the mild solutions h of the homogeneous Cauchy problem (in $X = L^q(\partial\Omega)$)

$$\begin{cases} \frac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda h(t) \ni 0 & \text{for } t \in (0,T), \\ h(0) = h_0, \end{cases} \tag{4.1}$$

for given inital value $h_0 \in L^q(\partial\Omega)$. As in the the last two preceding chapters, we assume throughout this chapter that Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 2$ with a Lipschitz-continuous boundary $\partial\Omega$. Further, for $1 < p < \infty$, suppose that the flux function $\boldsymbol{\sigma}$ is a Carathéodory function satisfying (2.2), (2.3), (2.4), (2.5) from Chapter 2.1.

In this chapter, we follow the convention introduced in Notation 1.5 (see Chapter 1.9) that Λ refers to the realization in the Banach space X of the Dirichlet-to-Neumann map associated with the Leray-Lions operator $A = \mathrm{div}(\boldsymbol{\sigma}(x,\nabla\cdot))$. Here, we restrict ourself to Banach spaces $X = L^q(\partial\Omega)$ for $1 \leq q < \infty$. Furthermore, $\{e^{-t\Lambda}\}_{t\geq 0}$ is the *semigroup generated by the negative Dirichlet-to-Neumann map* $-\Lambda$ (on X) and for a given $h \in L^q(\partial\Omega)$, we denote by $\overline{h}$ the mean-value $\fint_{\partial\Omega} h\,\mathrm{d}\mathcal{H}^{d-1}$ of h over $\partial\Omega$.

The following theorem is the main result of this chapter, and contains the statement of Theorem 1.6 as a special case. It provides the complete description of the L^q-L^∞-regularization effect of the semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ (on $L^q(\partial\Sigma)$).

Theorem 4.1 *Let $\sigma : \overline{\Omega}\times\mathbb{R}^d \to \mathbb{R}^d$ and $\mathcal{A} : \Omega\times\mathbb{R}^d \to \mathbb{R}$ be two Carathéodory functions satisfying (2.2)-(2.5). Then the semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ generated by the negative Dirichlet-to-Neumann map $-\Lambda$ (on $L^q(\partial\Sigma)$) satisfies the following global L^q-L^∞-regularization estimates for $1 \leq q < \infty$.*

1. *For $1 < p < d$, choose $q_0 \geq p$ minimal such that*

$$((d-1)/(d-p)-1)q_0 + p - 2 > 0. \tag{4.2}$$

Then for every $1 \leq q \leq (d-1)q_0/(d-p)$ satisfying $q > (2-p)(d-1)/(p-1)$, one has that

$$\|e^{t\Lambda}h_0 - \overline{h_0}\|_\infty \lesssim t^{-\alpha_q}\,\|h_0 - \overline{h_0}\|_q^{\gamma_q} \tag{4.3}$$

for every $t > 0$ and $h_0 \in L^q(\partial\Omega)$, where the exponents α_q and γ_q are given by

$$\begin{aligned} \alpha_q &= \frac{\alpha^*}{1-\gamma^*\left(1-\frac{q(d-p)}{(d-1)q_0}\right)}, \quad \gamma_q = \frac{\gamma^*\,\frac{q(d-p)}{(d-1)q_0}}{1-\gamma^*\left(1-\frac{q(d-p)}{(d-1)q_0}\right)},\\ \alpha^* &= \frac{d-p}{(p-1)q_0+(d-p)(p-2)}, \quad \gamma^* = \frac{(p-1)q_0}{(p-1)q_0+(d-p)(p-2)}. \end{aligned} \tag{4.4}$$

2. *For $\frac{2d}{d+1} < p < d$, condition (4.2) holds with $q_0 = p$. Thus, for every $1 \leq q \leq \frac{(d-1)p}{d-p}$ satisfying $q > \frac{(2-p)(d-1)}{p-1}$, the semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ satisfies the global L^q-L^∞-regularizing estimate (4.3) with exponents α_q and γ_q given by (4.4) with $q_0 = p$.*
3. *If $\frac{2d-1}{d} < p < d$, then for every $1 \leq q \leq \frac{(d-1)p}{d-p}$, the semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ satisfies the global L^q-L^∞-regularizing estimate (4.3) with exponents α_q and γ_q given by (4.4) with $q_0 = p$.*
4. *For $p = d \geq 2$, let $\theta \in (1-\frac{1}{p}, 1)$. Then, for every $1 \leq q \leq \frac{1}{1-\theta}$, the semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ satisfies the global L^q-L^∞-regularizing estimate (4.3) with exponents*

$$\begin{aligned} \alpha_q &= \frac{\alpha^*_\theta}{1-\gamma^*_\theta(1-q(1-\theta))}, \quad \gamma_q = \frac{\gamma^*_\theta q(1-\theta)}{1-\gamma^*_\theta(1-q(1-\theta))},\\ \alpha^*_\theta &= \frac{1}{\frac{1}{1-\theta}-2}, \quad \gamma^*_\theta = \frac{\frac{1}{1-\theta}-p}{\frac{1}{1-\theta}-2}. \end{aligned} \tag{4.5}$$

5. *Let $d < p < \infty$. Then, for every $1 \leq q \leq 2$, the semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ satisfies the L^q-L^∞-regularizing estimate (4.3) with exponents*

$$\alpha_q = \frac{1}{p-2+q}, \quad \gamma_q = \frac{q}{p-2+q}. \tag{4.6}$$

In order to prove this result, we need to introduce a few more auxilary tools, which we provide in the next four sections. In Section 4.7 we outline the proof of Theorem 4.1.

4.2 Preliminaries

In the next five section, we employ the same abstract notation as introduced in the preliminary Section 3.2 (of Chapter 3). Thus, for $1 \le q \le \infty$, we denote by L^q the standard Lebesgue space $L^q(\Sigma,\mu)$ of a σ-finite measure space $(\Sigma,\mathcal{B},\mu)$, and write $\langle\cdot,\cdot\rangle$ for the duality pairing on $(L^{q'},L^q)$. Further, A refers to an operator on L^q with domain $D(A) := \{u\,|\,Au \neq \emptyset\}$ and range $\mathrm{Rg}(A) := \bigcup_{u\in D(A)} Au$, and u_q denotes $|u|^{q-2}u$ for every $u \in L^q$.

Throughout this chapter, we employ for $1 \le q < \infty$, the pairing $[\cdot,\cdot]_q : L^q \times L^q \to \mathbb{R}$ defined by

$$[u,v]_q := \lim_{\lambda\to 0+} \frac{1}{q}\,\frac{\|u+\lambda v\|_q^q - \|u\|_q^q}{\lambda}$$

for every u, $v \in L^q$. We call the pairing $[\cdot,\cdot]_q$ the *q-brackets* in analogy to the so-called *produit* $<\cdot,\cdot>_s$(cf. [26], or also called *brackets* $[\cdot,\cdot]$, cf. [30, 23]).

For $1 \le q \le \infty$ and every $u, v \in L^q$, the number $[u,v]_q$ is the right-hand directional derivative of the function $u \mapsto \frac{1}{q}\|u\|_q^q$. Since the function $\lambda \mapsto \frac{1}{q}\|u+\lambda v\|_q^q$ is convex on $\mathbb{R}$, we can define $[\cdot,\cdot]_q$, equivalently, by

$$[u,v]_q = \inf_{\lambda>0} \frac{\frac{1}{q}\|u+\lambda v\|_q^q - \frac{1}{q}\|u\|_q^q}{\lambda} \tag{4.7}$$

for every u, $v \in L^q$. The q-bracket $[\cdot,\cdot]_q : L^q \times L^q \to \mathbb{R}$ is upper semicontinuous (respectively, continuous if $1 < q < \infty$) and

$$[u,v]_q = \langle u_q, v\rangle \qquad \text{for every } u,\ v \in L^q \text{ if } 1 < q < \infty, \tag{4.8}$$

while for $q = 1$, $[\cdot,\cdot]_1$ reduces to the classical *brackets* $[\cdot,\cdot]$ on L^1 given by

$$[u,v]_1 = \int_{\{u\neq 0\}} \mathrm{sign}_0(u)\, v\,\mathrm{d}\mu + \int_{\{u=0\}} |v|\,\mathrm{d}\mu \tag{4.9}$$

for every u, $v \in L^1$, where the *restricted signum* sign_0 is defined by

$$\mathrm{sign}_0(s) = \begin{cases} 1 & \text{if } s > 0,\\ 0 & \text{if } s = 0,\\ -1 & \text{if } s < 0 \end{cases}$$

for every $s \in \mathbb{R}$ (cf. [30, Section 2.2 & Example (2.8)] or [23, pp 102]).

Recall that by the characterizations (3.15) from Chapter 3.2.2.1, an operator A is accretive on L^q if and only if for every (u,v), $(\hat{u},\hat{v}) \in A$, there exists $\psi \in J(u-\hat{u})$ such taht $\langle \psi, v-\hat{v}\rangle \geq 0$. Now, if $q=1$, then the duality mapping J is given by (3.16) in Chapter 3.2.2.1, and if $1<q<\infty$, then J is given by (3.17). Therefore, and by (4.8) and (4.9), an operator A is accretive on L^q if and only if

$$[u-\hat{u}, v-\hat{v}]_q \geq 0 \qquad \text{for every } (u,v),\ (\hat{u},\hat{v}) \in A$$

(see [30, Corollary 2.14] or, alternatively, by [23, p 103 formula (3.15)]).

Further, the following two properties of the q-brackets will be helpful later:

$$[u,v]_q \leq \frac{1}{q}\|u+v\|_q^q - \frac{1}{q}\|u\|_q^q \qquad \text{for every } u,\ v \in L^q, \tag{4.10}$$

and

$$[u, \alpha\, v + \omega u]_q = \alpha\,[u,v]_q + \omega\,\|u\|_q^q \tag{4.11}$$

for every u, $v \in L^q$, ω, $\alpha \in \mathbb{R}$. Here, note that inequality (4.10) is an immediate consequence of (4.7). Property (4.11) is shown for $q=1$ in [30, Proposition (2.5)] (or, alternatively, [23, Proposition 3.7]) and if $1<q<\infty$ then (4.11) can be easily deduced from (4.8).

4.3 Sobolev implies L^q-L^r regularization estimates

Global L^q-L^r regularization estimates satisfied by the mild solution u of a homogeneous Cauchy problem

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t}(t) + A(u(t)) \ni 0 & \text{for } t \in (0,T),\\ u(0) = u_0 \end{cases} \tag{4.12}$$

are derived from the following stronger accretivity property satsified by the operator A.

Definition 4.1 Let $1 \leq q < \infty$ and $1 \leq r \leq \infty$. Then we say that an operator A on L^q satisfies an L^q-L^r *Sobolev inequality with differences* and with parameter$\sigma > 0$ if there is a constant $C > 0$ such that

$$\|u-\hat{u}\|_r^{\sigma} \leq C\,[u-\hat{u}, v-\hat{v}]_q \tag{4.13}$$

for every (u,v), $(\hat{u},\hat{v}) \in A$.

An operator A on L^q is said to satisfy an L^q-L^r *Sobolev inequality at* $(u_0,0) \in A$ with parameter $\sigma > 0$, if there is a constant $C > 0$ such that

$$\|u-u_0\|_r^{\sigma} \leq C\,[u-u_0, v]_q \tag{4.14}$$

for every $(u,v) \in A$.

Finally, if the underying measure space (Σ,μ) has a finite measure μ, then an operator A on L^q is said to satisfy an L^q-L^r *Sobolev inequality with mean values* with parameter $\sigma > 0$, if there is a constant $C > 0$ such that

$$\|u-\overline{u}\|_r^\sigma \le C\,[u-\overline{u},v]_q \tag{4.15}$$

for every $(u,v) \in A$, where $\overline{u} := \int_\Sigma u\,\mathrm{d}\mu/\mu(\Sigma)$ denotes the mean-value of u.

Remark 4.1 It is worth mentioning that every operator A on L^q satisfying an L^q-L^r Sobolev inequality (4.13) for $1 \le q < \infty$ and $1 \le r \le \infty$ with parameter $\sigma > 0$ is necessarily accretive in L^q.

Next, we intend to outline the global L^q-L^r regularization estimate which every mild solutions $u(t) = e^{-tA}u_0$ of the homogeneous Cauchy problem (4.1) satisfies with initial condition $u_0 \in L^q$ provided the operator A satisfies an L^q-L^r Sobolev inequality. We express this L^q-L^r regularization effect in terms of the semigroup $\{e^{-tA}\}_{t\ge 0}$ generated by $-A$. The following theorem is our first main result of this section.

Theorem 4.2 *Let $1 \le q < \infty$, $1 \le r \le \infty$, and let A be an m-accretive operator on L^q. Suppose, A satisfies the L^q-L^r Sobolev inequality (4.13) with parameter $\sigma > 0$ and the semigroup $\{e^{-tA}\}_{t\ge 0} \sim -A$ on $\overline{D(A)}^{L^q}$ is contractive in L^r. Then $\{e^{-tA}\}_{t\ge 0}$ satisfies the following L^q-L^r regularity estimate*

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_r \le \left(\frac{C}{q}\right)^{1/\sigma} t^{-\alpha}\,\|u-\hat{u}\|_q^\gamma \tag{4.16}$$

for every $t > 0$ and u, $\hat{u} \in \overline{D(A)}^{L^q}$ with exponents $\alpha = \frac{1}{\sigma}$ and $\gamma = \frac{q}{\sigma}$.

In application, a common situation is that the operator A is m-completely accretive on L^2 (see Definition 3.16) and satisfies an L^2-L^r Sobolev inequality. Therefore, the following special case of Theorem 4.2 is quite useful to mention.

Corollary 4.1 *Let $1 \le r \le \infty$ and for $\omega \ge 0$, let A be an m-completely accretive operator on L^2. If for $\sigma > 0$, A satisfies the L^2-L^r- Sobolev inequality*

$$\|u-\hat{u}\|_r^\sigma \le C\,[u-\hat{u},v-\hat{v}]_2 \tag{4.17}$$

for every (u,v), $(\hat{u},\hat{v}) \in A$, then the semigroup $\{e^{-tA}\}_{t\ge 0} \sim -A$ on $\overline{D(A)}^{L^2}$ satisfies

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_r \le \left(\frac{C}{2}\right)^{1/\sigma} t^{-\alpha}\,\|u-\hat{u}\|_2^\gamma$$

for every $t > 0$ and u, $\hat{u} \in \overline{D(A)}^{L^2}$ with $\alpha = \frac{1}{\sigma}$ and $\gamma = \frac{2}{\sigma}$.

Now, we turn to the proof of Theorem 4.2. For this, we first consider the case $q > 1$. According to (4.8), one has that in the L^q-L^r Sobolev inequality (4.13), the q-brackets

$$[u-\hat{u}, v-\hat{v}]_q = \langle (u-\hat{u})_q, v-\hat{v}\rangle$$

for every $u, \hat{u}, v, \hat{v} \in L^q$. Moreover, since L^q and its dual space are uniformly convex Banach spaces, one has that for every initial value $u_0 \in D(A)$, the corresponding mild solution $u(t) = e^{-tA}u_0$ of the homogeneous Cauchy problem (4.1) is differentiable with values in L^q at a.e. $t > 0$, and satisfies

$$\tfrac{\mathrm{d}}{\mathrm{d}t}_{+} u(t) = -A^0 u(t) \qquad \text{for every } t \geq 0 \tag{4.18}$$

(cf (3.29) in Chapter 3.2.2.1). Using this leads to the following short proof of Theorem 4.2 (cf [55]).

Proof (of Theorem 4.2, first proof for q > 1) First, let $u, \hat{u} \in D(A)$. By hypothesis,

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_{\tilde{q}} \leq \|e^{-sA}u - e^{-sA}\hat{u}\|_{\tilde{q}} \tag{4.19}$$

for every $t \geq s > 0$ and for every $\tilde{q} \in \{q, r\}$. Combining (4.19) with (4.18) and the L^q-L^r Sobolev inequality (4.13), one finds that

$$\begin{aligned}
\|u-\hat{u}\|_q^q &\geq \|u-\hat{u}\|_q^q - \|e^{-tA}u - e^{-tA}\hat{u}\|_q^q \\
&= -\int_0^t \tfrac{\mathrm{d}}{\mathrm{d}s}\|e^{-sA}u - e^{-sA}\hat{u}\|_q^q \,\mathrm{d}s \\
&= q\int_0^t \langle (e^{-sA}u - e^{-sA}\hat{u})_q, A^\circ e^{-sA}u - A^\circ e^{-sA}\hat{u}\rangle \,\mathrm{d}s \\
&\geq q\int_0^t \langle (e^{-sA}u - e^{-sA}\hat{u})_q, A^\circ e^{-sA}u - A^\circ e^{-sA}\hat{u}\rangle \,\mathrm{d}s \\
&\geq \frac{q}{C}\int_0^t \|e^{-sA}u - e^{-sA}\hat{u}\|_r^\sigma \,\mathrm{d}s \\
&\geq \frac{q}{C}\, t\, \|e^{-tA}u - e^{-tA}\hat{u}\|_r^\sigma
\end{aligned}$$

for every $u, \hat{u} \in D(A)$. Therefore, the semigroup $\{e^{-tA}\}_{t\geq 0}$ satisfies the global L^q-L^r regularization inequality (4.16) for initial values of the domain $D(A)$.

Now, let $u, \hat{u} \in \overline{D(A)}^{L^q}$. Then, there are sequences $(u_n)_{n\geq 1}$ and $(\hat{u}_n)_{n\geq 1}$ in $D(A)$ such that u_n converges to u and $\hat{u}_n$ converges to $\hat{u}$ in L^q. Since by assumption, the semigroup $\{e^{-tA}\}_{t\geq 0}$ is contractive in L^q, we have

$$\lim_{n\to\infty} (e^{-tA}u_n - e^{-tA}\hat{u}_n) = e^{-tA}u - e^{-tA}\hat{u} \qquad \text{in } L^q.$$

Moreover, by the first step of this proof, the L^q-L^r regularization inequality (4.16) implies that

$$\|e^{-tA}u_n - e^{-tA}\hat{u}_n\|_r \leq \left(\tfrac{C}{q}\right)^{1/\sigma} t^{-\alpha}\, \|u_n - \hat{u}_n\|_q^{\frac{q}{\sigma}}$$

for every $n \geq 1$. Since the L^r-norm is lower semicontinuous on L^q, sending $n \to \infty$ in the previous inequality yields $e^{-tA}u - e^{-tA}\hat{u} \in L^r$ and

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_r \le \left(\frac{C}{q}\right)^{1/\sigma} t^{-\alpha}\, \|u-\hat{u}\|_q^{\frac{q}{\sigma}}.$$

Therefore inequality (4.16) holds for every $u, \hat{u} \in \overline{D(A)}^{L^q}$, completing the proof of Theorem 4.2 for $q > 1$. □

Our second proof of Theorem 4.2 is rather technical and uses the definition of mild solutions (cf. [151]).

Proof (of Theorem 4.2, second proof) Let $u, \hat{u} \in \overline{D(A)}^{L^q}$. For given $t > 0$ and $N \ge 1$, set $t_n = n\frac{t}{N}$ for every $n = 0, \dots, N$. Since by hypothesis, A satisfies the range condition (3.14) for $X = L^q$ (see Chapter 3.2.2.1), there are $u_1, \hat{u}_1 \in D(A)$ solving

$$u_1 + \tfrac{t}{N} A u_1 \ni u \quad \text{ and } \quad \hat{u}_1 + \tfrac{t}{N} A\hat{u}_1 \ni \hat{u}.$$

Now, iteratively, for every $n = 1, \dots, N$, there are solutions u_n and $\hat{u}_n \in D(A)$ of

$$u_n + \tfrac{t}{N} A u_n \ni u_{n-1} \quad \text{ and } \quad \hat{u}_n + \tfrac{t}{N} A\hat{u}_n \ni \hat{u}_{n-1}, \tag{4.20}$$

respectively. Further, for $v_n = (u_{n-1} - u_n)\frac{N}{t}$ and $\hat{v}_n = (\hat{u}_{n-1} - \hat{u}_n)\frac{N}{t}$, both inclusions in (4.20) can be rewritten as $v_n \in Au_n$ and $\hat{v}_n \in A\hat{u}_n$, or as $J_{t/N}u_{n-1} = u_n$ and $J_{t/N}\hat{u}_{n-1} = \hat{u}_n$ for every $n = 1, \dots, N$, where $u_0 = u$ and $\hat{u}_0 = \hat{u}$. Hence by the L^q-L^r Sobolev inequality (4.13), and by (4.11) and (4.10), one sees that

$$\begin{aligned}
&\|u_n - \hat{u}_n\|_r^{\sigma}\\
&\le C\,[u_n - \hat{u}_n, v_n - \hat{v}_n]_q\\
&= C\,\tfrac{N}{t}\,[u_n - \hat{u}_n, (u_{n-1} - u_n) - (\hat{u}_{n-1} - \hat{u}_n)]_q\\
&= C\,\tfrac{N}{t}\,[u_n - \hat{u}_n, (u_{n-1} - \hat{u}_{n-1}) - (u_n - \hat{u}_n)]_q\\
&\le C\,\tfrac{N}{t}\left(\tfrac{1}{q}\|u_{n-1} - \hat{u}_{n-1}\|_q^q - \tfrac{1}{q}\|u_n - \hat{u}_n\|_q^q\right)
\end{aligned} \tag{4.21}$$

for every $n = 1, \dots, N$. By assumption, the resolvent $J_{t/N}$ of A is contractive on L^q. Thus, one has that

$$\begin{aligned}
\|u_n - \hat{u}_n\|_q &= \|J_{t/N}u_{n-1} - J_{t/N}\hat{u}_{n-1}\|_q\\
&\le \|u_{n-1} - \hat{u}_{n-1}\|_q\\
&\ \ \vdots\\
&\le \|u - \hat{u}\|_q
\end{aligned} \tag{4.22}$$

By using (4.22), in order to estimate the term $\|u_n - \hat{u}_n\|_q^{\varrho}$ in (4.21) and subsequently, multiplying the resulting inequality by $\frac{t}{N}$ yields that

$$\tfrac{t}{N}\|u_n - \hat{u}_n\|_r^{\sigma} \le C\left(\tfrac{1}{q}\|u_{n-1} - \hat{u}_{n-1}\|_q^q - \tfrac{1}{q}\|u_n - \hat{u}_n\|_q^q\right)$$

Rearranging the last inequality gives

$$\tfrac{1}{q}\|u_n - \hat{u}_n\|_q^q \le \tfrac{1}{q}\|u_{n-1} - \hat{u}_{n-1}\|_q^q + b_n$$

for every $n = 1, \dots, N$, where we set

$$b_n := -\tfrac{t}{N}\, \|u_n - \hat{u}_n\|_r^\sigma\, C^{-1} \tag{4.23}$$

It is easy to see that

$$\begin{cases} \text{for finite sequences } \{\lambda_n\}_{n=1}^N \subseteq [0,\infty) \text{ and } \{a_n\}_{n=1}^N,\, \{b_n\}_{n=1}^N \subseteq \mathbb{R} \text{ satisfying} \\ a_n \le \lambda_n a_{n-1} + b_n \text{ for all } n = 1, \dots, N, \text{ one has that} \\ \qquad a_N \le a_0 \left(\prod_{n=1}^N \lambda_n\right) + \sum_{n=1}^N b_n \left(\prod_{k=n+1}^N \lambda_n\right) \end{cases} \tag{4.24}$$

(cf. [30, Exercise E3.8]). Applying this to $\lambda_n = 1$, $a_n = \frac{1}{q}\|u_n - \hat{u}_n\|_q^q$ and b_n given by (4.23), then one obtains

$$\tfrac{1}{q}\|u_n - \hat{u}_n\|_q^q \le \tfrac{1}{q}\|u_0 - \hat{u}_0\|_q^q - \tfrac{t}{N}\sum_{n=1}^N \|u_n - \hat{u}_n\|_r^\sigma\, C^{-1},$$

or, equivalently

$$\tfrac{1}{q}\|u_n - \hat{u}_n\|_q^q + C^{-1}\sum_{n=1}^N \tfrac{t}{N}\|u_n - \hat{u}_n\|_r^\sigma \le \tfrac{1}{q}\|u - \hat{u}\|_q^q$$

Now, let

$$U_N(s) := u \cdot \mathbb{1}_{\{t_0=0\}}(s) + \sum_{n=1}^N u_n\, \mathbb{1}_{(t_{n-1},t_n]}(s) \tag{4.25}$$

and

$$\hat{U}_N(s) := \hat{u} \cdot \mathbb{1}_{\{t_0=0\}}(s) + \sum_{n=1}^N \hat{u}_n\, \mathbb{1}_{(t_{n-1},t_n]}(s)$$

for every $s \in [0,t]$. Then the latter inequality can be rewritten as

$$\tfrac{1}{q}\|U_N(t) - \hat{U}_N(t)\|_q^q + C^{-1}\int_0^t \|U_N(s) - \hat{U}_N(s)\|_r^\sigma\, ds \le \tfrac{1}{q}\|u - \hat{u}\|_q^q \tag{4.26}$$

By the Crandall-Liggett theorem, one has that $U_N \to e^{-tA}u$ in $C([0,T];L^q)$ and $\hat{U}_N \to e^{-tA}\hat{u}$ in $C([0,T];L^q)$. Thus, sending $N \to \infty$ in (4.26) and by using the lower semicontinuity of the L^r-norm on L^q gives that

$$\tfrac{1}{q}\|e^{-tA}u - e^{-tA}\hat{u}\|_q^q + C^{-1}\int_0^t \|e^{-sA}u - e^{-sA}\hat{u}\|_r^\sigma\, ds \le \tfrac{1}{q}\|u - \hat{u}\|_q^q.$$

Since by assumption, the semigroup $\{e^{-tA}\}_{t\ge 0}$ is contractive in L^r, the last inequality implies that the global L^q-L^r regularization estimate (4.16) holds. This proves the statement of Theorem 4.2. □

In the situation that an m-accretive operators A on L^q satisfies the L^q-L^r Sobolev inequality (4.14) at some $(u_0, 0) \in A$, we can state the following result.

Theorem 4.3 *For $1 \le q < \infty$, $1 \le r \le \infty$, let A be an m-accretive operator on L^q. Suppose, A satisfies the L^q-L^r Sobolev inequality (4.14) at some $(u_0, 0) \in A$, and the semigroup $\{e^{-tA}\}_{t\ge 0} \sim -A$ on $\overline{D(A)}^{L^q}$ is contractive on L^r. Then one has that*

$$\|e^{-tA}u - u_0\|_r \le \left(\frac{C}{q}\right)^{1/\sigma} t^{-\alpha}\,\|u - u_0\|_q^{\gamma}$$

for every $t > 0$, $u \in \overline{D(A)}^{L^q}$ with exponents $\alpha = \frac{1}{\sigma}$ and $\gamma = \frac{q}{\sigma}$.

We omit the proof of Theorem 4.3 since it proceeds along the lines of the second proof of Theorem 4.2 for fixed $\hat{u} = u_0$.

Now, we come to the last theorem of this section, which treats the case of semi-groups $\{e^{-tA}\}_{t\ge 0}$ satisfying

$$e^{-tAu}(c\mathbb{1}_\Sigma) = c\mathbb{1}_\Sigma \qquad \text{for every } t \ge 0 \tag{4.27}$$

and every $c \in \mathbb{R}$.

Theorem 4.4 *Let (Σ, μ) a finite measure space, and for $1 \le q < \infty$, $1 \le r \le \infty$, let A be an m-accretive operator on L^q satisfying $(c\mathbb{1}_\Sigma, 0) \in A$ for every $c \in \mathbb{R}$. Suppose, A satisfies the L^q-L^r Sobolev inequality (4.15) with mean values, and the semigroup $\{e^{-tA}\}_{t\ge 0} \sim -A$ on $\overline{D(A)}^{L^q}$ is contractive on L^r. Then, one has that*

$$\|e^{-tA}u - \overline{u}\|_r \le \left(\frac{C}{q}\right)^{1/\sigma} t^{-\alpha}\,\|u - \overline{u}\|_q^{\gamma} \tag{4.28}$$

for every $t > 0$, $u \in \overline{D(A)}^{L^q}$ with exponents $\alpha = \frac{1}{\sigma}$ and $\gamma = \frac{q}{\sigma}$.

Proof (Theorem 4.4) Due to the hypothesis that $(c\mathbb{1}_\Sigma, 0) \in A$ for every $c \in \mathbb{R}$, one has that the semigroup satisfies (4.27) for every $c \in \mathbb{R}$. Thus, if for given $u \in D(A)$, we choose $\hat{u}$ to be the mean-value $\overline{u}$ of u. Then, (4.19) yields that

$$\|e^{-tA}u - \overline{u}\|_{\tilde{q}} \le \|e^{-sA}u - \overline{u}\|_{\tilde{q}}$$

and $A^\circ e^{-tA}\overline{u} = 0$. Thus, the *first proof* of Theorem 4.2 (given on page 94) is also valid for Theorem 4.4 in the case $q > 1$. Next, if one chooses $\hat{u}_n = \overline{u}$ in (4.20), then one sees that also the *second proof* of Theorem 4.2 (given on page 95) remains true to prove Theorem 4.4. □

Analogously as above, the important case $q = 2$ and A being m-completely accretive operator on L^2 follows immediately from Theorem 4.3.

Corollary 4.2 *Let (Σ,μ) a finite measure space, and for $1 \le r \le \infty$, let A be an m-completely accretive operator on L^2 satisfying $(c\mathbb{1}_\Sigma, 0) \in A$ for every $c \in \mathbb{R}$. If for $\sigma > 0$, A satisfies the L^2-L^r Sobolev inequality with mean values*

$$\|u-\overline{u}\|_r^\sigma \le C\,[u-\overline{u},v]_2 \tag{4.29}$$

for every $(u,v) \in A$, then the semigroup $\{e^{-tA}\}_{t\ge 0} \sim -A$ on $\overline{D(A)}^{L^2}$ satisfies the L^2-L^r regularization estimate

$$\|e^{-tA}u-\overline{u}\|_r \le \left(\frac{C}{2}\right)^{1/\sigma}\; t^{-\alpha}\;\|u-\overline{u}\|_2^\gamma \tag{4.30}$$

for every $t > 0$ and $u \in \overline{D(A)}^{L^2}$ with exponents $\alpha = \frac{1}{\sigma}$ and $\gamma = \frac{2}{\sigma}$.

4.4 Extrapolation towards L^1

This section is dedicated to providing a nonlinear version of [54, Lemme 1] (see also [55, Section I]). The first extrapolation result of this subsection is adapted to semigroups generated by completely accretive operators (see Section 3.2.2.2) satisfying the L^q-L^r-regularizing effect (4.31) (with differences) for $1 < q < r \le \infty$.

Theorem 4.5 *For $1 \le s < q < r \le \infty$, let $\{e^{-tA}\}_{t\ge 0}$ be an C_0-semigroup of L^s-contractions e^{-tA} acting on some subset D of L^q. Suppose there exist $\alpha > 0$, $\gamma > 0$ and $C > 0$ such that $\{e^{-tA}\}_{t\ge 0}$ satisfies the following L^q-L^r regularity estimate*

$$\|e^{-tA}u-e^{-tA}\hat{u}\|_r \le C\,t^{-\alpha}\,\|u-\hat{u}\|_q^\gamma \tag{4.31}$$

for every $t > 0$ and u, $\hat{u} \in D$. For $\theta_s = \frac{(r-q)s}{q(r-s)} > 0$ if $r < \infty$ and $\theta_s = \frac{s}{q}$ if $r = \infty$, assume that

$$\gamma(1-\theta_s) < 1. \tag{4.32}$$

Then one has that

$$\|e^{-tA}u-e^{-tA}\hat{u}\|_r \le \left(C\,2^{\frac{\alpha}{1-\gamma(1-\theta_s)}}\right)^{\frac{1}{1-\gamma(1-\theta_s)}}\; t^{-\alpha_s}\;\|u-\hat{u}\|_s^{\gamma_s} \tag{4.33}$$

for every $t > 0$ and u, $\hat{u} \in D \cap L^s$ with exponents

$$\alpha_s = \frac{\alpha}{1-\gamma(1-\theta_s)}, \qquad \gamma_s = \gamma\frac{\theta_s}{1-\gamma(1-\theta_s)}. \tag{4.34}$$

Remark 4.2 The statement of Theorem 4.5 remains unchanged if one replaces the assumption that the semigroup $\{e^{-tA}\}_{t\ge 0}$ is contractive in L^s by

$$\|e^{-tA}u-e^{-tA}\hat{u}\|_s \le M\,\|u-\hat{u}\|_s, \qquad (t \ge 0,\ u, \hat{u} \in D \cap L^s),$$

for some constant $M > 0$. Then the constant C in (4.33) has to be changed accordingly.

Proof (of Theorem 4.5) We outline the proof only for $r < \infty$ since the case $r = \infty$ is treated similarly. Then, set $\theta_s = \frac{(r-q)s}{q(r-s)}$ and assume that (4.32) holds. For $\theta := 1-\gamma(1-\theta_s)$, u, $\hat{u} \in L^s \cap D$ satisfying $u \neq \hat{u}$ and $T > 0$, set

$$C_{u,\hat{u},T} := \sup_{t\in[0,T]} \frac{t^{\alpha/\theta}\|e^{-tA}u - e^{-tA}\hat{u}\|_r}{\|u-\hat{u}\|_s^{\gamma_s}}.$$

By (4.31) and since θ_s satisfies $\frac{1}{q} = \frac{(1-\theta_s)}{r} + \frac{\theta_s}{s}$, Hölder's inequality imply

$$\begin{aligned}\|e^{-tA}u - e^{-tA}\hat{u}\|_r &\leq C\left(\tfrac{t}{2}\right)^{-\alpha} \|e^{-\frac{t}{2}A}u - e^{-\frac{t}{2}A}\hat{u}\|_q^{\gamma}\\ &\leq C\left(\tfrac{t}{2}\right)^{-\alpha} \|e^{-\frac{t}{2}A}u - e^{-\frac{t}{2}A}\hat{u}\|_r^{\gamma(1-\theta_s)} \|e^{-\frac{t}{2}A}u - e^{-\frac{t}{2}A}\hat{u}\|_s^{\gamma\theta_s}\end{aligned}$$

Since $\{e^{-tA}\}_{t\geq 0}$ is contractive in L^s, one has that

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_r \leq C\left(\tfrac{t}{2}\right)^{-\alpha} \|e^{-\frac{t}{2}A}u - e^{-\frac{t}{2}A}\hat{u}\|_r^{\gamma(1-\theta_s)} \|u-\hat{u}\|_s^{\gamma\theta_s}$$

and so by definition of $C_{u,\hat{u},T}$,

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_r \leq C\left(\tfrac{t}{2}\right)^{-\alpha-\alpha\frac{\gamma(1-\theta_s)}{\theta}} C_{u,\hat{u},T}^{\gamma(1-\theta_s)} \|u-\hat{u}\|_s^{\gamma(\theta_s+\gamma_s(1-\theta_s))}$$

for every $t \in [0, 2T]$. Since $\gamma\theta_s + \gamma_s\gamma(1-\theta_s) = \gamma_s$ and $1 + \frac{\gamma(1-\theta_s)}{\theta} = \frac{1}{\theta}$, the last estimate can rewritten as

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_r \leq C\, 2^{\frac{\alpha}{\theta}}\, t^{-\frac{\alpha}{\theta}}\, C_{u,\hat{u},T}^{\gamma(1-\theta_s)} \|u-\hat{u}\|_s^{\gamma_s}$$

for every $t \in [0, T]$. Dividing this inequality by $t^{-\frac{\alpha}{\theta}} \|u-\hat{u}\|_s^{\gamma_s}$ and taking the supremum over $[0, T]$ on the left hand-side of the resulting inequality yields

$$C_{u,\hat{u},T} \leq C\, 2^{\frac{\alpha}{\theta}}\, C_{u,\hat{u},T}^{\gamma(1-\theta_s)}. \tag{4.35}$$

By (4.32), we can conclude from (4.35) that $C_{u,\hat{u},T}$ is uniformly bounded in u, $\hat{u}$ by constant $(C\,2^{\frac{\alpha}{\theta}})^{\frac{1}{\theta}} > 0$ where $\theta = 1-\gamma(1-\theta_s)$. Therefore, one has that

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_r \leq (C\,2^{\frac{\alpha}{\theta}})^{\frac{1}{\theta}}\; t^{-\frac{\alpha}{\theta}}\, \|u-\hat{u}\|_s^{\gamma_s}$$

for every $t \in [0, T]$ and u, $\hat{u} \in D \cap L^s$, where $T > 0$ was arbitrary. Taking $t = T$ in this inequality, we can conclude that inequality (4.33) holds for every $t > 0$ and u, $\hat{u} \in D \cap L^s$. □

Our second extrapolation result of this subsection is adapted to semigroups $\{e^{-tA}\}_{t\geq 0}$ enjoying the L^q-L^r-regularizing effect (4.36) for $1 < q < r \leq \infty$ at some *invariant element* $u_0 \in L^r \cap L^s$; that is, $e^{-tA}u_0 = u_0$.

Theorem 4.6 *For* $1 \leq s < q < r \leq \infty$*, let* $\{e^{-tA}\}_{t\geq 0}$ *be a* C_0*-semigroup of* L^s*-contractions* e^{-tA} *acting on a subset* D *of* L^q*. Suppose, there are* $C > 0$ *and exponents* $\alpha > 0$ $\gamma > 0$ *such that*

$$\|e^{-tA}u - u_0\|_r \le C\, t^{-\alpha}\, \|u - u_0\|_q^{\gamma} \tag{4.36}$$

for every $t > 0$ and $u \in D$, where $u_0 \in L^s \cap L^r$ is an invariant element of $\{e^{-tA}\}_{t\ge 0}$. For $\theta_s = \frac{(r-q)s}{q(r-s)} > 0$ if $r < \infty$ and $\theta_s = \frac{s}{q}$ if $r = \infty$, assume that (4.32) holds. Then, one has that

$$\|e^{-tA}u - u_0\|_r \le \left(C\, 2^{\frac{\alpha}{1-\gamma(1-\theta_s)}}\right)^{\frac{1}{1-\gamma(1-\theta_s)}}\, t^{-\alpha_s}\, \|u - u_0\|_s^{\gamma_s} \tag{4.37}$$

for every $t > 0$ and $u \in D \cap L^s$ with exponents (4.34).

Proof (of Theorem 4.6) By using the same arguments as outlined in the proof of Theorem 4.5, where one replaces $\hat{u}$ and $e^{-tA}\hat{u}$ by u_0 and uses that u_0 is an invariant element of the semigroup $\{e^{-tA}\}_{t\ge 0}$, one finds that the statement of Theorem 4.6 holds. □

We continue this chapter by establishing a new nonlinear interpolation theorem of independent interest.

4.5 A nonlinear interpolation theorem

In this section, we state our nonlinear interpolation theorem, which generalises both Peetre's ([123, Theorem 3.1]) and Tartar's (cf. [140, Théorème 4]) nonlinear interpolation results. Our nonlinear interpolation theorem complements the existing literature in three ways, namely, by introducing additional parameters p_0, r_0, r_1, by treating the borderline cases $p_0 = \infty$, $p_1 < \infty$ and $p_0 < \infty$, $p_1 = \infty$, and by giving exact constants.

We begin by recalling some basic definitions, notations and results from the classical interpolation theory (cf., for instance, [124] or [45, Chapter 3]). Let X_0 and X_1 be two real or complex Banach spaces such that both are continuously embedded into a Hausdorff topological vector space $\mathcal{X}$. A pair $\{X_0, X_1\}$ of Banach spaces X_0 and X_1 satisfying these conditions is called an *interpolation couple*. We equip the *intersection space* $X_0 \cap X_1$ and the *sum space*

$$X_0 + X_1 := \left\{ x \,\middle|\, \text{there are } x_0 \in X_0,\ x_1 \in X_1 \text{ s.t. } x = x_0 + x_1 \right\}$$

respectively with the norm $\|x\|_{X_0\cap X_1} := \max\{\|x\|_{X_0}, \|x\|_{X_1}\}$ and

$$\|x\|_{X_0+X_1} := \inf\left\{ \|x_0\|_{X_0} + \|x_1\|_{X_1} \,\middle|\, x = x_0 + x_1,\ x_0 \in X_0, x_1 \in X_1 \right\}.$$

Then $X_0 \cap X_1$ and $X_0 + X_1$ are Banach spaces and

$$X_0 \cap X_1 \hookrightarrow Z \hookrightarrow X_0 + X_1 \tag{4.38}$$

for $Z = X_0$ and $Z = X_1$ each with linear continuous embeddings (cf. [45, Proposition 3.2.1]). A Banach space Z satisfying (4.38) is called an *intermediate space* (of X_0 and X_1).

For any Banach space X equipped with norm $\|\cdot\|_X$ and for every $1 \leq q \leq \infty$, we denote by $L^q_*(X)$ the Banach space of all (classes of) strongly $\mathrm{d}t/t$-measurable functions $f : (0, \infty) \to X$ having finite norm

$$\|f\|_{L^q_*(X)} := \begin{cases} \left\{ \int_0^\infty \|f(t)\|^q_X \frac{\mathrm{d}t}{t} \right\}^{1/q} & \text{if } 1 \leq q < \infty, \\ \operatorname*{ess\,sup}_{t \in (0,\infty)} \|f(t)\|_X & \text{if } q = \infty. \end{cases}$$

We shall make use of the so-called *mean-method*, which was introduced by J.-L. Lions and Peetre ([105, 104]) and further elaborated, for instance, in [122, 94].

We begin by introducing the *mean spaces* (*espaces de moyennes*). Let (X_0, X_1) be an interpolation couple. Then for every $0 < \theta < 1$ and $1 \leq p_0, p_1 \leq \infty$, the *mean space* $(X_0, X_1)_{\theta, p_0, p_1}$ is defined by the space of all elements $u \in X_0 + X_1$ with the property

$$\begin{cases} \text{for } i = 0, 1, \text{ there is a measurable function } v_i : (0, \infty) \to X_i \\ \text{satisfying } u = v_0(t) + v_1(t) \text{ in } X_0 + X_1 \text{ for a.e. } t \in (0, \infty), \\ t^{-\theta} v_0 \in L^{p_0}_*(X_0) \text{ and } t^{1-\theta} v_1 \in L^{p_1}_*(X_1). \end{cases} \tag{4.39}$$

We equip the mean space $(X_0, X_1)_{\theta, p_0, p_1}$ with the norm

$$\|u\|_{\theta, p_0, p_1} := \inf_{u = v_0(t) + v_1(t)} \max \left\{ \|t^{-\theta} v_0\|_{L^{p_0}_*(X_0)}, \|t^{1-\theta} v_1\|_{L^{p_1}_*(X_1)} \right\},$$

where the infimum is taken of all representation pairs (v_0, v_1) satisfying (4.39). Then, it is not difficult to see that each mean space $(X_0, X_1)_{\theta, p_0, p_1}$ is an intermediate space (cf. [105, p. 9]). Moreover, the spaces $(X_0, X_1)_{\theta, p_0, p_1}$ admits the so-called *interpolation property* (cf. [146, p. 63]), that is, for every linear mapping $T : X_0 + X_1 \to X_0 + X_1$ such that its restriction to X_i yields a linear and bounded operator from X_i into itself, where $i = 0, 1$, one has that the restriction of T to $(X_0, X_1)_{\theta, p_0, p_1}$ yields a linear and bounded operator from $(X_0, X_1)_{\theta, p_0, p_1}$ into itself ([105, Théorème (3.1)]). In particular, one has

$$\|u\|_{\theta, p_0, p_1} = \inf_{a = v_0(t) + v_1(t)} \|t^{-\theta} v_0\|^{1-\theta}_{L^{p_0}_*(X_0)} \|t^{1-\theta} v_1\|^{\theta}_{L^{p_1}_*(X_1)}, \tag{4.40}$$

for every $u \in (X_0, X_1)_{\theta, p_0, p_1}$, where the infimum is taken of all representation pairs (v_0, v_1) satisfying (4.39) (cf. [105, Lemme (3.1)]). In addition, the following continuous embedding is valid.

Lemma 4.1 ([105, Théorème 5.3]) *Let $0 < \theta < 1$ and $1 \leq p_0, p_1, s_0, s_1 \leq \infty$. Then for $s_0 \leq p_0$ and $s_1 \leq p_1$, one has*

$$\|u\|_{\theta,p_0,p_1} \le C_{\theta,r_0,r_1}\,\|u\|_{\theta,s_0,s_1}$$

for all $u \in (X_0,X_1)_{\theta,s_0,s_1}$, *where the constant*

$$C_{\theta,r_0,r_1} := \begin{cases} 1 & \text{if } s_0 = p_0 \text{ and } s_1 = p_1,\\ \inf\limits_{\varphi\in D_+} \|t^{-\theta}\varphi\|^{1-\theta}_{L^{r_0}_*(\mathbb{R})} & \text{if } s_1 = p_1\\ \inf\limits_{\varphi\in D_+} \|t^{1-\theta}\varphi\|^{\theta}_{L^{r_1}_*(\mathbb{R})} & \text{if } s_0 = p_0\\ \inf\limits_{\varphi\in D_+} \|t^{-\theta}\varphi\|^{1-\theta}_{L^{r_0}_*(\mathbb{R})}\,\|t^{1-\theta}\varphi\|^{\theta}_{L^{r_1}_*(\mathbb{R})} & \text{if otherwise} \end{cases} \tag{4.41}$$

$$\text{with } \tfrac{1}{r_0} = 1 - \left[\tfrac{1}{s_0} - \tfrac{1}{p_0}\right] \quad \text{and} \quad \tfrac{1}{r_1} = 1 - \left[\tfrac{1}{s_1} - \tfrac{1}{p_1}\right],$$

and D_+ *denotes the set of all test functions* $\varphi \in C_c^\infty((0,\infty))$ *satisfying* $\varphi \ge 0$ *and* $\int_0^\infty \varphi(\frac{1}{t})\frac{\mathrm{d}t}{t} = 1$.

Due to the result [122, Théorème 3.1] by Peetre, for every $0<\theta<1$ and $1 \le p_0, p_1, p \le \infty$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, the mean space $(X_0,X_1)_{\theta,p_0,p_1}$ coincides with the (classical) *real interpolation space* $(X_0,X_1)_{\theta,p}$ with equivalent norms. For the definition of the interpolation space $(X_0,X_1)_{\theta,p}$ we refer, for instance, to [45, Definition 3.2.4]. Combining this together with the density result [145, Theorem 1.6.2], we can state the following generalized version of the density result [105, Théorème 2.1].

Lemma 4.2 *Let* (X_0,X_1) *be an interpolation couple and suppose that one of the following cases holds:*

1. $1 \le p_0, p_1 < \infty$;
2. $1 \le p_0 < \infty$ *and* $p_1 = \infty$;
3. $1 \le p_1 < \infty$ *and* $p_0 = \infty$.

Then, for every $0<\theta<1$, *the intersection space* $X_1 \cap X_2$ *is dense in* $(X_0,X_1)_{\theta,p_0,p_1}$.

Now, we are in a position to state our first nonlinear interpolation theorem.

Theorem 4.7 *Let* (X_0,X_1) *and* (Y_0,Y_1) *be two interpolation couples and* T *be a mapping from* X_0+X_1 *into* Y_0+Y_1 *with domain containing* $X_0\cap X_1$. *Suppose there are exponents* $0<\alpha_0$, $\alpha_1<\infty$ *and constants* M_0, $M_1 \ge 0$ *such that*

$$\|Tu - T\hat{u}\|_{Y_0} \le M_0\,\|u-\hat{u}\|^{\alpha_0}_{X_0} \tag{4.42}$$

for all u, $\hat{u} \in X_0\cap X_1$ *and*

$$\|Tu - T\hat{u}\|_{Y_1} \le M_1\,\|u-\hat{u}\|^{\alpha_1}_{X_1} \tag{4.43}$$

for all u, $\hat{u} \in X_0\cap X_1$. *For* $0<\theta<1$ *and* $1 \le p_0^*$, $p_1^* \le \infty$ *(excluding* $p_0^* = p_1^* = \infty$*) satisfying* $p_0^* \ge \frac{1}{\alpha_0}$ *and* $p_1^* \ge \frac{1}{\alpha_1}$, *let* $1 \le q, p_0, p_1 \le \infty$, $0<\eta<1$ *and* $0<\alpha<\infty$ *be given by*

$$\begin{aligned} &\tfrac{1}{q} = \tfrac{1-\theta}{p_0^*} + \tfrac{\theta}{p_1^*}, \qquad p_0 = \alpha_0 p_0^*, \quad p_1 = \alpha_1 p_1^*, \\ &\eta = \tfrac{\theta\,\alpha_1}{(1-\theta)\,\alpha_0 + \theta\,\alpha_1}, \; \alpha = (1-\theta)\alpha_0 + \theta\alpha_1 \end{aligned} \tag{4.44}$$

and let $1 \le s_0 \le p_0$ and $1 \le s_1 \le p_1$. Then the following statements hold.

1. *One has*

$$\|Tu - T\hat{u}\|_{(Y_0,Y_1)_{\theta,p_0^*,p_1^*}} \le \left(\tfrac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q}} M_0^{1-\theta}\, M_1^{\theta}\, C^{\alpha}_{\eta,r_0,r_1} \|u - \hat{u}\|^{\alpha}_{(X_0,X_1)_{\eta,s_0,s_1}} \tag{4.45}$$

   for every u, $\hat{u} \in X_0 \cap X_1$, where the constant C_{η,r_0,r_1} is given by (4.41).

2. *If there is a $u_0 \in X_0 \cap X_1$ such that $Tu_0 \in (Y_0,Y_1)_{\theta,p_0^*,p_1^*}$, then T can be uniquely extended to a mapping $T : (X_0,X_1)_{\eta,s_0,s_1} \to (Y_0,Y_1)_{\theta,p_0^*,p_1^*}$ satisfying inequality (4.45) for all u, $\hat{u} \in (X_0,X_1)_{\eta,s_0,s_1}$.*

From Theorem 4.7 we obtain as a special case the following result extending Browder's nonlinear interpolation theorem [42, Theorem 1] to Lipschitz-continuous mappings with different constants M_0 and M_1 on different spaces X_0 and Y_0.

Corollary 4.3 *Let (X_0,X_1) and (Y_0,Y_1) be two interpolation couples and T be a mapping from $X_0 + X_1$ into $Y_0 + Y_1$ with domain containing $X_0 \cap X_1$. Suppose there are constants M_0, $M_1 \ge 0$ such that T satisfies*

$$\|Tu - T\hat{u}\|_{Y_0} \le M_0\, \|u - \hat{u}\|_{X_0}$$

for all u, $\hat{u} \in X_0 \cap X_1$ and

$$\|Tu - T\hat{u}\|_{Y_1} \le M_1\, \|u - \hat{u}\|_{X_1}$$

for all u, $\hat{u} \in X_0 \cap X_1$. Then, for every $0 < \theta < 1$, $1 \le p_0$, $p_1 \le \infty$ (excluding $p_0 = p_1 = \infty$), and $1 \le s_0 \le p_0$, $1 \le s_1 \le p_1$, the following statements hold.

1. *One has*

$$\|Tu - T\hat{u}\|_{(Y_0,Y_1)_{\theta,p_0,p_1}} \le M_0^{1-\theta}\, M_1^{\theta}\, C_{\theta,r_0,r_1} \|u - \hat{u}\|_{(X_0,X_1)_{\theta,s_0,s_1}} \tag{4.46}$$

   for every u, $\hat{u} \in X_0 \cap X_1$, where the constant C_{η,r_0,r_1} is given by (4.41).

2. *If there is a $u_0 \in X_0 \cap X_1$ such that $Tu_0 \in (Y_0,Y_1)_{\theta,p_0,p_1}$, then T can be uniquely extended to a mapping $T : (X_0,X_1)_{\theta,s_0,s_1} \to (Y_0,Y_1)_{\theta,p_0,p_1}$ satisfying inequality (4.46) for all u, $\hat{u} \in (X_0,X_1)_{\theta,s_0,s_1}$.*

One important application of Corollary 4.3 is the following one. Let $(\Sigma,\mathcal{B},\mu)$ be a σ-finite measure space. Then, according to Corollary B.1 (stated in Appendix B.1), one has that the mean space

$$(L^1, L^\infty(\Sigma))_{\theta,1,\infty} = L^{\frac{1}{1-\theta}}$$

with equal norms for every $0 < \theta < 1$. Therefore, Corollary 4.3 yields the following interpolation property of a family $\{e^{-tA}\}_{t\ge 0}$ of ω-quasi contractive mappings e^{-tA} :

$L^1 \to L^1$ with $\omega \in \mathbb{R}$. This corollary might be of independent interest and generalizes a nonlinear interpolation result [42] due to Browder.

Corollary 4.4 *Let $(\Sigma, \mathcal{B}, \mu)$ be a σ-finite measure space and for $\omega \in \mathbb{R}$, let $\{e^{-tA}\}_{t\geq 0}$ be a family of ω-quasi contractive mappings $e^{-tA} : L^1 \to L^1$ and $u_0 \in L^1 \cap L^\infty$ such that $e^{-tA}u_0 = u_0$ for all $t \geq 0$. If $\{e^{-tA}\}_{t\geq 0}$ is ω-quasi contractive on $L^1 \cap L^\infty$ with respect to the L^∞-norm, then $\{e^{-tA}\}_{t\geq 0}$ is ω-quasi contractive on L^q for all $1 \leq q \leq \infty$.*

Next, we turn to the proof of main interpolation theorem (Theorem 4.7).

Proof (of Theorem 4.7) First, we fix $\hat{u} \in X_0 \cap X_1$ and show that

$$\|T(u+\hat{u}) - T\hat{u}\|_{(Y_0,Y_1)_{\theta,p_0^*,p_1^*}} \leq \left(\frac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q}} M_0^{1-\theta}\, M_1^{\theta}\, \|u\|^{\alpha}_{(X_0,X_1)_{\eta,p_0,p_1}} \tag{4.47}$$

for all $u \in X_0 \cap X_1$. To do so, let $u \in X_0 \cap X_1$. Since $X_0 \cap X_1$ is continuously injected into $(X_0, X_1)_{\eta,p_0,p_1}$, there is a pair (v_0, v_1) of measurable functions satisfying (4.39). Since $u \in X_0 \cap X_1$ and $u = v_0 + v_1$, it follows that $v_i(t) \in X_0 \cap X_1$ for a.e. $t \in (0,\infty)$ and every $i = 0, 1$. For $\lambda := \frac{\theta}{\eta\,\alpha_0} > 0$, we set

$$w_0(t) = T(v_0(t^\lambda) + \hat{u}) - T\hat{u} \quad \text{ and } \quad w_1(t) = T(u+\hat{u}) - T\hat{u} - w_0(t)$$

for a.e. $t \in (0,\infty)$. Then, $T(u+\hat{u}) - T\hat{u} = w_0(t) + w_1(t)$ for a.e. $t \in (0,\infty)$, and by using (4.42) and (4.43), one sees that the functions $w_i : (0,\infty) \to Y_i$ are measurable and satisfy

$$\|w_i(t)\|_{Y_i} \leq M_i\, \|v_i(t^\lambda)\|^{\alpha_i}_{X_i} \tag{4.48}$$

for a.e. $t \in (0,\infty)$ and each $i = 0, 1$. Since we have chosen $\lambda = \frac{\theta}{\eta\,\alpha_0}$ and $p_0 = p_0^*\alpha_0$, we obtain by applying inequality (4.48) and substituting $s = t^\lambda$ that

$$\|t^{-\theta} w_0\|_{L_*^{p_0^*}(Y_0)} \leq M_0 \left(\frac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{p_0^*}} \|s^{-\eta} v_0\|^{\alpha_0}_{L^{p_0}(X_0)}.$$

On the other hand, $\eta = \frac{\theta\,\alpha_1}{(1-\theta)\,\alpha_0 + \theta\,\alpha_1}$ is equivalent to $\frac{1-\eta}{\eta} = \frac{(1-\theta)\,\alpha_0}{\theta\,\alpha_1}$ hence $\lambda = \frac{1-\theta}{(1-\eta)\,\alpha_1}$. Using this together with inequality (4.48), the fact that $p_1 = p_1^*\alpha_1$, and applying the substitution $s = t^\lambda$, we see that

$$\|t^{1-\theta} w_1\|_{L_*^{p_1^*}(Y_1)} \leq M_1 \left(\frac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{p_1^*}} \|s^{1-\eta} v_1\|^{\alpha_1}_{L^{p_1}(X_1)}.$$

Thus $T(u+\hat{u}) - T\hat{u} \in (Y_1, Y_2)_{\theta,p_0^*,p_1^*}$. Combining the last two estimates together with (4.40) yields

$$\begin{aligned}
&\|T(u+\hat{u}) - T\hat{u}\|_{(Y_0,Y_1)_{\theta,p_0^*,p_1^*}} \\
&\quad\leq M_0^{1-\theta}\, M_1^{\theta} \left(\frac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q}} \|s^{-\eta} v_0\|^{(1-\theta)\,\alpha_0}_{L^{p_0}(X_0)}\, \|s^{1-\eta} v_1\|^{\theta\,\alpha_1}_{L^{p_1}(X_1)}
\end{aligned}$$

$$\leq M_0^{1-\theta} M_1^{\theta} \left(\frac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q}} \max\left\{\|s^{-\eta} v_0\|_{L^{p_0}(X_0)}, \|s^{1-\eta} v_1\|_{L^{p_1}(X_1)}\right\}^{\alpha}.$$

Taking the infimum over all representation pairs (v_0, v_1) satisfying (4.39) shows that inequality (4.47) holds. Now, for every $u, \hat{u} \in X_0 \cap X_1$, replacing u by $u - \hat{u} \in X_0 \cap X_1$ in (4.47) gives

$$\|Tu - T\hat{u}\|_{(Y_0,Y_1)_{\theta,p_0^*,p_1^*}} \leq \left(\frac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q}} M_0^{1-\theta} M_1^{\theta} \|u - \hat{u}\|^{\alpha}_{(X_0,X_1)_{\eta,p_0,p_1}} \tag{4.49}$$

for all u, $\hat{u} \in X_0 \cap X_1$. Applying Lemma 4.1 yields inequality (4.45) for every u, $\hat{u} \in X_0 \cap X_1$, proving that the first statement of Theorem 4.7 holds.

Under the assumption, there is a $u_0 \in X_0 \cap X_1$ such that $Tu_0 \in (Y_0, Y_1)_{\theta,p_0^*,p_1^*}$, inequality (4.45) implies that the mapping T maps from $X_0 \cap X_1$ equipped with the $(X_0, X_1)_{\eta,p_0,p_1}$-norm into $(Y_0, Y_1)_{\theta,p_0^*,p_1^*}$. Thus by Lemma (6.1) and since the spaces $(X_0, X_1)_{\eta,p_0,p_1}$ and $(Y_0, Y_1)_{\theta,p_0^*,p_1^*}$ are complete, we can conclude that T admits a unique Hölder-continuous extension from $(X_0, X_1)_{\eta,p_0,p_1}$ to $(Y_0, Y_1)_{\theta,p_0^*,p_1^*}$ satisfying (4.49) for all u, $\hat{u} \in (X_0, X_1)_{\eta,p_0,p_1}$. This completes the proof of Theorem 4.7. □

In our second nonlinear interpolation theorem, we consider the situation when the mapping T admits an element $u_0 \in X_0 \cap X_1$ such that $Tu_0 \in Y_0 \cap Y_1$.

Theorem 4.8 *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples and T a mapping from $X_0 + X_1$ into $Y_0 + Y_1$ with domain containing $X_0 \cap X_1$. Suppose T is continuous from $X_0 \cap X_1$ equipped with the X_0-norm to Y_0 and there are $u_0 \in X_0 \cap X_1$ satisfying $Tu_0 \in Y_0 \cap Y_1$, exponents $0 < \alpha_0$, $\alpha_1 < \infty$, and M_0, $M_1 \geq 0$ such that*

$$\|Tu - Tu_0\|_{Y_0} \leq M_0 \|u - u_0\|^{\alpha_0}_{X_0} \qquad \text{for all } u \in X_0 \cap X_1 \tag{4.50}$$

and

$$\|Tu - T\hat{u}\|_{Y_1} \leq M_1 \|u - \hat{u}\|^{\alpha_1}_{X_1} \qquad \text{for all } u,\ \hat{u} \in X_0 \cap X_1. \tag{4.43}$$

For $0 < \theta < 1$, $1 \leq p_0^*$, $p_1^* \leq \infty$ (excluding $p_0^* = p_1^* = \infty$) satisfying $p_0^* \geq \frac{1}{\alpha_0}$ and $p_1^* \geq \frac{1}{\alpha_1}$, let $1 \leq q, p_0, p_1 \leq \infty$, $0 < \eta < 1$ and $0 < \alpha < \infty$ be given by (4.44), and let $1 \leq s_0 \leq p_0$ and $1 \leq s_1 \leq p_1$. Then one has

$$\|Tu - Tu_0\|_{(Y_0,Y_1)_{\theta,p_0^*,p_1^*}} \leq \left(\frac{\eta\,\alpha_0}{\theta}\right)^{\frac{1}{q}} M_0^{1-\theta} M_1^{\theta} C^{\alpha}_{\eta,r_0,r_1} \|u - u_0\|^{\alpha}_{(X_0,X_1)_{\eta,s_0,s_1}} \tag{4.51}$$

for every $u \in X_0 \cap X_1$, where the constant C_{η,r_0,r_1} is given by (4.41).

Proof (of Theorem 4.8) Let $u \in X_0 \cap X_1$. Since $X_0 \cap X_1$ is continuously injected into $(X_0, X_1)_{\eta,p_0,p_1}$, there are measurable functions $v_i : (0, \infty) \longrightarrow X_i$ for $i = 0, 1$ satisfying $u - u_0 = v_0(t) + v_1(t)$ in $X_0 + X_1$ for a.e. $t \in (0, \infty)$,

$$t^{-\theta} v_0 \in L^{p_0}_*(X_0) \qquad \text{and} \qquad t^{1-\theta} v_1 \in L^{p_1}_*(X_1). \tag{4.52}$$

For $\lambda := \frac{\theta}{\eta\,\alpha_0} > 0$, we set

$$w_0(t) = T(v_0(t^\lambda) + u_0) - Tu_0 \quad \text{and} \quad w_1(t) = Tu - Tu_0 - w_0(t)$$

for a.e. $t \in (0,\infty)$. By construction, $Tu - Tu_0 = w_0(t) + w_1(t)$ for a.e. $t \in (0,\infty)$. Since by assumption, T is continuous from $X_0 \cap X_1$ equipped with the X_0-norm to Y_0, the function $w_0 : (0,\infty) \to Y_0$ is strongly measurable. By (4.43), T is Hölder-continuous from $X_0 \cap X_1$ equipped with the X_1-norm to Y_1. Thus, the function $w_1 : (0,\infty) \to Y_1$ is strongly measurable. Moreover, by (4.50) and (4.43), we have that the inequalities (4.48) hold for $i = 0, 1$. Now, we can proceed as in the proof of Theorem 4.7 to conclude that inequality (4.51) holds for all $u \in X_0 \cap X_1$. □

In some applications, the assumption that *the mapping T is continuous from $X_0 \cap X_1$ equipped with the X_0-norm topology to Y_0* in Theorem 4.8 is too strong. This can be circumvented, for instance, by the following result.

Theorem 4.9 *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples, Y_0 being a separable Banach space. Let T be a mapping from $X_0 + X_1$ into $Y_0 + Y_1$ with domain containing $X_0 \cap X_1$. Suppose there is some $u_0 \in X_0 \cap X_1$ such that $Tu_0 \in Y_0 \cap Y_1$ and T satisfies the following three conditions.*

- *T is continuous from $X_0 \cap X_1$ equipped with the X_0-norm to Y_0 equipped with the weak topology,*
- *there are exponents $0 < \alpha_0$, $\alpha_1 < \infty$ and constants M_0, $M_1 \geq 0$ such that T satisfies (4.50) and (4.43).*

For every $0 < \theta < 1$ and $1 \leq p_0^*$, $p_1^* \leq \infty$ (excluding $p_0^* = p_1^* = \infty$) satisfying $p_0^* \geq \frac{1}{\alpha_0}$ and $p_1^* \geq \frac{1}{\alpha_1}$, let $1 \leq q, p_0, p_1 \leq \infty$, $0 < \eta < 1$, $0 < \alpha < \infty$ given by (4.44), and let $1 \leq s_0 \leq p_0$ and $1 \leq s_1 \leq p_1$. Then T satisfies inequality (4.51) for every $u \in X_0 \cap X_1$, where the constant C_{η,r_0,r_1} is given by (4.41).

Remark 4.3 Consider the following situation: For $1 \leq q, r < \infty$, let $X_0 = L^q$, $X_1 = L^\infty$, $Y_0 = L^r$ and $Y_1 = L^\infty$, where one assumes that (Σ, μ) is a *separable measure space* (cf, [41, Definition on p.98]). Suppose T satisfy the assumptions of Theorem 4.9 and we choose

$$\begin{aligned} &p_0^* = r, \qquad p_1^* = \infty, \qquad\qquad p_0 = \beta p_0^* = \beta r > q \geq 1,\\ &p_1 = p_1^* = \infty,\ s_0 = q < \beta r = p_0,\ s_1 = \infty. \end{aligned}$$

Then, by Corollary B.1,

$$(X_0, X_1)_{\eta,s_0,s_1} = L^{\frac{q}{(1-\eta)}} \qquad \text{and} \qquad (Y_0, Y_1)_{\theta,p_0^*,p_1^*} = L^{\frac{r}{(1-\theta)}}$$

with equal norms for every $0 < \theta, \eta < 1$ and so Theorem 4.9 yields

$$\begin{aligned} &\|Tu - u_0\|_{\frac{r}{1-\theta}}\\ &\quad \leq \left[\frac{\beta}{(1-\theta)\beta+\theta}\right]^{\frac{1-\theta}{r}} M_0^{1-\theta} M_1^{\theta} C_{\eta,r_0,1}^{(1-\theta)\beta+\theta} \|u - u_0\|_{\frac{q}{1-\eta(\theta)}}^{(1-\theta)\beta+\theta} \end{aligned} \tag{4.53}$$

for every $u \in L^q \cap L^\infty$ and every $0 < \theta < 1$, where $r_0 = \frac{q\beta r}{\beta r(q-1)+q}$. In addition, to the above assumptions, we suppose

$$T \text{ is continuous from } L^{\frac{q}{1-\eta(\theta)}} \text{ to } L^{\frac{q}{1-\eta(\theta)}}.$$

Since $L^q \cap L^\infty$ is dense in $L^{\frac{q}{1-\eta(\theta)}}$, for every $u \in L^{\frac{q}{1-\eta(\theta)}}$, there is a sequence (u_n) in $L^q \cap L^\infty$ such that u_n converges to u in $L^{\frac{q}{1-\eta(\theta)}}$ and so Tu_n converges to Tu in $L^{\frac{q}{1-\eta(\theta)}}$. By (4.53), (Tu_n) is bounded in $L^{\frac{r}{1-\theta}}$ and hence, after eventually passing to a subsequence of (u_n), we may assume that Tu_n converges weakly to v in $L^{\frac{r}{1-\theta}}$ for some $v \in L^{\frac{r}{1-\theta}}$. Since $L^{\frac{q}{1-\eta(\theta)}}$ and $L^{\frac{r}{1-\theta}}$ are both continuously embedded into L^m_{loc}, with $m := \min\{\frac{q}{1-\eta(\theta)}, \frac{r}{1-\theta}\}$, we obtain $v = Tu$ a.e. on Σ and so, sending $n \to \infty$ in (4.53) for $u = u_n$ and using Fatou's lemma shows that (4.53) holds for all $u \in L^{\frac{q}{1-\eta(\theta)}}$.

Proof (of Theorem 4.9) Let $u \in X_0 \cap X_1$ and for $i = 0,\ 1$, let $v_i : (0,\infty) \to X_i$ be measurable such that $u - u_0 = v_0(t) + v_1(t)$ in $X_0 + X_1$ for a.e. $t \in (0,\infty)$ and (4.52) holds. For $\lambda := \frac{\theta}{\eta\,\alpha_0} > 0$, we set

$$w_0(t) = T(v_0(t^\lambda) + u_0) - Tu_0 \quad \text{and} \quad w_1(t) = Tu - Tu_0 - w_0(t)$$

for a.e. $t \in (0,\infty)$. By construction, $Tu - Tu_0 = w_0(t) + w_1(t)$ for a.e. $t \in (0,\infty)$. By assumption, T is continuous from $X_0 \cap X_1$ equipped with the X_0-norm topology to Y_0 equipped with the weak-topology. Hence w_0 is weakly measurable. But since by assumption, Y_0 is separable, the function $w_0 : (0,\infty) \to Y_0$ is strongly measurable due to Pettis's theorem ([90, Theorem 3.5.3]). By (4.43), T is Hölder-continuous from $X_0 \cap X_1$ equipped with the X_1-norm to Y_1. Thus, the function $w_1 : (0,\infty) \to Y_1$ is strongly measurable. Moreover, by (4.50) and (4.43), we have that the inequalities (4.48) hold for $i = 0, 1$. Now, we can proceed as in the proof of Theorem 4.7 and see that the statement of this theorem holds. □

4.6 Extrapolation towards L^∞ via interpolation of the semigroup

To the best of our knowledge, first extrapolation results towards L^∞ in the context of *linear semigroups* and employing Riesz-Thorin's or Stein's linear interpolation theorems go back to the pioneering work [136] by Simon and Høegh-Krohn (see also [67, Theorem 3.3]). An alternative approach using a duality argument has been given in [54, Lemme 1]. However, in this article, we are confronted with a more difficult situation, since the family of operators $\{e^{-tA}\}_{t\ge 0}$ are (in general) nonlinear. Hence neither a duality argument or a linear Riesz-Thorin interpolation theorem can be used. Our extrapolation result towards L^∞ is a nonlinear generalisation of the techniques developed in [136, 67, 54]. Our proof relies essentially on the nonlinear interpolation results Theorem 4.7 and Theorem 4.9, as well as the fact that the mean spaces involving L^{p_0} and L^{p_1} spaces are isometrically isomorphic to an appropriate L^p space (cf. Corollary B.1).

Our first extrapolation result towards L^∞ is adapted to the class of semigroups $\{e^{-tA}\}_{t\geq 0}$ of completely contractions e^{-tA} on L^q satisfying a global L^q-L^r-regularization estimate (4.31) (with differences) for $1 \leq q, r < \infty$.

Theorem 4.10 *Let $1 \leq q,\ r < \infty$, $\omega \geq 0$, and $\{e^{-tA}\}_{t\geq 0}$ be a semigroup acting on $L^q \cap L^\infty$. Suppose, the semigroup $\{e^{-tA}\}_{t\geq 0}$ is contractive in L^∞ and satisfies the L^q-L^r-regularity estimate*

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_r \leq C\, t^{-\alpha}\, \|u-\hat{u}\|_q^{\gamma} \tag{4.54}$$

for every $t > 0$ and $u,\ \hat{u} \in L^q \cap L^\infty$, for some exponents $\alpha,\ \gamma > 0$. If γ satisfies

$$\gamma r > q, \tag{4.55}$$

then for every $q_0 \geq q\gamma^{-1}$ satisfying

$$(\tfrac{\gamma r}{q} - 1)\, q_0 + q(\tfrac{1}{\gamma} - 1) > 0, \tag{4.56}$$

the semigroup $\{e^{-tA}\}_{t\geq 0}$ satisfies the $L^{\gamma r q^{-1} q_0}$-L^∞-regularity estimate

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_\infty \lesssim\ t^{-\alpha^*}\ \|u-\hat{u}\|_{\gamma r q^{-1} q_0}^{\gamma^*} \tag{4.57}$$

for every $t > 0$ and $u,\ \hat{u} \in L^{\gamma r q^{-1} q_0}$, with exponents

$$\alpha^* = \frac{\alpha q \gamma^{-1}}{(\frac{\gamma r}{q} - 1)\, q_0 + q(\frac{1}{\gamma} - 1)}, \qquad \gamma^* = \frac{(\frac{\gamma r}{q} - 1)\, q_0}{(\frac{\gamma r}{q} - 1)\, q_0 + q(\frac{1}{\gamma} - 1)}. \tag{4.58}$$

Remark 4.4 The idea of the proof of Theorem 4.10 is based on an iteration over the semigroup $\{e^{-tA}\}_{t\geq 0}$. To do this, the two conditions (4.55) and (4.56) are heavily involved in the *recursive construction* of the strictly increasing sequence $(q_n)_{n\geq 0} \subseteq (1, \infty)$ satisfying

$$\lim_{n\to+\infty} q_n = +\infty.$$

Namely, for $q_0 \geq q\gamma^{-1}$ satisfy (4.56), let $q_0 = q_0$ and define q_n by

$$q_{n+1} = q_n\, \kappa - r\, \kappa^{-1}\, (\gamma - 1), \qquad (n \geq 1), \tag{4.59}$$

Further, for κ given by

$$\kappa = \frac{\gamma r}{q}, \tag{4.60}$$

condition (4.55) is equivalent to $\kappa > 1$. By assumption, $\{e^{-tA}\}_{t\geq 0}$ is L^∞-contractive and satisfies the L^q-L^r-regularity estimate (4.54). Thus, one can *interpolate* between both inequalities (Theorem 4.7). In this way, we construct a family (in $\tilde{q} \geq q_0$) of $L^{\tilde{q}+r\kappa^{-1}(\gamma-1)}$-$L^{\tilde{q}\kappa}$-regularity inequalities (see (5.34) below). For every $n \geq 0$, choosing $\tilde{q} = q_n$ in the $L^{\tilde{q}+r\kappa^{-1}(\gamma-1)}$-$L^{\tilde{q}\kappa}$-regularity inequality and by using the semigroup property of $\{e^{-tA}\}_{t\ge 0}$, one obtains that $\{e^{-tA}\}_{t\ge 0}$ satisfies the $L^{\gamma r q^{-1} q_0}$-L^∞ regularization estimate (4.57).

We give now the details of the proof.

Proof (of Theorem 4.10) We apply the nonlinear interpolation result Theorem 4.7 to the following situation: let $X_0 = L^q$, $X_1 = L^\infty$, $Y_0 = L^r$, $Y_1 = L^\infty$, and for any fixed $t > 0$, let $T = e^{-tA}$.

By assumption, e^{-tA} satisfies L^q-L^r regularity estimate (4.54) and and is contractive in L^∞. Hence the mapping T satisfies inequality (4.42) with $\alpha_0 = \gamma > 0$, $M_0 = Ct^{-\alpha}$ and inequality (4.43) with $\alpha_1 = 1$, $M_1 = 1$. Further, we choose

$$\begin{array}{lll} p_0^* = r, & p_1^* = \infty, & p_0 = \gamma p_0^* = \gamma r > q \ge 1, \\ p_1 = p_1^* = \infty, & s_0 = q < \gamma r = p_0, & s_1 = \infty. \end{array}$$

Then, by Corollary B.1,

$$(X_0, X_1)_{\eta, s_0, s_1} = L^{\frac{q}{(1-\eta)}} \qquad \text{and} \qquad (Y_0, Y_1)_{\theta, p_0^*, p_1^*} = L^{\frac{r}{(1-\theta)}}$$

with equal norms for every $0 < \theta, \eta < 1$. Thus, Theorem 4.7 yields

$$\begin{aligned} \|e^{-tA}u - e^{-tA}\hat{u}\|_{\frac{r}{1-\theta}} \le \left[\tfrac{\gamma}{(1-\theta)\gamma+\theta}\right]^{\frac{1-\theta}{r}} \left[Ct^{-\alpha}\right]^{1-\theta} \times \\ \times \left[\inf_{\varphi\in D_+} \|s^{-\theta}\varphi\|^{1-\theta}_{L^{r_0}_*(\mathbb{R})}\right]^{(1-\theta)\gamma+\theta} \|u-\hat{u}\|^{(1-\theta)\gamma+\theta}_{\frac{q}{1-\eta(\theta)}} \end{aligned} \tag{4.61}$$

for every $t > 0$, u, $\hat{u} \in L^q \cap L^\infty$ and every $0 < \theta < 1$, where

$$r_0 = \frac{q\gamma r}{\gamma r(q-1)r + q}.$$

Next, we choose a test function $\varphi^* \in D_+$ with support $\text{supp}(\varphi^*) \subseteq [1,2]$. Then, there are constants $C_{\varphi^*,1}$, $C_{\varphi^*,2} > 0$ such that

$$C_{\varphi^*,1} \le \|s^{-\theta}\varphi^*\|_{L^{r_0}_*(\mathbb{R})} \le C_{\varphi^*,2} \qquad \text{for every } 0 < \theta < 1.$$

By applying this estimate to (4.61), one gets that

$$\begin{aligned} \|e^{-tA}u - e^{-tA}\hat{u}\|_{\frac{r}{1-\theta}} \le \left[\tfrac{\gamma}{(1-\theta)\gamma+\theta}\right]^{\frac{1-\theta}{r}} \left[Ct^{-\alpha}\right]^{1-\theta} \times \\ \times \|s^{-\theta}\varphi^*\|^{(1-\theta)((1-\theta)\gamma+\theta)}_{L^{r_0}_*(\mathbb{R})} \|u-\hat{u}\|^{(1-\theta)\gamma+\theta}_{\frac{q}{1-\eta(\theta)}} \end{aligned}$$

for every $t > 0$, u, $\hat{u} \in L^q \cap L^\infty$ and every $0 < \theta < 1$.

Now, let κ be given by (4.60) and for every $\tilde{q} > r\kappa^{-1} = \frac{q}{\gamma}$, set

$$\theta_{\tilde{q}} = 1 - \frac{1}{\tilde{q}}\frac{r}{\kappa}.$$

Then by hypothesis (4.55), one has that $\kappa > 1$, and for every $\tilde{q} > r\kappa^{-1}$,

$$0 < \theta_{\tilde{q}} < 1, \qquad\qquad 1-\theta_{\tilde{q}} = \frac{1}{\tilde{q}}\frac{r}{\kappa},$$
$$1-\eta(\theta_{\tilde{q}}) = \frac{r\kappa^{-1}\gamma}{\tilde{q}+r\kappa^{-1}(\gamma-1)}, \qquad \frac{\gamma}{(1-\theta_{\tilde{q}})\gamma+\theta_{\tilde{q}}} = \frac{\gamma\tilde{q}}{\tilde{q}+r\kappa^{-1}(\gamma-1)} > 0.$$

Further, for every $\tilde{q} > r\kappa^{-1}$, we set

$$C_{\varphi^*,\tilde{q}} := \|s^{\frac{1}{\tilde{q}}\frac{r}{\kappa}-1}\varphi^*\|_{L^{r_0}_*(\mathbb{R})}.$$

With this setting in mind, the previous inequality reduces to inequality (5.34) below holding for every $t > 0$, u, $\hat{u} \in L^q \cap L^\infty$ and $\tilde{q} > r\kappa^{-1}$.

Finally, we choose $q_0 \geq r\kappa^{-1}$ such that (4.56) holds (where one notes that with the setting of this proof, condition (4.56) coincides with (4.62) below) and $\tilde{q} > q_0$. The conditions on q_0 are sufficient to run an iteration in the time-variable, which is the statement of the following iteration lemma. From this lemma, we can conclude that the statement of this theorem holds. □

Lemma 4.3 *Let $q \leq q_0$, $r < \infty$, $\kappa > 1$, $\gamma > 0$, and $\{e^{-tA}\}_{t\geq 0}$ be a semigroup acting on $L^{\kappa q_0} \cap L^\infty$. Suppose, for $q_0 \geq r K^{-1}$ satisfying*

$$(\kappa-1)q_0 + r\kappa^{-1}(1-\gamma) > 0 \tag{4.62}$$

and for α, $\beta > 0$ and $\omega \geq 0$, the semigroup $\{e^{-tA}\}_{t\geq 0}$ satisfies

$$\begin{aligned}\|e^{-tA}u - e^{-tA}\hat{u}\|_{\tilde{q}\kappa} \leq \left[\frac{\gamma\tilde{q}}{\tilde{q}+r\kappa^{-1}(\gamma-1)}\right]^{\frac{1}{\tilde{q}\kappa}} \left[C t^{-\alpha}\right]^{\frac{1}{\tilde{q}}\frac{r}{\kappa}} \times \\ \times C_{\varphi^*,\tilde{q}}^{\frac{1}{\tilde{q}}\frac{r}{\kappa}(\frac{1}{\tilde{q}}\frac{r}{\kappa}(\gamma-1)+1)} \|u-\hat{u}\|_{\tilde{q}+r\kappa^{-1}(\gamma-1)}^{\frac{1}{\tilde{q}}r\kappa^{-1}(\gamma-1)+1}\end{aligned} \tag{4.63}$$

for every u, $\hat{u} \in L^{\kappa q_0} \cap L^\infty$, $t > 0$ and $\tilde{q} \geq q_0$, where $C_{\varphi^*,\tilde{q}}$ satisfies

$$C_{\varphi^*,1} \leq C_{\varphi^*,\tilde{q}} \leq C_{\varphi^*,2} \tag{4.64}$$

for some constants $C_{\varphi^*,1}$, $C_{\varphi^*,2} > 0$ independent of $\tilde{q} \geq q_0$. Then,

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_\infty \lesssim t^{-\frac{\alpha r\kappa^{-1}}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}} \|u-\hat{u}\|_{\kappa q_0}^{\frac{(\kappa-1)q_0}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}} \tag{4.65}$$

for every u, $\hat{u} \in L^{\kappa q_0}$ and every $t > 0$.

For the proof of this lemma, we simplify some techniques from [151] and extend an idea from the linear semigroup theory [117] to nonlinear one.

Proof For $q_0 \geq r\kappa^{-1}$ such that (4.62) holds, we construct a sequence $(q_n)_{n\geq 0}$ recursively by (4.59). Then

$$q_{n+1} = \kappa q_n + r\kappa^{-1}(1-\gamma) \tag{4.66}$$

for every integer $n \geq 0$ and so, an induction over $n \in \mathbb{N}_0$ yields

$$q_n = \kappa^n [q_0 + r\kappa^{-1}(\gamma - 1)] + r\kappa^{-1}(1-\gamma) \sum_{\nu=0}^{n} \kappa^\nu, \tag{4.67}$$

that is

$$q_n = \kappa^n \frac{(\kappa-1)q_0 + r\kappa^{-1}(1-\gamma)}{\kappa - 1} - \frac{r\kappa^{-1}(1-\gamma)}{\kappa-1}. \tag{4.68}$$

Using (4.67), we see that

$$q_{n+1} - q_n = \kappa^n \Big[(\kappa-1)q_0 + r\kappa^{-1}(1-\gamma)\Big]$$

hence the sequence $(q_n)_{n\geq 0}$ is strictly increasing if and only if q_0 satisfies condition (4.62). Moreover, by (4.68), since $\kappa > 1$, and by (4.62), we see that

$$\lim_{n\to\infty} q_n = \infty \tag{4.69}$$

and

$$\lim_{n\to\infty} \frac{q_n}{\kappa^n} = \frac{(\kappa-1)q_0 + r\kappa^{-1}(1-\gamma)}{\kappa-1}. \tag{4.70}$$

Since

$$\frac{1}{q_n} r\kappa^{-1}(\gamma-1) + 1 = \frac{\kappa\, q_{n-1}}{q_n} \qquad \text{and} \qquad \frac{\gamma\, q_n}{q_n + r\kappa^{-1}(\gamma-1)} = \frac{\gamma\, q_n}{q_{n-1}\kappa},$$

inserting the sequence $(q_n)_{n\geq 0}$ into (5.34) yields

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_{\kappa q_n} \leq C_{q_n}^{\frac{1}{q_n\kappa}}\, t^{-\frac{\alpha r}{q_n\kappa}}\, C_{\varphi^*,q_n}^{\frac{r}{q_n\kappa}\frac{\kappa q_{n-1}}{q_n}}\, \|u-\hat{u}\|_{q_{n-1}\kappa}^{\frac{\kappa q_{n-1}}{q_n}} \tag{4.71}$$

for every $t > 0$, u, $\hat{u} \in L^{\kappa q_0} \cap L^\infty$, and $n \geq 1$, where

$$C_{q_n} := \tfrac{\gamma\, q_n}{q_{n-1}\kappa}\, C^r.$$

Now, let $(t_\nu)_{\nu\geq 0}$ be a sequence in $[0,1]$ such that $\sum_{\nu=0}^\infty t_\nu = 1$ which we will specify below. By assumption, $\{e^{-tA}\}_{t\geq 0}$ is a semigroup and $e^{-tA}u$, $e^{-tA}\hat{u} \in L^{\kappa q_0} \cap L^\infty$ for every $t \geq 0$. Thus, we can iterate (4.71) and obtain

$$\begin{aligned} &\|T_{t\sum_{\nu=0}^n t_\nu} u - T_{t\sum_{\nu=0}^n t_\nu}\hat{u}\|_{\kappa q_{n+1}} \\ &\leq \prod_{\nu=1}^{n+1} C_{q_\nu}^{\frac{\kappa^{n-\nu}}{q_{n+1}}} \prod_{\nu=0}^{n} t_\nu^{-\frac{\alpha r}{\kappa}\frac{\kappa^{n-\nu}}{q_{n+1}}} \prod_{\nu=1}^{n+1} C_{\varphi^*,q_\nu}^{\frac{r}{q_\nu}\frac{\kappa^{n+1-\nu}q_{\nu-1}}{q_{n+1}}}\, t^{-\frac{\alpha r}{\kappa q_{n+1}}\sum_{\nu=0}^n \kappa^\nu}\, \|u-\hat{u}\|_{\kappa q_0}^{q_0\frac{\kappa^{n+1}}{q_{n+1}}}. \end{aligned} \tag{4.72}$$

Since by assumption, $\kappa > 1$, by (4.68), and by (4.62), we see that

$$\lim_{n\to\infty} \frac{1}{q_{n+1}} \sum_{\nu=0}^{n} \kappa^\nu = \frac{1}{(\kappa-1)q_0 + r\kappa^{-1}(1-\gamma)}. \tag{4.73}$$

Further, by (4.70),

$$\lim_{n\to\infty} t^{-\frac{\alpha r}{\kappa}\frac{1}{q_{n+1}}\sum_{\nu=0}^{n}\kappa^{n-\nu}} = t^{-\frac{\alpha r}{\kappa}\frac{1}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}} \qquad \text{for every } t>0. \tag{4.74}$$

Now, we choose, for instance, $t_\nu = 2^{-\nu-1}$. Then

$$\prod_{\nu=0}^{n} t_\nu^{-\frac{\alpha r}{\kappa}\frac{\kappa^{n-\nu}}{q_{n+1}}} = 2^{\frac{\alpha r}{\kappa}\frac{\kappa^n}{q_{n+1}}\sum_{\nu=0}^{n}(\nu+1)\kappa^{-\nu}}.$$

Using

$$\sum_{\nu=0}^{\infty}(\nu+1)\kappa^{-\nu} = \frac{\kappa^2}{(\kappa-1)^2}$$

and (4.70), one obtains

$$\lim_{n\to\infty}\frac{\kappa^n}{q_{n+1}}\sum_{\nu=0}^{n}(\nu+1)\kappa^{-\nu} = \frac{\kappa}{\kappa-1}\,\frac{1}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}, \tag{4.75}$$

therefore

$$\lim_{n\to\infty}\prod_{\nu=0}^{n} t_\nu^{-\frac{\alpha r}{\kappa}\frac{\kappa^{n-\nu}}{q_{n+1}}} = 2^{\frac{\alpha r}{\kappa}\frac{\kappa}{\kappa-1}\frac{1}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}}. \tag{4.76}$$

Next, by (4.64), one has

$$C_{\varphi^*,1}^{\frac{r\kappa^{n+1}}{q_{n+1}}\sum_{\nu=1}^{n+1}\frac{q_{\nu-1}}{\kappa^\nu q_\nu}} \le \prod_{\nu=1}^{n+1} C_{\varphi^*,q_\nu}^{\frac{r}{q_\nu}\frac{\kappa^{n+1-\nu}q_{\nu-1}}{q_{n+1}}} \le \prod_{\nu=1}^{n+1} C_{\varphi^*,2}^{\frac{r\kappa^{n+1}}{q_{n+1}}\sum_{\nu=1}^{n+1}\frac{q_{\nu-1}}{\kappa^\nu q_\nu}}. \tag{4.77}$$

Since by (4.68), one has that $a_\nu := \frac{q_{\nu-1}}{\kappa^\nu q_\nu}$ satisfies $\lim_{\nu\to\infty}|\frac{a_{\nu+1}}{a_\nu}| = \frac{1}{\kappa}$, the ratio test implies that the series $\sum_{\nu=1}^{\infty}\frac{q_{\nu-1}}{\kappa^\nu q_\nu}$ converges. Furthermore, (4.70) yields

$$\lim_{n\to\infty}\frac{\kappa^{n+1}}{q_{n+1}} = \frac{(\kappa-1)}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}. \tag{4.78}$$

Thus

$$\lim_{n\to\infty}\frac{r\kappa^{n+1}}{q_{n+1}}\sum_{\nu=1}^{n+1}\frac{q_{\nu-1}}{\kappa^\nu q_\nu} = \frac{r(\kappa-1)}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}\sum_{\nu=1}^{\infty}\frac{q_{\nu-1}}{\kappa^\nu q_\nu}$$

so that sending $n\to\infty$ in (4.77) yields

$$C_{\varphi^*,1}^{\frac{r(\kappa-1)}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}\sum_{\nu=1}^{\infty}\frac{q_{\nu-1}}{\kappa^\nu q_\nu}} \leq \liminf_{n\to\infty}\prod_{\nu=1}^{n+1} C_{\varphi^*,q_\nu}^{\frac{r}{q_\nu}\frac{\kappa^{n+1-\nu}q_{\nu-1}}{q_{n+1}}}$$
$$\leq \limsup_{n\to\infty}\prod_{\nu=1}^{n+1} C_{\varphi^*,q_\nu}^{\frac{r}{q_\nu}\frac{\kappa^{n+1-\nu}q_{\nu-1}}{q_{n+1}}} \tag{4.79}$$
$$\leq C_{\varphi^*,2}^{\frac{r(\kappa-1)}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}\sum_{\nu=1}^{\infty}\frac{q_{\nu-1}}{\kappa^\nu q_\nu}}.$$

Using again (4.78), we see that

$$\lim_{n\to\infty}\|u-\hat{u}\|_{\kappa q_0}^{q_0\frac{\kappa^{n+1}}{q_{n+1}}} = \|u-\hat{u}\|_{\kappa q_0}^{\frac{(\kappa-1)q_0}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}}. \tag{4.80}$$

It remains to control the product

$$\prod_{\nu=1}^{n+1} C_{q_\nu}^{\frac{\kappa^{n-\nu}}{q_{n+1}}} = \prod_{\nu=1}^{n+1}\left[\frac{\beta q_n}{q_{n-1}\kappa}\right]^{\frac{\kappa^{n-\nu}}{q_{n+1}}} \times C^{\frac{r\sum_{\nu=1}^{n+1}\kappa^{n-\nu}}{q_{n+1}}} \tag{4.81}$$

as $n\to\infty$. Since $\kappa>1$ and by (4.70),

$$\lim_{n\to\infty}\frac{1}{q_{n+1}}\sum_{\nu=1}^{n+1}\kappa^{n-\nu} = \frac{\kappa^{-1}}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)} \tag{4.82}$$

and so

$$\lim_{n\to\infty} C^{\frac{r\sum_{\nu=1}^{n+1}\kappa^{n-\nu}}{q_{n+1}}} = C^{\frac{r\kappa^{-1}}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}}. \tag{4.83}$$

For every $n\geq 1$, the quotient $\frac{\gamma q_n}{q_{n-1}\kappa} = \frac{\gamma q_n}{q_n+r\kappa^{-1}(\gamma-1)}$ can be controlled by

$$\gamma < \frac{\gamma q_n}{q_{n-1}\kappa} < \frac{\gamma}{1+\frac{r}{q_0}\kappa^{-1}(\gamma-1)} \qquad \text{if } 0<\gamma<1$$

and by

$$\frac{\gamma}{1+\frac{r}{q_0}\kappa^{-1}(\gamma-1)} < \frac{\gamma q_n}{q_{n-1}\kappa} < \gamma \qquad \text{if } \gamma\geq 1.$$

Thus for general $\gamma>0$, there are constants $C_1, C_2>0$ such that

$$C_1^{\frac{1}{q_{n+1}}\sum_{\nu=1}^{n+1}\kappa^{n-\nu}} \leq \prod_{\nu=1}^{n+1}\left[\frac{\gamma q_n}{q_{n-1}\kappa}\right]^{\frac{\kappa^{n-\nu}}{q_{n+1}}} \leq C_2^{\frac{1}{q_{n+1}}\sum_{\nu=1}^{n+1}\kappa^{n-\nu}} \tag{4.84}$$

for every $n\geq 0$ and so by (4.82), sending $n\to\infty$ in (4.84) yields

$$C_1^{\frac{\kappa^{-1}}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}} \leq \liminf_{n\to\infty}\prod_{\nu=1}^{n+1}\left[\frac{\gamma q_n}{q_{n-1}\kappa}\right]^{\frac{\kappa^{n-\nu}}{q_{n+1}}} \leq \limsup_{n\to\infty}\prod_{\nu=1}^{n+1} C_{q_\nu}^{\frac{\kappa^{n-\nu}}{q_{n+1}}} \leq C_2^{\frac{\kappa^{-1}}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}}.$$

Thus sending $n \to \infty$ in inequality (4.72) and using (4.74), (4.76), (4.80), (4.83), (4.79) together with the fact that $q_n \nearrow \infty$ as $n \to \infty$ yields

$$\|e^{-tA}u - e^{-tA}\hat{u}\|_\infty \le \Big[C_2\, C^r\Big]^{\frac{r\,\kappa^{-1}}{(\kappa-1)q_0 + r\,\kappa^{-1}(1-\gamma)}}\, t^{-\frac{\alpha\, r\,\kappa^{-1}}{(\kappa-1)q_0 + r\,\kappa^{-1}(1-\gamma)}} \times$$
$$\times\; C_{\varphi^*,2}^{\frac{r(\kappa-1)}{(\kappa-1)q_0 + r\,\kappa^{-1}(1-\gamma)} \sum_{\nu=1}^{\infty} \frac{q_{\nu-1}}{\kappa^\nu q_\nu}}\; \|u - \hat{u}\|_{\kappa q_0}^{\frac{(\kappa-1)\,q_0}{(\kappa-1)q_0 + r\,\kappa^{-1}(1-\gamma)}}$$

showing that inequality (4.65) holds for u, $\hat{u} \in L^{\kappa q_0} \cap L^\infty$. By hypothesis, the semigroup $\{e^{-tA}\}$ acts on $L^{\kappa q_0} \cap L^\infty$, that is, every e^{-tA} maps $L^{\kappa q_0} \cap L^\infty$ to $L^{\kappa q_0} \cap L^\infty$. Since $L^{\kappa q_0} \cap L^\infty$ is dense in $L^{\kappa q_0}$, a standard approximation argument shows that the first claim of this iteration lemma holds. This completes the proof. □

Our second extrapolation result towards L^∞ is adapted to the class of semigroups $\{e^{-tA}\}_{t\ge 0}$ of completely contractions e^{-tA} on L^q satisfying the L^q-L^r-regularization estimate (4.85) at some invariant element $u_0 \in L^\infty \cap L^q$ for $1 \le q, r < \infty$.

Theorem 4.11 *For $1 \le q$, $r < \infty$, let $\{e^{-tA}\}_{t\ge 0}$ be a semigroup acting on $L^q \cap L^\infty$. Suppose, $\{e^{-tA}\}_{t\ge 0}$ is contractive in L^∞ and at least one of the following conditions is satisfied.*

1. *$(\Sigma, \mathcal{B}, \mu)$ is a separable measure space and for every $t > 0$, e^{-tA} is continuous from $L^q \cap L^\infty$ equipped with the L^q-norm to L^r equipped with the weak-topology.*
2. *for every $t > 0$, e^{-tA} is continuous from $L^q \cap L^\infty$ quipped with the L^q-norm to L^r equipped with the L^r-norm.*

Further, suppose, there are α, $\gamma > 0$ satisfying (4.55) and

$$\|e^{-tA}u - u_0\|_r \le C\, t^{-\alpha}\, \|u - u_0\|_q^\gamma \tag{4.85}$$

for every $t > 0$ and $u \in L^q \cap L^\infty$, where $u_0 \in L^q \cap L^\infty$ is an invariant element of $\{e^{-tA}\}_{t\ge 0}$. Then the semigroup $\{e^{-tA}\}_{t\ge 0}$ satisfies the $L^{\gamma r q^{-1} q_0}$-L^∞-regularity estimate

$$\|e^{-tA}u - u_0\|_\infty \lesssim t^{-\alpha^*}\, \|u - u_0\|_{\gamma r q^{-1} q_0}^{\gamma^*}$$

for every $t > 0$ and $u \in L^{\gamma r q^{-1} q_0}$, where α^* and γ^* are given by (4.58) and $q_0 \ge q\,\gamma^{-1}$ such that (4.56) holds.

The proof of this theorem proceeds analogously as the one of Theorem 4.10, where one replaces the application of interpolation Theorem 4.7 by Theorem 4.8 or Theorem 4.9. Furthermore, one applies the extrapolation argument from Remark 4.3 and replaces Lemma 4.3 by the following one. We leave the details of the proof to the interested reader.

Lemma 4.4 *Let $1 \le q_0$, $r < \infty$, $\kappa > 1$, and $\{e^{-tA}\}_{t\ge 0}$ be a semigroup acting on $L^{\kappa q_0} \cap L^\infty$. Suppose, for $q_0 \ge r\,K^{-1}$ satisfying (4.62), exponents α, $\gamma > 0$, and some $u_0 \in L^{\kappa q_0} \cap L^\infty$, the semigroup $\{e^{-tA}\}_{t\ge 0}$ satisfies the family (in $\tilde{q} \ge q_0$) of $L^{\tilde{q} + r\,\kappa^{-1}(\gamma - 1)}$-$L^{\tilde{q}\,\kappa}$-regularity estimates*

$$\|e^{-tA}u-u_0\|_{\tilde{q}\kappa} \le \left[\frac{\gamma\tilde{q}}{\tilde{q}+r\kappa^{-1}(\gamma-1)}\right]^{\frac{1}{\tilde{q}\kappa}} \left[Ct^{-\alpha}\right]^{\frac{1}{\tilde{q}}\frac{r}{\kappa}} \times$$
$$\times\, C_{\varphi^*,\tilde{q}}^{\frac{1}{\tilde{q}}\frac{r}{\kappa}(\frac{1}{\tilde{q}}\frac{r}{\kappa}(\gamma-1)+1)} \|u-u_0\|_{\tilde{q}+r\kappa^{-1}(\gamma-1)}^{\frac{1}{\tilde{q}}r\kappa^{-1}(\gamma-1)+1}$$

for every $u \in L^{\kappa q_0} \cap L^\infty$, $t > 0$, $\tilde{q} > q_0$, where $C_{\varphi^*,\tilde{q}}$ satisfies (4.64) for some constants $C_{\varphi^*,1}$, $C_{\varphi^*,2} > 0$ independent of $\tilde{q}$. Then, the semigroup $\{e^{-tA}\}_{t\ge 0}$ satisfies the $L^{\kappa q_0}$-L^∞-regularization estimate

$$\|e^{-tA}u-u_0\|_\infty \lesssim t^{-\frac{\alpha r\kappa^{-1}}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}} \|u-u_0\|_{\kappa q_0}^{\frac{(\kappa-1)q_0}{(\kappa-1)q_0+r\kappa^{-1}(1-\gamma)}}.$$

for every $u \in L^{\kappa q_0}$ and every $t > 0$.

The proof of Lemma 4.4 proceeds as the one of Lemma 4.3. We omit the details.

4.7 Proof of Theorem 4.1

In this section, we apply the abstract extrapolation theory to the semigroup $\{e^{-t\Lambda}\}_{t\ge 0}$ generated by the Dirichlet-to-Neumann map Λ on L^q, and thereby outline the proof of Theorem 4.1.

The key inequality to the L^q-L^∞ regularization estimates are the following classical Sobolev-trace inequalities

$$\|\mathcal{T}r(u)\|_{\frac{pd}{d-1}} \le C_1\|u\|_{W^{1,p}(\Omega)} \tag{4.86}$$

if $1 < p < d$ (cf. [115, Théorème 4.2]) and

$$\|\mathcal{T}r(u)\|_q \le C_2\|u\|_{W^{1,p}(\Omega)} \tag{4.87}$$

for every $q \ge 1$ if $p = d$ (cf. [115, Théorème 4.6]), and the classical Sobolev-Morrey inequality

$$\|\mathcal{T}r(u)\|_\infty \le C_3\|u\|_{W^{1,p}(\Omega)} \tag{4.88}$$

holding for every u (cf. [115, Théorème 3.8]).

Proof (of Theorem 4.1) First, one applies Poincaré's inequality

$$\|\mathcal{T}r(u)-\overline{\mathcal{T}r(u)}\|_p \le C\,\||\nabla u|\|_p, \qquad (u \in W^{1,p}(\Omega)), \tag{4.89}$$

holding for every $u \in W^{1,p}(\Omega)$, where $\overline{u_{|\partial\Omega}} := \frac{1}{\mathcal{H}(\partial\Omega)}\int_{\partial\Omega} u\,\mathrm{d}\mathcal{H}^{d-1}$ denotes the *mean value* of u on $\partial\Omega$ (cf. [85, Lemma 2.5]), to (4.86) (respectively, to (4.87) and (4.88)). Then, one obtains

$$\|\mathcal{T}r(u)-\overline{\mathcal{T}r(u)}\|_{\frac{p(d-1)}{d-p}} \le C\||\nabla u|\|_p, \qquad (u \in W^{1,p}(\Omega)),$$

if $1 < p < d$,

$$\|\mathcal{T}r(u)-\overline{\mathcal{T}r(u)}\|_{\frac{1}{1-\theta}} \le C_\theta\,\||\nabla u|\|_p, \qquad (u\in W^{1,p}(\Omega),\ 0\le\theta<1),$$

if $p=d\ge 2$, and if $d<p<\infty$, then one gets that

$$\|\mathcal{T}r(u)-\overline{\mathcal{T}r(u)}\|_\infty \le C\,\||\nabla u|\|_p, \qquad (u\in W^1_{p,p}(\Sigma)).$$

Now, taking pth power on both sides of each of the three inequalities shows that the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ satisfies the L^2-L^r Sobolev inquality (**??**) in Corollary 4.2. Hence, the semigroup $\{e^{-t\Lambda_\sigma}\}_{t\ge0}$ satisfies the L^2-L^r regularization estimate (4.30) for every $h\in L^2(\partial\Omega)$ with parameters

$$\begin{aligned}
&r=\tfrac{p(d-1)}{d-p}, && \alpha=\frac{1}{p}, && \gamma=\frac{2}{p} && \text{if } 1<p<d,\\
&r=\tfrac{1}{1-\theta}, && \alpha=\frac{1}{p}, && \gamma=\frac{2}{p} && \text{if } p=d,\\
&r=\infty, && \alpha=\frac{1}{p}, && \gamma=\frac{2}{p} && \text{if } p=d.
\end{aligned}$$

Now, applying Theorem 4.8 and subsequently Theorem 4.6 to each of the three cases $1<p<d$, $p=d$, and $p>d$, one sees that statements of Theorem **??** holds. □

Chapter 5
Aronson-Bénilan type estimates

Abstract In this chapter, we establish Aronson-Bénilan type estimates, which are applicable to abstract nonlinear semigroups. These type of estimates provide additional regularity properties to mild solutions of an abstract Cauchy problem. Crandall and Bénilan [28] showed that such estimates in an abstract Banach space framework hold if the leading operator A in the Cauchy problem is homogeneous of order $\alpha > 0$ and $\alpha \neq 1$. This theory has been extended in the two papers [86] and [87] by allowing $\alpha \neq 1$ and by adding a Lipschitz perturbation F (which is not necessary homogeneous). We apply this theory to the perturbed Cauchy problem governed by the Dirichlet-to-Neumann map Λ and in particular, we show that the regularity estimates stated in Theorem 1.8 hold. The content of this chapter covers parts of the papers [86] and [87].

5.1 Main results

Throughout this chapter, let X be either $L^1(\partial\Omega)$, $L^\psi(\partial\Omega)$ for any N-function satisfying the Δ_2-condition, or $X = C(\partial\Omega)$. The aim of this chapter is to prove the regularization effect of mild solutions h of the perturbed Cauchy problem (in X)

$$\begin{cases} \frac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda h(t) + F(h(t)) \ni 0 & \text{for } t \in (0,T), \\ h(0) = h_0 \end{cases} \tag{5.1}$$

as it is stated in Theorem 1.8 (see Chapter 1.10). Since the homogeneity property of the Dirichlet-to-Neumann map Λ in (5.1) is crucial for this regularization effect of mild solutions, we follow here the convention introduced in Notation 1.5 (see Chapter 1.9) and refer to Λ being the realization $\Lambda_{|X}$ in the Banach space X of the Dirichlet-to-Neumann map Λ_{σ_0} associated with the weighted negative p-Laplace operator $A = -\operatorname{div}(\sigma_0|\nabla\cdot|^{p-2}\nabla\cdot)$.

Further, let $f : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ be a Lipschitz-continuous Carathéodory function, that is,

- $f(\cdot,h) : \partial\Omega \to \mathbb{R}$ is measurable on $\partial\Omega$ for every $h \in \mathbb{R}$, (5.2)
- $f(x,0) = 0$ for a.e. $x \in \partial\Omega$, and (5.3)
- there is a constant $\omega \geq 0$ such that
$$|f(x,h)-f(x,\hat{h})| \leq \omega\,|h-\hat{h}| \quad \text{for all } h,\,\hat{h}\in\mathbb{R}, \text{ a.e. } x\in\partial\Omega, \tag{5.4}$$

and $F : X \to X$ be the *Nemytskii operator* of f on X given by

$$(Fh)(x) := f(x,h(x)) \qquad \text{for a.e. } x \in \partial\Omega,$$

for every $h \in X$.

It follows from Proposition 3.3 (in Chapter 3.2.2.2) that the operator $\Lambda+F$ is an ω-quasi m-completely accretive operator on X. Thus, by the classical existence theory (cf. [30] or [23, Corollary 4.2]), the perturbed Cauchy problem (5.1) is well-posed in the sense of mild solutions. In particular, $\Lambda+F$ generates a C_0-semigroup $\{e^{-tA}\}_{t\geq 0}$ on X of Lipschitz-continuous mappings e^{-tA} on X with Lipschitz constant $e^{\omega t}$.

With this preliminary in mind, we can now state the outline the regularization effect of mild solutions h of the perturbed Cauchy problem (5.1) (in X). This result is summarized in Theorem 1.8 and has been first proved in [86]. For the convenience of the reader, we stated here again.

Theorem 5.1 *Assume that the conductivity coefficient σ_0 satisfies Assumption 1.1 and $f : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous Carathéodory function. Further, suppose, $1 < p < \infty$ but $p \neq 2$. Then every mild solution h of the perturbed Cauchy problem (5.1) admits the following regularization effect.*

1. *(*L^1 Aronson-Bénilan type estimates*) For every $h_0 \in X$, the mild solution h of (5.1) is a* strong solution *of (5.1) in X, and satisfies*
$$\left\|\frac{\mathrm{d}h}{\mathrm{d}t_+}(t)\right\|_X \leq \frac{[2+\omega t]\;e^{\omega t}}{|p-2|\,t}\|h_0\|_X \quad \textit{for every } t>0. \tag{5.5}$$
2. *(*Point-wise Aronson-Bénilan type estimates*) For every positive $h_0 \in X$, the strong solution h of the perturbed Cauchy problem (5.1) in X satisfies*
$$(p-2)\frac{\mathrm{d}h}{\mathrm{d}t_+}(t) \geq -\frac{h(t)}{t}+(p-2)\,g_0(t), \tag{5.6}$$
for a.e. $t > 0$, where $g_0 : (0,\infty) \to X$ is a measurable function.

The two inequalities (5.1) and (5.5) in Theorem 5.1 are obtained from a more abstract functional anlaytical framework, which we develop now.

5.2 Preliminaries

In this section, we gather some intermediate results to prove the main theorem of this chapter.

5.2.1 Nonlinear semigroup theory - Part II

To prove Theorem 5.1, we develop regularity results, which can be applied to abstract nonlinear semigroups. This requires us to build up on the theory about homogeneous operators summarized in Chapter 3.2.3.

5.2.1.1 Homogeneous operators - Part II

Let X be a linear vector space, $\|\cdot\|_X$ a semi-norm on X, and A a homogeneous operator on X of order $\alpha \neq 1$ (see Definition 3.18 in Chapter 3.2.3).

We begin by considering the inhomogeneous Cauchy problem (in X)

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) \ni g(t) & \text{for a.e. } t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{5.7}$$

and want to discuss the impact of the homogeneity of A on the solutions u of the inhomogeneous Cauchy problem (5.7). For this, suppose $g \in C([0,T];X)$, $u_0 \in X$, and $u \in C^1([0,T];X)$ be a *classical* solution of the inhomogeneous Cauchy problem (5.7). Further, for given $\lambda > 0$, set

$$w_\lambda(t) = \lambda^{\frac{1}{\alpha-1}} u(\lambda t)$$

for every $t \in \left[0, \frac{T}{\lambda}\right]$. Then, w_λ satisfies

$$\begin{aligned} \frac{\mathrm{d}w_\lambda}{\mathrm{d}t}(t) &= \lambda^{\frac{1}{\alpha-1}+1} \frac{\mathrm{d}u}{\mathrm{d}t}(\lambda t) \in \lambda^{\frac{\alpha}{\alpha-1}} \left[g(\lambda t) - A(u(\lambda t))\right] \\ &= -A(w_\lambda(t)) + \lambda^{\frac{\alpha}{\alpha-1}} g(\lambda t) \end{aligned} \tag{5.8}$$

for every $t \in (0, T/\lambda)$ with initial value

$$w_\lambda(0) = \lambda^{\frac{1}{\alpha-1}} u(0) = \lambda^{\frac{1}{\alpha-1}} u_0. \tag{5.9}$$

Next, assume that the inhomogeneous Cauchy problem (5.7) is well-posed for every $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0,T;X)$ in the sense that there is a semigroup $\{e^{-tA}\}_{t=0}^T$ of mappings $e^{-tA} : \overline{D(A)}^X \times L^1(0,T;X) \to \overline{D(A)}^X$ given by

$$e^{-tA}(u_0, g) := u(t) \qquad \text{for every } t \in [0,T], \tag{5.10}$$

$u_0 \in \overline{D(A)}^X$, and $g \in L^1(0,T;X)$, where u is the unique (mild) solution of the inhomogeneous Cauchy problem (5.7). Then, the computation (5.7) and (5.9) can be reformulated in terms of this semigroup $\{e^{-tA}\}_{t=0}^T$ as follows. One has that

$$e^{-tA}(0,0) = 0 \qquad \text{for every } t \in [0,T] \tag{5.11}$$

(i.e., $u(t) \equiv 0$ is the unique solution of (5.7) for $u_0 = 0$ and $g(t) \equiv 0$), and

$$\lambda^{\frac{1}{\alpha-1}} e^{-\lambda t A}(u_0, g(\lambda\cdot)) = e^{-tA}(\lambda^{\frac{1}{\alpha-1}} u_0, \lambda^{\frac{\alpha}{\alpha-1}} g(\lambda\cdot)) \tag{5.12}$$

for every $t \in [0,T/\lambda]$, $\lambda > 0$. Now, identity (5.12) together with the standard growth estimate

$$\begin{aligned} e^{-\omega t} &\|e^{-tA}(u_0,g) - e^{-tA}(\hat{u}_0,\hat{g})\|_X \\ &\le L e^{-\omega s}\|e^{-\omega s}(u_0,g) - e^{-\omega s}(\hat{u}_0,\hat{g})\|_X \\ &\qquad + L\int_s^t e^{-\omega r}\,\|g(r) - \hat{g}(r)\|_X\,\mathrm{d}r \end{aligned} \tag{5.13}$$

for every $0 \le s \le t \le T$, $u_0 \in \overline{D(A)}^X$, g, $\hat{g} \in L^1(0,T;X)$, holding for some $\omega \in \mathbb{R}$ and $L \ge 1$, are the main ingredients to obtain L^1 Aronson-Bénilan type estimates as the one given in (5.5). This leads to our first intermediate result. This lemma also generalizes the case of homogeneous operators of order zero (cf [87, Theorem 2.3]), and the case $\omega = 0$ treated in [28, Theorem 4].

Lemma 5.1 *Let $\{e^{-tA}\}_{t=0}^T$ be a family of mappings $e^{-tA} : C \times L^1(0,T;X) \to C$ defined on a subset $C \subseteq X$, and suppose there are $\omega \in \mathbb{R}$, $L \ge 1$, and $\alpha \neq 1$ such that $\{e^{-tA}\}_{t=0}^T$ satisfies (5.11)-(5.13). Then, the following statements hold.*

1. *For every $u_0 \in C$, $g \in L^1(0,T;X)$, $t \in (0,T]$ and $h > 0$ such that $t + h \in (0,T]$, one has that*

$$\begin{aligned} &\|e^{-(t+h)A}(u_0,g) - e^{-tA}(u_0,g)\|_X \\ &\le \left|\left(1+\tfrac{h}{t}\right) - \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}}\right| L\int_0^t e^{\omega(t-s)}\|g(s+\tfrac{h}{t}s)\|_X\,\mathrm{d}s \\ &\quad + \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L\int_0^t e^{\omega(t-s)}\|g(s+\tfrac{h}{t}s) - g(s)\|_X\,\mathrm{d}s \\ &\quad + L e^{\omega t}\left|\left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} - 1\right|\left(2\,\|u_0\|_X + \int_0^t e^{-\omega s}\|g(s)\|_X\,\mathrm{d}s\right). \end{aligned} \tag{5.14}$$

2. *If one denotes*

$$V_\omega(g,t) := \limsup_{h\to 0+}\int_0^t e^{-\omega s}\frac{\|g(s+hs) - g(s)\|_X}{h}\,\mathrm{d}s, \tag{5.15}$$

and $\{e^{-tA}\}_{t=0}^T$ satisfies (5.13), then for every $t > 0$ and $u_0 \in C$, one has that

$$\limsup_{h\to 0+}\left\|\frac{e^{-(t+h)A}(u_0,g)-e^{-tA}(u_0,g)}{h}\right\|_X \leq \frac{L}{t}\,e^{\omega t}\left[2\frac{\|u_0\|_X}{|1-\alpha|}+\frac{1}{|1-\alpha|}\int_0^t e^{-\omega s}\|g(s)\|_X\,\mathrm{d}s+V_\omega(g,t)\right], \tag{5.16}$$

and if $g\in W^{1,1}(0,T;X)$*, then*

$$\limsup_{h\to 0+}\left\|\frac{e^{-(t+h)A}(u_0,g)-e^{-tA}(u_0,g)}{h}\right\|_X \leq \frac{L}{t}\,e^{\omega t}\left[2\frac{\|u_0\|_X}{|1-\alpha|}+\frac{1}{|1-\alpha|}\int_0^t e^{-\omega s}\|g(s)\|_X\,\mathrm{d}s + \int_0^t e^{-\omega s}\|g'(s)\|_X\,s\,\mathrm{d}s\right]. \tag{5.17}$$

3. *If for given* $u_0\in C$ *and* $g\in W^{1,1}(0,T;X)$*,* $\frac{\mathrm{d}}{\mathrm{d}t+}e^{-tA}(u_0,g)$ *exists (in* X*) at a.e.* $t\in(0,T)$*, then*

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t+}e^{-tA}(u_0,g)\right\|_X \leq \frac{L}{t}\,e^{\omega t}\left[2\frac{\|u_0\|_X}{|1-\alpha|}+\frac{1}{|1-\alpha|}\int_0^t e^{-\omega s}\|g(s)\|_X\,\mathrm{d}s + \int_0^t e^{-\omega s}\|g'(s)\|_X\,s\,\mathrm{d}s\right]. \tag{5.18}$$

Our proof of Lemma 5.1 uses the same techniques as in [28].

Proof Let $u_0\in C$, $g\in L^1(0,T;X)$, $t>0$, and $h>0$ satisfying $t+h\leq T$. Further, let $\lambda=1+\frac{h}{t}$. Then, (5.12) becomes

$$\begin{aligned} e^{-(t+h)A}&(u_0,g)-e^{-tA}(u_0,g)\\ &= e^{-\lambda tA}(u_0,g)-e^{-tA}(u_0,g)\\ &= \lambda^{\frac{1}{1-\alpha}}e^{-tA}\left[\lambda^{\frac{1}{\alpha-1}}u_0,\lambda^{\frac{\alpha}{\alpha-1}}g(\lambda\cdot)\right]-e^{-tA}(u_0,g) \end{aligned} \tag{5.19}$$

and so,

$$\begin{aligned} e^{-(t+h)A}&(u_0,g)-e^{-tA}(u_0,g)\\ &= \lambda^{\frac{1}{1-\alpha}}\left[e^{-tA}\left[\lambda^{\frac{1}{\alpha-1}}u_0,\lambda^{\frac{\alpha}{\alpha-1}}g(\lambda\cdot)\right]-e^{-tA}(u_0,g(\lambda\cdot)\right]\\ &\quad+\lambda^{\frac{1}{1-\alpha}}\left[e^{-tA}\left[u_0,g(\lambda\cdot)\right]-e^{-tA}(u_0,g)\right]\\ &\quad+\left[\lambda^{\frac{1}{1-\alpha}}-1\right]e^{-tA}(u_0,g). \end{aligned} \tag{5.20}$$

Applying to this (5.13) and by using (5.11), one sees that

$$\|e^{-(t+h)A}(u_0,g)-e^{-tA}(u_0,g)\|_X$$

$$\begin{aligned}
&\leq \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} \left\| e^{-tA}\left[\lambda^{\frac{1}{\alpha-1}}u_0, \lambda^{\frac{\alpha}{\alpha-1}}g(\lambda\cdot)\right] - e^{-tA}(u_0, g(\lambda\cdot)\right\|_X \\
&\qquad + \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} \left\| e^{-tA}\left[u_0, g(\lambda\cdot)\right] - e^{-tA}(u_0, g)\right\|_X \\
&\qquad\qquad + \left| \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} - 1\right| \|e^{-tA}(u_0,g)\|_X \\
&\leq \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L e^{\omega t} \left\| \left(1+\tfrac{h}{t}\right)^{\frac{1}{\alpha-1}} u_0 - u_0\right\|_X \\
&\quad + \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L \int_0^t e^{\omega(t-s)} \left\| (1+\tfrac{h}{t})^{\frac{\alpha}{\alpha-1}} g(s+\tfrac{h}{t}s) - g(s+\tfrac{h}{t}s)\right\|_X \mathrm{d}s \\
&\quad + \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L \int_0^t e^{\omega(t-s)} \|g(s+\tfrac{h}{t}s) - g(s)\|_X \,\mathrm{d}s \\
&\qquad + L e^{\omega t} \left| \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} - 1\right| \left(\|u_0\|_X + \int_0^t e^{-\omega s}\|g(s)\|_X \,\mathrm{d}s\right) \\
&= \left| \left(1+\tfrac{h}{t}\right) - \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} \right| L \int_0^t e^{\omega(t-s)} \|g(s+\tfrac{h}{t}s)\|_X \,\mathrm{d}s \\
&\qquad + \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L \int_0^t e^{\omega(t-s)} \|g(s+\tfrac{h}{t}s) - g(s)\|_X \,\mathrm{d}s \\
&\qquad\qquad + L e^{\omega t} \left| \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} - 1\right| \left(2\,\|u_0\|_X + \int_0^t e^{-\omega s}\|g(s)\|_X \,\mathrm{d}s\right),
\end{aligned}$$

which is (5.14). It is clear that (5.16)-(5.18) follow from (5.14). □

Examples of measurable X-valued functions $g : [0,T] \to X$ with finite $V_\omega(g,t)$ at a.e. t and integrable on $L^1(0,T)$, are functions of *bounded variation* (cf [40, Appendice, Section 2.]).

Definition 5.1 For a measurable X-valued function $g : [0,T] \to X$, one calls

$$\mathrm{Var}(g;[0,T]) := \sup\left\{ \sum_{i=1}^{N} \|f(t_i) - f(t_{i-1})\|_X \;\middle|\; \begin{array}{c}\text{all partitions :}\\ 0 = t_0 < \cdots < t_N = T\end{array}\right\}$$

the *total variation of* g. Each X-valued function $g : [0,T] \to X$ is said to have *bounded variation* on $[0,T]$ if $\mathrm{Var}(g;[0,T])$ is finite. We denote by $BV(0,T;X)$ the space of all functions $g : [0,T] \to X$ of bounded variation and to simplify the notation, we set $V_g(t) = \mathrm{Var}(g;[0,t])$ for $t \in (0,T]$.

Functions of bounded variation have the following properties.

Proposition 5.1 *Let $g \in BV(0,T;X)$. Then the following statements hold.*

1. *$g \in L^\infty(0,T;X)$;*
2. *At every $t \in [0,T]$, the left-hand side limit $g(t-) := \lim_{s\to t-} g(s)$ and right-hand side limit $g(t+) := \lim_{s\to t+} g(s)$ exist in X; and the set of discontinuity points in $[0,T]$ is at most countable;*

3. *The mapping $t \mapsto V_g(t)$ is monotonically increasing on $[0,T]$, and*

$$\|g(t)-g(s)\|_X \le V_g(t)-V_g(s) \qquad \textit{for all } 0 \le s \le t \le T; \tag{5.21}$$

4. *For $\omega \ge 0$, one has that*

$$\int_0^t e^{-\omega s}\frac{\|g(s+hs)-g(s)\|_X}{h}\,\mathrm{d}s \le t\,V_g(t) \qquad \textit{for all } h \in (0,t],\ 0<t\le T.$$

5. *For $\omega \ge 0$, let $V_\omega(g,t)$ be given by (5.15). Then $V_\omega(g,t)$ belongs to $L^\infty(0,T)$ satisfying*

$$V_\omega(g,t) \le t\,V_g(t) \qquad \textit{for all } t \in [0,T].$$

The first three statements are standard and can be found, for example, in [40, Section 2., Lemme A.1]. Thus, we only outline the proof of statement (4) and (5).

Proof Obviously, (5) follows from (4). Thus, it remains to show that for given $g \in BV(0,T;X)$, (4) holds. To see this, let $t \in (0,T)$, $h \in (0,t]$ such that $t+h \le T$. Then, by (5.21) and since $\omega \ge 0$,

$$\begin{aligned}\int_0^t e^{-\omega s}\frac{\|g(s+hs)-g(s)\|_X}{h}\,\mathrm{d}s &\le \tfrac{1}{h}\int_0^t e^{-\omega s}\Big(V_g((1+h)s)-V_g(s)\Big)\mathrm{d}s\\ &\le \tfrac{1}{h}\int_0^t \Big(V_g((1+h)s)-V_g(s)\Big)\mathrm{d}s.\end{aligned}$$

By using the substitution $r=(1+h)s$, we get

$$\begin{aligned}&\tfrac{1}{h}\int_0^t V_g((1+h)s)\,\mathrm{d}s - \tfrac{1}{h}\int_0^t V_g(s)\,\mathrm{d}s\\ &\quad = \tfrac{1}{h(1+h)}\int_0^{(1+h)t} V_g(r)\,\mathrm{d}r - \tfrac{1}{h}\int_0^t V_g(s)\,\mathrm{d}s \le \tfrac{1}{h}\int_t^{t+ht} V_g(s)\,\mathrm{d}s\end{aligned}$$

and by the monotonicity of $t \mapsto V_g(t)$,

$$\tfrac{1}{h}\int_t^{t+ht} V_g(s)\,\mathrm{d}s \le t\,V_g(t).$$

This shows that (4) holds. □

In the case $g \equiv 0$, we can take $T=\infty$. Then the mapping e^{-tA} given by (5.10) only depends on the initial value u_0. In other words,

$$e^{-tA}u_0 = e^{-tA}(u_0,0) \qquad \text{for every } u_0 \in C \text{ and } t \ge 0. \tag{5.22}$$

In this case, Lemma 5.1 reads as follows (cf [29]).

Corollary 5.1 *Let $\{e^{-tA}\}_{t\ge 0}$ be a family of mappings $e^{-tA}: C \to C$ defined on a subset $C \subseteq X$, and suppose there are $\omega \in \mathbb{R}$, $L \ge 1$, and $\alpha \ne 1$ such that $\{e^{-tA}\}_{t\ge 0}$ satisfies*

$$\|e^{-tA}u_0 - e^{-tA}\hat{u}_0\|_X \le L\, e^{\omega t}\, \|u_0 - \hat{u}_0\|_X \tag{5.23}$$

for every $t \ge 0$, u, $\hat{u} \in C$,

$$\lambda^{\frac{1}{\alpha-1}}\, T_{\lambda t} u_0 = e^{-tA}[\lambda^{\frac{1}{\alpha-1}} u_0] \tag{5.24}$$

for every $\lambda > 0$, $t \ge 0$ and $u_0 \in C$. Further, suppose $e^{-tA}0 \equiv 0$ for all $t \ge 0$. Then, for every $u_0 \in C$,

$$\|e^{-(t+h)A}u_0 - e^{-tA}u_0\|_X \le 2\,L \left|1 - \left(1 + \tfrac{h}{t}\right)^{\frac{1}{1-\alpha}}\right| e^{\omega t} \|u_0\|_X. \tag{5.25}$$

$t > 0$, $h \ne 0$ satisfying $1 + \frac{h}{t} > 0$. In particular, the family $\{e^{-tA}\}_{t\ge 0}$ satisfies

$$\limsup_{h\to 0+} \frac{\|e^{-(t+h)A}u_0 - e^{-tA}u_0\|_X}{h} \le \frac{2Le^{\omega t}}{|1-\alpha|}\,\frac{\|u_0\|_X}{t} \tag{5.26}$$

for every $t > 0$, $u_0 \in C$. Moreover, if for $u_0 \in C$, the right-hand side derivative

$$\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}_{+} \quad \text{exists (in } X\text{) at } t > 0,$$

then one has that

$$\left\|\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t_+}\right\|_X \le \frac{2\,L\,e^{\omega t}}{|1-\alpha|}\,\frac{\|u_0\|_X}{t}. \tag{5.27}$$

NextFinally, we turn to the perturbed inhomogeneous Cauchy problem (in X)

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) + F(u(t)) \ni g(t) & \text{on } (0,T), \\ u(0) = u_0, \end{cases} \tag{5.28}$$

for given $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0,T;X)$, involving a homogenous operator A in X of order $\alpha \ne 1$, and a Lipschitz continuous perturbation $F : X \to X$ with Lipschitz constant $\omega \ge 0$ satisfying $F(0) = 0$. We assume that perturbed inhomogeneous Cauchy problem (5.28) is well-posed in X in the sense that for every $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0,T;X)$, there is a unique function $u \in C([0,T];X)$ satisfying $u(0) = u_0$ in X and by the relation (5.10) a semigroup $\{e^{-tA}\}_{t=0}^T$ is generated of mappings $e^{-tA} : \overline{D(A)}^X \times L^1(0,T;X) \to \overline{D(A)}^X$ satisfying (5.13) for every $0 \le s < t \le T$.

One important idea to obtain global L^1 Aronson-Bénilan type estimates for the (mild) solutions u of the perturbed inhomogeneous Cauchy problem (5.28) is the assumption that for given $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0,T;X)$, the unique solution u given by (5.10) is also the unique solution of the *unperturbed* inhomogeneous Cauchy problem (5.7) for $\tilde{g} : [0,T] \to X$ given by

$$\tilde{g}(t) := g(t) - F(e^{-tA}(u_0, g)), \qquad (t \in [0,T]). \tag{5.29}$$

This property can be expressed by

$$\tilde{T}_t(u_0,\tilde{g}) = e^{-tA}(u_0,g) \qquad \text{holds for every } t\in[0,T], \tag{5.30}$$

where $\{\tilde{T}_t\}_{t=0}^T$ denotes the semigroup associated with (5.7). The advantage of (5.30) is that one can employ inequality (5.13) satisfied by the family $\{\tilde{T}_t(\cdot,\tilde{g})\}_{t\geq 0}$. Thus, by Lemma 5.1, the following estimate holds.

Theorem 5.2 *Let $F : X \to X$ be a Lipschitz continuous mapping with Lipschitz constant $\omega_F \geq 0$ satisfying $F(0) = 0$. Given $T > 0$ and a subset $C \subseteq X$, assume there are families $\{e^{-tA}\}_{t=0}^T$ and $\{\tilde{T}_t\}_{t=0}^T$ of mappings e^{-tA}, $\tilde{T}_t : C\times L^1(0,T;X) \to C$ satisfying (5.11) and related through (5.30) for every $u_0 \in C$ and $g \in L^1(0,T;X)$ with $\tilde{g}$ given by (5.29). Further suppose, $\{\tilde{T}_t\}_{t=0}^T$ satisfies (5.12) and (5.13) for some $\omega \geq 0$ and $L \geq 1$, and $\{e^{-tA}\}_{t=0}^T$ satisfies (5.13) with $\tilde{\omega} = \omega+\omega_F$ and L.*

Then, if for $u_0 \in C$ and $g \in BV(0,T;X)$, the function $t \mapsto e^{-tA}(u_0,g)$ is locally Lipschitz continuous on $[0,T)$, then one has that

$$\begin{aligned}\limsup_{h\to 0+} &\frac{\|e^{-(t+h)A}(u_0,g) - e^{-tA}(u_0,g)\|}{h}\\ &\leq \frac{e^{\omega t}}{t}\left[a(t) + L\omega_F\int_0^t a(s)e^{L\,\omega_F(t-s)}\mathrm{d}s\right]\end{aligned} \tag{5.31}$$

for a.e. $t \in (0,T)$, where

$$\begin{aligned}a(t) := L\,V_\omega(g,t) + \frac{L}{|1-\alpha|}&\left[\left(2+\omega_F\,L\int_0^t e^{\omega_F s}\mathrm{d}s\right)\|u_0\|_X\right.\\ &\left.+\int_0^t e^{-\omega s}\|g(s)\|_X\,\mathrm{d}s + \omega_F\,L\int_0^t\int_0^s e^{-\omega_F r}\|g(r)\|_X\mathrm{d}r\,\mathrm{d}s\right].\end{aligned} \tag{5.32}$$

and $V_\omega(g,\cdot)$ is given by (5.15).

For the proof of this theorem, we still need the following version of Gronwall's lemma.

Lemma 5.2 ([134, Lemma D.2]) *Suppose $v \in L^1_{loc}([0,T))$ satisfies*

$$v(t) \leq a(t) + \int_0^t v(s)\,b(s)\mathrm{d}s \qquad \text{for a.e. } t\in(0,T), \tag{5.33}$$

where $b \in C([0,T))$ satisfying $b(t) \geq 0$, and $a \in L^1_{loc}([0,T))$. Then,

$$v(t) \leq a(t) + \int_0^t a(s)\,b(s)\,e^{\int_s^t b(r)\mathrm{d}r}\,\mathrm{d}s \qquad \text{for a.e. } t\in(0,T). \tag{5.34}$$

We are now ready to give the proof of Theorem 5.2.

Proof (Proof of Theorem 5.2) Let $u_0 \in C$ and $g \in BV(0,T;X)$. Fix $t > 0$, and let $h > 0$ such that $t+h < T$. Then, by the assumption that there is a family $\{\tilde{T}_t\}_{t=0}^T$ of mappings $\tilde{T}_t$ satisfying (5.30) for every $u_0 \in C$ and $g \in L^1(0,T;X)$ with $\tilde{g}$ given by (5.29), and $\{\tilde{T}_t\}_{t=0}^T$ satisfies (5.11)-(5.13) for some $\omega \geq 0$, L, we can apply Lemma 5.1 to $\tilde{T}_t(u_0, \tilde{g})$. Then by (5.14), since $\tilde{g}$ is given by (5.29), by (5.30), and by the triangle inequality,

$$\begin{aligned}
&\|e^{-(t+h)A}(u_0,g) - e^{-tA}(u_0,g)\|_X \\
&= \|\tilde{T}_{t+h}(u_0,\tilde{g}) - \tilde{T}_t(u_0,\tilde{g})\|_X \\
&\leq \left|\left(1+\tfrac{h}{t}\right) - \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}}\right| L \int_0^t e^{\omega(t-s)} \|g(s+\tfrac{h}{t}s) - F(e^{-(s+\frac{h}{t}s)A}(u_0,g))\|_X \,\mathrm{d}s \\
&+ \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L \int_0^t e^{\omega(t-s)} \|g(s+\tfrac{h}{t}s) - g(s)\|_X \,\mathrm{d}s \\
&+ \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L \int_0^t e^{\omega(t-s)} \|F(e^{-(s+\frac{h}{t}s)A}(u_0,g)) - F(e^{-sA}(u_0,g))\|_X \,\mathrm{d}s \\
&+ L e^{\omega t} \left|\left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} - 1\right| \left[2\,\|u_0\|_X + \int_0^t e^{-\omega s}\left[\|g(s)\|_X + \|F(e^{-sA}(u_0,g))\|_X\right] \mathrm{d}s\right]
\end{aligned}$$

Since F is globally Lipschitz continuous with constant ω_F, $F(0) = 0$, and since $\{e^{-tA}\}_{t=0}^T$ satisfies (5.11) and (5.13) with $\tilde{\omega} = \omega + \omega_F$ and L, one has that

$$\|F(e^{-sA}(u_0,g))\|_X \leq \omega_F L \left[e^{\tilde{\omega}s}\|u_0\|_X + \int_0^s e^{\tilde{\omega}(s-r)}\|g(r)\|_X \,\mathrm{d}r\right].$$

We apply this to the last integral on the right-hand side of the previous estimate, and substitute $y = (1+h/t)s$ into the first integral on the right-hand side of the previous estimate. Then, dividing by $h > 0$ both sides in the resulting inequality yields that

$$\begin{aligned}
&\frac{\|e^{-(t+h)A}(u_0,g) - e^{-tA}(u_0,g)\|_X}{h} \\
&\leq \left|\frac{(1+\frac{h}{t}) - (1+\frac{h}{t})^{\frac{1}{1-\alpha}}}{\frac{1}{t}h}\right| \frac{1+\frac{h}{t}}{t} L e^{\omega t} \times \\
&\qquad\times \int_0^{t+h} e^{-\frac{\omega}{1+h/t}y} \|g(y) - F(e^{-yA}(u_0,g))\|_X \,\mathrm{d}y \\
&\quad+ \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L \tfrac{e^{\omega t}}{t} \int_0^t e^{-\omega s} \frac{\|g(s+\frac{h}{t}s) - g(s)\|_X}{\frac{h}{t}} \,\mathrm{d}s \\
&\quad+ \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L e^{\omega t}\omega_F \int_0^t e^{-\omega s} \frac{\|e^{-(s+\frac{h}{t}s)A}(u_0,g) - e^{-sA}(u_0,g)\|_X}{\frac{s}{t}h} \tfrac{s}{t} \,\mathrm{d}s \\
&\quad+ \tfrac{Le^{\omega t}}{t} \left|\frac{(1+\frac{h}{t})^{\frac{1}{1-\alpha}} - 1}{\frac{1}{t}h}\right| \left[\left(2 + \omega_F L \int_0^t e^{\omega_F s}\mathrm{d}s\right) \|u_0\|_X \right. \\
&\qquad\left. + \int_0^t e^{-\omega s}\|g(s)\|_X \,\mathrm{d}s + \omega_F L \int_0^t \int_0^s e^{-\omega_F r}\|g(r)\|_X \mathrm{d}r\,\mathrm{d}s\right],
\end{aligned} \tag{5.35}$$

where we use twice that $e^{-\omega s}e^{\tilde{\omega}s} = e^{\omega_F s}$. Note that

$$\limsup_{h\to 0+}\int_0^t e^{-\omega s}\frac{\|g(s+\frac{h}{t}s)-g(s)\|_X}{\frac{h}{t}}\,\mathrm{d}s = V_\omega(g,t)$$

and by Proposition 5.1, one has that $V_\omega(g,\cdot)\in L^\infty([0,T))$. Since $t\mapsto e^{-tA}(u_0,g)$ is locally Lipschitz continuous on $[0,T)$, for every $\varepsilon\in(0,T)$ there is a constant $C_\varepsilon>0$ such that

$$\frac{\left\|e^{-(s+\frac{h}{t}s)A}(u_0,g)-e^{-sA}(u_0,g)\right\|_X}{\frac{s}{t}h}\le C$$

for every $s\in[0,T-\varepsilon]$ and $h>0$ satisfying $s+\frac{h}{t}s<T-\varepsilon$. Thus, by the reverse version of Fatou's lemma, taking in (5.35) the limit-superior as $h\to 0+$ gives

$$\begin{aligned}
&e^{-\omega t}t\limsup_{h\to 0+}\frac{\|e^{-(t+h)A}(u_0,g)-e^{-tA}(u_0,g)\|}{h}\\
&\quad\le LV_\omega(g,t)+L\omega_F\int_0^t e^{-\omega s}s\left[\limsup_{h\to 0+}\frac{\|e^{-(s+h)A}(u_0,g)-e^{-sA}(u_0,g)\|}{h}\right]\mathrm{d}s\\
&\qquad+\frac{L}{|1-\alpha|}\Bigg[\left(2+\omega_F L\int_0^t e^{\omega_F s}\mathrm{d}s\right)\|u_0\|_X\\
&\qquad\qquad+\int_0^t e^{-\omega s}\|g(s)\|_X\,\mathrm{d}s+\omega_F L\int_0^t\int_0^s e^{-\omega_F r}\|g(r)\|_X\mathrm{d}r\,\mathrm{d}s\Bigg].
\end{aligned}$$

Now, applying Gronwall's lemma (Lemma 5.2) to $a(t)$ given by (5.32),

$$\begin{aligned}
&b(t)\equiv L\omega_F,\ \text{and}\\
&v(t)=e^{-\omega t}t\limsup_{h\to 0+}\frac{\|e^{-(t+h)A}(u_0,g)-e^{-tA}(u_0,g)\|}{h},
\end{aligned}$$

then one obtains (5.31). This completes the proof. □

Next, we intend to extrapolate the regularity estimate (4.45) for $f\equiv 0$.

Corollary 5.2 *Let $\{e^{-tA}\}_{t\ge 0}$ be a semigroup of mappings $e^{-tA}: C\to C$ defined on a subset $C\subseteq X$ and suppose, there is a second vector space Y with semi-norm $\|\cdot\|_Y$ and constants M, γ, $\delta>0$ and $\hat{\omega}\in\mathbb{R}$ such that $\{e^{-tA}\}_{t\ge 0}$ satisfies the following Y-X-regularity estimate*

$$\|e^{-tA}u_0\|_X\le M\,e^{\hat{\omega}t}\frac{\|u_0\|_Y^\gamma}{t^\delta}\qquad\text{for every } t>0 \text{ and } u_0\in C\cap Y. \tag{5.36}$$

If for $\alpha\neq 1$, ω, $\omega_F\in\mathbb{R}$ and $L\ge 1$, $\{e^{-tA}\}_{t\ge 0}$ satisfies

$$\limsup_{h\to 0+} \frac{\|e^{-(t+h)A}u_0 - e^{-tA}u_0\|_X}{h} \le \frac{e^{\omega t}}{t}\,\frac{L}{|1-\alpha|}\left[b(t) + L\omega_F \int_0^t b(s)\, e^{L\,\omega_F(t-s)}\mathrm{d}s\right]\|u_0\|_X \tag{5.37}$$

for a.e. $t > 0$ *and* $u_0 \in C$*, with* $b(t) := 2 + \omega_F\, L \int_0^t e^{\omega_F s}\mathrm{d}s$*, then*

$$\limsup_{h\to 0+} \frac{\|e^{-(t+h)A}u_0 - e^{-tA}u_0\|_X}{h} \le \frac{2^{\delta+1}\, e^{\frac{\omega+\hat{\omega}}{2}t}}{t^{\delta+1}}\,\frac{L\,M}{|1-\alpha|}\left[b(\tfrac{t}{2}) + L\omega_F \int_0^{\frac{t}{2}} b(s)\, e^{L\,\omega_F(\frac{t}{2}-s)}\mathrm{d}s\right]\|u_0\|_Y^{\gamma}. \tag{5.38}$$

In particular, if the right-hand side derivative $\frac{\mathrm{d}}{\mathrm{d}t_+}e^{-tA}u_0$ *exists (in* X*) at* $t > 0$*, then*

$$\left\|\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t_+}\right\|_X \le \frac{2^{\delta+1}\, e^{\frac{\omega+\hat{\omega}}{2}t}}{t^{\delta+1}}\,\frac{L\,M}{|1-\alpha|}\left[b(\tfrac{t}{2}) + L\omega_F \int_0^{\frac{t}{2}} b(s)\, e^{L\,\omega_F(\frac{t}{2}-s)}\mathrm{d}s\right]\|u_0\|_Y^{\gamma}.$$

Proof Let $u_0 \in C$ and $t > 0$. Note, if $u_0 \notin Y$ then (5.38) trivially holds. Thus, it is sufficient to consider the case $u_0 \in C \cap Y$. By the semigroup property of $\{e^{-tA}\}_{t\ge 0}$ and by (5.37) and (5.36), one sees that

$$\begin{aligned}
&\limsup_{h\to 0+} \frac{\|e^{-(t+h)A}u_0 - e^{-tA}u_0\|_X}{h}\\
&= \limsup_{h\to 0+} \frac{\|e^{-(\frac{t}{2}+h)A}(e^{-\frac{t}{2}A}u_0) - e^{-\frac{t}{2}A}(e^{-\frac{t}{2}A}u_0)\|_X}{h}\\
&\le \frac{2\,e^{\omega\frac{t}{2}}}{t}\,\frac{L}{|1-\alpha|}\left[b(\tfrac{t}{2}) + L\omega_F \int_0^{\frac{t}{2}} b(s)\, e^{L\,\omega_F(\frac{t}{2}-s)}\mathrm{d}s\right]\|e^{-\frac{t}{2}A}u_0\|_X\\
&\le \frac{2^{\delta+1}\, e^{\frac{\omega+\hat{\omega}}{2}t}}{t^{\delta+1}}\,\frac{L\,M}{|1-\alpha|}\left[b(\tfrac{t}{2}) + L\omega_F \int_0^{\frac{t}{2}} b(s)\, e^{L\,\omega_F(\frac{t}{2}-s)}\mathrm{d}s\right]\|u_0\|_Y^{\gamma}.
\end{aligned}$$

Now, we suppose, there is a partial ordering "$\le$" on X such that $(X, \le)$ is an ordered vector space. Then, we can state the following theorem.

Theorem 5.3 *Let* $(X, \le)$ *be an ordered vector space and* $F : X \to X$ *a Lipschitz continuous mapping satisfying* $F(0) = 0$*. Suppose, there is a subset* $C \subseteq X$ *and two families* $\{e^{-tA}\}_{t\ge 0}$ *and* $\{\tilde{T}_t\}_{t\ge 0}$ *of mappings* $e^{-tA} : C \to C$ *and* $\tilde{T}_t : C \times L^1_{loc}([0,\infty); X) \to C$ *related by the equation*

$$e^{-tA}u_0 = \tilde{T}_t(u_0, \tilde{g}) \qquad \text{for all } t \ge 0,\ u_0 \in C, \tag{5.39}$$

where $\tilde{g}$ *is given by* $\tilde{g}(t) = -F(e^{-tA}u_0)$*. Further, suppose*

for every u_0, $\hat{u}_0 \in C$ satisfying $u_0 \le \hat{u}_0$, one has $e^{-tA}u_0 \le e^{-tA}\hat{u}_0$ for all $t \ge 0$
(5.40)

and $\{\tilde{T}_t\}_{t\ge 0}$ satisfies (5.11)-(5.13) for some $\omega \ge 0$ and $L \ge 1$. Then for every $u_0 \in C$ satisfying $u_0 \ge 0$, one has that

$$\frac{e^{-(t+h)A}u_0 - e^{-tA}u_0}{h} \ge \frac{(1+\frac{h}{t})^{\frac{1}{1-\alpha}} - 1}{h}\,\frac{e^{-tA}u_0}{t} + g_h(t) \tag{5.41}$$

for every t, $h > 0$ if $\alpha > 1$ and

$$\frac{e^{-(t+h)A}u_0 - e^{-tA}u_0}{h} \le \frac{(1+\frac{h}{t})^{\frac{1}{1-\alpha}} - 1}{h}\,\frac{e^{-tA}u_0}{t} + g_h(t) \tag{5.42}$$

for every t, $h > 0$ if $\alpha < 1$, where for every $h > 0$, $g_h : (0,\infty) \to X$ is a continuous function satisfies

$$\begin{aligned}\|g_h(t)\|_X \le \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L\times& \\ \times \int_0^t e^{\omega(t-r)} &\left\| \frac{F(e^{-rA}u_0) - \left(1+\frac{h}{t}\right)^{\frac{\alpha}{\alpha-1}} F(e^{-(r+\frac{h}{t}r)A}u_0)}{h} \right\|_X \mathrm{d}r\end{aligned} \tag{5.43}$$

for every $t > 0$.

Now, we turn to the proof of Theorem 5.3.

Proof (Proof of Theorem 5.3) First, let $\{\tilde{T}_t\}_{t\ge 0}$ be the family of operators related to $\{e^{-tA}\}_{t\ge 0}$ by (5.39), and for t, $h > 0$, let $\lambda := \left(1+\frac{h}{t}\right)$. Since $\lambda > 1$, $\lambda^{\frac{1}{\alpha-1}}u_0 \le u_0$ if $\alpha < 1$ and $\lambda^{\frac{1}{\alpha-1}}u_0 \ge u_0$ if $\alpha > 1$. Thus, if $\alpha < 1$, then by (5.19) and (5.40), one has that

$$\begin{aligned}\tilde{T}_{t+h}(u_0,\tilde{g}) - \tilde{T}_t(u_0,\tilde{g}) &= \lambda^{\frac{1}{1-\alpha}}\tilde{T}_t\left[\lambda^{\frac{1}{\alpha-1}}u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{g}(\lambda\cdot)\right] - \tilde{T}_t(u_0,\tilde{g}) \\ &= \lambda^{\frac{1}{1-\alpha}}\left[\tilde{T}_t\left[\lambda^{\frac{1}{\alpha-1}}u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{g}(\lambda\cdot)\right] - \tilde{T}_t\left[u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{g}(\lambda\cdot)\right]\right] \\ &\qquad + \lambda^{\frac{1}{1-\alpha}}\tilde{T}_t\left[u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{g}(\lambda\cdot)\right] - \tilde{T}_t(u_0,\tilde{g}) \\ &\le \lambda^{\frac{1}{1-\alpha}}\left[\tilde{T}_t\left[u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{g}(\lambda\cdot)\right] - \tilde{T}_t\left[u_0,\tilde{g}\right]\right] \\ &\qquad + \left[\lambda^{\frac{1}{1-\alpha}} - 1\right]\tilde{T}_t(u_0,\tilde{g})\end{aligned}$$

and, similarly, if $\alpha > 1$, then

$$\begin{aligned}\tilde{T}_{t+h}(u_0,\tilde{g}) - \tilde{T}_t(u_0,\tilde{g}) \ge \lambda^{\frac{1}{1-\alpha}}&\left[\tilde{T}_t\left[u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{g}(\lambda\cdot)\right] - \tilde{T}_t\left[u_0,\tilde{g}\right]\right] \\ &+ \left[\lambda^{\frac{1}{1-\alpha}} - 1\right]\tilde{T}_t(u_0,\tilde{g}).\end{aligned}$$

Now, by replacing $\tilde{g}(t)$ by $-F(e^{-tA}u_0)$ and by (5.30), we can rewrite the above two inequalities and arrive to (5.41) and (5.42), where $g_h(t)$ is given by

$$g_h(t) = \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} \frac{\tilde{T}_t\left[u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{g}(\lambda\cdot)\right] - \tilde{T}_t\left[u_0, \tilde{g}\right]}{h}.$$

Note, by (5.13), one has that g_h satisfies (5.43). □

By Theorem 5.3, if the derivative $\frac{\mathrm{d}}{\mathrm{d}t_+}e^{-tA}u_0$ belongs to $L^1_{loc}(0,T;X)$ for $T>0$, then we can state the following.

Corollary 5.3 *Under the hypotheses of Theorem 5.3, suppose that for $u_0 \in C$ satisfying $u_0 \geq 0$, the right hand-side derivative $\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}_+ \in L^1_{loc}([0,T);X)$ for some $T>0$. Then, one has that*

$$(\alpha-1)\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}_+ \geq -\frac{e^{-tA}u_0}{t} + (\alpha-1)g_0(t),$$

for a.e. $t\in(0,T)$, where $g_0 : (0,T) \to X$ is a measurable function satisfying

$$\|g_0(t)\|_X \leq \frac{L}{t}\int_0^t e^{\omega(t-r)}\left[\omega\left\|\frac{\mathrm{d}e^{-rA}u_0}{\mathrm{d}r}_+\right\|_X + \frac{|\alpha|}{|\alpha-1|}\|T_r u\|_X\right]\mathrm{d}r \tag{5.44}$$

for a.e. $t\in(0,T)$.

5.2.2 Homogeneous accretive operators

As for classical solutions, the fact that A is homogeneous of order $\alpha\neq 1$, is also reflected in the notion of mild solution and, in particular, in the semigroup $\{e^{-tA}\}_{t=0}^T$ as demonstrated in our next proposition.

Proposition 5.2 (Homogeneous accretive operators) *Let A be a quasi m-accretive operator on X and $\{e^{-tA}\}_{t=0}^T$ the semigroup generated by $-A$ on $\overline{D(A)}^X \times L^1(0,T;X)$. If A is homogeneous of order $\alpha\neq 1$, then for every $\lambda>0$, $\{e^{-tA}\}_{t=0}^T$ satisfies (5.12) for every $(u_0,g)\in \overline{D(A)}^X\times L^1(0,T;X)$.*

Proof Let $\lambda>0$, $g\in X$, and for $\mu>0$, denote by J_μ^{A+g} the resolvent operator $(I_X+\mu(A+g))^{-1}$ of $A+g$. Then, for every u, $v\in X$ and $\mu>0$, one has that

$$J_\mu^{A-\lambda^{\frac{\alpha}{\alpha-1}}g}\left[\lambda^{\frac{1}{\alpha-1}}v\right] = u \qquad \text{if and only if} \qquad u+\mu(Au-\lambda^{\frac{\alpha}{\alpha-1}}g)\ni\lambda^{\frac{1}{\alpha-1}}v.$$

Now, the hypothesis that A is *homogeneous of order* $\alpha\neq 1$ implies that the right-hand side in the previous characterization is equivalent to

$$\lambda^{\frac{1}{1-\alpha}}u+\lambda\mu(A(\lambda^{\frac{1}{1-\alpha}}u)-f)\ni v, \qquad \text{or} \qquad J_{\lambda\mu}^{A-f}v=\lambda^{\frac{1}{1-\alpha}}u.$$

Therefore, one has that

$$\lambda^{\frac{1}{\alpha-1}} J_{\lambda\mu}^{A-f} v = J_{\mu}^{A-\lambda^{\frac{\alpha}{\alpha-1}} f}\left[\lambda^{\frac{1}{\alpha-1}} v\right] \qquad \text{for all } \lambda, \mu > 0, \text{ and } v \in X. \tag{5.45}$$

Now, let $u_0 \in \overline{D(A)}^X$, $\pi : 0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0,T]$, and $g \in L^1(0,T;X)$ a step function given by

$$g = \sum_{i=1}^{N} g_i \, \mathbb{1}_{(t_{i-1},t_i]}. \tag{5.46}$$

Further, let u be the unique mild solution of the inhomogeneous Cauchy problem (5.7) with g given by (5.46). Then u is also a step function given by

$$u(t) = u_0 \, \mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^{N} u_i(t) \, \mathbb{1}_{(t_{i-1},t_i]}(t),$$

where on each subinterval $(t_{i-1},t_i]$, u_i is the unique mild solution of

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} + A(u_i(t)) \ni g_i \quad \text{on } (t_{i-1},t_i), \text{ and} \quad u_i(t_{i-1}) = u_{i-1}(t_{i-1})$$

for every $i = 1,\dots,N$ (cf [30, Chapter 4.3]). Next, let $\lambda > 0$ and set

$$w_\lambda(t) := \lambda^{\frac{1}{\alpha-1}} u(\lambda t) \qquad \text{for every } t \in \left[0, \tfrac{T}{\lambda}\right].$$

Then,

$$w_\lambda(t) = \lambda^{\frac{1}{\alpha-1}} u_0 \, \mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^{N} \lambda^{\frac{1}{\alpha-1}} u_i(\lambda t) \mathbb{1}_{\left(\frac{t_{i-1}}{\lambda}, \frac{t_i}{\lambda}\right]}(t)$$

for every $t \in \left[0, \frac{T}{\lambda}\right]$. Obviously, $w_\lambda(0) = \lambda^{\frac{1}{\alpha-1}} u_0$. Thus, to show that (5.12) holds, it remains to verify that w_λ is a mild solution of

$$\tfrac{\mathrm{d}w_\lambda}{\mathrm{d}t} + A(w_\lambda(t)) \ni \lambda^{\frac{\alpha}{\alpha-1}} g(\lambda t)$$

on $\left(0, \frac{T}{\lambda}\right)$. Or, in other words,

$$w_\lambda(t) = e^{-tA}(\lambda^{\frac{1}{\alpha-1}} u_0, \lambda^{\frac{\alpha}{\alpha-1}} g(\lambda \cdot)) \tag{5.47}$$

for every $t \in \left[0, \frac{T}{\lambda}\right]$. Let $t \in (0, t_1/\lambda]$ and $n \in \mathbb{N}$. We apply (5.45) to

$$\mu = \frac{t}{n} \qquad \text{and} \qquad v = J_{\frac{\lambda t}{n}}^{A-\lambda^{\frac{\alpha}{\alpha-1}} g_1}[\lambda^{\frac{1}{\alpha-1}} u_0].$$

Then, one finds that

$$\left[J_{\frac{t}{n}}^{A-\lambda^{\frac{\alpha}{\alpha-1}}g_1}\right]^2[\lambda^{\frac{1}{\alpha-1}}u_0] = J_{\frac{t}{n}}^{A-\lambda^{\frac{\alpha}{\alpha-1}}g_1}\left[\lambda^{\frac{1}{\alpha-1}}J_{\frac{\lambda t}{n}}^{A-g_1}u_0\right] = \lambda^{\frac{1}{\alpha-1}}\left[J_{\frac{\lambda t}{n}}^{A-g_1}\right]^2 u_0.$$

Applying (5.45) to $\lambda^{\frac{1}{\alpha-1}}\left[J_{\frac{\lambda t}{n}}^{A-g_1}\right]^i u_0$ iteratively for $i=2,\dots,n$ yields

$$\lambda^{\frac{1}{\alpha-1}}\left[J_{\frac{\lambda t}{n}}^{A-g_1}\right]^n u_0 = \left[J_{\frac{t}{n}}^{A-\lambda^{\frac{\alpha}{\alpha-1}}g_1}\right]^n\left[\lambda^{\frac{1}{\alpha-1}}u_0\right]. \tag{5.48}$$

Since the semigroup $\{e^{-tA}\}_{t=0}^T$ is obtained by the *exponential formula*

$$e^{-tA}(u(t_{i-1}),g_i) = u_i(t) = \lim_{n\to\infty}\left[J_{\frac{t-t_{i-1}}{n}}^{A-g_i}\right]^n u(t_{i-1}) \qquad \text{in } C([t_{i-1},t_i];X)$$

holding iteratively for every $i=1,\dots,N$, where for $\mu>0$, $J_\mu^{A-g_i} = (I+\mu(A-g_i))^{-1}$ is the *resolvent operator* of $A-g_i$, sending $n\to+\infty$ in (5.48) yields on the one side

$$\lim_{n\to+\infty}\lambda^{\frac{1}{\alpha-1}}\left[J_{\frac{\lambda t}{n}}^{A-g_1}\right]^n u_0 = \lambda^{\frac{1}{\alpha-1}}u_1(\lambda t) = w_\lambda(t),$$

and on the other side

$$\lim_{n\to+\infty}\left[J_{\frac{t}{n}}^{A-\lambda^{\frac{\alpha}{\alpha-1}}g_1}\right]^n\left[\lambda^{\frac{1}{\alpha-1}}u_0\right] = e^{-tA}(\lambda^{\frac{1}{\alpha-1}}u_0,\lambda^{\frac{\alpha}{\alpha-1}}g_1),$$

showing that (5.47) holds for every $t\in[0,\frac{t_1}{\lambda}]$. Repeating this argument on each subinterval $(\frac{t_{i-1}}{\lambda},\frac{t_i}{\lambda}]$ for $i=2,\dots,N$, where one replaces in (5.48) u_0 by $u(t_{i-1})$, and g_1 by g_i, then one sees that w_λ satisfies (5.47) on the whole interval $[0,\frac{T}{\lambda}]$. □

By the preceding proposition and by Lemma 5.1, we can now state the following result.

Corollary 5.4 *Let A be a quasi m-accretive operator on a Banach space X and $\{e^{-tA}\}_{t=0}^T$ the semigroup generated by $-A$ on $L^1(0,T;X)\times\overline{D(A)}^X$. If A is homogeneous of order $\alpha\neq 1$, then for every $(u_0,g)\in\overline{D(A)}^X\times L^1(0,T;X)$, $t\mapsto e^{-tA}(u_0,g)$ satisfies*

$$\begin{aligned}\limsup_{h\to0+}&\left\|\frac{e^{-(t+h)A}(u_0,g)-e^{-tA}(u_0,g)}{h}\right\|_X\\ &\le\frac{1}{t}e^{\omega t}\left[2\frac{\|u_0\|_X}{|1-\alpha|}+\frac{1}{|1-\alpha|}\int_0^t e^{-\omega s}\|g(s)\|_X\,\mathrm{d}s+V_\omega(g,t)\right],\end{aligned}\tag{5.49}$$

for a.e. $t\in(0,T]$, where $V_\omega(g,t)$ is defined by (5.15). In particular, if $g\in W^{1,1}(0,T;X)$ and $\frac{\mathrm{d}}{\mathrm{d}t}e^{-tA}(u_0,g)$ exists in X at a.e. $t\in(0,T)$, then $e^{-tA}(u_0,g)$ satisfies

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t_+}e^{-tA}(u_0,g)\right\|_X \le \frac{L}{t}\,e^{\omega t}\left[2\frac{\|u_0\|_X}{|1-\alpha|}+\frac{1}{|1-\alpha|}\int_0^t e^{-\omega s}\|g(s)\|_X\,\mathrm{d}s \right.$$
$$\left. +\int_0^t e^{-\omega s}\|g'(s)\|_X\,s\,\mathrm{d}s\right] \tag{5.50}$$

for a.e. $t\in(0,T)$.

The next theorem is our first main result in this section.

Theorem 5.4 (L^1 Aronson-Bénilan type estimates) *For given $\alpha\in\mathbb{R}\setminus\{1\}$, let A be an m-accretive operator in X which is homogeneous of order α and suppose, the mapping $F:X\to X$ is Lipschitz continuous on X with constant $\omega\ge 0$, $F(0)=0$, and let $g\in BV(0,T;X)$. Then for every $u_0\in D(A)$, the mild solution u of the perturbed inhomogeneous Cauchy problem (5.28) satisfies*

$$\limsup_{h\to 0+}\frac{\|u(t+h)-u(t)\|_X}{h}\le\frac{1}{t}\left[a_\omega(t)+\omega\int_0^t a_\omega(s)e^{\omega(t-s)}\mathrm{d}s\right] \tag{5.51}$$

for a.e. $t\in(0,T)$, where

$$a_\omega(t):=V_0(g,t)+\frac{1}{|1-\alpha|}\left[\left(1+e^{\omega t}\right)\|u_0\|_X\right.$$
$$\left.+\int_0^t\|g(s)\|_X\,\mathrm{d}s+\omega\int_0^t\int_0^s e^{-\omega r}\|g(r)\|_X\mathrm{d}r\,\mathrm{d}s\right]. \tag{5.52}$$

and $V_0(g,\cdot)$ is given by (5.15). In particular, if for $u_0\in D(A)$, the right-hand side derivative $\frac{\mathrm{d}u}{\mathrm{d}t+}$ exists, then

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}t_+}(t)\right\|_X\le\frac{1}{t}\left[a(t)+\omega\int_0^t a(s)e^{\omega(t-s)}\mathrm{d}s\right]\quad \textit{for a.e. } t\in(0,T). \tag{5.53}$$

Concerning the regularizing effect of mild solutions of the inhomogeneous Cauchy problem (5.7), we recall the following well-known result from the literature.

Lemma 5.3 ([30, Lemma 7.8]) *If $A+\omega I$ is accretive in X and $g\in BV(0,T;X)$, then for every $u_0\in D(A)$, the mild solution $u(t):=e^{-tA}(u_0,g)$, ($t\in[0,T]$), of the inhomogeneous Cauchy problem (5.7) is Lipschitz continuous on $[0,T]$ satisfying*

$$\limsup_{h\to 0+}\frac{\|u(t+h)-u(t)\|_X}{h}$$
$$\le e^{\omega t}\|g(0+)-y\|_X+\tilde V(g,t+)+\omega\int_0^t e^{\omega(t-s)}\tilde V(g,s+)\,\mathrm{d}s$$

for every $t\in[0,T]$ and $v\in Au_0$, where

$$\tilde V(g,t+):=\limsup_{h\to 0+}\int_0^t\frac{\|g(s+h)-g(s)\|_X}{h}\,\mathrm{d}s.$$

With the preceding Theorem 5.2, Proposition 5.2, and Lemma 5.3 in mind, we are in the position to outline the proof of our main Theorem 5.4.

Proof (Proof of Theorem 5.4) We begin by recalling that if A is m-accretive in X and F a Lipschitz continuous mapping with Lipschitz constant ω, then $A+F$ is ω-quasi m-accretive in X. Hence, for every $T>0$, there is a semigroup $\{e^{-tA}\}_{t=0}^{T}$ associated with the perturbed inhomogeneous Cauchy problem (5.28) of mappings $e^{-tA}:\overline{D(A)}^{X}\times L^1(0,T;X)\to\overline{D(A)}^{X}$ satisfying (5.11) and (5.13) with ω and $L=1$. Further, the semigroup $\{\tilde{T}_t\}_{t=0}^{T}$ generated by $-A$ satisfies (5.11) and (5.13) with $\omega=0$ and $L=1$, (5.30) for every $u_0\in\overline{D(A)}^{X}$ and $g\in L^1(0,T;X)$ with $\tilde{g}$ given by (5.29), and by Proposition 5.2, $\{\tilde{T}_t\}_{t=0}^{T}$ satisfies (5.12). Now, let $u_0\in D(A)$ and $g\in BV(0,T;X)$. Then by Lemma 5.3, the mild solution $u(t):=e^{-tA}(u_0,g)$, $(t\in[0,T])$, of the perturbed inhomogeneous Cauchy problem (5.28) is Lipschitz continuous on $[0,T]$. Thus, we can apply Theorem 5.2 and obtain that u satisfies (5.51). □

Note, in order that the semigroup $\{e^{-tA}\}_{t=0}^{T}$ generated by $-A$ satisfies regularity estimate (5.18) (respectively, (5.27)), one requires that every mild solution u of the inhomogeneous Cauchy problem (5.7) is *differentiable* at a.e. $t\in(0,T)$. In other words, u needs to be a *strong solution* of the inhomogeneous Cauchy problem (5.7) (see Definition 3.8 in Chapter 3.2.2.1).

Of course, every strong solution u of

$$\frac{du}{dt}(t)+A(u(t))\ni g(t)\qquad\text{for a.e. } t\in(0,T),\tag{5.54}$$

is a mild solution of (5.54), and u is absolutely continuous and differentiable with values in X at a.e. $t\in(0,T)$. The differential inclusion (5.54) admits mild and Lipschitz continuous solutions if A is ω-quasi m-accretive in X (cf [30, Lemma 7.8]). But, in general, an absolutely continuous functions $u:[0,T]\to X$ is not necessarily differentiable a.e. on $(0,T)$. Only if one assumes additional geometric properties on X, then the latter implication holds true. Our next definition is taken from [30, Definition 7.6] (cf [10, Chapter 1]).

Definition 5.2 A Banach space X is said to have the *Radon-Nikodým property* if every absolutely continuous function $F:[a,b]\to X$, $(a,b\in\mathbb{R}, a<b)$, is differentiable almost everywhere on (a,b).

Known examples of Banach spaces X admitting the Radon-Nikodým property are:

- **(Dunford-Pettis)** if $X=Y^*$ is separable, where Y^* is the dual space of a Banach space Y;
- if X is *reflexive*.

We emphasize that $X_1=L^1(\Sigma,\mu)$, $X_2=L^\infty(\Sigma,\mu)$, or $X_3=C(\mathcal{M})$ for a σ-finite measure space (Σ,μ), or respectively, for a compact metric space $(\mathcal{M},d)$ don't have, in general, the Radon-Nikodým property (cf [10]). Thus, it is even more surprising that there is a class of operators A (namely, the class of *completely* *accretive operators*, see Section 3.2.2.2 in Chapter 3), for which the inhomogeneous differential inclusion (5.54) admits strong solutions $u : [0,T] \to X$ (for $X = L^1(\Sigma,\mu)$ or $X = L^\psi(\Sigma,\mu)$).

Now, by Corollary 5.4 and Theorem 3.2 (in Chapter 3.2.2.1), we can conclude the following results. We emphasize that one crucial point in the statement of Corollary 5.5 below is that due to the uniform estimate (5.17), one has that every mild solution u of the inhomogeneous Cauchy problem (5.7) is strong.

Corollary 5.5 *Suppose A is a quasi m-accretive operator on a Banach space X admitting the Radon-Nikodým property, and $\{e^{-tA}\}_{t=0}^T$ is the semigroup generated by $-A$ on $\overline{D(A)}^X \times L^1(0,T;X)$. If A is homogeneous of order $\alpha \neq 1$, then for every $u_0 \in \overline{D(A)}^X$ and $g \in W^{1,1}(0,T;X)$, the unique mild solution $u(t) := e^{-tA}(u_0,g)$ of the inhomogeneous Cauchy problem (5.7) is strong and $\{e^{-tA}\}_{t=0}^T$ satisfies (5.50) for a.e. $t \in (0,T)$.*

We omit the proof of Corollary 5.5 since it is straightforward, and turn to the next corollary.

Corollary 5.6 *Let A be an m-accretive operator on a reflexive Banach space X, $F : X \to X$ a Lipschitz continuous mapping with Lipschitz-constant $\omega \geq 0$ satisfying $F(0) = 0$, and $\{e^{-tA}\}_{t=0}^T$ the semigroup generated by $-(A+F)$ on $\overline{D(A)}^X \times L^1(0,T;X)$. If A is homogeneous of order $\alpha \neq 1$, then for every $u_0 \in \overline{D(A)}^X$ and $g \in BV(0,T;X)$, the unique mild solution u of perturbed inhomogeneous Cauchy problem (5.28) is strong and satisfies (5.53) for a.e. $t \in (0,T)$.*

Proof First, let $u_0 \in D(A)$ and $g \in BV(0,T;X)$. Then, Lemma 5.3 implies that the mild solution $u(t) = e^{-tA}(u_0,g)$ of the perturbed inhomogeneous Cauchy problem (5.28) is Lipschitz continuous on $[0,T)$. Since every reflexive Banach space X admits the Radon-Nikodým property, u is differentiable a.e. on $(0,T)$. Thus, by Theorem 5.4, there is a function $\psi \in L^\infty(0,T)$ such that u satisfies

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}t_+}(t)\right\|_X \leq \frac{1}{t}\left[\frac{e^{\omega t}+1}{|1-\alpha|}\|u_0\|_X + \psi(t) + \omega\int_0^t \left[\frac{e^{\omega s}+1}{|1-\alpha|}\|u_0\|_X + \psi(s)\right] e^{\omega(t-s)}\mathrm{d}s\right]$$

for a.e. $t \in (0,T)$. Next, we square both sides of the last inequality and subsequently integrate over (a,b) for given $0 < a < b \leq T$. Then, one finds that

$$\int_a^b \left\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right\|_X^2 \mathrm{d}t \leq \int_a^b \frac{1}{t^2}\left\{\frac{e^{\omega t}+1}{|1-\alpha|}\|u_0\|_X + \psi(t) + \omega\int_0^t \left[\frac{e^{\omega s}+1}{|1-\alpha|}\|u_0\|_X + \psi(s)\right] e^{\omega(t-s)}\mathrm{d}s\right\}^2 \mathrm{d}t \tag{5.55}$$

Due to this estimate, we can now show that also for $u_0 \in \overline{D(A)}^X$ and $g \in BV(0,T;X)$, the corresponding mild solution u of the perturbed inhomogeneous Cauchy problem (5.28) is strong. To see this let $g \in BV(0,T;X)$ and $(u_{0,n})_{n\geq 1} \subseteq D(A)$ such that $u_{0,n} \longrightarrow u_0$ in X as $n \to \infty$ and set $u_n = e^{-tA}(u_{0,n}, g)$ for every $n \ge 1$. By the standard growth estimate (5.13) (which is satisfied with $L = 1$ by all u_n), $(u_n)_{n\ge1}$ is a Cauchy sequence in $C([0,T];X)$. Hence, there is a function $u \in C([0,T];X)$ satisfying $u(0) = u_0$ and $u_n \longrightarrow u$ in $C([0,T];X)$. In particular, one can show that u is the unique mild solution of the perturbed inhomogeneous Cauchy problem (5.28) with initial value u_0. Now, by inequality (5.55), $(\mathrm{d}u_n/\mathrm{d}t)_{n\ge1}$ is bounded in $L^2(a,b;X)$ for any $0 < a < b \le T$. Since X is reflexive, also $L^2(a,b;X)$ is reflexive and hence, there is a $v \in L^2(a,b;X)$ and a subsequence of $(u_{0,n})_{n\ge1}$, which, for simplicity, we denote again by $(u_{0,n})_{n\ge1}$, such that $\frac{\mathrm{d}u_n}{\mathrm{d}t} \rightharpoonup v$ weakly in $L^2(a,b;X)$ as $n \to +\infty$. Since $u_n \longrightarrow u$ in $C([a,b];X)$, it follows by a standard argument that $v = \frac{\mathrm{d}u}{\mathrm{d}t}$ in the sense of vector-valued distributions $\mathcal{D}'((a,b);X)$. Since $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^2(a,b;X)$, the mild solution u of Cauchy problem (5.28) is absolutely continuous on (a,b), and since X is reflexive, u is differentiable a.e. on (a,b). Since $0 < a < b < \infty$ were arbitrary, $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^1_{\mathrm{loc}}((0,\infty);X)$.

To see that u satisfies inequality (5.53), note that for $\varepsilon > 0$, the function $t \mapsto \tilde{u}(t) := u(t+\varepsilon)$ on $[0,T-\varepsilon]$ satisfies the hypotheses of Theorem 5.4 with $\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} \in L^1([0,T-\varepsilon);X)$ and so, we find that

$$\left\|\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t_+}(t)\right\|_X \le \frac{1}{t}\Bigg[\frac{e^{\omega t}+1}{|1-\alpha|}\|u(\varepsilon)\|_X + \psi(t) \\ + \omega\int_0^t \left[\frac{e^{\omega s}+1}{|1-\alpha|}\|u(\varepsilon)\|_X + \psi(s)\right] e^{\omega(t-s)}\mathrm{d}s\Bigg]$$

for every $t \in (0,T-\varepsilon]$ and $\varepsilon \in (0,t)$. Sending $\varepsilon \to 0+$ shows that u satisfies (5.53). Since $u_0 \in \overline{D(A)}^X$ was arbitrary, this completes the proof of this corollary. □

Next, we consider the case when the Banach space X and its dual space X^* are *uniformly convex* and $A+F$ is quasi m-accretive in X. Then by the classical nonlinear semigroup theory (cf [23, Theorem 4.6]), for every $u_0 \in D(A)$, $g \in W^{1,1}(0,T;X)$, the mild solution $u(t) = e^{-tA}(u_0, g)$, $(t \in [0,T])$, of the perturbed inhomogeneous Cauchy problem (5.28) is a strong one, u is everywhere differentiable from the right, $\frac{\mathrm{d}u}{\mathrm{d}t_+}$ is right continuous, and

$$\frac{\mathrm{d}u}{\mathrm{d}t_+}(t) + (A+F-g(t))^\circ u(t) = 0 \qquad \text{for every } t \ge 0,$$

where for every $t \in [0,T]$, $(A+F-g(t))^\circ$ denotes the *minimal selection* of $A+F-g(t)$ defined by (3.28) in Chapter 3.2.2.1. Thus, under those assumptions on X and by the preceding Corollary 5.6, we can conclude the following two corollaries. We begin by stating the inhomogeneous case.

Corollary 5.7 *Suppose X and its dual space X^* are uniformly convex, A be an m-accretive operator on X, $F : X \to X$ a Lipschitz continuous mapping with Lipschitz-constant $\omega \ge 0$, and satisfying $F(0) = 0$. Further, let $\{e^{-tA}\}_{t=0}^T$ be the semigroup on $\overline{D(A)}^X \times L^1(0,T;X)$ generated by $-(A+F)$. If A is homogeneous of order $\alpha \neq 1$, then for every $u_0 \in \overline{D(A)}^X$ and $g \in W^{1,1}(0,T;X)$, the mild solution $u(t) = e^{-tA}(u_0, g)$,*

($t \in [0,T]$) of the perturbed inhomogeneous Cauchy problem (5.28) is strong and

$$\left\|(A+F-g(t))^{\circ}e^{-tA}(u_0,g)\right\|_X \le \frac{1}{t}\left[a_{\omega}(t)+\omega\int_0^t a_{\omega}(s)e^{\omega(t-s)}\mathrm{d}s\right]$$

for every $t \in (0,T]$, where $a_{\omega}(t)$ is defined by (5.52).

Our next corollary considers mild solutions to the perturbed homogeneous Cauchy problem (in X)

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}+A(u(t)+F(u(t))) \ni 0 & \text{for a.e. } t \in (0,T), \\ u(0)=u_0. \end{cases} \tag{5.56}$$

Corollary 5.8 *Suppose X and its dual space X^* are uniformly convex, A be an m-accretive operator on X, $F : X \to X$ a Lipschitz continuous mapping with Lipschitz-constant $\omega \ge 0$, and satisfying $F(0)=0$. Further, let $\{e^{-tA}\}_{t\ge0}$ be the semigroup on $\overline{D(A)}^X$ generated by $-(A+F)$. If A is homogeneous of order $\alpha \neq 1$, then for every $u_0 \in \overline{D(A)}^X$, the mild solution $u(t)=e^{-tA}u_0$, ($t \ge 0$) of Cauchy problem (5.56) (for $f \equiv 0$) is a strong solution satisfying*

$$\left\|(A+F)^{\circ}e^{-tA}u_0\right\|_X \le \frac{e^{\omega t}+1}{|1-\alpha|\,t}\left[1+\omega\int_0^t e^{\omega(t-s)}\,\mathrm{d}s\right]\|u_0\|_X$$

for every $t>0$.

Our second main result in this section is concerned with a point-wise estimate on the time-derivative $\frac{du}{dt}$ of *positive* strong solutions u of the homogeneous Cauchy problem (5.56) in X, under the additional hypothesis that the underlaying Banach space X is equipped with a partial ordering "$\le$" such that the triple $(X,\|\cdot\|_X,\le)$ defines an Banach lattice, and if for this ordering "$\le$", every mild solution u of (5.56) is *order-preserving* (see Definition 3.13 in Section 3.2.2.2).

Theorem 5.5 (Point-wise Aronson-Bénilan type estimates) *Let A be an m-accretive operator on X, $(X,\|\cdot\|_X,\le)$ a Banach lattice, and let $F : X \to X$ be a Lipschitz continuous mapping on X with constant $\omega \ge 0$ satisfying $F(0)=0$. Suppose, for $\alpha \in \mathbb{R}\setminus\{1\}$, A is homogeneous of order α and every mild solution u of (5.56) is* order-preserving. *Then, for every positive $u_0 \in \overline{D(A)}^X$, the mild solution u of (5.56) satisfies*

$$\frac{u(t+h)-u(t)}{h} \ge \frac{(1+\frac{h}{t})^{\frac{1}{1-\alpha}}-1}{h}\,\frac{u(t)}{t}+g_h(t) \qquad \text{if } \alpha>1$$

and

$$\frac{u(t+h)-u(t)}{h} \le \frac{(1+\frac{h}{t})^{\frac{1}{1-\alpha}}-1}{h}\,\frac{u(t)}{t}+g_h(t) \qquad \text{if } \alpha<1,$$

for every t, $h>0$, where $g_h : (0,\infty) \to X$ is a continuous function. Further, for positive $u_0 \in \overline{D(A)}^X$, if the right hand-side derivative $\frac{\mathrm{d}u}{\mathrm{d}t+}$ belongs to $L^1_{loc}([0,\infty);X)$, then

$$(\alpha-1)\frac{\mathrm{d}u}{\mathrm{d}t_+}(t) \geq -\frac{u(t)}{t} + (\alpha-1)g_0(t), \tag{5.57}$$

for a.e. $t > 0$, where $g_0 : (0,\infty) \to X$ is a measurable function.

Proof (Proof of Theorem 5.5) In the case A is an m-accretive operator on a Banach lattice X and F a Lipschitz continuous perturbation with constant $\omega \geq 0$, then the statements of Theorem 5.5 immediately follow from Theorem 5.3 and Corollary 5.3 with constants $L = 1$. □

5.2.3 Homogeneous completely accretive operators

In [29], Bénilan and Crandall introduced the class of *completely accretive* operators A (see Section 3.2.2.2) and showed: even though the underlying (normal) Banach spaces $X \subseteq L_0(\Sigma,\mu)$ might not admit the Radon-Nikodým property, but A is an m-completely accretive on X and homogeneous of order $\alpha > 0$ with $\alpha \neq 1$, then for every $u_0 \in \overline{D(A)}^X$, the mild solution u of homogeneous Cauchy problem Cauchy problem (in X)

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) \ni 0 & \text{for a.e. } t \in (0,T), \\ \qquad u(0) = u_0, \end{cases} \tag{5.58}$$

is a strong one. This was extended in [87] to the zero-order case. Here, we provide a generalization to the case of completely accretive operators A, which are homogeneous of order $\alpha \neq 1$ and perturbed by a Lipschitz nonlinearity.

Here, we employ the same notation and function spaces as introduced in Chapter 3.2.1 and Chapter 3.2.2.2. Thus, the partial ordering " $\leq$ " is the standard one on $M(\Sigma,\mu)$; that is, for given u, $v \in M(\Sigma,\mu)$, we write $u \leq v$ provided $u(x) \leq v(x)$ for μ-a.e. $x \in \Sigma$. Further, for normed space X, $Y \subseteq M(\Sigma,\mu)$, we write $X \hookrightarrow Y$ for indicating that the space X is continuously embedded into the space Y.

Lemma 5.4 *Let $X \subseteq L_0(\Sigma,\mu)$ be a normal Banach space satisfying (3.33). For $\omega \in \mathbb{R}$, let $\{e^{-tA}\}_{t\geq 0}$ be a family of mappings $e^{-tA} : C \to C$ defined on a subset $C \subseteq X$ of ω-quasi complete contractions satisfying (5.12) and $e^{-tA}0 = 0$ for all $t \geq 0$. Then, for every $u_0 \in C$ and $t > 0$, the set*

$$\left\{ \frac{e^{-(t+h)A}u_0 - e^{-tA}u_0}{h} \,\middle|\, h \neq 0, t+h > 0 \right\} \tag{5.59}$$

is $\sigma(L_0, L^{1\cap\infty})$-weakly sequentially compact in $L_0(\Sigma,\mu)$.

The proof of this lemma is essentially the same as in the case $\omega = 0$ (cf [29]). For the convenience of the reader, we include here the proof.

Proof Let $u_0 \in C$, $t > 0$, and $h \neq 0$ such that $t + h > 0$. Then by taking $\lambda = 1 + \frac{h}{t}$ in (5.12), one sees that

$$\begin{aligned}|e^{-(t+h)A}u_0 - e^{-tA}u_0| &= |\lambda^{\frac{1}{1-\alpha}}\, e^{-tA}\left[\lambda^{\frac{1}{\alpha-1}}\, u_0\right] - e^{-tA}u_0| \\ &\le \lambda^{\frac{1}{1-\alpha}}\left|e^{-tA}\left[\lambda^{\frac{1}{\alpha-1}}\, u_0\right] - e^{-tA}u_0\right| + |\lambda^{\frac{1}{1-\alpha}} - 1|\,|e^{-tA}u_0|.\end{aligned}$$

Since e^{-tA} is an ω-quasi complete contraction and since $e^{-tA}0 = 0$, $(t \ge 0)$, claim (3) and (5) of Proposition 3.2 (see Chapter 3.2.2.2) imply that

$$\lambda^{\frac{1}{1-\alpha}}\, e^{-\omega t}\left|e^{-tA}\left[\lambda^{\frac{1}{\alpha-1}}\, u_0\right] - e^{-tA}u_0\right| \ll |1 - \lambda^{\frac{1}{1-\alpha}}|\,|u_0|$$

and

$$|\lambda^{\frac{1}{1-\alpha}} - 1|\, e^{-\omega t}|e^{-tA}u_0| \ll |\lambda^{\frac{1}{1-\alpha}} - 1|\,|u_0|.$$

Since the set $\{w\,|\,w \ll |\lambda^{\frac{1}{1-\alpha}} - 1|\,|u_0|\}$ is convex (cf (6) of Proposition 3.2), the previous inequalities imply that

$$\frac{1}{2}e^{-\omega t}|e^{-(t+h)A}u_0 - e^{-tA}u_0| \ll |\lambda^{\frac{1}{1-\alpha}} - 1|\,|u_0|.$$

Using again (3) of Proposition 3.2, gives

$$\frac{|e^{-(t+h)A}u_0 - e^{-tA}u_0|}{|\lambda^{\frac{1}{1-\alpha}} - 1|} \ll 2\, e^{\omega t}\,|u_0|. \tag{5.60}$$

Since for every $u \in M(\Sigma,\mu)$, one always has that $u^+ \ll |u|$, the transitivity of " $\ll$ " (cf (4) of Proposition 3.2) implies that

$$f_h := \frac{e^{-(t+h)A}u_0 - e^{-tA}u_0}{\lambda^{\frac{1}{1-\alpha}} - 1} \quad \text{satisfies} \quad f_h^+ \ll 2\, e^{\omega t}\,|u_0|.$$

Therefore and since $|u_0| \in X$, (1) of Proposition 3.4 yields that the two sets $\{f_h^+|\, h \neq 0, t+h > 0\}$ and $\{|f_h|\,|\, h \neq 0, t+h > 0\}$ are $\sigma(L_0, L^{1\cap\infty})$- weakly sequentially compact in $L_0(\Sigma,\mu)$. Since $f_h^- = |f_h| - f_h^+$ and $f_h = f_h^+ - f_h^-$, and since

$$\lim_{h\to 0+}\frac{\lambda^{\frac{1}{1-\alpha}} - 1}{h} = \lim_{h\to 0+}\frac{(1+\frac{h}{t})^{\frac{1}{1-\alpha}} - 1}{h} = \frac{1}{t(1-\alpha)} \neq 0, \tag{5.61}$$

we can conclude that the claim of this lemma holds. □

With these preliminaries in mind, we can now state the regularization effect of the semigroup $\{e^{-tA}\}_{t\ge 0}$ generated by a ω-quasi m-completely accretive operator of homogeneous order $\alpha \neq 1$.

Theorem 5.6 *Let $X \subseteq L_0(\Sigma,\mu)$ be a normal Banach space satisfying (3.33), and $\|\cdot\|$ denote the norm on X. For $\omega \in \mathbb{R}$, let A be ω-quasi m-completely accretive in X, and $\{e^{-tA}\}_{t\ge 0}$ be the semigroup generated by $-A$ on $\overline{D(A)}^X$. If A is homogeneous of order $\alpha \neq 1$, then for every $u_0 \in \overline{D(A)}^X$ and $t > 0$, $\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}$ exists in X and*

$$|A^{\circ}e^{-tA}u_0| \le \frac{2e^{\omega t}}{|\alpha-1|}\,\frac{|u_0|}{t} \qquad \mu\text{-a.e. on } \Sigma. \tag{5.62}$$

In particular, for every $u_0 \in \overline{D(A)}^{X}$,

$$\left\|\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}_{+}\right\| \le \frac{2e^{\omega t}}{|\alpha-1|}\,\frac{\|u_0\|}{t} \qquad \text{for every } t>0, \tag{5.63}$$

and

$$\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}_{+} \ll \frac{2e^{\omega t}}{|\alpha-1|}\,\frac{|u_0|}{t} \qquad \text{for every } t>0. \tag{5.64}$$

Proof Let $u_0 \in \overline{D(A)}^{X}$, $t>0$, and $(h_n)_{n\ge1} \subseteq \mathbb{R}$ be a zero sequence such that $t+h_n>0$ for all $n \ge 1$. Then, by Proposition 5.2, we can apply the compactness result stated in Lemma 5.4. Thus, there is a $z \in L_0(\Sigma,\mu)$ and a subsequence $(h_{k_n})_{n\ge1}$ of $(h_n)_{n\ge1}$ such that

$$\lim_{n\to\infty}\frac{e^{-(t+h_{k_n})A}u_0 - e^{-tA}u_0}{h_{k_n}} = z \qquad \text{weakly in } L_0(\Sigma,\mu). \tag{5.65}$$

By (2d) of Theorem 3.4 (in Chapter 3.2.2.2), one has that $(e^{-tA}u_0, -z) \in A$. Thus (2f) of Theorem 3.4 yields that $z = -A^{\circ}e^{-tA}u_0$ and

$$\lim_{n\to\infty}\frac{e^{-(t+h_{k_n})A}u_0 - e^{-tA}u_0}{h_{k_n}} = -A^{\circ}e^{-tA}u_0 \qquad \text{strongly in } L_0(\Sigma,\mu). \tag{5.66}$$

After possibly passing to another subsequence, the limit (5.66) also holds μ-a.e. on Σ. The argument shows that the limit (5.66) is independent of the choice of the initial zero sequence $(h_n)_{n\ge1}$. Thus

$$\lim_{h\to0}\frac{e^{-(t+h)A}u_0 - e^{-tA}u_0}{h} = -A^{\circ}e^{-tA}u_0 \qquad \text{exists } \mu\text{-a.e. on } \Sigma. \tag{5.67}$$

Since $2e^{-\omega t}|u_0| \in X$, by (5.60), and by (5.61), it follows from (2) of Proposition 3.4 (in Chapter 3.2.2.2) that

$$\lim_{h\to0}\frac{e^{-(t+h)A}u_0 - e^{-tA}u_0}{h} = -A^{\circ}e^{-tA}u_0 \qquad \text{exists in } X \tag{5.68}$$

and with $\lambda = 1+\frac{h}{t}$,

$$\frac{|e^{-(t+h)A}u_0 - e^{-tA}u_0|}{|\lambda^{\frac{1}{1-\alpha}}-1|} \le 2\,e^{-\omega t}\,|u_0|$$

for all $h \neq 0$ satisfying $t+h>0$. Sending $h \to 0$ in the last inequality and applying (5.68) gives (5.62). In particular, by Corollary 5.1, one has that (5.63) holds for the norm $\|\cdot\|_X$ on X. Moreover, (5.60) is equivalent to

$$\int_\Sigma j\left(\frac{|e^{-(t+h)A}u_0 - e^{-tA}u_0|}{|\lambda^{\frac{1}{1-\alpha}} - 1|}\right) \mathrm{d}\mu \le \int_\Sigma j\left(2\, e^{-\omega t}\, |u_0|\right) \mathrm{d}\mu \tag{5.69}$$

for all $h \neq 0$ satisfying $t + h > 0$, and every $j \in J_0$. By the lower semicontinuity of $j \in J_0$ and by the μ-a.e. limit (5.67), we have that

$$j\left(\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}(x)\,|\alpha - 1|\,t\right) \le \liminf_{h\to 0} j\left(\frac{|e^{-(t+h)A}u_0(x) - e^{-tA}u_0(x)|}{|\lambda^{\frac{1}{1-\alpha}} - 1|}\right)$$

for μ-a.e. $x \in \Sigma$. Thus, taking the limit inferior as $h \to 0+$ in (5.69) and applying Fatou's lemma yields

$$\int_\Sigma j\left(\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}(x)\,|\alpha - 1|\,t\right) \mathrm{d}\mu \le \int_\Sigma j\left(2\, e^{-\omega t}\, |u_0|\right) \mathrm{d}\mu$$

Since $j \in J_0$ was arbitrary and by (3) of Proposition 3.2, this shows that (5.64) holds and thereby completes the proof of this theorem. □

5.3 Proof of Theorem 5.1

In this section, we outline the proof of Theorem 5.1.

Proof We recall that according to Proposition 3.5 (in Chapter 3.3), the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ in $L^2(\partial\Omega)$ is homogeneous of order $\alpha := p - 1 > 0$ and $\alpha \neq 1$ (since by assumption, $p \neq 2$). Thus also the realization Λ in the other Banach spaces X of the Dirichlet-to-Neumann map Λ_{σ_0} is homogeneous of the same order (cf the Proof of Theorem 3.1 in Chapter 3).

Further, we recall that by Remark 3.6, the every Banach spaces X, except of $C(\partial\Omega)$, considered in this theorem satisfies the property (3.33) from Proposition 3.4 (in Chapter 3.2.2.2). Therefore, by Theorem 5.6, for every $h_0 \in X$, the function $h(t) = e^{-tA}h_0$ is differentiable with values in X a.e. on $(0, \infty)$ and satisfies

$$\left\|\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}_+\right\|_X \le \frac{\|u_0\|_X}{|p-2|\,t}\left[1 + e^{\omega t} + \omega \int_0^t \left(1 + e^{\omega s}\right) e^{\omega(t-s)}\mathrm{d}s\right]$$

for every $t > 0$. Rearranging the right hand side of the last estimate yields the L^1 Aronson-Bénilan type estimate (5.5) for every Banach spaces X considered in this theorem except of $C(\partial\Omega)$. Moreover, it follows from Theorem 5.6 that for every positive $h_0 \in X$, the point-wise Aronson-Bénilan type estimate (5.6) holds.

To see that (5.5) also holds for $X = L^\infty(\partial\Omega)$, let $h_0 \in L^\infty(\partial\Omega)$ and $t > 0$. We assume $\|\frac{\mathrm{d}e^{-tA}h_0}{\mathrm{d}t}_+\|_\infty > 0$ (otherwise, there is nothing to show). Then, for every $s \in (0, \|\frac{\mathrm{d}e^{-tA}h_0}{\mathrm{d}t}_+\|_\infty)$ and $2 \le q < \infty$, Chebyshev's inequality yields

$$\mathcal{H}^{d-1}\left(\left\{\left|\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}_+\right| \geq s\right\}\right)^{1/q} \leq \frac{\left\|\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}_+\right\|_q}{s}$$

and so, by (5.5) for $X = L^q(\partial\Omega)$

$$s\mathcal{H}^{d-1}\left(\left\{\left|\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}\right| \geq s\right\}\right)^{1/q} \leq \frac{[2+\omega t]\; e^{\omega t}}{|p-2|\,t}\|u_0\|_q$$

Thus and since $\lim_{q\to\infty}\|h_0\|_q = \|h_0\|_\infty$, sending $q \to +\infty$ in the last inequality, yields

$$s \leq \frac{[2+\omega t]\; e^{\omega t}}{|p-2|\,t}\|h_0\|_\infty$$

and since $s \in (0, \left\|\frac{\mathrm{d}e^{-tA}u_0}{\mathrm{d}t}_+\right\|_\infty)$ was arbitrary, we have thereby shown that (5.5) also holds for $X = L^\infty(\partial\Omega)$. This completes the proof of Theorem 5.1. □

Chapter 6
The Dirichlet-to-Neumann map on open sets

Abstract In this chapter, we outline the theory of *j-elliptic functionals* and, in particular, the main Theorem 1.1 in Chapter 1.6.1. Then, we apply this theory to show that the Dirichlet-to-Neumann map Λ_σ associated with a Leray-Lions operator can be realized as the $\mathcal{T}r$-subgradient in $L^2(\partial\Omega)$, where Ω is a general open subset in $\mathbb{R}^d$ (Theorem 6.1). This proves the statement of Theorem 1.3. Moreover, we show that in this framework, the negative Dirichlet-to-Neumann map $-\Lambda_\sigma$ generates a strongly continuous semigroup of contractions on $L^2(\partial\Omega)$, which is order-preserving, L^∞-contractive, and which can be extrapolated on $L^\psi(\partial\Omega)$ for every N-function ψ. The content of this chapter covers parts of the paper [51].

6.1 Main results

The theory of energy functionals on Hilbert spaces and their subgradients has proven to be a powerful tool for the study nonlinear elliptic and parabolic partial differential equations [22, 23, 40, 154]. Not only can existence and uniqueness of solutions be established with minimal effort by variational principles, the variational approach also allows one to prove results about the regularity of solutions, maximum or comparison principles and the large-time behaviour of solutions in the case of parabolic problems. Very often, this theory is a natural generalisation to the nonlinear case of the corresponding theory of sesquilinear forms used in the study of *linear* elliptic and parabolic equations [63, 65, 66, 103]; in that case, the form is defined on a Hilbert space V and induces an operator on another Hilbert space H. A key point in the whole theory is that the space V is canonically embedded in H, that is, that there exists a bounded injection $i : \mathrm{V} \to \mathrm{H}$. One can however find a plethora of examples which do not fit into this framework, although one would expect (or hope) that variational methods should still be applicable; as a prototype consider the case where H is a trace space of V.

Recently, Arendt and ter Elst developed a general theory of *j-elliptic forms* [12, 13], see also [15], where the embedding i is replaced with a closed linear map

$j : \mathrm{V} \supseteq D(j) \to \mathrm{H}$, which is however not necessarily injective or even bounded. This allowed them to develop a rich variational theory of the *Dirichlet-to-Neumann map* acting on functions defined on the boundary $\partial\Omega$ of a general (bounded) open set $\Omega \subseteq \mathbb{R}^d$. This linear theory was recently extended to a nonlinear framework in [51] by Chill, Kennedy and the author to treat possibly nonlinear quasilinear operators such as the weighted p-Laplace operator $\Delta_{p,\sigma}$.

In this chapter, we shall outline the general theory of *j-elliptic energy functionals* developed in [51], which will allow us to incorporate and treat the p-Dirichlet-to-Neumann map on general open domains.

To do this, we assume throughout this chapter that Ω is a open subset of $\mathbb{R}^d$ satisfying Assumption 1.3. Further, for $1 < p, r < \infty$, suppose that the flux function $\boldsymbol{\sigma}$ is a Carathéodory function satisfying (2.2), (2.3), (2.4), (2.5) from Chapter 2.1.

For given $h \in L^r(\partial\Omega)$ and $H \in V_{p,r}(\Omega)$, the variational problem

$$\min\left\{\varphi(\hat{u})\,\mathrm{d}x \,\middle|\, \hat{u} \in V_{p,r}(\Omega),\ \text{ with } \hat{u} - H \in \ker(\mathcal{T}r)\right\}$$

for $\varphi : V_{p,r}(\Omega) \to [0,\infty)$ given by

$$\varphi(\hat{u}) = \tfrac{1}{p}\int_\Omega \sigma_0 |\nabla\hat{u}|^p\,\mathrm{d}x \tag{6.1}$$

for every $\hat{u} \in V_{p,2}(\Omega)$, admits a minimizer $\hat{u}_h \in V_{p,r}(\Omega)$, which necessarily satisfies $\mathcal{T}r(\hat{u}_h - H) = 0$ and

$$\langle\varphi(\hat{u}),\xi\rangle = \int_\Omega \boldsymbol{\sigma}(x,\nabla\hat{u})\,\nabla\xi\mathrm{d}x = 0 \tag{6.2}$$

for every test function $\xi \in \ker(\mathcal{T}r)$.

Definition 6.1 For given $h \in L^r(\partial\Omega)$ and $H \in V_{p,r}(\Omega)$, we call a function $\hat{u}_h \in V_{p,r}(\Omega)$ a *weak solution* of Dirichlet problem

$$\begin{cases} -\mathrm{div}(\boldsymbol{\sigma}(x,\nabla u)) = 0 & \text{in } \Omega,\\ \qquad\qquad\qquad u = h & \text{on } \partial\Omega \end{cases} \tag{6.3}$$

if $\mathcal{T}r(\hat{u}_h - H) = 0$ and (6.2) for every test function $\xi \in \ker(\mathcal{T}r)$, where $\mathcal{T}r$ denotes the weak trace operator from Definition 1.9.

By using this notation of a weak solution u_h of Dirichlet problem (6.3), we can now define the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ in $L^2(\partial\Omega)$ associated with the quasi-linear operator A given by (2.20).

Definition 6.2 We define the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ in $L^2(\partial\Omega)$ by setting

$$\Lambda_{|L^2} = \left\{(h,g) \in L^2(\partial\Omega)\times L^2(\partial\Omega) \,\middle|\, \exists\, \hat{u} \in V_{p,2}(\Omega) \text{ s.t. } \mathcal{T}r(\hat{u}) = u \text{ satisfying (6.4)}\right\}$$

where

$$\int_{\Omega} \sigma(x, \nabla \hat{u}) \nabla \hat{v} \,\mathrm{d}x = \int_{\partial\Omega} g \mathcal{T}r(\hat{v}) \,\mathrm{d}\mathcal{H}^{d-1} \tag{6.4}$$

for every $\hat{v} \in V_{p,2}(\Omega)$.

Due to the subsequent lemma, an element $g \in \Lambda_{|L^2(\partial\Omega)}$ is uniquely determined by the integral equation (6.4). Thus, the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ is a well-defined mapping on $L^2(\partial\Omega)$.

Lemma 6.1 *Let $1 \le p, r < \infty$ and suppose Assumption 1.3 holds. Then the set $\{\hat{u}|_{\partial\Omega} : \hat{u} \in C_c^\infty(\mathbb{R}^d)\}$ is a subset of $\mathcal{T}r(V_{p,r}(\Omega))$ and is dense in $L^r(\partial\Omega)$. In particular, $\mathcal{T}r(V_{p,r}(\Omega))$ is dense in $L^r(\partial\Omega)$.*

Proof By the Stone-Weierstraß theorem [153, Chapter 0], the set of restrictions of $C_c^\infty(\mathbb{R}^d)$ functions to $\partial\Omega$ is dense in $C_c(\partial\Omega)$. Since $\mathcal{H}^{d-1}$ is a Borel regular measure [73], which is finite on every compact set, $C_c(\partial\Omega)$ is dense in $L^r(\partial\Omega)$ (see [132, Theorem 3.14]). Now, since we may identify every $\hat{u} \in C_c^\infty(\mathbb{R}^d)$ with an element of $V_{p,r}(\Omega)$ as in Remark 1.4 (c) in such a way that $\mathcal{T}r(\hat{u}) = \hat{u}|_{\partial\Omega}$, we may identify the set $\{\hat{u}|_{\partial\Omega} : \hat{u} \in C_c^\infty(\mathbb{R}^d)\}$ with a subset of $\mathcal{T}r(V_{p,r}(\Omega))$ and conclude that $\mathcal{T}r(V_{p,r}(\Omega))$ is dense in $L^r(\partial\Omega)$. □

Next, we note that a density argument shows that $V_{p,r}(\Omega)$ has a lattice structure whose ordering is induced by that of the space V_0 defined in (1.30) in the natural way. We omit the easy proof, which follows directly from the fact that V_0 inherits the lattice structure of $W^{1,p}(\Omega)$ and $C_c(\overline{\Omega})$.

Lemma 6.2 *The space $V_{p,r}(\Omega)$ is a lattice for any $1 \le p$, $r < \infty$, and the lattice operations are continuous. Moreover, assuming p, q, $r \ge 1$ satisfy (1.28) (from Chapter 1.6.2), then ι is a lattice isomorphism onto its range, and in particular $\iota(V_{p,r}(\Omega))$ is a sublattice of $W^1_{p,q}(\Omega) \times L^r(\partial\Omega)$ equipped with its natural ordering. As a consequence, j and $\mathcal{T}r$ from (1.32) and (1.33) (from Chapter 1.6.2) are lattice homomorphisms.*

Now, we turn to the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ as given in Definition 6.2. We show that it can be realised as the $\mathcal{T}r$-subgradient of the functional φ defined by (6.1). This result includes the statement of Theorem 1.3.

Theorem 6.1 *Assume that Ω is a open subset of $\mathbb{R}^d$ satisfying Assumption 1.3. Then, the functional $\varphi : V_{p,2}(\Omega) \to \mathbb{R}$ defined by (6.1) is convex, continuously differentiable, and $\mathcal{T}r$-elliptic. Moreover, the $\mathcal{T}r$-subgradient $\partial_{\mathcal{T}r}\varphi$ of φ is densely defined and coincides with the Dirichlet-to-Neumann map $\Lambda_{|L^2}$ introduced in Definition 6.2.*

We give the proof of Theorem 6.1 in Section 6.4.

Along the way, we shall show that many known, or "classical", results from the theory of energy functionals and nonlinear semigroups on a Hilbert space can be readily adapted to this setting. Our desired generation result now follows from Theorem 6.6.

Theorem 6.2 *The Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ generates a strongly continuous semigroup S of contractions on $L^2(\partial\Omega)$.*

We wish to study the order properties of this semigroup, and in particular show that it extrapolates to $L^q(\partial\Omega)$ for $q \in [1, \infty)$.

Theorem 6.3 *The semigroup S generated by $\Lambda_{|L^2}$ on $L^2(\partial\Omega)$ is order preserving and L^∞-contractive. In particular, the semigroup S is also positive and extrapolates to a strongly continuous semigroup of contractions on $L^\psi(\partial\Omega)$ for every N-function ψ.*

We give the proof of Theorem 6.3 in Section 6.4.

6.2 The j-subgradient and basic properties

6.2.1 Definition and characterisation as a classical gradient

Throughout, let V be a real locally convex topological vector space and H a real Hilbert space equipped with inner product $\langle\cdot,\cdot\rangle_{\mathrm{H}}$ and associated norm denoted by $\|\cdot\|_{\mathrm{H}}$. Further, let $j : \mathrm{V} \to \mathrm{H}$ be a linear operator which is weak-to-weak continuous.

Definition 6.3 Given a functional $\varphi : \mathrm{V} \to (-\infty, \infty]$, we call the set $\mathrm{D}(\varphi) := \{\varphi < +\infty\}$ its *effective domain*, and we say that φ is *proper* if the effective domain is non-empty.

Definition 6.4 Given a functional $\varphi : \mathrm{V} \to (-\infty, \infty]$, we call the operator

$$\partial_j \varphi := \left\{ (u, f) \in \mathrm{H} \times \mathrm{H} \;\middle|\; \begin{array}{c} \exists \hat{u} \in \mathrm{D}(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{v} \in \mathrm{V} \\ \liminf_{t \searrow 0} \frac{\varphi(\hat{u}+t\hat{v})-\varphi(\hat{u})}{t} \geq \langle f, j(\hat{v})\rangle_{\mathrm{H}} \end{array} \right\}$$

the *j-subgradient* of φ.

As in Section 3.2.2, we shall usually view operators on H as relations $A \subseteq \mathrm{H} \times \mathrm{H}$, but we shall also use the notation

$$A(u) := \Big\{ f \in \mathrm{H} \,\Big|\, (u, f) \in A \Big\},$$

which suggests that A is a mapping from H into 2^{H}, the power set of H, that is, A is a so-called multivalued operator. We take the usual definition of the *domain* of an operator $A \subseteq \mathrm{H} \times \mathrm{H}$ as the set

$$D(A) := \Big\{ u \in \mathrm{H} \,\Big|\, \exists f \in \mathrm{H} \text{ s.t. } (u, f) \in A \Big\},$$

and similarly for the *range* of A.

Definition 6.5 We say that the functional φ is *j-semiconvex* if there exists $\omega \in \mathbb{R}$ such that the "shifted" functional $\varphi_\omega : \mathrm{V} \to (-\infty, \infty]$ given by

$$\varphi_\omega(\hat{u}) = \varphi(\hat{u}) + \frac{\omega}{2} \|j(\hat{u})\|_{\mathrm{H}}^2$$

for every $\hat{u} \in \mathrm{V}$ is convex. and we say that the functional φ is *j-elliptic* if there exists $\omega \geq 0$ such that φ_ω is convex and coercive. Saying that a functional φ defined on a locally convex topological vector space is *coercive* means that sublevels $\{\varphi \leq c\}$ are relatively weakly compact for every $c \in \mathbb{R}$. Finally, we say that the functional φ is *lower semicontinuous* if the sublevels $\{\varphi \leq c\}$ are closed in the topology of V for every $c \in \mathbb{R}$.

Remark 6.1 In the important special case when V is a Banach space, j is weak-to-weak continuous if and only if j is continuous. Moreover, in this case, the "shifted" functional φ_ω is lower semicontinuous if and only if φ itself is lower semicontinuous. Finally, if V is a *reflexive* Banach space, then φ is coercive if and only if the sublevels $\{\varphi \leq c\}$ are (norm-) bounded.

Lemma 6.3 *Let* V, H, j *and* φ *be as above.*

(a) If φ_ω is convex for some $\omega \in \mathbb{R}$, then

$$\partial_j \varphi = \left\{ (u, f) \in \mathrm{H} \times \mathrm{H} \;\middle|\; \begin{array}{c} \exists \hat{u} \in \mathrm{D}(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{v} \in \mathrm{V} \\ \varphi_\omega(\hat{u} + \hat{v}) - \varphi_\omega(\hat{u}) \geq \langle f + \omega j(\hat{u}), j(\hat{v}) \rangle_{\mathrm{H}} \end{array} \right\}.$$

(b) If φ is Gâteaux differentiable with Gâteaux derivative φ', then

$$\partial_j \varphi = \left\{ (u, f) \in \mathrm{H} \times \mathrm{H} \;\middle|\; \begin{array}{c} \exists \hat{u} \in \mathrm{D}(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{v} \in \mathrm{V} \\ \varphi'(\hat{u})\hat{v} = \langle f, j(\hat{v}) \rangle_{\mathrm{H}} \end{array} \right\}.$$

Proof Let $\omega \in \mathbb{R}$. Then from the limit

$$\lim_{t \searrow 0} \frac{\omega}{2} \frac{\|j(\hat{u} + t\hat{v})\|_{\mathrm{H}}^2 - \|j(\hat{u})\|_{\mathrm{H}}^2}{t} = \omega \langle j(\hat{u}), j(\hat{v}) \rangle_{\mathrm{H}}, \tag{6.5}$$

we obtain first that

$$\partial_j \varphi = \left\{ (u, f) \in \mathrm{H} \times \mathrm{H} \;\middle|\; \begin{array}{c} \exists \hat{u} \in \mathrm{D}(\varphi) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{v} \in \mathrm{V} \\ \liminf_{t \searrow 0} \frac{\varphi_\omega(\hat{u} + t\hat{v}) - \varphi_\omega(\hat{u})}{t} \geq \langle f + \omega j(\hat{u}), j(\hat{v}) \rangle_{\mathrm{H}} \end{array} \right\},$$

which holds for general φ. Now claim (a) follows from the assumption that φ_ω is convex. Claim (b) is a straightforward consequence of the definition of the j-subgradient and the Gâteaux differentiability of φ. □

Remark 6.2 (a) There exists a well-established classical setting in which subgradients of functionals have been defined. This is the setting $\mathrm{V} = \mathrm{H}$ and $j = I$ the identity operator. The j-subgradient then coincides with the usual subgradient defined in the

literature; see, for example, Brezis [40], Rockafellar [128]. In this classical situation, we call j-elliptic functionals simply *elliptic functionals*, we call the j-subgradient simply *subgradient*, and we write $\partial\varphi$ instead of $\partial_j\varphi$.
(b) Another setting frequently encountered in the literature is the case where V is a Banach space and $j : \mathrm{V} \to \mathrm{H}$ is a bounded, *injective* operator with *dense range* (see, for example, J.-L. Lions [103]). In other words, V is a Banach space which is continuously and densely embedded into a Hilbert space H. For simplicity, V may then be identified with a subspace of H (the range of j), so that j reduces to the identity operator which is usually neglected in the notation. Identify H with its dual space, so that we have a Gelfand triple

$$\mathrm{V} \hookrightarrow \mathrm{H} = \mathrm{H}' \hookrightarrow \mathrm{V}'.$$

Let $\varphi : \mathrm{V} \to (-\infty, \infty]$ be a Gâteaux differentiable functional with Gâteaux derivative $\varphi' : \mathrm{V} \to \mathrm{V}'$. By Lemma 6.3 (b), the j-subgradient of φ is then a *single-valued operator* on the Hilbert space H in the sense that for every $u \in \mathrm{H}$ there is at most one $f \in \mathrm{H}$ such that $(u, f) \in \partial_j\varphi$. It is then natural to identify $\partial_j\varphi$ with an operator $\mathrm{H} \supseteq D(\partial_j\varphi) \to \mathrm{H}$. By Lemma 6.3 (b), this operator coincides with the part of the Gâteaux derivative φ' in H.
(c) Conversely, in the setting of (b) above, we may also "extend" the functional $\varphi$to the functional $\varphi^{\mathrm{H}} : \mathrm{H} \to (-\infty, \infty]$ given by

$$\varphi^{\mathrm{H}}(u) := \begin{cases} \varphi(\hat{u}) & \text{if } j(\hat{u}) = u, \\ +\infty & \text{else;} \end{cases}$$

this extension is well defined by the injectivity of j. A straightforward calculation shows that

$$\partial_j\varphi = \partial\varphi^{\mathrm{H}}.$$

Hence, the situation from (b) can be reduced to the situation from (a), that is, the situation of classical subgradients. We shall see below that this remains true in more general situations.

Definition 6.6 A finite sequence $((u_i, f_i))_{0 \le i \le n}$ is called *cyclic* if $(u_0, f_0) = (u_n, f_n)$. An operator $A \subseteq \mathrm{H} \times \mathrm{H}$ is called *cyclically monotone* if for every cyclic sequence $((u_i, f_i))_{0 \le i \le n}$ in A one has

$$\sum_{i=1}^{n} \langle f_i, u_i - u_{i-1} \rangle_{\mathrm{H}} \ge 0.$$

Clearly, every cyclically monotone operator is *monotone* in the sense that for every (u_1, f_1), $(u_2, f_2) \in A$ one has

$$\langle f_2 - f_1, u_2 - u_1 \rangle \ge 0;$$

simply choose $n = 2$ in the previous inequality.

Lemma 6.4 *Assume that $\varphi : \mathrm{V} \to (-\infty, \infty]$ is convex. Then the j-subgradient $\partial_j\varphi$ is cyclically monotone.*

Proof Let $((u_i, f_i))_{0\le i\le n}$ be a cyclic sequence in $\partial_j\varphi$. Then there exists a cyclic sequence $(\hat{u}_i)_{0\le i\le n}$ in V such that

$$j(\hat{u}_i) = u_i \text{ for every } 0 \le i \le n,$$

and, by Lemma 6.3 (a), for every $\hat{v} \in \mathrm{V}$ one has

$$\begin{aligned} \varphi(\hat{u}_1 + \hat{v}) - \varphi(\hat{u}_1) &\ge \langle f_1, j(\hat{v})\rangle_{\mathrm{H}}, \\ \vdots \quad & \quad \vdots \\ \varphi(\hat{u}_n + \hat{v}) - \varphi(\hat{u}_n) &\ge \langle f_n, j(\hat{v})\rangle_{\mathrm{H}}. \end{aligned}$$

Choosing $\hat{v} = \hat{u}_{i-1} - \hat{u}_i$ in the i-th inequality, we obtain

$$\begin{aligned} \varphi(\hat{u}_0) - \varphi(\hat{u}_1) &\ge \langle f_1, j(\hat{u}_0) - j(\hat{u}_1)\rangle_{\mathrm{H}}, \\ \vdots \quad & \quad \vdots \\ \varphi(\hat{u}_{n-1}) - \varphi(\hat{u}_n) &\ge \langle f_n, j(\hat{u}_{n-1}) - j(\hat{u}_n)\rangle_{\mathrm{H}}. \end{aligned}$$

Summing the inequalities and using the cyclicity of $(\hat{u}_i)_{0\le i\le n}$ we obtain

$$0 \ge \sum_{i=1}^{n} \langle f_i, u_{i-1} - u_i\rangle_{\mathrm{H}},$$

which implies the claim. □

By [40, Théorème 2.5, p.38], every cyclically monotone operator A on a Hilbert space H is already contained in a classical subgradient (that is, $j = I$; see Remark 6.2 (a) above). More precisely, [40, Théorème 2.5, p.38] and Lemma 6.4 imply the following result.

Corollary 6.1 *Assume that $\varphi : \mathrm{V} \to (-\infty, \infty]$ is convex. Then there exists a convex, proper, lower semicontinuous functional $\varphi^{\mathrm{H}} : \mathrm{H} \to (-\infty, \infty]$ such that $\partial_j\varphi \subseteq \partial\varphi^{\mathrm{H}}$.*

We shall identify the functional φ^{H} under somewhat stronger assumptions on φ in Section 6.2.3 below; see Theorem 6.5.

Theorem 6.4 *Assume that $\varphi : \mathrm{V} \to (-\infty, \infty]$ is convex, proper, lower semicontinuous and j-elliptic. Then the j-subgradient $\partial_j\varphi$ is maximal monotone.*

Proof Since φ is convex, Lemma 6.4 implies that the j-subgradient is monotone. By Minty's theorem, it suffices to prove that $\omega' I + \partial_j\varphi$ is surjective for some $\omega' > 0$. By assumption, we can choose $\omega \ge 0$ such that φ_ω is convex, proper, lower semicontinuous and coercive. Now, fix $\omega' > \omega$ and let $f \in \mathrm{H}$. Then for every $\hat{u} \in \mathrm{V}$ and $u := j(\hat{u})$ we have by Definition 6.4 (or more precisely Lemma 6.3(a))

$$f \in \omega' u + \partial_j \varphi(u) \tag{6.6}$$

if and only if

$$\varphi_{\omega'}(\hat{u}+\hat{v}) - \varphi_{\omega'}(\hat{u}) \geq \langle f, j(\hat{v})\rangle_{\mathrm{H}} \qquad \text{for all } \hat{v} \in V,$$

or

$$\varphi_{\omega'}(\hat{u}+\hat{v}) - \langle f, j(\hat{u}) + j(\hat{v})\rangle_{\mathrm{H}} \geq \varphi_{\omega'}(\hat{u}) - \langle f, j(\hat{u})\rangle_{\mathrm{H}} \qquad \text{for all } \hat{v} \in V.$$

The latter property is equivalent to

$$\hat{u} = \arg\min \{\varphi_{\omega'}(\cdot) - \langle f, j(\cdot)\rangle_{\mathrm{H}}\}.$$

In other words, finding a solution of the stationary problem (6.6) is equivalent to finding a minimiser of the functional $\varphi_{\omega'}(\cdot) - \langle f, j(\cdot)\rangle_{\mathrm{H}}$. By choice of ω, φ_ω is convex, lower semicontinuous and coercive. Moreover, since $\omega' > \omega$, the functional

$$\mathrm{V} \to \mathbb{R}, \quad \hat{u} \mapsto \frac{\omega' - \omega}{2} \|j(\hat{u})\|^2_{\mathrm{H}} - \langle f, j(\hat{u})\rangle_{\mathrm{H}}$$

is convex, lower semicontinuous, and bounded from below. As a consequence, $\varphi_{\omega'}(\cdot) - \langle f, j(\cdot)\rangle_{\mathrm{H}}$ is convex, lower semicontinuous, and coercive. Hence, sublevels of this functional are convex, closed, and relatively weakly compact. By the Hahn-Banach theorem in the form of [131, Theorem 3.12, p.66], the closure in the topology on V and the weak closure of any convex set are identical. Hence sublevels of this functional are weakly compact. A standard compactness argument using a decreasing sequence of sublevels now implies that the functional above attains its minimum, and the claim follows. □

Corollary 6.2 *Assume that φ is j-semiconvex. Then there exists a proper, lower semicontinuous, elliptic functional $\varphi^{\mathrm{H}} : \mathrm{H} \to (-\infty, \infty]$ such that $\partial_j \varphi \subseteq \partial \varphi^{\mathrm{H}}$. If, in addition, $\varphi$is proper, lower semicontinuous and j-elliptic, then $\partial_j \varphi = \partial \varphi^{\mathrm{H}}$, and $\omega I + \partial_j \varphi$ is maximal monotone for some $\omega \geq 0$.*

Proof By assumption, there exists $\omega \geq 0$ such that φ_ω is convex. Thus by Lemma 6.3 (a) and by definition of the j-subgradient of φ_ω,

$$\omega I + \partial_j \varphi = \partial_j \varphi_\omega. \tag{6.7}$$

Since φ_ω is convex, Corollary 6.1 implies that there is a convex, proper and lower semicontinuous functional $\varphi^H : H \to \mathbb{R} \cup \{+\infty\}$ such that $\partial_j \varphi_\omega \subseteq \partial \varphi^H$ and so by identity (6.7), $\partial_j \varphi \subseteq \partial \varphi^{\mathrm{H}} - \omega I$ holds. Then the functional $\tilde{\varphi}^H := \varphi^H - \frac{\omega}{2}\|\cdot\|^2_H$ defined on H is obviously proper, lower semicontinuous and elliptic with subgradient $\partial \tilde{\varphi}^H = \partial \varphi^H - \omega I$. Hence, replacing φ^H with $\tilde{\varphi}^H$ shows that the first statement of the corollary holds. Further, the inclusion $\partial_j\varphi_\omega \subseteq \partial\varphi^H$ means that $\partial\varphi^H$ is a monotone extension in $H\times H$ of $\partial_j\varphi_\omega$. The additional assumptions that $\varphi$is proper, lower semicontinuous and j-elliptic imply that $\partial_j\varphi_\omega$ is a maximal monotone set in $H\times H$ and hence $\partial_j\varphi_\omega = \partial\varphi^H$. Using again identity (6.7), we obtain that $\omega I + \partial_j\varphi = \partial\varphi^{\mathrm{H}}$ is maximal monotone, completing the proof of this corollary. □

6.2.2 Elliptic extensions

In order to identify the functional φ^{H} from Corollaries 6.1 and 6.2, it is convenient to consider first the set $\hat{E}_u$ of all *elliptic extensions* $\hat{u} \in D(\varphi)$ of an element $u \in \mathrm{H}$, which is defined by

$$\hat{E}_u = \left\{\hat{u} \in D(\varphi)\;\middle|\; j(\hat{u}) = u \text{ and } \liminf_{t\searrow 0} \frac{\varphi(\hat{u}+t\hat{v})-\varphi(\hat{u})}{t} \geq 0 \text{ for every } \hat{v} \in \ker j\right\}.$$

By using the limit (6.5) and since $\langle j(\hat{u}), j(\hat{v})\rangle_{\mathrm{H}} = 0$ for any $\hat{v} \in \ker j$, we see that

$$\hat{E}_u = \left\{\hat{u} \in D(\varphi)\;\middle|\; j(\hat{u}) = u \text{ and } \liminf_{t\searrow 0} \frac{\varphi_\omega(\hat{u}+t\hat{v})-\varphi_\omega(\hat{u})}{t} \geq 0 \text{ for every } \hat{v} \in \ker j\right\}$$

for every $u \in H$ and $\omega \in \mathbb{R}$. Thus, if φ_ω is convex for some $\omega \in \mathbb{R}$, then

$$\hat{E}_u = \left\{\hat{u} \in D(\varphi)\;\middle|\; j(\hat{u}) = u \text{ and } \varphi_\omega(\hat{u}+\hat{v})-\varphi_\omega(\hat{u}) \geq 0 \text{ for every } \hat{v} \in \ker j\right\}$$

for every $u \in H$ and by using the fact that for every $\hat{v} \in \ker j$,

$$\varphi_\omega(\hat{u}+\hat{v}) = \varphi(\hat{u}+\hat{v}) + \frac{\omega}{2}\,\|j(\hat{u}+\hat{v})\|_{\mathrm{H}}^2 = \varphi(\hat{u}+\hat{v}) + \frac{\omega}{2}\,\|j(\hat{u})\|_{\mathrm{H}}^2, \tag{6.8}$$

we can conclude that if φ is j-semiconvex then

$$\hat{E}_u = \left\{\hat{u} \in D(\varphi)\;\middle|\; j(\hat{u}) = u \text{ and } \varphi(\hat{u}+\hat{v})-\varphi(\hat{u}) \geq 0 \text{ for every } \hat{v} \in \ker j\right\} \tag{6.9}$$

for every $u \in H$.

On the one hand, the set $\hat{E}_u$ is motivated by the definition of the j-subgradient $\partial_j\varphi$ (see Definition 6.4). In fact, if $(u,f) \in \partial_j\varphi$, and if $\hat{u} \in D(\varphi)$ is such that $j(\hat{u}) = u$ and

$$\liminf_{t\searrow 0} \frac{\varphi(\hat{u}+t\hat{v})-\varphi(\hat{u})}{t} \geq \langle f, j(\hat{v})\rangle_{\mathrm{H}} \text{ for every } \hat{v} \in \mathrm{V},$$

as in the definition of $\partial_j\varphi$, then $\hat{u}$ is necessarily an elliptic extension of u. Hence,

$$\partial_j\varphi = \left\{(u,f) \in \mathrm{H}\times\mathrm{H}\;\middle|\; \begin{array}{c} \exists\,\hat{u} \in \hat{E}_u \text{ such that for every } \hat{v} \in \mathrm{V} \\ \liminf_{t\searrow 0} \frac{\varphi(\hat{u}+t\hat{v})-\varphi(\hat{u})}{t} \geq \langle f, j(\hat{v})\rangle_{\mathrm{H}} \end{array}\right\}$$

and if φ_ω is convex for some $\omega \in \mathbb{R}$, then we obtain in a similar manner to claim (a) of Lemma 6.3 that

$$\partial_j \varphi = \left\{ (u,f) \in \mathrm{H}\times\mathrm{H} \;\middle|\; \begin{array}{c} \exists\, \hat{u} \in \hat{E}_u \text{ such that for every } \hat{v} \in \mathrm{V} \\ \varphi_\omega(\hat{u}+\hat{v}) - \varphi_\omega(\hat{u}) \geq \langle f + \omega j(\hat{u}), j(\hat{v})\rangle_{\mathrm{H}} \end{array} \right\}.$$

In other words, for the identification of the j-subgradients $f \in \partial_j\varphi(u)$ at a point $u \in H$, we only need to consider elliptic extensions $\hat{u} \in \hat{E}_u$ of u (instead of general $\hat{u} \in D(\varphi)$).

Often, these elliptic extensions are obtained as solutions of an elliptic problem with input data u, explaining why we call them *elliptic extensions*; compare also with Caffarelli and Silvestre [46].

Lemma 6.5 *Let* V, H, j *and* φ *be as above. Then:*

(a) If, for some $\omega \in \mathbb{R}$, *the functional* φ_ω *is convex (resp. coercive, resp. lower semicontinuous), then for every* $\hat{u} \in \mathrm{V}$ *the restriction* $\varphi|_{\hat{u}+\ker j}$ *is convex (resp. coercive, resp. lower semicontinuous).*

(b) If φ *is* j*-semiconvex and* $u = j(\hat{u})$ *for some* $u \in \mathrm{H}$ *and* $\hat{u} \in \mathrm{V}$, *then*

$$\hat{E}_u = \left\{ \hat{v} \in \hat{u} + \ker j \mid \hat{v} \textit{ minimizes } \varphi|_{\hat{u}+\ker j} \right\}.$$

(c) If φ *is* j*-semiconvex, then for every* $u \in D(\partial_j\varphi)$ *and every* $\hat{u} \in \hat{E}_u$ *one has*

$$\varphi(\hat{u}) = \inf_{j(\hat{v})=u} \varphi(\hat{v})$$

In particular, φ *is constant on* $\hat{E}_u$ *for every* $u \in H$.

Proof Claim (a) follows from the trivial observation (6.8), (b) directly from (6.9), and (c) follows from (b). □

6.2.3 Identification of φ^{H}

We shall now identify the functional φ^{H} from Corollaries 6.1 and 6.2 (only up to a constant, of course). Throughout this section, $\varphi$is assumed to be proper and j-semiconvex. For the identification in Theorem 6.5 we will asume, in addition, that φ is in fact convex, lower semicontinuous and j-elliptic.

Consider first the two functionals $\varphi_0, \varphi_1 : H \to (-\infty,\infty]$ given by

$$\begin{aligned} \varphi_0(u) &:= \inf_{j(\hat{u})=u} \varphi(\hat{u}), \text{ and} \\ \varphi_1(u) &:= \sup_{\substack{U\subseteq \mathrm{H} \text{ open} \\ u\in U}} \inf_{j(\hat{v})\in U} \varphi(\hat{v}) \qquad (u \in \mathrm{H}). \end{aligned}$$

By definition of φ_0 and by definition of the j-subgradient,

$$D(\varphi_0) = j(D(\varphi)) \supseteq D(\partial_j \varphi), \tag{6.10}$$

and in particular $\varphi_0(u)$ is finite for every $u \in D(\partial_j \varphi)$. Now choose $(u_0, f_0) \in \partial_j \varphi$, and consider in addition the functionals $\varphi_2, \varphi_3 : H \to (-\infty, \infty]$ given by

$$\varphi_2(u) := \sup \left\{ \sum_{i=0}^{n} \langle f_i, u_{i+1} - u_i \rangle_{\mathrm{H}} + \varphi_0(u_0) \,\middle|\, \begin{array}{c} n \in \mathbb{N},\ (u_i, f_i) \in \partial_j \varphi \\ \text{for } i = 1, \dots, n,\ u_{n+1} = u \end{array} \right\}$$

$$\varphi_3(u) := \sup \left\{ \langle f, u - v \rangle_{\mathrm{H}} + \varphi_0(v) \,\middle|\, (v, f) \in \partial_j \varphi \right\}.$$

Note that formally the definition of the functional φ_2 depends on the choice of the pair (u_0, f_0). However, under somewhat stronger but natural assumptions on φ, it is in fact independent of this choice.

Theorem 6.5 (Identification of φ^{H} for convex φ) *Assume that $\varphi$is convex, proper, lower semicontinuous and j-elliptic, and let φ^{H} be the functional from Corollary 6.1. Then we have*

$$\varphi^{\mathrm{H}} = \varphi_0 = \varphi_1 = \varphi_2 = \varphi_3,$$

where the first equality holds modulo an additive constant, and

$$D(\varphi^{\mathrm{H}}) = j(D(\varphi)).$$

Proof 1*st step.* We claim that the functionals φ_1, φ_2, and φ_3 are convex and lower semicontinuous. The functionals φ_2 and φ_3 are convex and lower semicontinuous because they are pointwise suprema of families of continuous, convex functionals. In order to see that φ_1 is lower semicontinuous, we show that the superlevel sets $\{\varphi_1 > c\}$ are open for every $c \in \mathbb{R}$. If $c \in \mathbb{R}$ and if $u \in \{\varphi_1 > c\}$, then, by definition of the supremum, there exists an open neighbourhood U of u such that

$$\inf_{j(\hat{v}) \in U} \varphi(\hat{v}) > c.$$

However, by definition of φ_1, this means $U \subseteq \{\varphi_1 > c\}$. Hence, the superlevel set $\{\varphi_1 > c\}$ is open for every $c \in \mathbb{R}$, and φ_1 is lower semicontinuous. Convexity of φ_1 is shown by restricting the supremum in the definition of φ_1 to the supremum over *convex*, open neighbourhoods U of the origin 0, by replacing the infimum over all $\hat{v} \in V$ satisfying $j(\hat{v}) \in U$ with the infimum over all $\hat{v} \in V$ satisfying $j(\hat{v}) \in u + U$, and by using a similar argument as for φ_2 and φ_3.

2*nd step.* We prove that

$$\varphi_0 = \varphi_1.$$

The inequality $\varphi_0 \geq \varphi_1$ follows immediately from the definition of both functionals. In order to prove the converse inequality, fix u such that $\varphi_1(u) < \infty$ (if $\varphi_1(u) = \infty$, then the inequality $\varphi_0(u) \leq \varphi_1(u)$ is trivial). By definition of φ_1 and by choosing a filter of open neighbourhoods of u, we find a sequence $(\hat{u}_n)$ in $D(\varphi)$ such that

$$\begin{aligned} u &= \lim_{n\to\infty} j(\hat{u}_n) \quad \text{and} \\ \varphi_1(u) &= \lim_{n\to\infty} \varphi(\hat{u}_n). \end{aligned} \tag{6.11}$$

By assumption, there exists $\omega \geq 0$ such that φ_ω is lower semicontinuous and coercive. The preceding two equalities imply that $(\varphi_\omega(\hat{u}_n))$ is a convergent and thus bounded sequence in $\mathbb{R}$. By coercivity, there exists a weakly convergent subnet $(\hat{u}_\alpha)$ of $(\hat{u}_n)$. Let $\hat{u}$ be its weak limit point. Since j is weak-to-weak continuous, we have $j(\hat{u}) = u$. By definition of φ_0, since $\varphi$is lower semicontinuous, also with respect to the weak topology, and by the second limit in (6.11), we obtain

$$\varphi_0(u) \leq \varphi(\hat{u}) \leq \liminf_\alpha \varphi(\hat{u}_\alpha) = \varphi_1(u) < \infty.$$

3*rd step.* We show that

$$\varphi_0(u) = \varphi_3(u) \tag{6.12}$$

for every $u \in D(\partial_j\varphi)$,

$$\varphi_0(u) \geq \langle f, u - v\rangle + \varphi_0(v) \tag{6.13}$$

for every $u \in H$, $(v, f) \in \partial_j\varphi$, and

$$\varphi_3(u) \geq \langle f, u - v\rangle + \varphi_3(v) \tag{6.14}$$

for every $u \in H$, $(v, f) \in \partial_j\varphi$. Fix $u \in D(\partial_j\varphi)$. The inequality $\varphi_3(u) \geq \varphi_0(u)$ follows by taking $v = u$ in the supremum in the definition of φ_3. Now, let $u \in D(\varphi_0)$ and $(v, f) \in \partial_j\varphi$. By the definition of the j-subgradient and by Lemma 6.5 (c), for every $\hat{v} \in \hat{E}_v$ and every $\hat{u} \in V$ with $j(\hat{u}) = u$,

$$\varphi(\hat{u}) \geq \langle f, u - v\rangle + \varphi(\hat{v}) = \langle f, u - v\rangle + \varphi_0(v).$$

Taking the infimum on the left-hand side of this inequality over all $\hat{u} \in V$ with $j(\hat{u}) = u$, we obtain (6.13). Taking then the supremum on the right-hand side of the inequality (6.13) over all $(v, f) \in \partial_j\varphi$, we obtain

$$\varphi_0(u) \geq \varphi_3(u). \tag{6.15}$$

Since $D(\partial_j\varphi) \subseteq D(\varphi_0)$ (see (6.10)), we obtain that equality (6.12) holds for $u \in D(\partial_j\varphi)$. The inequality (6.14) follows from the definition of φ_3 and inequality (6.15).
4*th step.* We have

$$\begin{aligned} &\partial_j\varphi \subseteq \partial\varphi^{\mathrm{H}}, \\ &\partial_j\varphi \subseteq \partial\varphi_0, \\ &\partial_j\varphi \subseteq \partial\varphi_2, \quad \text{and} \\ &\partial_j\varphi \subseteq \partial\varphi_3. \end{aligned}$$

The first inclusion follows from Corollary 6.1, and the third inclusion from the proof of [40, Théorème 2.5, p.38] and Lemma 6.4. The second and the fourth inclusions follow from (6.13) and (6.14), respectively. By Theorem 6.4, the j-subgradient on the left-hand side of these four inclusions is maximal monotone, that is, it has no proper monotone extension. On the other hand, the subgradients on the right-hand sides are monotone by Step 1, Step 2 and Lemma 6.4. We thus conclude that

$$\partial_j \varphi = \partial \varphi^{\mathrm{H}} = \partial \varphi_2 = \partial \varphi_3 = \partial \varphi_0 (= \partial \varphi_1).$$

Since the functions φ^{H}, $\varphi_0 = \varphi_1$, φ_2 and φ_3 are convex, proper and lower semicontinuous, we can deduce by applying [127, Theorem 3] that

$$\varphi^{\mathrm{H}} = \varphi_0 = \varphi_1 = \varphi_2 = \varphi_3$$

modulo an additive constant. By Steps 2 and 3, and since $\varphi_2(u_0) = \varphi_3(u_0)$, the equalities $\varphi_0 = \cdots = \varphi_3$ hold without adding a constant. The equality $D(\varphi^{\mathrm{H}}) = j(D(\varphi))$ follows from (6.10), and we have proved the claim. □

By using again the equality (6.7) as in the proof of Corollary 6.2, we obtain immediately the following corollary to Theorem 6.5.

Corollary 6.3 (Identification of φ^{H} for j-elliptic φ) *Assume that φ is proper, lower semicontinuous and j-elliptic, and let φ^{H} be the functional from Corollary 6.2. Then one has that*

$$\varphi^{\mathrm{H}} = \varphi_0 = \varphi_1,$$

where the first equality holds modulo an additive constant, and

$$D(\varphi^{\mathrm{H}}) = j(D(\varphi)).$$

6.2.4 The case when j is a weakly closed operator

We shall now briefly discuss a case which is formally more general than the setting considered up to now. As before, we let V be a real locally convex topological vector space and H a real Hilbert space. However,

$$j : \mathrm{V} \supseteq D(j) \to \mathrm{H}$$

is now merely a weakly closed, linear operator, that is, its *graph*

$$G(j) := \left\{ (\hat{u}, j(\hat{u})) \,\middle|\, \hat{u} \in D(j) \right\}$$

is weakly closed in $\mathrm{V} \times \mathrm{H}$, which is equipped with the natural, locally convex product topology. The definition of the j-subgradient of a functional $\varphi : \mathrm{V} \to (-\infty, \infty]$ then admits the following straightforward generalisation:

$$\partial_j \varphi := \left\{ (u,f) \in \mathrm{H} \times \mathrm{H} \;\middle|\; \begin{array}{c} \exists \hat{u} \in \mathrm{D}\varphi) \cap D(j) \text{ s.t. } j(\hat{u}) = u \text{ and for every } \hat{v} \in D(j) \\ \liminf_{t \searrow 0} \frac{\varphi(\hat{u}+t\hat{v}) - \varphi(\hat{u})}{t} \geq \langle f, j(\hat{v}) \rangle_{\mathrm{H}} \end{array} \right\}.$$

This formally more general setting can however be reduced to the setting considered up to now; indeed, it suffices to consider the space

$$\bar{\mathrm{V}} := G(j),$$

equipped with the natural, locally convex topology induced from $\mathrm{V} \times \mathrm{H}$, the operator

$$\bar{j} : \bar{V} \to \mathrm{H}, \\ (\hat{u}, j(\hat{u})) \mapsto j(\hat{u}),$$

and the functional

$$\bar{\varphi} : \bar{V} \to (-\infty, \infty], \\ (\hat{u}, j(\hat{u})) \mapsto \varphi(\hat{u}).$$

Then $\bar{\mathrm{V}}$ is a locally convex topological vector space, and $\bar{j}$ is weak-to-weak continuous. Moreover, one easily verifies that

$$\partial_{\bar{j}} \bar{\varphi} = \partial_j \varphi,$$

where the subgradient on the left-hand side of this equality is the $\bar{j}$-subgradient initially defined and studied throughout this section while the subgradient on the right-hand side of this equality is the j-subgradient defined as above, when j is only a weakly closed, linear operator. Note that it may happen that φ is proper while $\bar{\varphi}$ is not; it is therefore convenient to replace the definition and to say that φ is *proper* if the *effective domain* $D(\varphi) \cap D(j)$ is non-empty. On the other hand, we can make the following simple but useful observations.

Lemma 6.6 *Assume that* V, H, j, φ, $\bar{\mathrm{V}}$, $\bar{j}$ *and* $\bar{\varphi}$ *are as above. If* φ *is convex (resp. coercive, resp. lower semicontinuous), then the same is true of* $\bar{\varphi}$.

So up to changing the definition of properness and of effective domain, all results on j-subgradients from this section remain true, and the same is true for the results below.

6.3 Semigroups and invariance of convex sets

The main results from Section 6.2 and the classical theory of evolution equations governed by subgradients imply the following well-posedness or generation theorem, which is the starting point of this section.

Theorem 6.6 *Let* V *be a real locally convex topological vector space,* H *a real Hilbert space and* $j : \mathrm{V} \to \mathrm{H}$ *a linear, weak-to-weak continuous operator. Let* $\varphi : \mathrm{V} \to (-\infty, \infty]$ *be proper, lower semicontinuous and* j*-elliptic. Then for every initial value* $u_0 \in \overline{D(\varphi^{\mathrm{H}})} = \overline{j(D(\varphi))}$ *the gradient system*

$$\begin{cases} \dot{u} + \partial_j \varphi(u) \ni 0 & \text{on } (0,\infty) \\ \qquad u(0) = u_0 \end{cases} \tag{6.16}$$

admits a unique solution

$$u \in C(\mathbb{R}_+; \mathrm{H}) \cap W^{1,\infty}_{loc}((0,\infty); \mathrm{H})$$

satisfying the differential inclusion (6.16) *for almost every* $t \in (0,\infty)$*. In particular, this also means* $u(t) \in D(\partial_j \varphi)$ *for almost every* $t \in (0,\infty)$.

Corollary 6.4 *Assume that the hypotheses as Theorem* 6.6 *hold. Then, denoting by* u *the unique solution of* (6.16) *corresponding to the initial value* $u_0 \in \mathrm{H}$*, setting* $S(\cdot)u_0 := u$ *defines a strongly continuous semigroup* $S = (S(t))_{t \geq 0}$ *of nonlinear Lipschitz continuous mappings on* $\overline{D(\varphi^{\mathrm{H}})}$.

Notation 6.4 We call the semigroup S the semigroup *generated by* (φ, j) and we write $S \sim (\varphi, j)$.

In what follows, it will be convenient to assume that S is always defined on the entire Hilbert space H. This can be achieved by replacing $S(t)$ by $S(t)P$, if necessary, where P denotes the orthogonal projection onto the closed, convex subset $\overline{D(\varphi^{\mathrm{H}})}$ of H. Note that in this way, the semigroup S is in general only strongly continuous for $t > 0$.

Proof By Corollary 6.2, the j-subgradient of φ is equal to the classical subgradient of a proper, lower semicontinuous, elliptic functional on H. Moreover, up to adding a multiple of the identity, the subgradient is maximal monotone. Well-posedness of the gradient system and generation of a semigroup on the closure $\overline{D(\partial\varphi^{\mathrm{H}})}$ of $D(\partial\varphi^{\mathrm{H}})$ in H follow from [40, Théorème 3.1] while the regularity of solutions is stated in [40, Théorème 3.2]. The characterisation of $\overline{D(\partial\varphi^{\mathrm{H}})}$ used in the statement follows from [40, Proposition 2.11] and Theorem 6.5. □

In the context of gradient systems governed by j-subgradients, one might be interested in the lifting of solutions with values in the reference Hilbert space H to *solutions* with values in the energy space V. By a solution in the energy space we mean a function $\hat{u} : \mathbb{R}_+ \to \mathrm{V}$ such that $u := j(\hat{u})$ coincides almost everywhere with a solution of the gradient system (6.16). It is always possible to find such a lifting, since, by Theorem 6.6, problem (6.16) admits a solution u taking values in $D(\partial_j \varphi)$ almost everywhere. Now it suffices, for almost every $t \in \mathbb{R}_+$, to choose an elliptic extension $\hat{u}(t) \in E_{u(t)} \neq \emptyset$. The measurability or – in Banach spaces – the integrability questions which arise in this context, will not be discussed here. We only mention that if there exists $\omega \in \mathbb{R}$ such that φ_ω is *strictly* convex, or if $\varphi$is strictly convex in each affine subspace $\hat{v} + \ker j$, then the sets $E_{u(t)}$ are singletons, and thus the solution $\hat{u}$ in the energy space is uniquely determined.

We point out that among evolution equations governed by maximal monotone operators, gradient systems play a prominent role which is comparable to the role of evolution equations governed by self-adjoint linear operators among the class of all linear evolution equations. Gradient systems exhibit a regularising effect in the sense that the solution to an arbitrary initial value immediately moves into the domain of the subgradient (see Theorem 6.6 above). Moreover, the non-autonomous gradient system

$$\begin{cases} \dot{u} + \partial_j \varphi(u) \ni f & \text{on } (0,\infty) \\ \qquad u(0) = u_0 \end{cases}$$

has L^2-maximal regularity in the sense that for every initial value $u_0 \in D(\varphi^{\mathrm{H}}) = j(D(\varphi))$ and every right-hand side $f \in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathrm{H})$ there exists a unique solution $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+;\mathrm{H})$ satisfying the differential inclusion almost everywhere [40, Théorème 3.6]. These well-known facts are fundamental for the corresponding solution theory, but are not the central focus of the present article.

The purpose of the rest of this section is to collect some qualitative results for the semigroup S generated by (φ, j) under the additional assumption that the energy functional $\varphi$is convex. In this case, S is a semigroup of contractions [40, Théorème 3.1]. We first characterise invariance of closed, convex sets under the semigroup generated by (φ, j) in terms of the functional φ. We then apply this abstract result in order to characterise positive semigroups, a comparison principle for two semigroups, order preserving semigroups, domination of semigroups, L^∞-contractivity of semigroups and extrapolation, in the case when the underlying Hilbert space H is of the form $L^2(\Sigma,\mu)$ for a suitable measure space $(\Sigma,\mathcal{B},\mu)$. Similar results are known in the literature for semigroups generated by classical subgradients; see Barthélemy [24] (except for the extrapolation result), and indeed, the following results will be obtained as a consequence of the results in the literature together with our identification theorem (Theorem 6.5). Before giving the theorem, we introduce the following notion.

Definition 6.7 We say that a not necessarily densely defined, nonlinear operator S on the Hilbert space H *leaves a subset* $C \subseteq \mathrm{H}$ *invariant* if $SC \subseteq C$. Accordingly, we say that a semigroup S leaves C invariant if $S(t)$ leaves C invariant for every $t \geq 0$.

Our next theorem extends [24, Théorème 1.1].

Theorem 6.7 *Assume that $\varphi$is convex, proper, lower semicontinuous and j-elliptic, and let S be the semigroup on* H *generated by (φ, j). Let $C \subseteq \mathrm{H}$ be a closed, convex set, and denote by P_C the orthogonal projection of* H *onto C. Then the following assertions are equivalent:*

1. *The semigroup S leaves C invariant.*

2. *For every $\lambda > 0$ the resolvent J_λ of $\partial_j\varphi$ leaves C invariant.*
3. *For every $u \in H$ one has*
$$\varphi^{\mathrm{H}}(P_C u) \le \varphi^{\mathrm{H}}(u).$$
4. *For every $\hat{u} \in D(\mathrm{e})$ there is a $\hat{v} \in D(\varphi)$ such that $P_C j(\hat{u}) = j(\hat{v})$ and*
$$\varphi(\hat{v}) \le \varphi(\hat{u}).$$

Proof The equivalence between the assertions (1), (2) and (3) follows from [40, Proposition 4.5] and [24, Théorème 1.1]; we wish to prove that (3) and (4) are equivalent. Without loss of generality, we may assume that the equalities in Theorem 6.5 hold without adding a constant to φ^{H}, that is, in particular, $\varphi^{\mathrm{H}} = \varphi_0$.

Suppose (4) holds and take $u \in H$ such that $\varphi^{\mathrm{H}}(u)$ is finite (otherwise (3) is obviously true). By the characterisation of φ^{H} (Theorem 6.5), and the fact that the infimum in the definition of φ_0 is a minimum, there is a $\hat{u} \in D(\varphi)$ such that $j(\hat{u}) = u$ and $\varphi^{\mathrm{H}}(u) = \varphi_0(u) = \varphi(\hat{u})$. In addition, we can deduce from the hypothesis that there is a $\hat{v} \in D(\varphi)$ satisfying $j(\hat{v}) = P_C u$ and
$$\varphi(\hat{v}) \le \varphi(\hat{u}).$$
Again applying Theorem 6.5 yields
$$\varphi^{\mathrm{H}}(P_C u) = \varphi_0(P_C u) \le \varphi(\hat{v}) \le \varphi(\hat{u}) = \varphi^{\mathrm{H}}(u),$$
and so we have proved (3).

Conversely, suppose that (3) is true. Let $\hat{u} \in D(\varphi)$ such that $j(\hat{u}) = u$. Then the hypothesis, Theorem 6.5, and the fact that the infimum in the definition of φ_0 is a minimum imply that there is a $\hat{v} \in D(\varphi)$ such that $j(\hat{v}) = P_C u$ and
$$\varphi(\hat{v}) = \varphi^{\mathrm{H}}(P_C u) \le \varphi^{\mathrm{H}}(u) \le \varphi(\hat{u}).$$
This proves that (4) is true and thus completes the proof. □

The next theorem is equivalent to Theorem 6.7 and extends [24, Théorème 1.9].

Theorem 6.8 *Assume that φ is convex, proper, lower semicontinuous and j-elliptic, and let $C_1, C_2 \subseteq \mathrm{H}$ be two closed, convex sets such that*
$$P_{C_2} C_1 \subseteq C_1,$$
where, as before, P_{C_2} denotes the orthogonal projection of H *onto C_2. Suppose that the semigroup S generated by (φ, j) leaves C_1 invariant. Then the following assertions are equivalent:*

1. *$S(t)(C_1 \cap C_2) \subseteq C_2$ for every $t \ge 0$.*
2. *For every $u \in C_1$, one has*
$$\varphi^{\mathrm{H}}(P_{C_2} u) \le \varphi^{\mathrm{H}}(u).$$

3. *For every* $\hat{u} \in D(\mathrm{e})$ *with* $j(\hat{u}) \in C_1$ *there is a* $\hat{v} \in D(\mathrm{e})$ *such that* $P_{C_2} j(\hat{u}) = j(\hat{v})$ *and*

$$\varphi(\hat{v}) \leq \varphi(\hat{u}).$$

Indeed, if we take $C_1 = H$ then we see that Theorem 6.7 is a special case of Theorem 6.8. However, with a little bit more effort we also see that Theorem 6.7 implies Theorem 6.8.

Proof The equivalence between assertions (1) and (2) follows from [24, Théorème 1.9] and the equivalence between (2) and (3) is shown by using the same arguments as given in the proof of Theorem 6.7. □

6.3.1 Positive semigroups

Throughout the rest of this section, $(\Sigma, \mathcal{B}, \mu)$ is a measure space and the underlying Hilbert space is $\mathrm{H} = L^2(\Sigma, \mu)$. This Hilbert space is equipped with the natural ordering, the positive cone $L^2(\Sigma, \mu)^+$ being the set of all elements which are positive almost everywhere, which turns it into a Hilbert lattice. The lattice operations are denoted as usual, that is, we write $u \vee v$ and $u \wedge v$ for the supremum and the infimum, respectively, $u^+ = u \vee 0$ is the positive part, $u^- = (-u) \vee 0$ the negative part, and $|u| = u^+ + u^-$ the absolute value of an element $u \in L^2(\Sigma, \mu)$.

Definition 6.8 A semigroup S on $L^2(\Sigma, \mu)$ is called *positive* if $S(t)u \geq 0$ for every $u \geq 0$ and every $t \geq 0$.

In other words, the semigroup S is positive if and only if S leaves the closed positive cone $C := L^2(\Sigma)^+$ invariant. Since the positive cone is also convex, and since the projection onto this cone is given by

$$P_{L^2(\Sigma,\mu)^+} u = u^+,$$

we immediately obtain from Theorem 6.7 the following characterisation of positivity.

Theorem 6.9 (Positive semigroups) *Assume that* φ *is convex, proper, lower semi-continuous and* j*-elliptic, that* $j(D(\varphi))$ *is dense in* $\mathrm{H} = L^2(\Sigma, \mu)$*, let* S *be the semi-group on* $L^2(\Sigma, \mu)$ *generated by* (φ, j)*. Then the following assertions are equivalent:*

1. *The semigroup* S *is positive.*
2. *For every* $u \in L^2(\Sigma, \mu)$*, one has that*

$$\varphi^{\mathrm{H}}(u^+) \leq \varphi^{\mathrm{H}}(u).$$

3. *For every* $\hat{u} \in D(\varphi)$*, there is a* $\hat{v} \in D(\varphi)$ *such that* $j(\hat{u})^+ = j(\hat{v})$ *and*

$$\varphi(\hat{v}) \leq \varphi(\hat{u}).$$

6.3.2 Comparison and domination of semigroups

Theorem 6.10 (Comparison of semigroups) *Let* V_1 *and* V_2 *be two real locally convex topological vector spaces,* $H = L^2(\Sigma,\mu)$ *and let* $j_1 : \mathrm{V}_1 \to L^2(\Sigma,\mu)$ *and* $j_2 : \mathrm{V}_2 \to L^2(\Sigma,\mu)$ *be two linear operators which are weak-to-weak continuous. Further, let* $\varphi_1 : \mathrm{V}_1 \to \mathbb{R}\cup\{+\infty\}$ *and* $\varphi_2 : \mathrm{V}_2 \to \mathbb{R}\cup\{+\infty\}$ *be two convex, proper functionals, which are, respectively,* j_1*- and* j_2*-elliptic, assume that* $j_1(D(\varphi_1))$ *and* $j_2(D(\varphi_2))$ *are dense in* $L^2(\Sigma,\mu)$*, and let* S_1 *and* S_2 *be the semigroups on* $L^2(\Sigma,\mu)$ *generated by* (φ_1, j_1) *and* (φ_2, j_2)*, respectively. In addition, suppose that* $C \subseteq L^2(\Sigma,\mu)$ *is a closed, convex set satisfying*

$$u \wedge v \in C \quad \text{and} \quad u \vee v \in C \quad \text{for every } u,\, v \in C \tag{6.17}$$

and that the semigroups S_1 *and* S_2 *leave* C *invariant. Then the following assertions are equivalent:*

1. *For every* $u,\, v \in C$ *with* $u \le v$ *one has* $S_1(t)u \le S_2(t)v$ *for every* $t \ge 0$.
2. *For every* $u_1,\, u_2 \in C$ *one has*
$$\varphi_1^{\mathrm{H}}(u_1 \wedge u_2) + \varphi_2^{\mathrm{H}}(u_1 \vee u_2) \le \varphi_1^{\mathrm{H}}(u_1) + \varphi_2^{\mathrm{H}}(u_2).$$
3. *For every* $\hat{u}_1 \in D(\varphi_1)$*,* $\hat{u}_2 \in D(\varphi_2)$ *with* $u_1 := j_1(\hat{u}_1) \in C$ *and* $u_2 := j_2(\hat{u}_2) \in C$*, there are* $\hat{v}_1 \in D(\varphi_1)$*,* $\hat{v}_2 \in D(\varphi_2)$ *such that* $u_1 \wedge u_2 = j_1(\hat{v}_1)$*,* $u_1 \vee u_2 = j_2(\hat{v}_2)$ *and*
$$\varphi_1(\hat{v}_1) + \varphi_2(\hat{v}_2) \le \varphi_1(\hat{u}_1) + \varphi_2(\hat{u}_2).$$

Proof Although the equivalence between (1) and (2) follows from [24, Théorème 2.1], we believe it is instructive to show how this can be derived from Theorem 6.7 if one considers the product Hilbert space $\mathcal{H} := L^2(\Sigma,\mu) \times L^2(\Sigma,\mu)$ equipped with the natural inner product, and the product space $\mathcal{V} := \mathrm{V}_1 \times \mathrm{V}_2$ equipped with the natural, locally convex product topology. Let $j : \mathcal{V} \to \mathcal{H}$ be the bounded linear operator and $\Phi : \mathcal{V} \to (-\infty,\infty]$ the functional given respectively by

$$\begin{aligned} &j(\hat{u}_1,\hat{u}_2) := (j_1(\hat{u}_1), j_2(\hat{u}_2)) \text{ and} \\ &\Phi(\hat{u}_1,\hat{u}_2) := \varphi_1(\hat{u}_1) + \varphi_2(\hat{u}_2) \text{ for every } (\hat{u}_1,\hat{u}_2) \in \mathcal{V}. \end{aligned}$$

Then Φ is convex, proper, lower semicontinuous, j-elliptic, and the semigroup $\mathcal{S}$ generated by (Φ, j) is just the diagonal semigroup given by

$$\mathcal{S}(t)(u_1,u_2) = (S_1(t)u_1, S_2(t)u_2) \tag{6.18}$$

for every $t \ge 0$ and every $(u_1,u_2) \in D(S_1) \times D(S_2)$. With these definitions, assertion (1) is equivalent to the property that the product semigroup $\mathcal{S}$ leaves the closed, convex set

$$\mathcal{C} := \{(u,v) \in C \times C : u \le v\}$$

invariant. Note that the orthogonal projection of $\mathcal{H}$ onto $\mathcal{C}$ is not given by $(u_1,u_1) \mapsto (u_1 \wedge u_2, u_1 \vee u_2)$, as one might be led from assertion (2) to assume. However, if we take $C_1 = \mathcal{H}$ and $C_2 = C$, then by Theorem 6.8 and by following the same convexity argument as given in [24, p.247-250], one sees that the property that $\mathcal{S}$ leaves C invariant and assertion (2) are equivalent.

For us, it suffices to show that the assertions (2) and (3) are equivalent. So assume that (2) is true, and let $\hat{u}_i \in D(\varphi_i)$ be such that $u_i := j(\hat{u}_i) \in C$ for $i = 1$, 2. By Theorem 6.5, $u_i \in D(\varphi_i^{\mathrm{H}})$. By hypothesis, $u_1 \wedge u_2 \in D(\varphi_1^{\mathrm{H}})$, $u_1 \vee u_2 \in D(\varphi_2^{\mathrm{H}})$. Since the infimum in the definition of φ_0 is a minimum, it follows that there are $\hat{v}_i \in D(\varphi_i)$ for $i = 1, 2$ such that $j(\hat{v}_1) = u_1 \wedge u_2$ and $j(\hat{v}_2) = u_1 \vee u_2$ satisfying

$$\varphi_1(\hat{v}_1) = \varphi_1^{\mathrm{H}}(u_1 \wedge u_2) \qquad \text{and} \qquad \varphi_2(\hat{v}_2) = \varphi_2^{\mathrm{H}}(u_1 \vee u_2).$$

Combining this together with the inequality from the hypothesis and again the characterisation of φ^{H} (Theorem 6.5) yields

$$\varphi_1(\hat{v}_1) + \varphi_2(\hat{v}_2) = \varphi_1^{\mathrm{H}}(u_1 \wedge u_2) + \varphi_2^{\mathrm{H}}(u_1 \vee u_2) \leq \varphi_1^{\mathrm{H}}(u_1) + \varphi_2^{\mathrm{H}}(u_2) \leq \varphi_1(\hat{u}_1) + \varphi_2(\hat{u}_2).$$

Hence we have proved that (3) holds.

Conversely, assume that (3) is true, and let $u_1 \in D(\varphi_1^{\mathrm{H}}) \cap C$ and $u_2 \in D(\varphi_2^{\mathrm{H}}) \cap C$. Then Theorem 6.5 and the fact that the infimum in the definition of φ_0 is a minimum imply that there are $\hat{u}_i \in D(\varphi_i)$ such that $j(\hat{u}_i) = u_i$ and $\varphi_i(\hat{u}_i) = \varphi^{\mathrm{H}}(u_i)$ for $i = 1$, 2. Let $\hat{v}_1 \in D(\varphi_1)$ and $\hat{v}_2 \in D(\varphi_2)$ be as in the hypothesis. Recalling the identity $\varphi^{\mathrm{H}} = \varphi_0$ from Theorem 6.5, we obtain

$$\begin{aligned} \varphi_1^{\mathrm{H}}(u_1 \wedge u_2) + \varphi_2^{\mathrm{H}}(u_1 \vee u_2) &\leq \varphi_1(\hat{v}_1) + \varphi_2(\hat{v}_2) \\ &\leq \varphi_1(\hat{u}_1) + \varphi_2(\hat{u}_2) = \varphi_1^{\mathrm{H}}(u_1) + \varphi_2^{\mathrm{H}}(u_2). \end{aligned}$$

We formulate two consequences of Theorem 6.10. First, recall from Definition 3.13 that a semigroup $S = (S(t))_{t \geq 0}$ on $L^2(\Sigma, \mu)$ is called *order preserving on* $C \subseteq L^2(\Sigma, \mu)$ if for every $u, v \in C$ with $u \leq v$ one has $S(t)u \leq S(t)v$ for every $t \geq 0$. By taking the semigroup $S := S_1 = S_2$ (and $\varphi := \varphi_1 = \varphi_2$) in the previous theorem, we obtain the characterisation in terms of the functional φ of the property that the semigroup S is order preserving on C. This extends [24, Corollaire 2.2].

Corollary 6.5 (Order-preserving semigroups) *Assume that φ is convex, proper, lower semicontinuous and j-elliptic, and that $j(D(\varphi))$ is dense in $L^2(\Sigma, \mu)$. Suppose that $C \subseteq L^2(\Sigma, \mu)$ is a closed convex set satisfying (6.17) and that the semigroup S on $L^2(\Sigma, \mu)$ generated by (φ, j) leaves C invariant. Then the following assertions are equivalent:*

1. *The semigroup S is order preserving on C.*
2. *For every u_1, $u_2 \in C$ one has*

$$\varphi^{\mathrm{H}}(u_1 \wedge u_2) + \varphi^{\mathrm{H}}(u_1 \vee u_2) \leq \varphi^{\mathrm{H}}(u_1) + \varphi^{\mathrm{H}}(u_2).$$

3. *For every $\hat{u}_1$, $\hat{u}_2 \in D(\varphi)$ with $u_1 := j(\hat{u}_1) \in C$ and $u_2 := j(\hat{u}_2) \in C$, there are $\hat{v}_1$, $\hat{v}_2 \in D(\mathrm{e})$ such that $u_1 \wedge u_2 = j(\hat{v}_1)$, $u_1 \vee u_2 = j(\hat{v}_2)$ and*

$$\varphi(\hat{v}_1)+\varphi(\hat{v}_2) \le \varphi(\hat{u}_1)+\varphi(\hat{u}_2).$$

Definition 6.9 Let S_1 and S_2 be two semigroups on $L^2(\Sigma,\mu)$. We say that the semigroup S_1 *is dominated by* S_2, and we write $S_1 \preccurlyeq S_2$, if S_2 is positive and

$$|S_1(t)u| \le S_2(t)|u|$$

for every $u \in L^2(\Sigma)$ and every $t \ge 0$.

Our next result extends [24, Théorème 3.3].

Corollary 6.6 (Domination of semigroups) *Take the assumptions of Theorem 6.10, and suppose that S_2 is positive and order preserving on $L^2(\Sigma,\mu)^+$. Then the following assertions are equivalent:*

1. *S_1 is dominated by S_2.*
2. *For every $u_1 \in L^2(\Sigma,\mu)$, $u_2 \in L^2(\Sigma,\mu)^+$ one has*

$$\varphi_1^{\mathrm{H}}((|u_1| \wedge u_2)\,\mathrm{sign}(u_1))+\varphi_2^{\mathrm{H}}(|u_1| \vee u_2) \le \varphi_1^{\mathrm{H}}(u_1)+\varphi_2^{\mathrm{H}}(u_2).$$

3. *For every $\hat{u}_1 \in D(\varphi_1)$ with $u_1 := j_1(\hat{u}_1)$, $\hat{u}_2 \in D(\varphi_2)$ with $u_2 := j_2(\hat{u}_2) \in L^2(\Sigma,\mu)^+$ there are $\hat{v}_1 \in D(\varphi_1)$, $\hat{v}_2 \in D(\varphi_2)$ such that*

$$(|u_1| \wedge u_2)\,\mathrm{sign}(u_1) = j_1(\hat{v}_1), \qquad |u_1| \vee u_2 = j_2(\hat{v}_2)$$

and

$$\varphi_1(\hat{v}_1)+\varphi_2(\hat{v}_2) \le \varphi_1(\hat{u}_1)+\varphi_2(\hat{u}_2).$$

Proof (Proof of Corollary 6.6) The equivalence of the assertions (1) and (2) follows from [24, Théorème 3.3] and the equivalence between (2) and (3) is proved by using the same arguments as given above in the proof of Theorem 6.10. □

6.3.3 L^∞-contractivity and extrapolation of semigroups

We begin this subsection with the following definition.

Definition 6.10 Let $\psi : \mathrm{H} \to (-\infty,\infty]$ be a convex, proper and lower semicontinuous functional on a Hilbert space H. Then, a maximal monotone operator $A \subseteq \mathrm{H}\times\mathrm{H}$ is called *ψ-accretive* if for all (u_1,v_1), $(u_2,v_2) \in A$ and all $\lambda > 0$ one has

$$\psi(u_1-u_2+\lambda(v_1-v_2)) \ge \psi(u_1-u_2).$$

Similarly, we introduce the following notion for semgroups.

Definition 6.11 Let $\psi : \mathrm{H} \to (-\infty,\infty]$ be a convex, proper and lower semicontinuous functional on a Hilbert space H. Then, a semigroup S on the Hilbert space H is *ψ-contractive*, if for all $u_1, u_2 \in D(S) \subseteq \mathrm{H}$ and all $t \ge 0$, one has that

$$\psi(S(t)u_1 - S(t)u_2) \leq \psi(u_1 - u_2).$$

In what follows, a family of typical examples of functionals on the Hilbert space $\mathrm{H} = L^2(\Sigma)$ will be the L^p-norms (with effective domain $L^2 \cap L^p(\Sigma)$), and we then also speak of L^p-accretivity of the operator A, or of L^p-contractivity of the semigroup S.

The following result will be useful in the sequel.

Lemma 6.7 ([40, Proposition 4.7]) *Let $A \subseteq \mathrm{H}\times\mathrm{H}$ be a maximal monotone operator on a Hilbert space* H*, and let S be the semigroup generated by $-A$. Further, let $\psi : \mathrm{H} \to (-\infty, \infty]$ be a convex, proper and lower semicontinuous functional. Then A is ψ-accretive if and only if S is ψ-contractive.*

We first characterise L^∞-contractivity of semigroups. The equivalence of assertions (1) and (2) in the following theorem follows from Cipriani and Grillo [53, Section 3] and relies again on Theorem 6.7 and the same product semigroup construction as described in the proof of Theorem 6.10 (see also Bénilan and Picard [31] and Bénilan and Crandall [29]), while the proof of the equivalence of assertions (2) and (3) is similar to the proof of the corresponding equivalence in Theorem 6.7; we omit the details.

Theorem 6.11 (**L^∞-contractivity of semigroups**) *Assume that φ is convex, proper, lower semicontinuous and j-elliptic, that $j(D(\varphi))$ is dense in $L^2(\Sigma,\mu)$, and let S be the semigroup on $L^2(\Sigma,\mu)$ generated by (φ, j). Then the following assertions are equivalent:*

1. *The semigroup S is L^∞-contractive on $L^2(\Sigma,\mu)$.*
2. *For every u_1, $u_2 \in \mathrm{H}$ and for every $\alpha > 0$, one has*

$$\varphi^{\mathrm{H}}\left((u_1 \vee \tfrac{u_1+u_2-\alpha}{2}) \wedge (\tfrac{u_1+u_2+\alpha}{2})\right) + \varphi^{\mathrm{H}}\left((u_2 \wedge \tfrac{u_1+u_2+\alpha}{2}) \vee (\tfrac{u_1+u_2-\alpha}{2})\right)$$
$$\leq \varphi^{\mathrm{H}}(u_1) + \varphi^{\mathrm{H}}(u_2).$$

3. *For every $\hat{u}_1$, $\hat{u}_2 \in D(\varphi)$ with $u_1 = j(\hat{u}_1)$ and $u_2 = j(\hat{u}_2)$, and for every $\alpha > 0$, there are $\hat{v}_1$, $\hat{v}_2 \in D(\varphi)$ such that*

$$\left(u_1 \vee \tfrac{u_1+u_2-\alpha}{2}\right) \wedge \left(\tfrac{u_1+u_2+\alpha}{2}\right) = j(\hat{v}_1),$$
$$\left(u_2 \wedge \tfrac{u_1+u_2+\alpha}{2}\right) \vee \left(\tfrac{u_1+u_2-\alpha}{2}\right) = j(\hat{v}_2),$$

and

$$\varphi(\hat{v}_1) + \varphi(\hat{v}_2) \leq \varphi(\hat{u}_1) + \varphi(\hat{u}_2).$$

If in Theorem 6.11 the semigroup S is in addition order preserving, then we obtain a large number of additional equivalent statements. To that end, we use the notion of *Orlicz spaces* introduced in Definition 3.1 and the set $\mathcal{J}_0$ defined in (3.30) from Chapter 3.2.2.2.

Theorem 6.12 *Suppose in addition to the assumptions of Theorem 6.11 that $j(D(\varphi))$ is dense in H and the semigroup S is order preserving. Then the assertions (1), (2) and (3) from Theorem 6.11 are equivalent to each of the following assertions:*

4. *$\partial_j\varphi$ is L^∞-accretive on $L^2(\Sigma,\mu)$.*
5. *$\partial_j\varphi$ is L^1-accretive on $L^2(\Sigma,\mu)$.*
6. *$\partial_j\varphi$ is L^q-accretive on $L^2(\Sigma,\mu)$ for all $q\in(1,\infty)$.*
7. *$\partial_j\varphi$ is L^ψ-accretive on $L^2(\Sigma,\mu)$ for all N-functions ψ.*
8. *$\partial_j\varphi$ is completely accretive (in the sense of Definition 3.16), that is,*

$$\int_\Sigma \psi(u_1-u_2)\,\mathrm{d}\mu \le \int_\Sigma \psi(u_1-u_2+\lambda(v_1-v_2))\,\mathrm{d}\mu \tag{6.19}$$

for all $\psi\in\mathcal{J}_0$ and all (u_1,v_1), $(u_2,v_2)\in\partial_j\varphi$.

9. *The semigroup S is L^1-contractive on $L^2(\Sigma,\mu)$.*
10. *The semigroup S is L^q-contractive on $L^2(\Sigma,\mu)$ for all $q\in(1,\infty)$.*
11. *The semigroup S is L^ψ-contractive on $L^2(\Sigma,\mu)$ for all N-functions ψ.*
12. *The semigroup S is* completely contractive*, that is,*

$$\int_\Sigma \psi(S(t)u_1-S(t)u_2)\,\mathrm{d}\mu \le \int_\Sigma \psi(u_1-u_2)\,\mathrm{d}\mu \tag{6.20}$$

for all $\psi\in\mathcal{J}_0$, $t\ge 0$ and all u_1, $u_2\in L^2(\Sigma,\mu)$.

Moreover, if one of the equivalent conditions (1)-(12) holds, and if there exists $u_0\in L^1\cap L^\infty(\Sigma)$ such that the orbit $S(\cdot)u_0$ is locally bounded on $\mathbb{R}_+$ with values in $L^1\cap L^\infty(\Sigma)$, then, for every N-function ψ, the semigroup S can be extrapolated to a strongly continuous, order-preserving semigroup S_ψ of contractions on $L^\psi(\Sigma,\mu)$.

Following the convention of [53], we introduce the following definition.

Definition 6.12 A convex, proper and lower semicontinuous functional φ on $L^2(\Sigma,\mu)$ is called a (nonlinear) *semi-Dirichlet form* if φ satisfies property (2) of Corollary 6.5, and one calls φ a (nonlinear) *Dirichlet form* if φ satisfies, in addition, property (2) of Theorem 6.11 above.

Accordingly, a pair (φ,j) consisting of a weak-to-weak continuous operator $j: \mathrm{V}\to L^2(\Sigma)$ and a convex, proper and j-elliptic functional $\varphi:\mathrm{V}\to(-\infty,\infty]$ is called a *Dirichlet form* if (φ,j) satisfies the assertions (3) of Corollary 6.5 and (3) of Theorem 6.11.

By Corollary 6.5 and Theorem 6.11, Dirichlet forms are exactly those energy functionals on $L^2(\Sigma)$, respectively, pairs (φ,j), which generate order preserving, L^∞-contractive semigroups. This characterisation goes back to Bénilan and Picard [31], who also used the term *Dirichlet form* in the nonlinear context. Bénilan and Picard also proved in [31] that semigroups generated by Dirichlet forms extrapolate to contraction semigroups on all $L^q(\Sigma,\mu)$-spaces ($q\in[1,\infty]$) and, more generally, on Orlicz spaces; see also [53, Theorem 3.6] for the L^q case. This result is somewhat parallel to the theory of sesquilinear Dirichlet forms; see, for example, [117, Corollary 2.16]. Theorem 6.12 includes these results from [31, 53, 117].

For the proof of Theorem 6.12, we need first the so-called *duality principle* for subgradients established by Bénilan and Picard [31].

Lemma 6.8 (Duality Principle, [31, Corollaire 2.1 and subsequent Example]) *Let $\varphi^{\mathrm{H}} : L^2(\Sigma,\mu) \to (-\infty,\infty]$ be convex, proper and lower semicontinuous. Further, let $\psi : L^2(\Sigma,\mu) \to [0,\infty]$ be sublinear, proper and lower semicontinuous, and let $\hat{\psi} : L^2(\Sigma,\mu) \to [0,\infty]$ be defined by*

$$\hat{\psi}(u) = \sup_{\psi(v)\le 1} \langle u, v\rangle_H$$

for every $u \in H$. Then the subgradient $\partial\varphi^{\mathrm{H}}$ is ψ-accretive in $L^2(\Sigma,\mu)$ if and only if $\partial\varphi^{\mathrm{H}}$ is $\hat{\psi}$-accretive in $L^2(\Sigma,\mu)$.

Second, we need the following nonlinear interpolation theorem due to Bénilan and Crandall [29]. As in Section 3.2.2.2, let $M(\Sigma,\mu)$ be the space of equivalence classes of measurable functions $f : \Sigma \to \mathbb{R}$, equivalence meaning equality μ-a.e. on Σ, and $L^{1\cap\infty}(\Sigma,\mu)$ the intersection space $L^1(\Sigma,\mu) \cap L^\infty(\Sigma,\mu)$.

Lemma 6.9 ([29, Proposition 1.2]) *Let $S : M(\Sigma) \supseteq D(S) \to M(\Sigma,\mu)$ be an operator such that, for every u, $v \in D(S)$ and every $k \ge 0$, one has either $u \wedge (v+k) \in D(S)$ or $(u-k) \vee v \in D(S)$. Then S satisfies*

$$\int_\Sigma \psi(Su - Sv)\,\mathrm{d}\mu \le \int_\Sigma \psi(u-v)\,\mathrm{d}\mu \quad \text{for all } \psi \in \mathcal{J}_0 \text{ and all } u,\ v \in D(S)$$

if and only if S is order preserving and contractive for the L^1- and L^∞-norms.

Now, we can give the proof of Theorem 6.12.

Proof (Proof of Theorem 6.12) By Lemma 6.7, assertion (4) is equivalent to assertion (1) from Theorem 6.11, and for the same reason assertions (5) and (9), (6) and (10), (7) and (11), and (8) and (12) are equivalent. By the duality principle (Lemma 6.8), assertions (4) and (5) are equivalent.

By Lemma 6.9, and by the assumption that S is order preserving, the now equivalent assertions (1) and (9) imply the assertion (12).

Now assume that assertion (12) holds. Then the inequality in (6.20) holds for every N-function ψ, as well as for every dilation $\psi_\alpha := \psi(\frac{\cdot}{\alpha})$ of an N-function ψ ($\alpha > 0$), and for all $t \ge 0$, and u, $v \in L^2(\Sigma,\mu)$. In other words, if ψ is an N-function, then

$$\int_\Sigma \psi\Big(\frac{S(t)u - S(t)v}{\alpha}\Big)\,\mathrm{d}\mu \le \int_\Sigma \psi\Big(\frac{u-v}{\alpha}\Big)\,\mathrm{d}\mu$$

for every $\alpha > 0$, $t \ge 0$ and all u, $v \in L^2(\Sigma,\mu)$. Taking the infimum over all $\alpha > 0$, one finds that

$$\|S(t)u - S(t)v\|_{L^\psi} \le \|u-v\|_{L^\psi}$$

for all $t \ge 0$ and all u, $v \in L^2(\Sigma,\mu)$, that is, the semigroup S is L^ψ-contractive. Hence, assertion (12) implies assertion (11). The implication (11)⇒(10) follows by choosing $\psi(s) = s^q$ ($q \in (1,\infty)$), and the implication (10)⇒(9) follows from a passage to the limit ($q \to 1$). We have thus proved the equivalence of the assertions (1)-(12).

Now, assume that one of the equivalent assertions (1)-(12) holds, and assume that there exists $u_0 \in L^{1\cap\infty}(\Sigma,\mu)$ such that the orbit $S(\cdot)u_0$ is locally bounded from $\mathbb{R}_+$ with values in $L^1 \cap L^\infty(\Sigma,\mu)$. The latter assumption together with the fact that S is both L^1-contractive and L^∞-contractive implies that for every $u_1 \in L^{1\cap\infty}(\Sigma,\mu)$, the orbit $S(\cdot)u_1$ is locally bounded from $\mathbb{R}_+$ with values in $L^{1\cap\infty}(\Sigma,\mu)$. Now let ψ be an N-function. Since $L^{1\cap\infty}(\Sigma,\mu)$ is contained and dense in $L^\psi(\Sigma,\mu)$, since the semigroup S leaves this subspace of $L^\psi(\Sigma,\mu)$ invariant, and since S is L^ψ-contractive by assertion (11) and order preserving by assumption, the semigroup S extends to an order-preserving semigroup S_ψ of contractions on $L^\psi(\Sigma,\mu)$. In order to see that it is strongly continuous, it suffices to prove strong continuity on the subspace $L^{1\cap\infty}(\Sigma,\mu)$.

Let $u_1 \in L^{1\cap\infty}(\Sigma,\mu)$. Since the orbit $S(\cdot)u_1$ is locally bounded with values in $L^{1\cap\infty}(\Sigma,\mu)$, there exists a constant $C \geq 0$ such that

$$\sup_{t\in[0,1]} (\|S(t)u_1\|_{L^1} + \|S(t)u_1\|_{L^\infty}) \leq C.$$

Let $\varepsilon > 0$. Since ψ is an N-function, there exists $\delta > 0$ such that

$$\psi(s) \leq \varepsilon\, s \text{ for every } s \in [0,\delta].$$

Since the function ψ is bounded on $[\delta, C]$, there exists $C_\delta \geq 0$ such that

$$\psi(s) \leq C_\delta\, s^2 \text{ for every } s \in [\delta, C].$$

Hence,

$$\begin{aligned}
&\limsup_{t\searrow 0} \int_\Sigma \psi(|S(t)u_1 - u_1|)\,\mathrm{d}\mu \\
&\leq \limsup_{t\searrow 0} \left[\int_{|S(t)u_1-u_1|<\delta} \varepsilon\, |S(t)u_1 - u_1|\,\mathrm{d}\mu + \int_{|S(t)u_1-u_1|\geq\delta} C_\delta\, |S(t)u_1 - u_1|^2\,\mathrm{d}\mu \right] \\
&\leq \varepsilon \limsup_{t\searrow 0} \|S(t)u_1 - u_1\|_{L^1} + C_\delta \limsup_{t\searrow 0} \|S(t)u_1 - u_1\|_{L^2}^2 \leq \varepsilon\, 2C.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\lim_{t\searrow 0} \int_\Sigma \psi(|S(t)u_1 - u_1|)\,\mathrm{d}\mu = 0.$$

Replacing ψ by $\psi(\alpha^{-1}\cdot)$ ($\alpha > 0$) in this equality and using the definition of the L^ψ-norm, we deduce

$$\lim_{t\searrow 0} \|S(t)u_1 - u_1\|_{L^\psi} = 0.$$

This completes the proof. □

Remark 6.3 If we assume in Theorem 6.12 that the underlying measure space $(\Sigma, \mathcal{B}, \mu)$ is finite, then the semigroup S is easily seen to extrapolate to a *strongly continuous* contraction semigroup on $L^1(\Sigma, \mu)$, too (contractivity holds in general and is stated in assertion (9)).

Actually, strong continuity in $L^1(\Sigma, \mu)$ also holds for general measure spaces, if there is an element $u_0 \in L^{1\cap\infty}(\Sigma, \mu)$ such that the semigroup S leaves $\{u_0\}$ invariant. We only sketch the proof. Since the resolvent J_1 of $\partial_j\varphi$ is L^1-contractive on $L^1(\Sigma, \mu) \cap L^2(\Sigma, \mu)$ and since by assumption, $J_1 u_0 \in L^1(\Sigma, \mu) \cap L^2(\Sigma, \mu)$, the inverse triangle inequality implies that J_1 maps $L^1(\Sigma, \mu) \cap L^2(\Sigma, \mu)$ into $L^1(\Sigma, \mu)$. Thus J_1 has a unique extension on $L^1(\Sigma, \mu)$ (again denoted by J_1), and so the operator $A := J_1^{-1} - I$ is m-accretive on $L^1(\Sigma, \mu)$. By the Crandall-Liggett Theorem [59], $-A$ generates a strongly continuous contraction semigroup on $L^1(\Sigma, \mu)$, which by construction of A and the concrete form of its resolvent coincides with S on $L^1(\Sigma, \mu)$ by the exponential formula.

6.4 Application: The Dirichlet-to-Neumann map

In this section, we outline the proof of Theorem 6.1 and of Theorem 6.3.

Proof (of Theorem 6.1) It is easily checked that the functional φ defined by (6.1) is proper, convex, and continuously differentiable on $V_{p,2}(\Omega)$ with derivative

$$\varphi'(\hat{u})\hat{v} = \int_\Omega |\nabla \hat{u}|^{p-2} \nabla \hat{u} \nabla \hat{v} \,\mathrm{d}x$$

every $\hat{u}$, $\hat{v} \in V_{p,2}(\Omega)$. Using the definition of the space $V_{p,2}(\Omega)$ together with Maz'ya's inequality 1.29 (from Section 1.6), one sees that the shifted functional

$$\varphi_\omega(\hat{u}) := \varphi(\hat{u}) + \tfrac{\omega}{2} \|\mathcal{T}r(\hat{u})\|^2_{L^2(\partial\Omega)}$$

is coercive for every $\omega > 0$. In other words, φ is $\mathcal{T}r$-elliptic. The fact that $\mathcal{T}r$-subgradient $\partial_{\mathcal{T}r}\varphi$ coincides with the Dirichlet-to-Neumann map Λ_σ follows from the identification of the Fréchet derivative of φ above and from Lemma 6.3 (b). Finally, since the effective domain of φ is the entire space $V_{p,2}(\Omega)$, and since by Lemma 6.1, the trace operator $\mathcal{T}r$ has dense range in $L^2(\partial\Omega)$, it follows from Corollary 6.2, Theorem 6.5 and [40, Proposition 2.11, p.39]) that the $\mathcal{T}r$-subgradient of φ is densely defined. □

To conclude this chapter, we outline the proof of Theorem 6.3.

Proof (of Theorem 6.3) In order to show that the semigroup S generated by $-\Lambda_{|L^2}$ is order preserving, we apply Corollary 6.5. Let $\hat{u}_1$, $\hat{u}_2 \in V_{p,2}(\Omega) = D(\varphi)$. Then, since $j = \mathcal{T}r$ is a lattice homomorphism, one has that

$$j(\hat{u}_1) \wedge j(\hat{u}_2) = j(\hat{u}_1 \wedge \hat{u}_2) \quad \text{and} \quad j(\hat{u}_1) \vee j(\hat{u}_2) = j(\hat{u}_1 \vee \hat{u}_2)$$

with $\hat{u}_1 \wedge \hat{u}_2$, $\hat{u}_1 \vee \hat{u}_2 \in V_{p,2}(\Omega)$, and, noting that the orderings on $V_{p,2}(\Omega)$ and $L^2(\partial\Omega)$ are consistent (cf Lemma 6.2),

$$\begin{aligned}
&\varphi(\hat{u}_1 \wedge \hat{u}_2) + \varphi(\hat{u}_1 \vee \hat{u}_2) \\
&\quad = \tfrac{1}{p}\int_\Omega |\nabla \hat{u}_1|^p \mathbb{1}_{\{\hat{u}_1 \le \hat{u}_2\}}\,\mathrm{d}x + \tfrac{1}{p}\int_\Omega |\nabla \hat{u}_2|^p \mathbb{1}_{\{\hat{u}_1 > \hat{u}_2\}}\,\mathrm{d}x \\
&\qquad\qquad + \tfrac{1}{p}\int_\Omega |\nabla \hat{u}_1|^p \mathbb{1}_{\{\hat{u}_1 > \hat{u}_2\}}\,\mathrm{d}x + \tfrac{1}{p}\int_\Omega |\nabla \hat{u}_2|^p \mathbb{1}_{\{\hat{u}_1 \le \hat{u}_2\}}\,\mathrm{d}x \\
&\quad = \varphi(\hat{u}_1) + \varphi(\hat{u}_2).
\end{aligned}$$

By Corollary 6.5, the semigroup S is order preserving. Next, we show that the semigroup S is L^∞-contractive. Let $\hat{u}_1, \hat{u}_2 \in V_{p,2}(\Omega)$ and $\alpha > 0$ a real number. Then

$$\hat{v}_1 = \left(\hat{u}_1 \vee \tfrac{\hat{u}_1+\hat{u}_2-\alpha}{2}\right) \wedge \left(\tfrac{\hat{u}_1+\hat{u}_2+\alpha}{2}\right) = \begin{cases} \hat{u}_1 & \text{if } |\hat{u}_1 - \hat{u}_2| \le \alpha \\ \frac{\hat{u}_1+\hat{u}_2-\alpha}{2} & \text{if } \hat{u}_1 - \hat{u}_2 < -\alpha \\ \frac{\hat{u}_1+\hat{u}_2+\alpha}{2} & \text{if } \hat{u}_1 - \hat{u}_2 > \alpha \end{cases}$$

and

$$\hat{v}_2 = \left(\hat{u}_2 \wedge \tfrac{\hat{u}_1+\hat{u}_2+\alpha}{2}\right) \vee \left(\tfrac{\hat{u}_1+\hat{u}_2-\alpha}{2}\right) = \begin{cases} \hat{u}_2 & \text{if } |\hat{u}_2 - \hat{u}_1| \le \alpha \\ \frac{\hat{u}_1+\hat{u}_2-\alpha}{2} & \text{if } \hat{u}_2 - \hat{u}_1 < -\alpha \\ \frac{\hat{u}_1+\hat{u}_2+\alpha}{2} & \text{if } \hat{u}_2 - \hat{u}_1 > \alpha \end{cases}$$

are in $V_{p,2}(\Omega)$ and satisfy the first two equalities in Theorem 6.11, assertion (3), with $u_1 = j(\hat{u}_1)$ and $u_2 = j(\hat{u}_2)$; here again, we have used that $j = \mathcal{T}r$ is a lattice homomorphism. It remains to check that

$$\varphi(\hat{v}_1) + \varphi(\hat{v}_2) \le \varphi(\hat{u}_1) + \varphi(\hat{u}_2)$$

in order to see that assertion (3) of Theorem 6.11 is fulfilled. As this is an argument analogous to the one above, we omit it. The fact that the semigroup extrapolates to the whole scale of L^q- and L^ψ-spaces for every N-function ψ now follows immediately from the preceding two steps and Theorem 6.12. □

Chapter 7
The Dirichlet-to-Neumann map associated with the 1-Laplacian

Abstract This chapter is dedicated to the limiting case $p = 1$; we show that that the Dirichlet-to-Neumann map Λ associated with the singular (unweighted) 1-*Laplace operator* can be realized in $L^1(\partial\Omega)$ where Ω is a bounded domain with a boundary $\partial\Omega \in C^1$. We show that the Dirichlet-to-Neumann map Λ comes from a subdifferential in $L^1(\partial\Omega) \times L^\infty(\partial\Omega)$ and that $-\Lambda$ generates a strongly continuous semigroup on $L^1(\partial\Omega)$ and on $L^2(\partial\Omega)$. In particular, we establish the proofs of our main results Theorem 1.9, Theorem 1.10, and Theorem 1.11 in Chapter 1.11. The content of this chapter covers parts of [88].

7.1 Preliminaries

We begin by summarizing some fundamental notions, definitions, and results which we will apply later in this paper.

7.1.1 Functions of bounded variation

We begin by recalling some fundamental facts about functions of bounded variation. For more details on this topic, we refer the interested reader to [1], or [156].

Let Ω an open subset of $\mathbb{R}^d$, $d \geq 1$. Then, a function $u \in L^1(\Omega)$ is said to be a *function of bounded variation in* Ω, if the distributional partial derivatives $D_1 u := \frac{\partial u}{\partial x_1}, \dots, D_d u := \frac{\partial u}{\partial x_d}$ are finite Radon measures in Ω, that is, if

$$\int_\Omega u\, D_i \xi \,\mathrm{d}x = -\int_\Omega \xi \,\mathrm{d}D_i u$$

for all $\xi \in C_c^\infty(\Omega)$, $i = 1, \dots, d$. The linear vector space of functions $u \in L^1(\Omega)$ of bounded variation in Ω is denoted by $BV(\Omega)$. Further, we set $Du = (D_1 u, \dots, D_d u)$

for the *distributional gradient* of u. Then, Du belongs to the class $M^b(\Omega,\mathbb{R}^d)$ of $\mathbb{R}^d$-valued bounded Radon measure on Ω, and throughout this paper, we either write $|Du|(\Omega)$ or $\int_\Omega |Du|$ to denote the *total variation measure* of Du. By (cf [1, Proposition 3.6]), we have

$$|Du|(\Omega) = \sup\left\{ \int_\Omega u \operatorname{div} \mathbf{z}\, dx \,\middle|\, \mathbf{z} \in C_0^\infty(\Omega,\mathbb{R}^d), |\mathbf{z}(x)| \le 1 \text{ for } x \in \Omega \right\}.$$

In addition, it is worth noting that $u \mapsto |Du|(\Omega)$ is lower semicontinuous with respect to the L^1_{loc}-topology.

The space $BV(\Omega)$ equipped with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega),$$

forms a Banach space.

The following result due to Modica [113] is crucial for the minimization problem related to the Dirichlet problem for the 1-Laplace operator.

Proposition 7.1 ([113, Proposition 1.2]) *Let Ω be a bounded domain with a boundary $\partial\Omega$ of class C^1, and $\tau : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be a contraction in the second variable, uniformly with respect to the first one. Then, the functional $F : BV(\Omega) \to \mathbb{R}$ given by*

$$F(u) = \int_\Omega |Du| + \int_{\partial\Omega} \tau(x, \mathcal{T}r(u))\, \mathrm{d}\mathcal{H}^{d-1}$$

is lower semicontinuous on $BV(\Omega)$ with respect to the topology of $L^1(\Omega)$.

According to [73, Theorem 5.3.1] and [1, Theorem 3.87], if Ω is an open and bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$, then there is a bounded linear mapping $\mathcal{T}r : BV(\Omega) \to L^1(\partial\Omega)$ assigning to each $u \in BV(\Omega)$ an element $\mathcal{T}r(u) \in L^1(\partial\Omega)$ such that for $\mathcal{H}^{d-1}$-almost every $x \in \partial\Omega$, one has that $\mathcal{T}r(u)(x) \in \mathbb{R}$ and

$$\lim_{\rho\downarrow 0} \rho^{-d} \int_{\Omega\cap B_\rho(x)} |u(y) - \mathcal{T}r(u)(x)| dy = 0.$$

Moreover, $\mathcal{T}r$ is surjective, and for every $u \in BV(\Omega)$,

$$\int_\Omega u \operatorname{div}\xi\, \mathrm{d}x = -\int_\Omega \xi \cdot \mathrm{d}Du + \int_{\partial\Omega} (\xi\cdot\nu)\, \mathcal{T}r(u)\, \mathrm{d}\mathcal{H}^{d-1} \tag{7.1}$$

for all $\xi \in C^1(\mathbb{R}^d,\mathbb{R}^d)$, where ν denotes the outer unit normal vector on $\partial\Omega$. We call $\mathcal{T}r(u)$ the (weak) *trace* of u and $\mathcal{T}r$ the *trace operator* on $BV(\Omega)$. Note, if there is no danger of confusion, we sometimes also write simply u.

An important notion of convergence of measures in $M^b(\Omega)$ is the *strict convergence*; we say that a sequence $(u_n)_{n\ge1}$ in $BV(\Omega)$ *converges strictly* to some $u \in BV(\Omega)$ if $\int_\Omega |Du_n|$ converges to $\int_\Omega |Du|$ and u_n converges to u in $L^1(\Omega)$. We have the following useful result.

Proposition 7.2 ([1, Theorem 3.88]) *Let Ω be an open bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$. Then, the trace operator $\mathcal{T}r : BV(\Omega) \to L^1(\partial\Omega)$ is continuous from $BV(\Omega)$ equipped with the strict topology to $L^1(\partial\Omega)$, and surjective. Moreover, there exists a constant $C > 0$ such that*

$$\|\mathcal{T}r(u)\|_1 \leq \|u\|_{BV(\Omega)} \quad \textit{for all } u \in BV(\Omega). \tag{7.2}$$

The next proposition on Poincaré's inequality for BV-functions can be deduced from [156, Lemma 4.1.3] (see [52]). Here, we use the notation $\overline{h}$ to denote the *mean value* of a function $h \in L^1(\partial\Omega)$, defined by

$$\overline{h} = \frac{1}{\mathcal{H}^{d-1}(\partial\Omega)} \int_{\partial\Omega} h \,\mathrm{d}\mathcal{H}^{d-1}.$$

Proposition 7.3 *Let Ω be an open bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$. Then, there is a constant $C > 0$ such that*

$$\|\mathcal{T}r(u) - \overline{\mathcal{T}r(u)}\|_1 \leq C \int_\Omega |Du| \qquad \textit{for all } u \in BV(\Omega). \tag{7.3}$$

Next, we recall the following embedding theorems as stated in [109, Theorem 1 in Sect. 6.5.7, Theorem 1 in Sect. 9.5.7] and [129].

Theorem 7.1 *Suppose that $\Omega \subset \mathbb{R}^d$ is an open bounded set with Lipchitz boundary. Then for every $1 \leq p < \infty$, there is a constant $C_{p,d} > 0$ such that*

$$\|u\|_{L^{\frac{pd}{d-p}}(\Omega)} \leq C_{p,d} \left[\|\nabla u\|_{L^p(\Omega)} + \|\mathcal{T}r(u)\|_{L^p(\partial\Omega)}\right]$$

function $u \in W^{1,p}(\Omega)$. Moreover,

$$\|u\|_{L^{\frac{d}{d-1}}(\Omega)} \leq C_d \left[|Du|(\Omega) + \|\mathcal{T}r(u)\|_{L^1(\partial\Omega)}\right] \tag{7.4}$$

for every $u \in BV(\Omega)$.

For the rest of this subsection, we recall several results from [8] (see also cf [5]). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$.

For $1 \leq p \leq d$ and p' given by $1 = \frac{1}{p} + \frac{1}{p'}$, we introduce the following spaces

$$X_p(\Omega) := \left\{\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^d) \,\middle|\, \operatorname{div}(\mathbf{z}) \in L^p(\Omega)\right\}, \text{ and}$$
$$BV(\Omega)_{p'} := BV(\Omega) \cap L^{p'}(\Omega).$$

Then, by the Maz'ya-Sobolev embedding (7.4), one has that

$$BV(\Omega) = BV(\Omega)_{d/(d-1)}.$$

Now, for given $w \in C^1(\Omega)$, $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^d)$, and open subset A of Ω, the integral

$$\mu(A) := \int_A \mathbf{z}\cdot\nabla w \,\mathrm{d}x \tag{7.5}$$

defines a signed Radon measure on Ω. Inspired by (7.5), one can define a bilinear mapping $(\cdot, D\cdot) : X_p(\Omega)\times BV(\Omega)_{p'} \to M^b(\Omega)$ by

$$\langle(\mathbf{z}, Dw), \xi\rangle = -\int_\Omega w\,\xi\,\mathrm{div}(\mathbf{z})\,\mathrm{d}x - \int_\Omega w\,\mathbf{z}\cdot\nabla\xi\,\mathrm{d}x \tag{7.6}$$

for all $\xi\in C_0^\infty(\Omega)$, $\mathbf{z}\in X_p(\Omega)$ and $w\in BV(\Omega)_{p'}$. From (7.6), one obtains the following.

Proposition 7.4 ([8, Theorem 1.5]) , *For every open set $A\subseteq\Omega$ and for all $\xi\in C_0^\infty(A)$, one has that*

$$|\langle(\mathbf{z}, Dw), \xi\rangle| \le \|\xi\|_{L^\infty(A)}\,\|\mathbf{z}\|_{L^\infty(A)}\int_A |Dw|. \tag{7.7}$$

In particular, for given $\mathbf{z}\in X_p(\Omega)$ and $w\in BV(\Omega)_{p'}$, the linear functional $(\mathbf{z}, Dw) : C_0^\infty(\Omega)\to\mathbb{R}$ is a signed Radon measure in Ω with total variation measure $|(\mathbf{z}, Dw)|$.

We shall denote by

$$\int_A (\mathbf{z}, Dw) \quad\text{and}\quad \int_A |(\mathbf{z}, Dw)|$$

the value of the measures $(\mathbf{z}, Dw)$ and $|(\mathbf{z}, Dw)|$ on Borel subsets A of Ω. In fact, the measure $(\mathbf{z}, Dw)$ represents an extension of (7.5); namely, one has that

$$\int_\Omega (\mathbf{z}, Dw) = \int_\Omega \mathbf{z}\cdot\nabla w\,\mathrm{d}x \tag{7.8}$$

for every $w\in W^{1,1}(\Omega)\cap L^\infty(\Omega)$ and $\mathbf{z}\in X_p(\Omega)$.

Proposition 7.5 ([8, Corollary 1.6]) *Let Ω be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$ and for $1\le p\le d$ and p' given by $1 = \frac{1}{p}+\frac{1}{p'}$, let $u\in BV(\Omega)_{p'}$ and $\mathbf{z}\in X_p(\Omega)$. Then the measures $(\mathbf{z}, Du)$ and $|(\mathbf{z}, Du)|$ are absolutely continuous with respect to the measure $|Du|$ in Ω and*

$$\left|\int_B (\mathbf{z}, Du)\right| \le \int_B |(\mathbf{z}, Du)| \le \|\mathbf{z}\|_{L^\infty(A;\mathbb{R}^d)}\int_B |Du| \tag{7.9}$$

for every Borel set B and all open sets A such that $B\subseteq A\subseteq\Omega$.

Thus, there is a density function

$$\theta(\mathbf{z}, Dw, \cdot) = \frac{\mathrm{d}(\mathbf{z}, Dw)}{\mathrm{d}|Dw|} \in L^\infty(\Omega, |Dw|)$$

satisfying

$$|\theta(\mathbf{z}, Dw, x)| = 1 \text{ for } |Dw|\text{-a.e. } x \in \Omega. \tag{7.10}$$

The function $\theta(\mathbf{z}, Dw, \cdot)$ is called the Radon–Nikodým derivative of $(\mathbf{z}, Dw)$ with respect to $|Dw|$. Moreover, the following results holds.

Proposition 7.6 ([8], Chain rule for $(\mathbf{z}, D\cdot)$) *Let Ω be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$ and for $1 \le p \le d$ and p' given by $1 = \frac{1}{p} + \frac{1}{p'}$, let $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega)_{p'}$. Then, for every Lipschitz continuous, monotonically increasing function $T : \mathbb{R} \to \mathbb{R}$, one has that*

$$\theta(\mathbf{z}, D(T \circ w), x) = \theta(\mathbf{z}, Dw, x) \qquad \text{for } |Dw|\text{-a.e. } x \in \Omega. \tag{7.11}$$

Further, there is a unique linear extension $\gamma : X_p(\Omega) \to L^\infty(\partial\Omega)$ satisfying

$$\|\gamma(\mathbf{z})\|_\infty \le \|\mathbf{z}\|_\infty \tag{7.12}$$

and

$$\gamma(\mathbf{z})(x) = \mathbf{z}(x) \cdot \nu(x) \text{ for every } x \in \partial\Omega \text{ and } \mathbf{z} \in C^1(\overline{\Omega}, \mathbb{R}^d).$$

Definition 7.1 ([8]) For every $\mathbf{z} \in X_p(\Omega)$, we write $[\mathbf{z}, \nu]$ for $\gamma(\mathbf{z})$ and call $[\mathbf{z}, \nu]$ the *weak trace* of the normal component of $\mathbf{z}$.

With these preliminaries in mind, we can now state the *generalized integration by parts formula* for functions $w \in BV(\Omega)$.

Proposition 7.7 ([8], Generalized integration by parts) *Let Ω be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$ and let $1 \le p \le d$ and p' be given by $1 = \frac{1}{p} + \frac{1}{p'}$. Then*

$$\int_\Omega w \,\mathrm{div}(\mathbf{z})\,\mathrm{d}x + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z}, \nu] w \,\mathrm{d}\mathcal{H}^{d-1}. \tag{7.13}$$

for every $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega)_{p'}$.

We conclude this section on BV-functions with the following proposition on convergence results.

Proposition 7.8 *Let Ω be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$ and for $1 \le p \le d$ and p' given by $1 = \frac{1}{p} + \frac{1}{p'}$, suppose $(\mathbf{z}_n)_{n\ge1}$ and $\mathbf{z}$ are elements of $X_p(\Omega)$ such that*

$$\lim_{n\to\infty} \mathbf{z}_n = \mathbf{z} \qquad \text{weakly}^* \text{ in } L^\infty(\Omega; \mathbb{R}^d), \text{ and} \tag{7.14}$$

$$\lim_{n\to\infty} \mathrm{div}(\mathbf{z}_n) = \mathrm{div}(\mathbf{z}) \qquad \text{weakly in } L^p(\Omega). \tag{7.15}$$

Then, the following statements hold.

1. *For every $v \in BV(\Omega)_{p'}$,*

$$\lim_{n\to\infty} (\mathbf{z}_n, Dv) = (\mathbf{z}, Dv) \qquad \text{weakly}^* \text{ in } M^b(\Omega) \tag{7.16}$$

2. *For* $v \in BV(\Omega)_{p'}$, (7.16) *implies that*

$$\lim_{n\to\infty} \int_\Omega (\mathbf{z}_n, Dv) = \int_\Omega (\mathbf{z}, Dv). \tag{7.17}$$

3. *If, in addition, there is an* $C > 0$ *such that*

$$\sup_{n\geq 1} \|\mathrm{div}(\mathbf{z}_n)\|_\infty \leq C \tag{7.18}$$

and if there are $(v_n)_{n\geq 1}$, v *in* $BV(\Omega)$ *such that*

$$\lim_{n\to\infty} v_n = v \qquad \text{weakly}^* \text{ in } BV(\Omega), \tag{7.19}$$

then

$$\lim_{n\to\infty} (\mathbf{z}_n, Du_n) = (\mathbf{z}, Dv) \qquad \text{weakly}^* \text{ in } M^b(\Omega). \tag{7.20}$$

The first limit (7.16) is obtained by a light modification of the proof of [8, Proposition 2.1] and for the proofs of (7.17), we were inspired by the proof of [8, Lemma 1.8]. For convenience, we give here the details.

Proof Let $v \in BV(\Omega)_{p'}$. By (7.14), one has that

$$\sup_{n\geq 1} \|\mathbf{z}_n\|_\infty =: M \qquad \text{is finite and } \|\mathbf{z}\|_\infty \leq M. \tag{7.21}$$

Applying (7.21) to (7.9), one sees that

$$\left|\int_\Omega (\mathbf{z}_n, Dv)\right| \leq \int_\Omega |(\mathbf{z}_n, Dv)| \leq M \int_\Omega |Dv|. \tag{7.22}$$

Thus and by (7.7), for verifying that (7.16) holds; that is,

$$\lim_{n\to\infty} \langle (\mathbf{z}_n, Dv), \xi\rangle = \langle (\mathbf{z}, Dv), \xi\rangle \tag{7.23}$$

for every $\xi \in C_0(\Omega)$, it is sufficient to check this limit holds for every test functions $\xi \in C_c^\infty(\Omega)$. But for $\xi \in C_c^\infty(\Omega)$, (7.6) holds, and so by (7.14) and (7.15), one has that

$$\begin{aligned}
\langle (\mathbf{z}_n, Dv), \xi\rangle &= -\int_\Omega v\,\xi\,\mathrm{div}(\mathbf{z}_n)\,\mathrm{d}x - \int_\Omega v\,\mathbf{z}_n \cdot \nabla\xi\,\mathrm{d}x\\
&\to -\int_\Omega v\,\xi\,\mathrm{div}(\mathbf{z})\,\mathrm{d}x - \int_\Omega v\,\mathbf{z}\cdot\nabla\xi\,\mathrm{d}x\\
&= \langle (\mathbf{z}, Dv), \xi\rangle
\end{aligned}$$

as $n \to \infty$, which proves (7.23). Next, to see that (7.17) holds, we perform a 2ε-argument. For this, let $\varepsilon > 0$. Since the total variational measure $|Dv|$ is a bounded Radon measure on Ω, there is a subset $U \Subset \Omega$ such that

$$\int_{\Omega\setminus U} |Dv| \le \frac{\varepsilon}{4M} \tag{7.24}$$

and for every $\xi \in C_c^\infty(\Omega)$, there is an $N(\varepsilon,\xi) \in \mathbb{N}$ such that

$$|\langle(\mathbf{z}_n, Dv), \xi\rangle - \langle(\mathbf{z}, Dv), \xi\rangle| < \frac{\varepsilon}{2} \tag{7.25}$$

for all $n \ge N(\varepsilon,\xi)$. Now, we choose a test function $\xi \in C_c^\infty(\Omega)$ with the properties that $\xi \equiv 1$ on $\overline{U}$ and $0 \le \xi \le 1$ on Ω. Then, by (7.9), (7.21), (7.24) and (7.25), one finds that

$$\begin{aligned}
\left|\int_\Omega (\mathbf{z}_n, Dv) - \int_\Omega (\mathbf{z}, Dv)\right| &\le |\langle(\mathbf{z}_n, Dv), \xi\rangle - \langle(\mathbf{z}, Dv), \xi\rangle| \\
&\qquad \int_\Omega (1-\xi)\, \mathrm{d}|(\mathbf{z}_n, Dv)| + \int_\Omega (1-\xi)\, \mathrm{d}|(\mathbf{z}, Dv)| \\
&\le \frac{\varepsilon}{2} + \int_{\Omega\setminus U} |(\mathbf{z}_n, Dv)| + \int_{\Omega\setminus U} |(\mathbf{z}, Dv)| \\
&\le \frac{\varepsilon}{2} + 2M \int_{\Omega\setminus U} |Dv| \\
&\le \frac{\varepsilon}{2} + 2M \frac{\varepsilon}{4M} = \varepsilon
\end{aligned}$$

for all $n \ge N(\varepsilon,\xi)$, proving (7.17).

To see that the final claim (3) holds, we first note that by (7.6), the limits (7.14), (7.15), (7.18) and (7.19) yield that

$$\begin{aligned}
\langle(\mathbf{z}_n, Dv_n), \xi\rangle &= -\int_\Omega v_n \xi \operatorname{div}(\mathbf{z}_n)\, \mathrm{d}x - \int_\Omega v_n \mathbf{z}_n \cdot \nabla\xi\, \mathrm{d}x \\
&\to -\int_\Omega v \xi \operatorname{div}(\mathbf{z})\, \mathrm{d}x - \int_\Omega v\, \mathbf{z} \cdot \nabla\xi\, \mathrm{d}x \\
&= \langle(\mathbf{z}, Dv), \xi\rangle
\end{aligned}$$

for every $\xi \in C_0(\Omega)$, showing that $((\mathbf{z}_n, Dv_n))_{n\ge1}$ converges to $(\mathbf{z}, Dv)$ in the distributional sense. But since $(v_n)_{n\ge1}$ is bounded in $BV(\Omega)$, (7.22) applied to $w = v_n$ gives that

$$\left|\int_\Omega (\mathbf{z}_n, Dv_n)\right| \le \int_\Omega |(\mathbf{z}_n, Dv_n)| \le M \sup_{n\ge1} \int_\Omega |Dv_n| \le M\,C.$$

Thus and by (7.7), convergence of $((\mathbf{z}_n, Dv_n))_{n\ge1}$ in the distributional sense yields (7.20). □

7.1.2 Nonlinear semigroup theory - Part III

Throughout this second part of the preliminary section, suppose that X is a Banach space with norm $\|\cdot\|_X$, X' its dual space, $\langle\cdot,\cdot\rangle_{X',X}$ the duality brackets on $X'\times X$, and let I denote the *identity* on X.

If $A = \partial_H\varphi$ is the subgradient of a convex, proper, lower semicontinuous function φ on a Hilbert space H, then A is cyclically monotone (cf [127, (2.1) in Section 2]). We recall this result in the next theorem.

Theorem 7.2 (Rockafellar [127], cf [40, Théorème 2.5 & Corollaire 2.8]) *Let A be a monotone operator on a Hilbert space H. Then, the following statements hold.*

1. *A is cyclically monotone if and only if there is a proper, convex and lower semicontinuous function* $\varphi : H \to (-\infty,+\infty]$ *such that* $A \subseteq \partial_H\varphi$.
2. *A is maximal cyclically monotone if and only if there is a proper, convex and lower semicontinuous function* $\varphi : H \to (-\infty,+\infty]$ *such that*

$$A = \partial_H\varphi. \tag{7.26}$$

Moreover, the function φ *in* (7.26) *is unique up to an arbitrary additive constant.*

One can sharpen the statement (2) in Theorem 7.2 for homogeneous operators A of degree $\alpha\in\mathbb{R}$.

Theorem 7.3 *Let A be a homogeneous operator on a Hilbert space H of order* $\alpha\in\mathbb{R}$. *Then A is maximal cyclically monotone if and only if* (7.26) *holds for a unique proper, convex, lower semicontinuous* $\varphi : H \to [0,+\infty]$ *satisfying*

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(\lambda u) = \lambda^{\alpha+1}\varphi(u) \quad \text{for all } u \in D(A). \tag{7.27}$$

Proof By Theorem 7.2, we have that A is cyclically monotone if and only if there is a proper, convex and lower semicontinuous function $\varphi : H \to [0,+\infty]$ such that (7.26) holds. Moreover, the functional φ is given by

$$\varphi(u) := \sup_{n\in\mathbb{N}} \sup_{((u_i,v_i))_{i=0}^n\subseteq A} \Big\{(u-u_n,v_n)_H + \sum_{j=1}^{n}(u_j-u_{j-1},v_{j-1})_H\Big\}, u \in H,$$

(cf. Rockafellar [127]). Thus, it remains to verify that this functional φ satisfies $\varphi\geq 0$ on H and (7.27). To see this, let $(u_0,v_0) = (0,0)$ in the definition of φ. Then by the cyclic monotonicity of A, one has that $\varphi(0) = 0$. Moreover, since $(0,0)\in\partial_H\varphi$, it follows from the convexity of φ that $\varphi\geq 0$ on H. It is left to verify that

$$\varphi(\lambda u) = \lambda^{\alpha+1}\varphi(u) \quad \text{for all } u \in D(A). \tag{7.28}$$

Note, since $D(A)\subseteq D(\varphi)$, it follows from the homogeneity of A that for every $u\in D(A)$ and $\lambda\geq 0$, one has $\lambda u\in D(\varphi)$. Now, fix $u\in D(A)$ and $\lambda>0$. Since for every finite sequence $((u_i, v_i))_{i=0}^n \subseteq A$, one has that $((\lambda u_i, \lambda^\alpha v_i))_{i=0}^n \subseteq A$, it follows that

$$\begin{aligned}
&(\lambda u - \lambda u_n, \lambda^{\alpha+1} v_n)_H + \sum_{j=1}^n (\lambda u_j - \lambda u_{j-1}, \lambda^{\alpha+1} v_{j-1})_H \\
&= \lambda^{\alpha+1} \left\{ (u - u_n, v_n)_H + \sum_{j=1}^n (u_j - u_{j-1}, v_{j-1})_H \right\} \\
&\le \lambda^{\alpha+1} \varphi(u)
\end{aligned}$$

for every finite sequence $((u_i, v_i))_{i=0}^n \subseteq A$. Hence, by taking the supremum over all $((u_i, v_i))_{i=0}^n \subseteq A$ in the above inequality yields that

$$\varphi(\lambda u) \le \lambda^{\alpha+1} \varphi(u).$$

On the other hand, for every finite sequence $((u_i, v_i))_{i=0}^n \subseteq A$,

$$\begin{aligned}
&\lambda^{\alpha+1} \left\{ (u - u_n, v_n)_H + \sum_{j=1}^n (u_j - u_{j-1}, v_{j-1})_H \right\} \\
&= (\lambda u - \lambda u_n, \lambda^{\alpha+1} v_n)_H + \sum_{j=1}^n (\lambda u_j - \lambda u_{j-1}, \lambda^{\alpha+1} v_{j-1})_H \\
&\le \varphi(\lambda u).
\end{aligned}$$

Taking again the supremum over all $((u_i, v_i))_{i=0}^n \subseteq A$ in this inequality leads to the reverse inequality $\lambda^{\alpha+1} \varphi(u) \le \varphi(\lambda u)$. The uniqueness of a convex, proper, lower semicontinuous functional φ satisfying (7.27) follows from the fact that $\varphi(0) = 0$. This completes the proof of this theorem. □

Convex functionals, which are homogeneous of order $\alpha + 1$, $\alpha \in \mathbb{R}$, admit the following important property.

Proposition 7.9 *Let $\varphi : X \to [0, +\infty]$ be a convex, proper, and lower semicontinuous functional on a Banach space X and suppose, there is an $\alpha \in \mathbb{R}$ such that (7.27) holds. Then, one has that*

$$(\alpha + 1)\varphi(u) = \langle x', u \rangle_{X',X} \qquad \textit{for every } (u, x') \in \partial_{X \times X'} \varphi. \tag{7.29}$$

Proof Let $(u, x') \in \partial_{X \times X'} \varphi$. Then, by the definition of the sub-differential $\partial_{X \times X'} \varphi$, one has that

$$\langle x', w - u \rangle_{X',X} \le \varphi(w) - \varphi(u)$$

for every $w \in H$. For $t \in (-1, 1]$, let $w = (1 + t)u$. Then by (7.27), $w \in D(\varphi)$, the previous inequality reduces to

$$t \langle x', u \rangle_{X',X} \le \left((1 + t)^{\alpha+1} - 1 \right) \xi(u).$$

From this, we can deduce that (7.29) holds by first taking $t > 0$ then dividing by t and subsequently sending $t \longrightarrow 0+$, and the proceed in a similar way for $t < 0$. □

7.2 The Dirichlet problem for the 1-Laplace operator

In this section, we review the current state of knowledge about existence and uniqueness to the singular Dirichlet problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) = 0 & \text{in } \Omega, \\ \qquad\qquad u = h & \text{on } \partial\Omega, \end{cases} \tag{7.30}$$

for given boundary data $h \in L^1(\partial\Omega)$. As mentioned in the introduction of this paper, we assume that Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 2$ with a C^1-boundary $\partial\Omega$.

In order to obtain existence of solutions to Dirichlet problem (7.30), it is natural, to study the existence of a minimizer of the famous *least gradient problem*

$$\inf\left\{\int_\Omega |Dv| \,\Big|\, v \in BV(\Omega),\ v = h \text{ on } \partial\Omega\right\}. \tag{7.31}$$

Existence of solutions to the minimizing problem (7.31) was obtained by Parks [118, 119] under the hypotheses Ω is strictly convex and the boundary data h satisfies the bounded slope condition. Sternberg, Williams and Ziemer [138] improved this result by establishing existence and uniqueness of a minimizer $u \in BV(\Omega) \cap C(\overline{\Omega})$ of (7.31) for boundary data $u \in C(\partial\Omega)$ on bounded domains Ω with a Lipschitz boundary $\partial\Omega$ of non-negative mean curvature (in the weak sense) and not being locally area-minimizing.

On $BV(\Omega)$, there is a continuous trace operator $\mathcal{T}r : BV(\Omega) \longrightarrow L^1(\partial\Omega)$ available (see Proposition 7.2). Thus Sternberg, Williams and Ziemer called in [139] a function $u \in BV(\Omega)$ to be of *least gradient* if

$$\int_\Omega |Du| = \min\left\{\int_\Omega |Dv| \,\Big|\, v \in BV(\Omega),\ \mathcal{T}r(u) = \mathcal{T}r(v)\right\}.$$

Since for given $h \in L^1(\partial\Omega)$, there is a $H \in BV(\Omega)$ satisfying $\mathcal{T}r(H) = h$, a function $u \in BV(\Omega)$ satisfies the boundary constrain

$$u = h \qquad \text{on } \partial\Omega \tag{7.32}$$

in the *traces sense* if $\mathcal{T}r(u) = \mathcal{T}r(H)$. In many elliptic boundary-value problems (as for example, the Dirichlet problem associated with the p-Laplace operator, see, e.g., [85]), it is standard that the solution attains the boundary condition (7.32) merely in the sense of traces. However, by using this weak notion of attaining the boundary condition (7.32), a function $u \in BV(\Omega)$ is a minimizer of (7.31) if u minimizes the total variation $\int_\Omega |Dv|$ on the affine space $\mathcal{T}r(H) + BV_0(\Omega)$ (cf. [139, Theorem 2.2]), where $BV_0(\Omega)$ is the closure of the BV-norm of the set of test functions $C_c^\infty(\Omega)$. But this last problem has the two challenges that the trace operator $\mathcal{T}r$ is only continuous with respect to the strict topology and of missing compactness results on $BV(\Omega)$. Thus, to establish existence and uniqueness of a minimizer to (7.31) and related problems, the continuity condition on the boundary data h was used by many authors, including Miranda [112], Parks and Ziemer [120], Bombieri, De Giorgi, Giusti [36], or more recently, Jerrard, Moradifam, and Nachman [91].

Recently, Spradlin and Tamasan [137] constructed an essentially bounded boundary function h on the unit circle S^1 in $\mathbb{R}^2$ for which the minimizing problem (7.31) has no solution $u \in BV(\Omega)$ satisfy (7.32) in the sense of traces. If the set of discontinuities is countable, then in the planar case, Górny [80] (see also [82, 81], and Rybka and Sabra [82]) could establish existence of a minimizer to problem (7.31).

This suggests that for discontinuous boundary data $h \in L^1(\partial\Omega)$, the notion of traces for the boundary condition (7.32) might not be the right one for establishing existence of a minimizer to problem (7.31). Thus, Rossi, Segura and Mazón [110] studied for given $h \in L^1(\partial\Omega)$, the following *relaxed* functional $\Phi_h : L^{\frac{d}{d-1}}(\Omega) \to (-\infty, +\infty]$ given by

$$\Phi_h(v) = \begin{cases} \displaystyle\int_\Omega |Dv| + \int_{\partial\Omega} |h - v| \,\mathrm{d}\mathcal{H}^{d-1} & \text{if } v \in BV(\Omega), \\ +\infty & \text{if } v \in L^{\frac{d}{d-1}}(\Omega) \setminus BV(\Omega). \end{cases} \tag{7.33}$$

The functional Φ_h is convex, lower semicontinuous on $L^{\frac{d}{d-1}}(\Omega)$, and thanks to the Sobolev inequality for BV-functions (see (7.4) in Section 7.1.1), Φ_h is coercive. Thus, there is a $u \in BV(\Omega)$ solving the variational problem

$$\min_{v \in BV(\Omega)} \Phi_h(v). \tag{7.34}$$

One easily verifies that for given $h \in L^1(\partial\Omega)$ and $H \in BV(\Omega)$ satisfying $\mathcal{T}r(H) = h$, if $u \in BV(\Omega)$ is a function of least gradient satisfying the boundary condition (7.32) in the trace sense, then u is also a minimizer of problem (7.34). Moreover, every minimizer u of (7.34) satisfies the following inclusion of the first variation

$$0 \in \partial_{L^{\frac{d}{d-1}} \times L^d(\Omega)} \Phi_h(u) \qquad \text{in } L^{\frac{d}{d-1}}(\Omega) \times L^d(\Omega), \tag{7.35}$$

which is directly related to notion of *weak solutions* to Dirichlet problem (7.30).

By characterizing the subdifferential $\partial_{L^{\frac{d}{d-1}} \times L^d(\Omega)} \Phi_h$, the authors of [110] discovered that for boundary data $h \in L^1(\partial\Omega)$, a minimizer u_h of (7.34) satisfies the Dirichlet boundary condition (7.32) in problem (7.30) merely in the following *weaker sense*: there is a divergence free vector field $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ such that $\|\mathbf{z}_h\|_\infty \le 1$ and

$$[\mathbf{z}_h, \nu] \in \operatorname{sign}(h - \mathcal{T}r(u_h)) \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega, \tag{7.36}$$

where $[\mathbf{z}_h, \nu]$ denotes Anzellotti's generalized *normal trace*, ν the outward-pointing unit normal vector (see Section 7.1.1), and $\text{sign}(\cdot)$ is the accretive graph in $\mathbb{R}^2$ of the *signum* given by

$$\text{sign}(r) := \begin{cases} 1 & \text{if } r > 0, \\ [-1, 1] & \text{if } r = 0, \\ -1 & \text{if } r < 0. \end{cases}$$

More precisely, they obtained the following one.

Proposition 7.10 ([110, Theorem 2.5]) *For $h \in L^1(\partial\Omega)$ and $u \in BV(\Omega)$, the following statements are equivalent:*

(i) $0 \in \partial_{L^{\frac{d}{d-1}} \times L^d(\Omega)} \Phi_h(u)$.
(ii) there exists a vector field $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (7.36),

$$\|\mathbf{z}_h\|_\infty \leq 1, \tag{7.37}$$
$$-\text{div}(\mathbf{z}_h) = 0 \qquad \text{in } \mathcal{D}'(\Omega), \text{ and} \tag{7.38}$$
$$(\mathbf{z}_h, Du) = |Du| \qquad \text{as Radon measures.} \tag{7.39}$$

Having this characterization in mind, every solution $u \in BV(\Omega)$ of the least gradient problem (7.31) is a *weak solution* to the Dirichlet problem (7.30), and vice versa.

Definition 7.2 For given $h \in L^1(\partial\Omega)$, we call a function $u_h \in BV(\Omega)$ a *weak* solution to Dirichlet problem (7.30) if there exists a vector field $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (7.36)–(7.39).

By using Definition 7.2, further examples could be constructed showing the phenomenon of non-uniqueness in Dirichlet problem (7.30).

Example 7.1 In [110], the following counter example to the uniqueness of solutions to Dirichlet problem (7.30) on the unit ball $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ was constructed for discontinuous boundary data. Let the boundary function $h \in L^\infty(\partial\Omega)$ be given (in polar coordinates) by

$$h(\theta) := \begin{cases} \cos(2\theta) + 1, & \text{if } \cos(2\theta) > 0; \\ \cos(2\theta) - 1, & \text{if } \cos(2\theta) < 0; \end{cases}$$

for every $\theta \in (-\pi, \pi]$. Now, for every $-1 \leq \lambda \leq 1$, let $u^\lambda : \overline{\Omega} \to \mathbb{R}$ be given by

$$u^\lambda(x, y) = \begin{cases} 2x^2, & \text{if } |x| > \frac{\sqrt{2}}{2}, |y| < \frac{\sqrt{2}}{2}; \\ \lambda, & \text{if } |x| < \frac{\sqrt{2}}{2}, |y| < \frac{\sqrt{2}}{2}; \\ -2y^2, & \text{if } |x| < \frac{\sqrt{2}}{2}, |y| > \frac{\sqrt{2}}{2}. \end{cases}$$

Then, each u^λ is a weak solution of Dirichlet problem (7.30) satisfying the boundary conditions (7.32) in the weaker sense (7.36) with h.

Example 7.1 and the one given in [79] demonstrate well that smoothness of the boundary $\partial\Omega$ and other nice geometric properties of Ω (as, for instance, convexity of Ω) are not sufficient to establish uniqueness of solutions to the Dirichlet problem (7.30) for discontinuous boundary data $h \in L^\infty(\partial\Omega)$. This justifies the notation of differential inclusion used in (7.35). But, in particular, shows that the Dirichlet-to-Neumann operator Λ might be multi-valued.

Next, we turn to the following observation (cf., [111, Remark 2.9], where a similar observation has been made for weak solutions u of the 1-Laplace equation $-\Delta_1 u = 0$ equipped with some Robin-type boundary conditions).

Theorem 7.4 *For given $h \in L^1(\partial\Omega)$, let u_h and $\hat{u}_h$ be two weak solutions of Dirichlet problem (7.30) for the same boundary data h and let $\mathbf{z}_h$ and $\hat{\mathbf{z}}_h \in L^\infty(\Omega;\mathbb{R}^d)$ be two divergence-free vector fields satisfying (7.37). If $\mathbf{z}_h$ satisfies (7.36) and (7.39) with respect to u_h and $\hat{\mathbf{z}}_h$ satisfies (7.36) and (7.39) with respect to $\hat{u}_h$, then $\hat{\mathbf{z}}_h$ also satisfies (7.36) and (7.39) with respect to u_h and $\mathbf{z}_h$ satisfies (7.36) and (7.39) with respect to $\hat{u}_h$.*

Due to the statement of Theorem 7.4, it is useful to introduce the set

$$\mathcal{Z}_h := \left\{ \mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d) \,\middle|\, \begin{array}{l} \mathbf{z}_h \text{ satisfying (7.36)–(7.39) for a weak} \\ \text{solution } u_h \text{ of Dirichlet problem (7.30)} \end{array} \right\} \tag{7.40}$$

for every $h \in L^1(\partial\Omega)$. Since for given $h \in L^1(\partial\Omega)$, the minimization problem (7.34) admits a solution u_h, which by Proposition 7.10 is a weak solution of Dirichlet problem (7.30), the set $\mathcal{Z}_h$ is non-empty.

Before we give the proof of Theorem 7.4, let us first mention the following consequence (cf., [114, Theorem 1.2]). This corollary follows directly from the fact that the Dirichlet problem (7.30) always admits a weak solution and by using Theorem 7.4.

Corollary 7.1 *For given boundary data $h \in L^1(\partial\Omega)$, there is a divergence-free vector field $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying $\|\mathbf{z}_h\|_\infty \le 1$ such that every weak solution u_h of Dirichlet problem (7.30) satisfies (7.39) and (7.36) for the same $\mathbf{z}_h$.*

Corollary 7.1 says that there is a divergence-free vector field $\mathbf{z}_h$, which determines the level set of all weak solutions u_h of Dirichlet problem (7.30). More precisely, since $\|\mathbf{z}_h\|_\infty \le 1$ on Ω, (7.9) yields that

$$\int_\Omega (\mathbf{z}_h, Dw) \le 1 \tag{7.41}$$

for every $w \in BV(\Omega)$ with $|Dw|(\Omega) = 1$. Note, that (7.41) can well be interpreted as a *Radon-measure version* of the point-wise inequality

$$\mathbf{z}_h \cdot \xi \le 1 \qquad \text{a.e. on } \Omega$$

holding for any vector fields $\xi \in S^{d-1}$. Thus, (7.39) says that for every weak solutions u_h of Dirichlet problem (7.30), the vector field $Dw = Du_h/|Du_h|$ maximizes (7.41) in the sense of Radon measures. Recall, for given vector fields $\mathbf{z} \in \mathbb{R}^d$ with $|\mathbf{z}| \le 1$ and $\xi \in S^{d-1}$, the equality $z \cdot \xi = 1$ implies that $\mathbf{z}$ and ξ are parallel to each other and $|\mathbf{z}| = 1$. Thus, and since $|Dw|(\Omega) = 1$, (7.39) one be can understood as a condition implying that the two vector fields $\mathbf{z}_h$ and Dw are parallel to each other in some weak sense.

Further, as outlined in [114], (7.36) describes the set of possible jumps on the boundary $\partial\Omega$ of a weak solution u_h of (7.30). More precisely, it follows from (7.36) that up to a set of $\mathcal{H}^{d-1}$-measure zero, one has that

$$\begin{aligned}
\{x \in \partial\Omega \,|\, \mathcal{T}r(u_h)(x) > h(x)\} &\subseteq \{x \in \partial\Omega \,|\, [\mathbf{z}_h, \nu] = -1\},\\
\{x \in \partial\Omega \,|\, \mathcal{T}r(u_h)(x) < h(x)\} &\subseteq \{x \in \partial\Omega \,|\, [\mathbf{z}_h, \nu] = 1\},
\end{aligned}$$

and

$$\{x \in \partial\Omega \,|\, \mathcal{T}r(u_h)(x) = h(x)\} \subseteq \{x \in \partial\Omega \,|\, -1 \le [\mathbf{z}_h, \nu] \le 1\}.$$

Now, we turn to the proof of Theorem 7.4.

Proof (Proof of Theorem 7.4) Let u_h and $\hat{u}_h$ be two solutions of Dirichlet problem (7.30) for the same given boundary function $h \in L^1(\partial\Omega)$. Further, let $\mathbf{z}_h$ and $\hat{\mathbf{z}}_h \in L^\infty(\Omega;\mathbb{R}^d)$ be two vector fields satisfying (7.36)–(7.39) with respect to u_h and $\hat{u}_h$, respectively.

Note, by (7.38), the two vector fields $\mathbf{z}_h$ and $\hat{\mathbf{z}}_h$ belong to $X_d(\Omega)$ and by Sobolev-inequality (7.4), $(u_h - \hat{u}_h) \in BV_{d/(d-1)}(\Omega)$. Thus, the generalized integration by parts formula (7.13) yields

$$\int_\Omega (\mathbf{z}_h, D(u_h - \hat{u}_h)) - \int_{\partial\Omega} [\mathbf{z}_h, \nu]\big(\mathcal{T}r(u_h) - \mathcal{T}r(\hat{u}_h)\big)\,\mathrm{d}\mathcal{H}^{d-1} = 0$$

and

$$\int_\Omega (\hat{\mathbf{z}}_h, D(u_h - \hat{u}_h)) - \int_{\partial\Omega} [\hat{\mathbf{z}}_h, \nu]\big(\mathcal{T}r(u_h) - \mathcal{T}r(\hat{u}_h)\big)\,\mathrm{d}\mathcal{H}^{d-1} = 0\,.$$

Subtracting these two equations from each other and using the fact that the pairing $(\mathbf{z}_h, Dw)$ is bilinear yields

$$\begin{aligned}
&\int_\Omega (\mathbf{z}_h - \hat{\mathbf{z}}_h, D(u_h - \hat{u}_h))\\
&\quad + \int_{\partial\Omega} \big([\mathbf{z}_h, \nu] - [\hat{\mathbf{z}}_h, \nu]\big)\big(h - \mathcal{T}r(u_h) - (h - \mathcal{T}r(\hat{u}_h))\big)\,\mathrm{d}\mathcal{H}^{d-1} = 0\,.
\end{aligned} \tag{7.42}$$

Since $\mathbf{z}_h$ and $\hat{z}_h$ satisfy $\|\mathbf{z}_h\|_\infty \le 1$, $\|\hat{\mathbf{z}}_h\|_\infty \le 1$, it follows from (7.9) that

$$\left|\int_\Omega (\mathbf{z}_h, D\hat{u})\right| \le |D\hat{u}|\,(\Omega) \quad \text{and} \quad \left|\int_\Omega (\hat{\mathbf{z}}_h, Du)\right| \le |Du|(\Omega).$$

Thus, the bilinearity of the pairing $(\cdot, D\cdot)$ yields that

$$\int_\Omega (\mathbf{z}_h - \hat{\mathbf{z}}_h, D(u_h - \hat{u}_h)) = |D\hat{u}_h|(\Omega) - \int_\Omega (\mathbf{z}_h, D\hat{u}_h) + |Du_h|(\Omega) - \int_\Omega (\hat{\mathbf{z}}_h, Du_h) \geq 0. \tag{7.43}$$

Further, by the monotonicity of the sign-graph in $\mathbb{R}^2$, and since $\mathbf{z}_h$ and $\hat{\mathbf{z}}_h$ satisfy (7.36), one has that

$$\big([\mathbf{z}_h, \nu] - [\hat{\mathbf{z}}_h, \nu]\big)\big((h - \mathcal{T}r(u_h)) - (h - \mathcal{T}r(\hat{u}_h))\big) \geq 0 \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega$$

and so,

$$\int_{\partial\Omega} \big([\mathbf{z}_h, \nu] - [\hat{\mathbf{z}}_h, \nu]\big)\big((h - \mathcal{T}r(u_h)) - (h - \mathcal{T}r(\hat{u}_h))\big)\, \mathrm{d}\mathcal{H}^{d-1} \geq 0$$

Thus, (7.42) implies that

$$\int_\Omega (\mathbf{z}_h - \hat{\mathbf{z}}_h, D(u_h - \hat{u}_h)) = 0$$

or, equivalently,

$$\int_\Omega (\mathbf{z}_h, D\hat{u}_h) = |D\hat{u}_h|(\Omega) \quad \text{and} \quad \int_\Omega (\hat{\mathbf{z}}_h, Du_h) = |Du_h|(\Omega), \tag{7.44}$$

and

$$\big([\mathbf{z}_h, \nu] - [\hat{\mathbf{z}}_h, \nu]\big)\big((h - \mathcal{T}r(u_h)) - (h - \mathcal{T}r(\hat{u}_h))\big) = 0 \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega.$$

Then,

$$\begin{aligned} 0 &= \big([\mathbf{z}_h, \nu] - [\hat{\mathbf{z}}_h, \nu]\big)\,(h - \mathcal{T}r(u_h)) - (h - \mathcal{T}r(\hat{u}_h)) \\ &= |h - \hat{u}_h| - [\mathbf{z}_h, \nu](h - \mathcal{T}r(\hat{u}_h)) + |h - \mathcal{T}r(u_h)| - [\hat{\mathbf{z}}_h, \nu](h - \mathcal{T}r(u_h)) \end{aligned}$$

$\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. Since $\|[\mathbf{z}_h, \nu]\|_\infty \leq 1$ and $\|[\hat{\mathbf{z}}_h, \nu]\|_\infty \leq 1$, the previous equation yields that

$$[\hat{\mathbf{z}}_h, \nu](h - \mathcal{T}r(u_h)) = |h - \mathcal{T}r(u_h)| \ \text{ and } \ [\mathbf{z}_h, \nu](h - \mathcal{T}r(\hat{u}_h)) = |h - \mathcal{T}r(\hat{u}_h)|$$

$\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. From this, we can conclude that

$$[\hat{\mathbf{z}}_h, \nu] \in \operatorname{sign}(h - \mathcal{T}r(u_h)) \ \text{ and } \ [\mathbf{z}_h, \nu] \in \operatorname{sign}(h - \mathcal{T}r(\hat{u}_h))$$

$\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. Further, by recalling (7.10), there are Radon-Nikodým derivatives

$$\theta(\mathbf{z}_h, D\hat{u}_h, \cdot) = \frac{\mathrm{d}(\mathbf{z}_h, D\hat{u}_h)}{\mathrm{d}|D\hat{u}_h|} \quad \text{and} \quad \theta(\hat{z}_h, Du_h, \cdot) = \frac{\mathrm{d}(\hat{\mathbf{z}}_h, Du_h)}{\mathrm{d}|Du_h|}$$

satisfying $|\theta(\mathbf{z}_h, D\hat{u}_h, x)| = 1$ for $|D\hat{u}_h|$-a.e. $x \in \Omega$ and $|\theta(\hat{\mathbf{z}}_h, Du_h, x)| = 1$ for $|Du_h|$-a.e. $x \in \Omega$. Applying this to (7.44) yields that

$$\int_\Omega \theta(\mathbf{z}_h, D\hat{u}_h, \cdot)\,\mathrm{d}|D\hat{u}_h| = |D\hat{u}_h|(\Omega)$$

and

$$\int_\Omega \theta(\hat{\mathbf{z}}_h, Du_h, \cdot)\,\mathrm{d}|D\hat{u}_h| = |Du_h|(\Omega),$$

implying that $\theta(\mathbf{z}_h, D\hat{u}_h, \cdot) = 1$ and $\theta(\hat{\mathbf{z}}_h, Du_h, \cdot) = 1$ a.e. on Ω. This shows that

$$(\mathbf{z}_h, D\hat{u}_h) = |D\hat{u}_h| \quad \text{and} \quad (\hat{\mathbf{z}}_h, Du_h) = |Du_h|$$

as Radon measures and hence, completes the proof of showing that $\hat{\mathbf{z}}_h$ satisfies (7.36)-(7.39) with respect to u_h and $\mathbf{z}_h$ satisfies (7.36)-(7.39) with respect to $\hat{u}_h$. □

Even though there might be infinitely many divergence-free vector fields $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ to a given boundary data $h \in L^1(\partial\Omega)$, the value of the integral

$$\int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\, \mathrm{d}\mathcal{H}^{d-1}$$

remains the same for all vector fields $\mathbf{z}_h \in \mathcal{Z}_h$, where $\mathcal{Z}_h$ given by (7.40).

Theorem 7.5 *For every given boundary data $h \in L^1(\partial\Omega)$, one has that*

$$\int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\, \mathrm{d}\mathcal{H}^{d-1} = \min_{v \in BV(\Omega)} \Phi_h(v). \tag{7.45}$$

for every vector fields $\mathbf{z}_h \in \mathcal{Z}_h$.

Proof (Proof of Theorem 7.5) Let $h \in L^1(\partial\Omega)$, $\mathbf{z}_h \in \mathcal{Z}_h$, and u_h a weak solution of Dirichlet problem (7.30) with Dirichlet boundary data h. Then, by Proposition 7.10, u_h satisfies

$$\min_{v \in BV(\Omega)} \Phi_h(v) = \int_\Omega |Du_h| + \int_{\partial\Omega} |h - \mathcal{T}r(u_h)|\,\mathrm{d}\mathcal{H}^{d-1}. \tag{7.46}$$

On the other hand, by (7.39), the generalized integration by parts formula (7.13), (7.38), and (7.36), one sees that

$$\begin{aligned}
&\int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\, \mathrm{d}\mathcal{H}^{d-1} - \int_\Omega |Du_h| \\
&\quad = \int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\, \mathrm{d}\mathcal{H}^{d-1} - \int_\Omega (\mathbf{z}_h, Du_h) \\
&\quad = \int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h\, \mathrm{d}\mathcal{H}^{d-1} - \int_{\partial\Omega} [\mathbf{z}_h, \nu]\, \mathcal{T}r(u_h)\, \mathrm{d}\mathcal{H}^{d-1}
\end{aligned}$$

$$= \int_{\partial\Omega} [\mathbf{z}_n, \nu]\,(h - \mathcal{T}r(u_h))\,\mathrm{d}\mathcal{H}^{d-1}$$
$$= \int_{\partial\Omega} |h - \mathcal{T}r(u_h)|\,\mathrm{d}\mathcal{H}^{d-1}$$

and so,

$$\int_{\partial\Omega} [\mathbf{z}_h, \nu]\,h\,\mathrm{d}\mathcal{H}^{d-1} = \int_{\Omega} |Du_h| + \int_{\partial\Omega} |h - \mathcal{T}r(u_h)|\,\mathrm{d}\mathcal{H}^{d-1}. \tag{7.47}$$

Clearly, (7.45) follows from combining (7.46) with (7.47). □

7.3 A Robin-type problem for the 1-Laplace operator

In order to show that the Dirichlet-to-Neumann operator Λ associated with the 1-Laplace operator Δ_1 satisfies the range condition

$$\mathrm{Rg}(I + \lambda\Lambda) = X \tag{7.48}$$

for $X = L^1(\partial\Omega)$, we recall some recent results obtain by the second author with collaborators [111] on the following inhomogeneous *Robin-type boundary-value problem*

$$\begin{cases} -\Delta_1 u = 0 & \text{in } \Omega,\\ \dfrac{Du}{|Du|}\cdot\nu = T_1(g - \alpha u) & \text{on } \partial\Omega, \end{cases} \tag{7.49}$$

for the 1-Laplace operator Δ_1, for given $\alpha > 0$ and $g \in L^2(\partial\Omega)$.

In the boundary condition of problem (7.49), the function $T_1 : \mathbb{R} \to \mathbb{R}$ given by $T_1(s) = s$ if $|s| \le 1$ and $T_1(s) = \mathrm{sign}(s)$ if $|s| \ge 1$, denotes the *truncator operator*, which is necessary to add in (7.49), since it preserves the condition

$$\left\| \frac{Du}{|Du|}\cdot\nu \right\|_{\infty} \le 1 \tag{7.50}$$

satisfied by every solution u of problem (7.49) (cf., (7.12) and the fact that every vector field $\mathbf{z}$ associated with a weak solution u of (7.49) satisfies $\|\mathbf{z}\|_\infty \le 1$). We emphasize that the use of a truncator T_1 in the Robin-type boundary condition (7.49) is a phenomenon, which is exclusively generated by the structure of the 1-Laplace operator Δ_1 (and its co-normal derivative).

Another reason supporting the use of the truncator T_1 in the singular boundary-value problem (7.49) is provided by studying the correct associated (energy) functional; intuitively, the natural functional associated with problem (7.49) (without T_1) is given by

$$I_{\alpha,g}(u) := \int_{\Omega} |Du| + \int_{\partial\Omega} \left[\tfrac{\alpha}{2}|\mathcal{T}r(u)|^2 - g\,\mathcal{T}r(u)\right]\mathrm{d}\mathcal{H}^{d-1}, \qquad u \in V_2(\Omega),$$

where the space $V_2(\Omega)$ is given by

$$V_2(\Omega) = \Big\{u \in BV(\Omega)\,|\,\mathcal{T}r(u) \in L^2(\partial\Omega)\Big\}.$$

But the functional $I_{\alpha,g}$ is, in general, not lower semicontinuous with respect to the $L^1(\Omega)$-topology (cf., [113]). Thus, one employs instead the L^1-lower semicontinuous envelope

$$\Theta_{\alpha,g}(u) := \int_\Omega |Du| + \int_{\partial\Omega} \Gamma_g(x, \mathcal{T}r(u))\,\mathrm{d}\mathcal{H}^{d-1}, \tag{7.51}$$

$u \in V_2(\Omega)$, where $\Gamma_g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Borel function, which is convex and contractive with respect to the second variable, uniformly with respect to the first one, and satisfies $\frac{\partial}{\partial u}\Gamma_g(x,u) = T_1(g(x) - \alpha u)$. Here, for the L^1-lower semicontinuity of the functional Θ_g, the contractivity property of the mapping $u \mapsto \Gamma_g(x,u)$ is crucial (cf., Proposition 7.1).

To find the correct notion and the existence of *weak solutions* u to the inhomogeneous Robin-type problem (7.49), the authors of [111] start from the more regular Robin-type problem associated with the p-Laplace operator (for $p > 1$)

$$\begin{cases} -\Delta_p u_p = 0 & \text{in } \Omega, \\ |\nabla u_p|^{p-2}\nabla u_p \cdot \nu = T_1(g - \alpha u) & \text{on } \partial\Omega. \end{cases} \tag{7.52}$$

It is not hard to see that for every given $g \in L^2(\partial\Omega)$ and $\alpha > 0$, problem (7.52) admits a unique weak solution $u_p \in W^{1,p}(\Omega)$. After deriving *a priori*-estimates for $p \in (1,2)$, they establish in [111, Theorem 1.1.] the existence of the following type of solutions.

Definition 7.3 For given $g \in L^2(\partial\Omega)$ and $\alpha > 0$, we say that $u \in V_2(\Omega)$ is a *weak solution* to the inhomogeneous Robin-type problem (7.49) for the 1-Laplace operator if for u, there is a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (7.37)–(7.39), and

$$[\mathbf{z}, \nu] = T_1(g - \alpha u) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{7.53}$$

For later use, we restate the existence result [111, Theorem 1.1.] with more details to the convergence by the approximate problem (7.52).

Theorem 7.6 ([111, Theorem 1.1.]) *Let Ω be a bounded domain with a boundary $\partial\Omega$ of class C^1. Then, for every $g \in L^2(\partial\Omega)$ and $\alpha > 0$, there is a weak solution $u \in V_2(\Omega)$ of the inhomogeneous Robin-type problem (7.49) for the 1-Laplace operator. Moreover, for every sequence $(p_n)_{n\ge1}$ in $(1,2)$ converging to 1, there is a subsequence $(p_{k_n})_{n\ge1}$ and a weak solution $u \in V_2(\Omega)$ the inhomogeneous Robin-type problem (7.49) for the 1-Laplace operator such that*

$$\lim_{n\to\infty} u_{p_{k_n}} = u \qquad \textit{in } L^q(\Omega) \textit{ for all } 1 \le q < \frac{d}{d-1},$$

where $u_{p_{k_n}}$ is the unique solution of the Robin-type problem (7.52) associated with the p_{k_n}-Laplace operator.

Further, the following relation between the the inhomogeneous Robin-type problem (7.49) and the Dirichlet problem (7.30) was obtained in [111].

Proposition 7.11 ([111, Proposition 2.13]) *Let g, $h \in L^2(\partial\Omega)$, $\alpha > 0$, and $u \in BV(\Omega)$. Then the following statements hold. If u is a weak solution to the inhomogeneous Robin-type problem (7.49), then u is a weak solution to the Dirichlet problem (7.30) with Dirichlet boundary data*

$$h = g - \alpha\,[\mathbf{z}, \nu] \qquad \text{on } \partial\Omega, \tag{7.54}$$

in the weak sense (7.36), where $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^d)$ is some vector field associated with u via the conditions (7.36)-(7.39).

7.4 Proofs of the main results

This section is dedicated to outline the proofs of our main results Theorem 1.9, Theorem 1.10, and Theorem 1.11. The proofs of these results are obtained in several steps, which we fix respectively in a separate proposition. We begin by introducing the Dirichlet-to-Neumann operator Λ associated with the 1-Laplace operator Δ_1 as an operator in $L^1(\partial\Omega)$.

7.4.1 The Dirichlet-to-Neumann operator in L^1

We start this subsection with the following definition.

Definition 7.4 For given $h \in L^1(\partial\Omega)$, let $\mathcal{Z}_h$ be the set of divergence free vector fields defined by (7.40). Then, we define the *Dirichlet-to-Neumann operator Λ in $L^1(\partial\Omega)$ associated with the 1-Laplace operator Δ_1* by the set of all pairs $(h, g) \in L^1(\partial\Omega) \times L^1(\partial\Omega)$ with the property that there is a divergence free vector field $\mathbf{z}_h \in \mathcal{Z}_h$ such that

$$g = [\mathbf{z}, \nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{7.55}$$

Remark 7.1 (a) Since the minimization problem (7.34) admits a solution for every boundary data $h \in L^1(\partial\Omega)$ and by Proposition 7.10, the Dirichlet-to-Neumann operator Λ associated with Δ_1 has the effective domain

$$D(\Lambda) = L^1(\partial\Omega).$$

(b) The Dirichlet-to-Neumann operator Λ associated with Δ_1 satisfies

$$\Lambda \subseteq L^1(\partial\Omega) \times L^\infty(\partial\Omega) \tag{7.56}$$

since for every pair $(h,g)\in\Lambda$, one has that

$$\|g\|_\infty = \|[\mathbf{z}_h,\nu]\|_\infty \le 1,$$

for every vector field $\mathbf{z}_h\in\mathcal{Z}_h$.

We come to the first property of the Dirichlet-to-Neumann operator Λ.

Proposition 7.12 *The Dirichlet-to-Neumann operator Λ associated with the 1-Laplace operator Δ_1 is completely accretive in $L^1(\partial\Omega)$.*

Proof We aim to show that

$$\int_{\partial\Omega}(g-\hat{g})\,p(h-\hat{h})\mathrm{d}\mathcal{H}^{d-1}\ge 0 \tag{7.57}$$

for every (h,g), $(\hat{h},\hat{g})\in\Lambda$ and $p\in P_0$, where P_0 is the set of smooth truncator functions given by (4.49). Note, even if the function p in (7.57) would be the identity on $\mathbb{R}$, the integral in (7.57) would exist due to (7.56). Now, let (h,g), $(\hat{h},\hat{g})\in\Lambda$ and $p\in P_0$. Then by the definition of Λ, for each pair (h,g), $(\hat{h},\hat{g})$, there are weak solutions u_h, $u_{\hat{h}}$ of Dirichlet problem (7.30) with Dirichlet data h and $\hat{h}$, respectively, and associated vector fields $\mathbf{z}_h\in\mathcal{Z}_h$ and $\mathbf{z}_{\hat{h}}\in\mathcal{Z}_{\hat{h}}$. By the chain rule for BV-functions (see [1, Theorem 3.96]), the function $w:=p(u_h-u_{\hat{h}})$ belongs to $BV(\Omega)$. In addition, by the Lipschitz continuity of p and since

$$\mathcal{T}r v = v \qquad \text{for every } v\in BV(\Omega)\cap C(\overline{\Omega}),$$

one has that

$$\mathcal{T}r(w) = p(\mathcal{T}r(u_h)-\mathcal{T}r(u_{\hat{h}})) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega.$$

Thus, by applying the generalized integration by parts formula (7.13) to w with the two vector fields $\mathbf{z}_h$ and $\mathbf{z}_{\hat{h}}$, respectively, and by (7.38), one finds that

$$\int_\Omega(\mathbf{z}_h, D(p(u_h-u_{\hat{h}}))) = \int_{\partial\Omega}[\mathbf{z}_h,\nu]\,p(\mathcal{T}r(u_h)-\mathcal{T}r(u_{\hat{h}}))\,\mathrm{d}\mathcal{H}^{d-1}$$

and

$$\int_\Omega(\mathbf{z}_{\hat{h}}, D(p(u_h-u_{\hat{h}}))) = \int_{\partial\Omega}[\mathbf{z}_{\hat{h}},\nu]\,p(\mathcal{T}r(u_h)-\mathcal{T}r(u_{\hat{h}}))\,\mathrm{d}\mathcal{H}^{d-1}.$$

Since $g=[\mathbf{z},\nu]$ and $\hat{g}=[\mathbf{z}_{\hat{h}},\nu]$, we can conclude from these two integral equations that

$$\int_{\partial\Omega}(g-\hat{g})\,p(\mathcal{T}r(u_h)-\mathcal{T}r(u_{\hat{h}}))\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega(\mathbf{z}_h-\mathbf{z}_{\hat{h}}, D(p(u_h-u_{\hat{h}}))). \tag{7.58}$$

Since p is Lipschitz continuous and monotonically increasing, the chain rule for $(\mathbf{z}_h, D\cdot)$ (see Proposition 7.6) yields that the Radon-Nikodým derivative $\theta(\mathbf{z}_h, D(p(u_h-$

$u_{\hat{h}})),x)$ of $(\mathbf{z}_h, D(p(u_h - u_{\hat{h}})))$ with respect to the total variational measure $|D(p(u_h - u_{\hat{h}}))|$ satisfies

$$\theta(\mathbf{z}_h, D(p(u_h - u_{\hat{h}})),x) = \theta(\mathbf{z}_h, D(u_h - u_{\hat{h}}),x)$$

for $|D(u_h - u_{\hat{h}})|$-a.e. $x \in \Omega$. Moreover, the Radon-Nikodým derivative $\theta(\mathbf{z}_h - \mathbf{z}_{\hat{h}}, D(u_h - u_{\hat{h}}),x)$ of $(\mathbf{z}_h - \mathbf{z}_{\hat{h}}, D(u_h - u_{\hat{h}}))$ is positive since by the bilinearity of the pairing $(\cdot, D\cdot)$ and by (7.39), (7.37) and (7.9), one has that

$$\begin{aligned}\int_\Omega (\mathbf{z}_h - \mathbf{z}_{\hat{h}}, D(u_h - u_{\hat{h}})) = \int_\Omega |Du_h| - \int_\Omega (\mathbf{z}_{\hat{h}}, Du_h) \\ + \int_\Omega |Du_{\hat{h}}| - \int_\Omega (\mathbf{z}_{\hat{h}}, Du_h) \geq 0.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_\Omega (\mathbf{z} - \mathbf{z}_{\hat{h}}, D(p(u_h - u_{\hat{h}}))) \\ &\qquad = \int_\Omega \theta(\mathbf{z}_h - \mathbf{z}_{\hat{h}}, D(p(u_h - u_{\hat{h}})),x)\, |D(p(u_h - u_{\hat{h}}))| \\ &\qquad = \int_\Omega \theta(\mathbf{z}_h - \mathbf{z}_{\hat{h}}, D((u_h - u_{\hat{h}}),x)\, |D(p(u_h - u_{\hat{h}}))| \geq 0.\end{aligned}$$

Applying this to (7.58), shows that

$$\int_{\partial\Omega} (g - \hat{g})\, p(\mathcal{T}r(u_h) - \mathcal{T}r(u_{\hat{h}}))\, \mathrm{d}\mathcal{H}^{d-1} \geq 0, \tag{7.59}$$

which would complete the proof of this proposition if the weak solutions u_h and $u_{\hat{h}}$ of Dirichlet problem (7.30) would satisfy the Dirichlet boundary condition (7.32) in the sense of traces. But our notion of solutions to Dirichlet problem (7.30) assumes only that u_h and $u_{\hat{h}}$ satisfy Dirichlet boundary condition (7.32) in the weak sense (7.36). Thus, we still need to provide an argument, why (7.59) implies the desired inequality (7.57). Now, by (7.59),

$$\begin{aligned}&\int_{\partial\Omega} (g - \hat{g})\, p(h - \hat{h}) \mathrm{d}\mathcal{H}^{d-1} \\ &\quad \geq \int_{\partial\Omega} (g - \hat{g}) \Big(p(h - \hat{h}) - p(\mathcal{T}r(u_h) - \mathcal{T}r(u_{\hat{h}})) \Big)\, \mathrm{d}\mathcal{H}^{d-1} \\ &\quad = \int_{\partial\Omega} (g - \hat{g}) \int_0^1 p'\Big(s(h - \hat{h}) + (1-s)(\mathcal{T}r(u_h) - \mathcal{T}r(u_{\hat{h}})) \Big)\, \mathrm{d}s \times \\ &\qquad\qquad \times \Big[(h - \hat{h}) - (\mathcal{T}r(u_h) - \mathcal{T}r(u_{\hat{h}})) \Big] \mathrm{d}\mathcal{H}^{d-1}.\end{aligned} \tag{7.60}$$

Using again that $g = [\mathbf{z}, \nu]$ and $\hat{g} = [\mathbf{z}_{\hat{h}}, \nu]$, and since u_h and $u_{\hat{h}}$ satisfy the Dirichlet boundary condition (7.32) in the weak sense (7.36), one finds that

$$\begin{aligned}\big(g-\hat{g}\big)\,\big((h-\mathcal{T}r(u_h))-(\hat{h}-\mathcal{T}r(u_{\hat{h}}))\big)&\\ =|h-\mathcal{T}r(u_h)|+|\hat{h}-\mathcal{T}r(u_{\hat{h}})|&\\ -[\mathbf{z},\nu](\hat{h}-\mathcal{T}r(u_{\hat{h}}))-[\mathbf{z}_{\hat{h}},\nu](h-\mathcal{T}r(u_h))\ge 0&\end{aligned}$$

for $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. Moreover, since $p'\ge 0$, the integral

$$\int_0^1 p'\Big(s(h-\hat{h})+(1-s)(\mathcal{T}r(u_h)-\mathcal{T}r(u_{\hat{h}}))\Big)\,\mathrm{d}s\ge 0$$

$\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$, the last integral on the right hand-side in (7.60) is positive, implying that (7.57) holds. □

Proposition 7.13 *The Dirichlet-to-Neumann operator* Λ *associated with the* 1-*Laplace operator* Δ_1 *is homogeneous of order zero.*

Proof Let $\lambda>0$ and $(h,g)\in\Lambda$. Then, there are $u_h\in BV(\Omega)$ and a vector field $\mathbf{z}_h\in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (7.36)-(7.39) and $g=[\mathbf{z}_h,\nu]$. First, we show that the function λu_h is a weak solution of Dirichlet problem (7.30) for the boundary data λh. To see this, we begin by using the linearity of $w\mapsto(\mathbf{z}_h,Dw)$ and the homogeneity of $w\mapsto|Dw|$. Then and by (7.39), one has that

$$(\mathbf{z}_h,D(\lambda u_h))=\lambda(\mathbf{z}_h,Du_h)=\lambda\,|Du_h|=|D(\lambda u_h)|$$

in the sense of measures. Further, for the same vector field $\mathbf{z}_h$, which satisfies (7.36)-(7.38), one has that

$$[\mathbf{z}_h,\nu]\in\operatorname{sign}\Big(h-\mathcal{T}r(u_h)\Big)=\operatorname{sign}\Big(\lambda h-\lambda\mathcal{T}r(u_h)\Big).$$

Moreover, since $g=[\mathbf{z}_h,\nu]$, the latter computation shows that

$$g\in\operatorname{sign}\Big(\lambda h-\lambda\mathcal{T}r(u_h)\Big).$$

Thus, λu_h is a weak solution of Dirichlet problem (7.30) for the boundary data λh with the same value g as the generalized Neumann derivative $[\mathbf{z}_h,\nu]$ associated with the weak solution u_h of Dirichlet problem (7.30). Since $(h,g)\in\Lambda$ were arbitrary, we have thereby shown that

$$\Lambda(\lambda h)=\Lambda h\qquad\text{for all } h\in D(\Lambda)\text{ and }\lambda>0,$$

establishing the claim of this proposition. □

One of our aims is to relate the closure $\overline{\Lambda}^{L^1\times L^\infty_\sigma}$ in $L^1\times L^\infty_\sigma(\partial\Omega)$ of the Dirichlet-to-Neumann operator Λ with a sub-differential structure $\partial\varphi$ in $L^1\times L^\infty(\partial\Omega)$. Here, we write $L^\infty_\sigma(\partial\Omega)$ to denote $L^\infty(\partial\Omega)$ equipped with the weak*-topology $\sigma(L^\infty(\partial\Omega),L^1(\partial\Omega))$. For this, we introduce the following potential candidate of a convex function.

Proposition 7.14 *For given $h \in L^1(\partial\Omega)$, let $\mathcal{Z}_h$ be the set of divergence-free vector fields defined by (7.40). Then the functional $\varphi : L^1(\partial\Omega) \to [0,\infty)$ given by*

$$\varphi(h) = \int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h \,\mathrm{d}\mathcal{H}^{d-1} \qquad \text{for every } h \in L^1(\partial\Omega), \tag{7.61}$$

where the value of the integral is independent of the choice of $\mathbf{z}_h \in \mathcal{Z}_h$, is a well-defined convex and continuous functional on $L^1(\partial\Omega)$, which is homogeneous of order one and even.

Proof First, we note that thanks to Theorem 7.5, for given boundary value $h \in L^1(\partial\Omega)$, the value $\varphi(h)$ given by (7.61) is independent of the vector field $\mathbf{z}_h \in \mathcal{Z}_h$ and given by

$$\varphi(h) = \min_{v \in BV(\Omega)} \Phi_h(v),$$

where Φ_h is defined by (7.33). Therefore, φ is a well-defined, proper mapping. Next, we show that φ is homogeneous of order one. For this, let $h \in L^1(\partial\Omega)$ and $u_h \in BV(\Omega)$ a weak solution of Dirichlet problem (7.30) with Dirichlet data h and corresponding vector field $\mathbf{z}_h \in \mathcal{Z}_h$. Then by Proposition 7.10,

$$\lambda\varphi(h) = \lambda \min_{v \in BV(\Omega)} \Phi_h(v) = \lambda\Phi_h(u_h) = \Phi_{\lambda h}(\lambda u_h)$$

for every $\lambda > 0$. Since

$$\operatorname{sign}(h - \mathcal{T}r(u_h)) = \operatorname{sign}(\lambda(h - \mathcal{T}r(u_h))) = \operatorname{sign}(\lambda h - \mathcal{T}r(\lambda u_h))$$

for every $\lambda > 0$, and since $\mathbf{z}_h$ satisfies (7.36)-(7.38) with respect to u_h and h, it follows that the same vector field $\mathbf{z}_h$ satisfies (7.36)-(7.38) with respect to λu_h and Dirichlet data λh. Moreover, by the linearity of the mappings $(\mathbf{z}_h, D\cdot)$ and $|D\cdot|$, (7.39) yields that

$$(\mathbf{z}_h, D(\lambda u_h)) = \lambda\,(\mathbf{z}_h, D(u_h)) = \lambda\,|Du_h| = |D(\lambda u_h)|,$$

showing that $\mathbf{z}_h$ and λu_h also satisfy (7.39). Therefore, for every $\lambda > 0$, one has that $\mathbf{z}_h \in \mathcal{Z}_{\lambda h}$, implying that

$$\lambda\varphi(h) = \varphi(\lambda h) \tag{7.62}$$

for every $\lambda > 0$. Note, if $\lambda = 0$, then $u_0 \equiv 0$ is certainly a minimizer of Φ_0 over $BV(\Omega)$ and a weak solution of Dirichlet problem (7.30) with Dirichlet data $h = 0$ with corresponding vector field $\mathbf{z}_0 \equiv 0 \in \mathbb{R}^d$. Therefore, φ also satisfies (7.62) for $\lambda = 0$, completing the proof of homogeneity of φ.

To see that φ is convex, let h_1, $h_2 \in L^1(\partial\Omega)$, and $\lambda \in (0,1)$. Then, there are weak solutions u_1, $u_2 \in BV(\Omega)$ of Dirichlet problem (7.30) with Dirichlet data h_1, h_2 and corresponding vector fields $\mathbf{z}_{h_1}$, $\mathbf{z}_{h_2} \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (7.36)-(7.39) with respect to u_1 and u_2. Then by the homogeneity of φ, we have that

$$\lambda\,\varphi(h_1) = \Phi_{\lambda h_1}(\lambda u_1) \quad \text{ and } \quad (1-\lambda)\,\varphi(h_2) = \Phi_{(1-\lambda)h_2}((1-\lambda)\,u_2)$$

and so, by the convexity of Φ, and by Theorem 7.5,

$$\begin{aligned}
\lambda\,\varphi(h_1)+(1-\lambda)\,\varphi(h_2)&\\
&=\lambda\,\Phi_{\lambda h_1}(\lambda u_1)+(1-\lambda)\,\Phi_{(1-\lambda)h_2}((1-\lambda)\,u_2)\\
&\ge \Phi_{\lambda h_1+(1-\lambda)h_2}(\lambda u_1+(1-\lambda)\,u_2)\\
&\ge \min_{v\in BV(\Omega)}\Phi_{\lambda h_1+(1-\lambda)h_2}(v)=\varphi(\lambda h_1+(1-\lambda)h_2).
\end{aligned}$$

To see that the convex, proper functional φ given by (7.61) is continuous on $L^1(\partial\Omega)$, it is sufficient to show (cf., [135, Lemma 7.1]) that for every $h\in L^1(\partial\Omega)$, φ is bounded on a neighborhood of h. For this, let $h\in L^1(\partial\Omega)$, $r>0$ and $\hat{h}$ an element of the open ball $B_{L^1}(h,r)$ in $L^1(\partial\Omega)$ centered at h of radius r. Then, as for every vector field $\mathbf{z}_{\hat{h}}\in L^\infty(\Omega,\mathbb{R}^d)$ related to a weak solution $u_{\hat{h}}$ of Dirichlet problem (7.30) with Dirichlet data $\hat{h}$, one has that $\|[\mathbf{z}_{\hat{h}},\nu]\|_\infty\le 1$, it follows that

$$\varphi(\hat{h})\le\|\hat{h}\|_1\le r+\|h\|_1.$$

Finally, we show that φ is even. For this let $h\in L^1(\partial\Omega)$. Since the effective domain $D(\varphi)$ is the whole space $L^1(\partial\Omega)$, we also have that $-h\in D(\varphi)$. Moreover, let $u_h\in BV(\Omega)$ and $\mathbf{z}_h\in L^\infty(\Omega;\mathbb{R}^d)$ satisfy (7.36)-(7.38) with respect to u_h and h, and set

$$u_{-h}:=-u_h\quad\text{ and }\quad\mathbf{z}_{-h}:=-\mathbf{z}_h.$$

Then, obviously, $\mathbf{z}_{-h}$ satisfies $\|\mathbf{z}_{-h}\|_\infty\le 1$, $-\mathrm{div}(\mathbf{z}_{-h})=0$ in $\mathcal{D}'(\Omega)$ and by the bilinearity of the measure $(\cdot,D\cdot)$, it follows that

$$(\mathbf{z}_{-h},Du_{-h})=(-\mathbf{z}_h,D(-u_h))=(\mathbf{z}_h,Du_h)=|Du_h|=|D(-u_h)|=|Du_{-h}|$$

as Radon measures. In addition, by (7.36), one has that

$$[\mathbf{z}_{-h},\nu]=-[\mathbf{z}_h,\nu]\in-\mathrm{sign}(h-\mathcal{T}r(u_h))=\mathrm{sign}(-h-\mathcal{T}r(u_{-h}))$$

$\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. Hence, we have shown that the pair $(u_{-h},\mathbf{z}_{-h})$ satisfy (7.36)-(7.38). Thus, and by the linearity of the weak trace $\mathbf{z}\mapsto[\mathbf{z},\nu]$, one sees that

$$\varphi(-h)=\int_{\partial\Omega}[\mathbf{z}_{-h},\nu](-h)\,\mathrm{d}\mathcal{H}^{d-1}=\int_{\partial\Omega}[\mathbf{z}_h,\nu]\,h\,\mathrm{d}\mathcal{H}^{d-1}=\varphi(h).$$

This completes the proof of this proposition. □

Next, we turn to the relation of the closure

$$\overline{\Lambda}^{L^1\times L^\infty_\sigma}=\left\{(h,g)\in L^1\times L^\infty(\partial\Omega)\;\middle|\;\begin{array}{c}\text{there exists }((h_n,g_n))_{n\ge1}\subseteq\Lambda\text{ s.t.}\\ \lim_{n\to\infty}(h_n,g_n)=(h,g)\text{ in }L^1\times L^\infty_\sigma(\partial\Omega)\end{array}\right\}$$

of the Dirichlet-to-Neumann operator Λ in $L^1(\partial\Omega)\times L^\infty_\sigma(\partial\Omega)$ with the sub-differential operator $\partial_{L^1\times L^\infty(\partial\Omega)}\varphi$ in $L^1\times L^\infty(\partial\Omega)$.

Proposition 7.15 *For the closure $\overline{\Lambda}^{L^1\times L^\infty_\sigma}$ in $L^1\times L^\infty_\sigma(\partial\Omega)$ of the Dirichlet-to-Neumann operator Λ associated with the 1-Laplace operator, one has that*

$$\overline{\Lambda}^{L^1\times L^\infty_\sigma}\subseteq\partial\varphi,$$

where $\partial\varphi$ denotes the sub-differential operator in $L^1(\partial\Omega)\times L^\infty(\partial\Omega)$ of the functional $\varphi: L^1(\partial\Omega)\to[0,\infty)$ given by (7.61).

Proof We begin by taking $(h,g)\in\Lambda$ and $\hat{h}\in L^1(\partial\Omega)$. By definition of Λ and since the variational problem (7.34) for Dirichlet data $\hat{h}$ admits a solution which is characterized by Proposition 7.10, there are $u_{\hat{h}}\in BV(\Omega)$ and $\mathbf{z}_h$, $\mathbf{z}_{\hat{h}}\in L^\infty(\Omega;\mathbb{R}^d)$ such that $(u_{\hat{h}},\mathbf{z}_{\hat{h}})$ satisfies (7.36)-(7.39), and, in addition, g and $\mathbf{z}_h$ satisfy (7.55). Then, multiply g by $(\hat{h}-h)$ and integrating over $\partial\Omega$. Then by (7.55), the definition of φ, since $\|g\|_\infty\le 1$, and by the generalized integration by parts formula (7.13), one sees that

$$\begin{aligned}
&\int_{\partial\Omega} g(\hat{h}-h)\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad=\int_{\partial\Omega} g\,\hat{h}\,\mathrm{d}\mathcal{H}^{d-1}-\varphi(h)\\
&\quad=\int_{\partial\Omega} g\,(\hat{h}-\mathcal{T}r(u_{\hat{h}}))\,\mathrm{d}\mathcal{H}^{d-1}+\int_{\partial\Omega} g\,\mathcal{T}r(u_{\hat{h}})\,\mathrm{d}\mathcal{H}^{d-1}-\varphi(h)\\
&\quad\le\int_{\partial\Omega}|\hat{h}-\mathcal{T}r(u_{\hat{h}})|\,\mathrm{d}\mathcal{H}^{d-1}+\int_{\Omega}(\mathbf{z}_{\hat{h}},Du_{\hat{h}})\,\mathrm{d}x-\varphi(h)\\
&\quad\le\int_{\partial\Omega}|\hat{h}-\mathcal{T}r(u_{\hat{h}})|\,\mathrm{d}\mathcal{H}^{d-1}+\int_{\Omega}|Du_{\hat{h}}|-\varphi(h)\\
&\quad=\varphi(\hat{h})-\varphi(h).
\end{aligned}$$

Therefore, one has that $(h,g)\in\partial\varphi$, showing that $\Lambda\subseteq\partial\varphi$.

Next, let $(h,g)\in\overline{\Lambda}^{L^1\times L^\infty_\sigma}$, $((h_n,g_n))_{n\ge1}\subseteq\Lambda$ such that $h_n\to h$ in $L^1(\partial\Omega)$ and $g_n\to g$ in $L^\infty_\sigma(\partial\Omega)$. Then by the first part of this proof, we have that each $(h_n,g_n)\in\partial\varphi$ and hence,

$$\varphi(\hat{h})-\int_{\partial\Omega} g_n(\hat{h}-h_n)\,\mathrm{d}\mathcal{H}^{d-1}\ge\varphi(h_n)$$

for every $\hat{h}\in L^1(\partial\Omega)$ and every $n\ge1$. Taking the limit inferior as $n\to\infty$ on both sides of this inequality and using that φ is lower semicontinuous in $L^1(\partial\Omega)$, one finds that

$$\varphi(\hat{h})-\int_{\partial\Omega} g(\hat{h}-h)\,\mathrm{d}\mathcal{H}^{d-1}\ge\varphi(h),$$

showing that $(h,g)\in\partial\varphi$ and thereby, completing the proof of this proposition. □

With the help of the functional φ, we can now show that the Dirichlet-to-Neumann operator Λ is closed in $L^1\times L^1(\partial\Omega)$.

Proposition 7.16 *The Dirichlet-to-Neumann operator Λ is closed in $L^1\times L^\infty_\sigma(\partial\Omega)$.*

Proof (of Proof 7.16) Let $(h,g)\in\overline{\Lambda}^{L^1\times L^\infty_\sigma}$ the closure of the Dirichlet-to-Neumann operator Λ in $L^1\times L^\infty_\sigma(\partial\Omega)$. Then, there is a sequence

$$((h_n,g_n))_{n\ge 1}\subseteq \Lambda \qquad \text{such that} \qquad (h_n,g_n)\to(h,g) \text{ in } L^1(\partial\Omega)\times L^\infty_\sigma(\partial\Omega).$$

By definition of Λ, for every pair (h_n,g_n), there are $u_n\in BV(\Omega)$ and $\mathbf{z}_n\in L^\infty(\Omega;\mathbb{R}^d)$ satisfying

$$\|\mathbf{z}_n\|_\infty \le 1, \tag{7.63}$$

$$-\operatorname{div}(\mathbf{z}_n)=0 \qquad \text{in } \mathcal{D}'(\Omega), \text{ and} \tag{7.64}$$

$$(\mathbf{z}_n, Du_n)=|Du_n| \qquad \text{as Radon measures,} \tag{7.65}$$

$$g_n=[\mathbf{z}_n,\nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega, \tag{7.66}$$

and

$$[\mathbf{z}_n,\nu]\in \operatorname{sign}\Big(h_n-\mathcal{T}r(u_n)\Big) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{7.67}$$

Now, (7.63) yields that there is a vector field $\mathbf{z}_g\in L^\infty(\Omega;\mathbb{R}^d)$ satisfying $\|\mathbf{z}_g\|_\infty\le 1$ and, after possibly passing to a subsequence of $((h_n,g_n))_{n\ge1}$, one has that

$$\mathbf{z}_n \rightharpoonup \mathbf{z}_g \qquad \text{weakly}^* \text{ in } L^\infty(\Omega,\mathbb{R}^d). \tag{7.68}$$

Therefore and by (7.64), it follows that also the vector field $\mathbf{z}_g$ satisfies

$$-\operatorname{div}(\mathbf{z}_g)=0 \qquad \text{in } \mathcal{D}'(\Omega). \tag{7.69}$$

Thanks to (7.66), (7.63), and since $g_n\to g$ weakly* in $L^\infty(\partial\Omega)$, we can pass to a subsequence, if necessary, to conclude that

$$\|g\|_{L^\infty(\partial\Omega)} \le \liminf_{n\to\infty} \|g_n\|_{L^\infty(\partial\Omega)} \le 1.$$

Now, by (7.64), since $\mathbf{z}_g$ satisfies (7.69), and (7.68), it follows from Proposition 7.8 that

$$\lim_{n\to\infty}\int_\Omega (\mathbf{z}_n,Dw)=\int_\Omega(\mathbf{z}_g,Dw). \tag{7.70}$$

for every $w\in BV(\Omega)$. Thus, if $\xi\in L^1(\partial\Omega)$ and $w\in BV(\Omega)$ such that $\mathcal{T}r(w)=\xi$, then by the generalized integration by parts formula (7.13), by (7.66), since $\mathbf{z}_g$ satisfies (7.69), and by (7.70), one sees that

$$\begin{aligned}
\int_{\partial\Omega} g\,\xi\,\mathrm{d}\mathcal{H}^{d-1} &= \lim_{n\to\infty}\int_{\partial\Omega} g_n\,\xi\,\mathrm{d}\mathcal{H}^{d-1}\\
&= \lim_{n\to\infty}\int_{\partial\Omega}[\mathbf{z}_n,\nu]\,\xi\,\mathrm{d}\mathcal{H}^{d-1}\\
&= \lim_{n\to\infty}\int_\Omega(\mathbf{z}_n,Dw)+\int_\Omega \operatorname{div}(\mathbf{z}_n)\,w\,\mathrm{d}x\\
&= \int_\Omega(\mathbf{z}_g,Dw)\\
&= \int_{\partial\Omega}[\mathbf{z}_g,\nu]\,\xi\,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}$$

Since $\xi \in L^1(\partial\Omega)$ was arbitrary, we have thereby shown that

$$g = [\mathbf{z}_g, \nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \tag{7.71}$$

and

$$\int_\Omega (\mathbf{z}_g, Dw) = \int_{\partial\Omega} [\mathbf{z}_g, \nu]\, \mathcal{T}r(w)\, \mathrm{d}\mathcal{H}^{d-1} \qquad \text{for every } w \in BV(\Omega). \tag{7.72}$$

On the other hand, since each u_n is a weak solution of Dirichlet problem (7.30) with boundary data h_n, Proposition 7.10 yields that

$$\begin{aligned} &\int_\Omega |Du_n| + \int_{\partial\Omega} |\mathcal{T}r(u_n) - h_n|\, \mathrm{d}\mathcal{H}^{d-1} \\ &\qquad \le \int_\Omega |Dw| + \int_{\partial\Omega} |\mathcal{T}r(w) - h_n|\, \mathrm{d}\mathcal{H}^{d-1} \end{aligned} \tag{7.73}$$

for every $w \in BV(\Omega)$. Combining this estimate for some fixed $w \in BV(\Omega)$ together with the triangle inequality and the fact that $(h_n)_{n\ge 1}$ is bounded in $L^1(\partial\Omega)$, one finds a constant M such that

$$\int_\Omega |Du_n| + \int_{\partial\Omega} |\mathcal{T}r(u_n)|\, \mathrm{d}\mathcal{H}^{d-1} \le M \qquad \text{for all } n \ge 1.$$

Therefore and by the Maz'ya inequality (7.4), the sequence $(u_n)_{n\ge 1}$ is bounded in $BV(\Omega)$. Hence, there is a $u_h \in BV(\Omega)$ such that after possibly passing to a subsequence, $u_n \rightharpoonup u_h$ weakly* in $BV(\Omega)$. Now, let $w \in BV(\Omega)$. Then by (7.73),

$$\begin{aligned} &\int_\Omega |Du_n| + \int_{\partial\Omega} |\mathcal{T}r(u_n) - h|\, \mathrm{d}\mathcal{H}^{d-1} \\ &\quad \le \int_\Omega |Du_n| + \int_{\partial\Omega} |\mathcal{T}r(u_n) - h_n|\, \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\Omega} |h_n - h|\, \mathrm{d}\mathcal{H}^{d-1} \\ &\quad \le \int_\Omega |Dw| + \int_{\partial\Omega} |\mathcal{T}r(w) - h_n|\, \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\Omega} |h_n - h|\, \mathrm{d}\mathcal{H}^{d-1} \end{aligned}$$

and so, by the limit $h_n \to h$ in $L^1(\partial\Omega)$ and by Modica's convergence result (Proposition 7.1), one gets that

$$\int_\Omega |Du_h| + \int_{\partial\Omega} |\mathcal{T}r(u_h) - h|\, \mathrm{d}\mathcal{H}^{d-1} \le \int_\Omega |Dw| + \int_{\partial\Omega} |\mathcal{T}r(w) - h|\, \mathrm{d}\mathcal{H}^{d-1}$$

for every $w \in BV(\Omega)$, showing that u_h is a minimizer of the relaxed functional Φ_h given by (7.33). Thus, by Proposition 7.10, u_h is a weak solution of the Dirichlet problem (7.30) with boundary data h. Hence, there is a vector field $\mathbf{z}_h \in L^\infty(\Omega; \mathbb{R}^d)$ satisfying (7.36)-(7.39) with respect to u_h.

Now, by (7.64)-(7.67) and the generalized integration by parts formula, one sees that

$$\begin{aligned}
\int_{\partial\Omega} [\mathbf{z}_n,\nu]\, h_n \,\mathrm{d}\mathcal{H}^{d-1} - \int_\Omega |Du_n| \\
&= \int_{\partial\Omega} [\mathbf{z}_n,\nu]\, h_n \,\mathrm{d}\mathcal{H}^{d-1} - \int_{\partial\Omega} [\mathbf{z}_n,\nu]\, \mathcal{T}r(u_n) \,\mathrm{d}\mathcal{H}^{d-1} \\
&= \int_{\partial\Omega} [\mathbf{z}_n,\nu]\, (h_n - \mathcal{T}r(u_n)) \,\mathrm{d}\mathcal{H}^{d-1} \\
&= \int_{\partial\Omega} |h_n - \mathcal{T}r(u_n)| \,\mathrm{d}\mathcal{H}^{d-1},
\end{aligned}$$

or, equivalently,

$$\int_{\partial\Omega} [\mathbf{z}_n,\nu]\, h_n \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega |Du_n| + \int_{\partial\Omega} |h_n - \mathcal{T}r(u_n)| \,\mathrm{d}\mathcal{H}^{d-1}.$$

Note that the left-hand side in the above equation is $\varphi(h_n)$ for the functional φ given by (7.61). Since $(h_n, g_n) \to (h, g)$ in $L^1(\partial\Omega) \times L^\infty_\sigma(\partial\Omega)$, and since by Proposition 7.14, φ is continuous, one has that

$$\begin{aligned}
\int_{\partial\Omega} g\, h \,\mathrm{d}\mathcal{H}^{d-1} &= \lim_{n\to\infty} \int_{\partial\Omega} g_n\, h_n \,\mathrm{d}\mathcal{H}^{d-1} \\
&= \lim_{n\to\infty} \int_{\partial\Omega} [\mathbf{z}_n,\nu]\, h_n \,\mathrm{d}\mathcal{H}^{d-1} \\
&= \lim_{n\to\infty} \varphi(h_n) = \varphi(h) = \int_{\partial\Omega} [\mathbf{z}_h,\nu]\, h \,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}$$

Hence and by (7.71), we have shown that

$$\int_{\partial\Omega} [\mathbf{z}_g,\nu]\, h \,\mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\Omega} g\, h \,\mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\Omega} [\mathbf{z}_h,\nu]\, h \,\mathrm{d}\mathcal{H}^{d-1}. \tag{7.74}$$

Next, we intend to show that

$$(\mathbf{z}_g, Du_h) = |Du_h| \qquad \text{as Radon measures.} \tag{7.75}$$

To see this, recall that by (7.74) and since the pair $(\mathbf{z}_h, u_h)$ satisfy (7.36) and (7.39), one has that

$$\begin{aligned}
\int_{\partial\Omega} [\mathbf{z}_g,\nu]\, h \,\mathrm{d}\mathcal{H}^{d-1} &= \int_{\partial\Omega} [\mathbf{z}_h,\nu]\, h \,\mathrm{d}\mathcal{H}^{d-1} \\
&= \int_\Omega |Du_h| + \int_{\partial\Omega} |h - \mathcal{T}r(u_h)| \,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}$$

On the other hand, an integration by parts gives

$$\int_{\partial\Omega} [\mathbf{z}_g,\nu] h \mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\Omega} [\mathbf{z}_g,\nu] \mathcal{T}r(u_h) \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\Omega} [\mathbf{z}_g,\nu] \Big(h - \mathcal{T}r(u_h)\Big) \,\mathrm{d}\mathcal{H}^{d-1}$$

$$= \int_\Omega (\mathbf{z}_g, Du_h) + \int_{\partial\Omega} [\mathbf{z}_g, \nu]\Big(h - \mathcal{T}r(u_h)\Big)\,\mathrm{d}\mathcal{H}^{d-1}$$

Combining those two equations, one finds that

$$\begin{aligned}\int_\Omega (\mathbf{z}_g, Du_h) + \int_{\partial\Omega} [\mathbf{z}_g, \nu]\Big(h - \mathcal{T}r(u_h)\Big)\,\mathrm{d}\mathcal{H}^{d-1} \\ = \int_\Omega |Du_h| + \int_{\partial\Omega} |h - \mathcal{T}r(u_h)|\,\mathrm{d}\mathcal{H}^{d-1}\end{aligned}$$

or, equivalently,

$$\begin{aligned}\int_\Omega (\mathbf{z}_g, Du_h) - \int_\Omega |Du_h| \\ = \int_{\partial\Omega} |h - \mathcal{T}r(u_h)| - [\mathbf{z}_g, \nu]\Big(h - \mathcal{T}r(u_h)\Big)\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned} \tag{7.76}$$

Now,

$$[\mathbf{z}_g, \nu]\Big(h - \mathcal{T}r(u_h)\Big) \le |h - \mathcal{T}r(u_h)| \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \tag{7.77}$$

and by (7.9) and $\|\mathbf{z}_g\|_\infty \le 1$, one has that

$$\left|\int_\Omega (\mathbf{z}_g, Du_h)\right| \le \|\mathbf{z}_g\|_\infty \int_\Omega |Du_h| \le \int_\Omega |Du_h|.$$

Thus at both sides in (7.76), one has that

$$\begin{aligned}0 \ge \int_\Omega (\mathbf{z}_g, Du_h) - \int_\Omega |Du_h| \\ = \int_{\partial\Omega} |h - \mathcal{T}r(u_h)| - [\mathbf{z}_g, \nu]\Big(h - \mathcal{T}r(u_h)\Big)\,\mathrm{d}\mathcal{H}^{d-1} \ge 0,\end{aligned}$$

which implies that

$$\int_\Omega (\mathbf{z}_g, Du_h) = \int_\Omega |Du_h| \tag{7.78}$$

and

$$\int_{\partial\Omega} |h - \mathcal{T}r(u_h)| - [\mathbf{z}_g, \nu]\Big(h - \mathcal{T}r(u_h)\Big)\,\mathrm{d}\mathcal{H}^{d-1} = 0. \tag{7.79}$$

Since $(\mathbf{z}_g, Du)$ is absolutely continuous w.r.t. $|Du_h|$, (7.78) implies that (7.75) holds. Further, by (7.77), (7.79) implies that

$$[\mathbf{z}_g, \nu](h - \mathcal{T}r(u_h)) - |h - \mathcal{T}r(u_h)| = 0 \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega.$$

Since $[\mathbf{z}_h, \nu]$ and u_h satisfy (7.36), this means that

$$[\mathbf{z}_g, \nu] = [\mathbf{z}_h, \nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \{h \ne \mathcal{T}r(u_h)\}.$$

Since $\|[\mathbf{z}_g,\nu]\|_\infty \le 1$, we have thereby shown that

$$[\mathbf{z}_g,\nu] \in \operatorname{sign}(h - \mathcal{T}r(u_h)) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{7.80}$$

Summarizing, we have shown that for every $(h,g) \in \overline{\Lambda}^{L^1\times L^\infty_\sigma}$, there are $u_h \in BV(\Omega)$ and $\mathbf{z}_g \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying $\|\mathbf{z}_g\|_\infty \le 1$, (7.69), (7.71), (7.75), and (7.80), proving that $(h,g) \in \Lambda$. □

To complete the proof of Theorem 1.9, it remains to show that the Dirichlet-to-Neumann operator Λ associated with the 1-Laplace operator satisfies the *range condition* (7.48) for $X = L^1(\partial\Omega)$ and some (or, equivalently, for all) $\lambda > 0$. To obtain this, we use that the *restriction* Λ_{L^2} *of* Λ *on* $L^2(\partial\Omega)\times L^\infty(\partial\Omega)$ is m-accretive in $L^2(\partial\Omega)$. This property of $\Lambda_{|L^2}$ and the proof of its sub-differential structure is outlined in the following subsection.

7.4.2 The Dirichlet-to-Neumann operator in L^2

In this subsection, we focus on the Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ in $L^2(\partial\Omega)$.

Definition 7.5 We define the *Dirichlet-to-Neumann operator* $\Lambda_{|L^2}$ *in* $L^2(\partial\Omega)$ *associated with the* 1*-Laplace operator* Δ_1 by

$$\Lambda_{|L^2} = \Lambda \cap \Big(L^2(\partial\Omega)\times L^2(\partial\Omega)\Big);$$

or equivalent, by the set of all pairs $(h,g) \in L^2(\partial\Omega)\times L^2(\partial\Omega)$ with the property that there is a weak solution $u \in BV(\Omega)$ of Dirichlet problem (7.30) with Dirichlet data h and there is a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^d)$ associated with u (satisfying (7.36)-(7.39)) and

$$g = [\mathbf{z},\nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega.$$

Remark 7.2 Since $L^2(\partial\Omega) \subseteq L^1(\partial\Omega)$, it follows from Remark 7.1 that the effective domain $D(\Lambda_{|L^2})$ of the Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ associated with Δ_1 satisfies

$$D(\Lambda_{|L^2}) = L^2(\partial\Omega)$$

and the operator

$$\Lambda_{|L^2} \subseteq L^2(\partial\Omega)\times \overline{B}_{L^\infty(\partial\Omega)}. \tag{7.81}$$

It is clear that $\Lambda_{|L^2}$ is completely accretive in $L^2(\partial\Omega)$ since Λ admits this property in $L^1(\partial\Omega)$. We can say more about $\Lambda_{|L^2}$.

Proposition 7.17 *The Dirichlet-to-Neumann operator* $\Lambda_{|L^2}$ *in* $L^2(\partial\Omega)$ *associated with the* 1*-Laplace operator* Δ_1 *is cyclically monotone.*

Proof Let $(h_j)_{j=0}^n \subseteq D(\Lambda_{|L^2})$ be a finite cyclic sequence with $h_0 = h_n$ and $(g_j)_{j=0}^n$ a corresponding sequence of elements $g_j \in \Lambda_{|L^2} h_j$. Then, for every $j = 0,\dots,n$, there is a weak solution $u_j \in BV(\Omega)$ of Dirichlet problem (7.30) with Dirichlet data h_j, (w.l.g., we may assume $u_0 = u_n$), and there is a vector field $\mathbf{z}_j \in L^\infty(\Omega;\mathbb{R}^d)$ associated with u_j (satisfying (7.36)-(7.39)) and

$$g_j = [\mathbf{z}_j, \nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{7.82}$$

By applying the generalized integration by parts formula (7.13) to $w = (u_j - u_{j-1})$ and the vector field $\mathbf{z}_j$, and by using (7.38), gives

$$\int_\Omega (\mathbf{z}_j, D(u_j - u_{j-1})) = \int_{\partial\Omega} [\mathbf{z}_j, \nu]\Big(\mathcal{T}r(u_j) - \mathcal{T}r(u_{j-1})\Big)\,\mathrm{d}\mathcal{H}^{d-1}$$

for $j = 1, \dots, n$. Therefore, by (7.82), the bilinearity of the pairing $(\cdot, D\cdot)$, and since $u_0 = u_n$, it follows from the last integral equation that

$$\begin{aligned}
\sum_{j=1}^n \int_\Omega (\mathbf{z}_j, D(u_j - u_{j-1})) &= \sum_{j=1}^n \left[\int_\Omega |Du_j| - \int_\Omega (\mathbf{z}_j, Du_{j-1})\right] \\
&= \sum_{j=1}^{n-1} \left[\int_\Omega |Du_j| - \int_\Omega (\mathbf{z}_{j+1}, Du_j)\right] \\
&\quad + \int_\Omega |Du_n| - \int_\Omega (\mathbf{z}_1, Du_0).
\end{aligned}$$

By (7.9), (7.39), and since $u_0 = u_n$, we can conclude that the right hand-side in the last equation is non-negative and hence, we have shown that

$$\sum_{j=1}^n \int_{\partial\Omega} g_j \Big(\mathcal{T}r(u_j) - \mathcal{T}r(u_{j-1})\Big)\mathrm{d}\mathcal{H}^{d-1} \ge 0.$$

By using now this inequality, one sees that

$$\begin{aligned}
&\sum_{j=1}^n \int_{\partial\Omega} g_j \Big(h_j - h_{j-1}\Big)\mathrm{d}\mathcal{H}^{d-1} \\
&\quad \ge \int_{\partial\Omega} \sum_{j=1}^n g_j \Big((h_j - \mathcal{T}r(u_j)) - (h_{j-1} - \mathcal{T}r(u_{j-1}))\Big)\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned} \tag{7.83}$$

By (7.82), since u_j satisfies the Dirichlet boundary condition (7.32) in the weak sense (7.36) with $h = h_j$, and since $h_0 = h_n$ and $u_0 = u_n$, one finds that

$$\begin{aligned}
&\sum_{j=1}^n g_j \Big((h_j - \mathcal{T}r(u_j)) - (h_{j-1} - \mathcal{T}r(u_{j-1}))\Big) \\
&\quad = \sum_{j=1}^n g_j\,(h_j - \mathcal{T}r(u_j)) - \sum_{j=1}^n g_j\,(h_{j-1} - \mathcal{T}r(u_{j-1}))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} g_j\,(h_j - \mathcal{T}r(u_j)) - \sum_{j=0}^{n-1} g_{j+1}\,(h_j - \mathcal{T}r(u_j))\\
&= \sum_{j=1}^{n-1} |h_j - \mathcal{T}r(u_j)| - [\mathbf{z}_{j+1},\nu]\,(h_j - \mathcal{T}r(u_j))\\
&\qquad\qquad + |h_n - \mathcal{T}r(u_n)| - [\mathbf{z}_1,\nu]\,(h_0 - \mathcal{T}r(u_0)) \geq 0
\end{aligned}
$$

for $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$. Applying this to (7.83) yields that

$$\sum_{j=1}^{n} \int_{\partial\Omega} g_j\left(h_j - h_{j-1}\right)\mathrm{d}\mathcal{H}^{d-1} \geq 0,$$

and since the cyclic sequence $(h_j)_{j=0}^{n} \subseteq D(\Lambda_{|L^2})$ was arbitrary, we have thereby shown that $\Lambda_{|L^2}$ is cyclically monotone. □

Proposition 7.18 *The Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ associated with the 1-Laplace operator satisfies the range condition (7.48) for $X = L^2(\partial\Omega)$.*

Proof Let $g \in L^2(\partial\Omega)$ and $\lambda > 0$. Then, our aim is to find a boundary function $h \in L^2(\partial\Omega)$ such that the inclusion

$$h + \lambda\Lambda_{|L^2} h \ni g \tag{7.84}$$

holds. By the definition of $\Lambda_{|L^2}$, inclusion (7.84) is equivalent to the fact that there is a vector field $\mathbf{z}_h \in \mathcal{Z}_h$ such that the weak trace $[\mathbf{z}_h,\nu]$ of the normal component of $\mathbf{z}_h$ is given by

$$\frac{g-h}{\lambda} = [\mathbf{z}_h,\nu] \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega.$$

Since $\|[\mathbf{z}_h,\nu]\|_\infty \leq 1$, it is natural to impose on the vector field $\mathbf{z}_h$ the condition

$$[\mathbf{z}_h,\nu] = T_1\left(\frac{g-h}{\lambda}\right) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega, \tag{7.85}$$

where T_1 denotes the truncator introduced in Section 7.3. Thus, if we find a boundary function h such that there is a *weak solution* u to the elliptic boundary-value problem

$$
\begin{cases}
-\Delta_1 u = 0 & \text{in } \Omega,\\
u = h & \text{on } \partial\Omega,\\
\dfrac{Du}{|Du|}\cdot\nu = T_1\left(\dfrac{g-h}{\lambda}\right) & \text{on } \partial\Omega,
\end{cases}
\tag{7.86}
$$

then h is a solution to the inclusion (7.84).

Definition 7.6 For given $g \in L^2(\partial\Omega)$, $h \in L^1(\partial\Omega)$, and $\lambda > 0$, we call a function $u \in BV(\Omega)$ a *weak solution* of boundary problem (7.86) if there is a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (7.37)-(7.39) and the weak trace $[\mathbf{z},\nu]$ satisfies (7.85) and

$$\frac{g-h}{\lambda} \in \operatorname{sign}(h - \mathcal{T}r(u)) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{7.87}$$

According to Theorem 7.6, for given $g \in L^2(\partial\Omega)$ and $\lambda > 0$, there is a weak solution u of the Robin-type boundary-value problem (7.49) with $\alpha = \lambda$; that is, $u \in BV(\Omega)$ with trace $\mathcal{T}r(u) \in L^2(\partial\Omega)$, and there is a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (7.37)-(7.39), and

$$[\mathbf{z},\nu] = T_1(g - \lambda u) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega.$$

Now, according to Proposition 7.11, u is also a weak solution of Dirichlet problem (7.30) for Dirichlet data

$$h := g - \lambda\,[\mathbf{z},\nu].$$

Since for this choice of h, one trivially has that $\frac{g-h}{\lambda} = [\mathbf{z},\nu]$, it follows that u, h and $\mathbf{z}$ satisfy (7.85) and (7.87). Moreover, since $h \in L^2(\partial\Omega)$, we have thereby shown that there is a h satisfying the inclusion (7.84), completing the proof of Proposition 7.18. □

We conclude this section with the following characterization of the Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ on $L^2(\partial\Omega)$.

Proposition 7.19 *The Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ in $L^2(\partial\Omega)$ can be characterized as the sub-differential operator $\partial_{L^2(\partial\Omega)}\varphi_{|L^2}$ in $L^2(\partial\Omega)$; that is,*

$$\Lambda_{|L^2} = \partial_{L^2(\partial\Omega)}\varphi_{|L^2}$$

where $\varphi_{|L^2}$ denotes the restriction on $L^2(\partial\Omega)$ of the functional φ given by (7.61).

We provide two different proofs of Proposition 7.19.

Proof (of Proposition 7.19, 1st Proof) By Proposition 7.17 and Proposition 7.18, the Dirichlet-to-Neumann operator Λ_{L^2} is a maximal cyclically monotone operator in $L^2(\partial\Omega)$. Moreover, by Proposition 7.13 and since $\Lambda_{L^2} \subseteq \Lambda$, we have that Λ_{L^2} is homogeneous of order zero. Therefore by Theorem 7.3, there is a unique proper, convex, lower semicontinuous functional $\hat{\varphi}$ on $L^2(\partial\Omega)$, which is homogeneous of order one satisfying $\Lambda_{L^2} = \partial_{L^2(\partial\Omega)}\hat{\varphi}$. Since $\hat{\varphi}$ is homogeneous of order one, it follows from (7.29) that $\hat{\varphi}$ satisfies

$$\hat{\varphi}(h) = \int_{\partial\Omega} g\,h\,\mathrm{d}\mathcal{H}^{d-1} \qquad \text{for every } (h,g) \in \Lambda_{L^2}.$$

By definition of Λ_{L^2}, for every $(h,g) \in \Lambda_{L^2}$, there is a vector field $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ and $u_h \in BV(\Omega)$ satisfying (7.36)-(7.39), and $g = [\mathbf{z}_h,\nu]$. From this, we can conclude that

$$\hat{\varphi}(h) = \int_{\partial\Omega} [\mathbf{z}_h,\nu]\,h\,\mathrm{d}\mathcal{H}^{d-1} = \varphi(h)$$

for every $h \in L^2(\partial\Omega)$, which identifies the functional $\hat{\varphi}$ with φ given by (7.61). □

The argument of our second proof of Proposition 7.19 is a bit shorter.

Proof (of Proposition* 7.19, 2*nd* *Proof) On the other hand, the restriction $\varphi_{|L^2}$ of the functional φ given by (7.61) on $L^2(\partial\Omega)$ is by Proposition 7.14, convex, proper, lower semicontinuous on $L^2(\partial\Omega)$ and homogeneous of order one. Moreover, by following the same argument as in the proof of Proposition 7.15, one easily sees that $\Lambda_{L^2} \subseteq \partial_{L^2(\partial\Omega)}\varphi_{|L^2}$, which means that $\partial_{L^2(\partial\Omega)}\varphi_{|L^2}$ is a monotone extension of Λ_{L^2}. But since Λ_{L^2} is maximal monotone, this is only possible if $\Lambda_{L^2} = \partial_{L^2(\partial\Omega)}\varphi_{|L^2}$ (see [40, Proposition 2.2]), which proves the claim of Proposition 7.19. □

7.4.3 The Dirichlet-to-Neumann operator in L^1 (continued)

With this in mind, we can now complete the proof of Theorem 1.9.

Proof (of Theorem* 1.9.*) We only show that Λ satisfies the range condition (7.48) since then the characterization (1.71) follows from Theorem 3.4 (in Chapter 3.2.2.2) and the other statements of this theorem were proved in the before-going propositions. By Proposition 7.18, the restriction $\Lambda_{|L^2}$ of Λ on $L^2(\partial\Omega)$ satisfies the range condition (7.48) for $X = L^2(\partial\Omega)$. Thus and since $\Lambda_{|L^2} \subseteq \Lambda$, we have that Λ satisfies the range condition (7.84) for every $g \in L^2(\partial\Omega)$. Now, let $g \in L^1(\partial\Omega)$ and choose a sequence $(g_n)_{n\ge 1}$ in $L^2(\partial\Omega)$ such that $g_n \to g$ in $L^1(\partial\Omega)$. If J_λ denotes the resolvent operator of Λ, then for every $n \ge 1$, $h_n = J_\lambda g_n$ belongs to $L^2(\partial\Omega)$ and satisfies (7.84) with right hand-side g_n. By Proposition 7.12, $(h_n)_{n\ge 1}$ is a Cauchy sequence in $L^1(\partial\Omega)$. Hence, there is an $h \in L^1(\partial\Omega)$ such that $h_n \to h$ in $L^1(\partial\Omega)$. Now,

$$\Lambda_{|L^2} h_n \ni \frac{g_n - h_n}{\lambda} \to \frac{g-h}{\lambda} \qquad \text{in } L^1(\partial\Omega) \text{ as } n \to \infty.$$

Note, that the sequence $((g_n - h_n)/\lambda)_{n\ge 1}$ is also bounded in $L^\infty(\partial\Omega)$. Thus, after passing to a subsequence, we also have that $(g_n - h_n)/\lambda \to (g-h)/\lambda$ weakly* in $L^\infty(\partial\Omega)$. Since by Proposition 7.16, Λ is closed in $L^1 \times L^\infty_\sigma(\partial\Omega)$, we have thereby shown that

$$\Lambda h = \frac{g-h}{\lambda},$$

which is equivalent to the range condition (7.48) for $X = L^1(\partial\Omega)$. This completes the proof of this theorem. □

Next, we outline the proof of Theorem 1.10.

Proof (Theorem* 1.10*) By Proposition 7.19 and the Hilbert space theory on maximal monotone operators (see [40]), for every $h_0 \in L^2(\partial\Omega)$ and $g \in L^2(0,T;L^2(\partial\Omega))$, there is a unique strong solution

$$h \in W^{1,2}([\delta,T);L^2(\partial\Omega)) \cap C([0,\infty);L^2(\partial\Omega)),\ \delta \in (0,T),$$

of the perturbed homogeneous Cauchy problem (in $L^2(\partial\Omega)$)

$$\begin{cases} \frac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda h(t) + F(h(t)) \ni g(t) & \text{for } t \in (0,T), \\ h(0) = h_0. & \text{on } \partial\Omega. \end{cases} \tag{7.88}$$

Under the hypothesis that $f(\cdot,h)$ satisfies either (1.76) or (1.80) from Chapter 1.11, the function $G : L^2(\partial\Omega) \to \mathbb{R}$ defined by

$$G(h) := \int_{\partial\Omega} \int_0^{h(x)} f(x,r)\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^{d-1} \tag{7.89}$$

is $C^1(L^2(\partial\Omega);\mathbb{R})$ with derivative $G'(h) = F(h)$. Hence, the following *chain rule* holds

$$\frac{d}{dt}G(h(t)) = (G'(h(t)), \frac{dh}{dt}(t))_{L^2(\partial\Omega)} = (F(h(t)), \frac{dh}{dt}(t))_{L^2(\partial\Omega)}$$

for every $h \in W^{1,2}(0,T;L^2(\partial\Omega))$. Thus and by [40, Lemme 3.3] applied to the second functional φ_f defined in (1.79) (in Chapter 1.11), we get

$$\left\|\frac{\mathrm{d}h}{\mathrm{d}t}(t)\right\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\varphi_f(h(t)) = \left(g(t), \frac{\mathrm{d}h}{\mathrm{d}t}(t)\right)_{L^2(\partial\Omega)} \tag{7.90}$$

for a.e. $t > 0$.

Now, since $\varphi_f(h)$ is defined for all $h \in L^2(\partial\Omega)$, we can integrate the latter equation over the whole interval $(0,t)$ for every $t \in (0,T)$. Hence, we find that

$$\int_0^t \left\|\frac{\mathrm{d}h}{\mathrm{d}s}(s)\right\|_2^2 \mathrm{d}s + \varphi_f(h(t)) = \varphi_f(h_0) + \int_0^t \left(g(s), \frac{\mathrm{d}h}{\mathrm{d}s}(s)\right)_{L^2(\partial\Omega)} \mathrm{d}s.$$

Next, applying Young's inequality to compensate the term $\frac{\mathrm{d}h}{\mathrm{d}s}(s)$ on the right-hand side, one sees that the global estimate (1.81) holds. This proves statement (2) of Theorem 1.10.

Next, suppose that $f(x,h)$ satisfies the growth estimate (1.80), and let $g \in L^2(0,T;L^2(\partial\Omega))$, $h_0 \in L^1(\partial\Omega)$, and $(h_{0,n})_{n\geq 1}$ a sequence in $L^2(\partial\Omega)$ converging to h_0 in $L^1(\partial\Omega)$. Since $\partial\Omega$ has finite measure, each strong solutions h_n of Cauchy problem (in $L^2(\partial\Omega)$)

$$\begin{cases} \frac{\mathrm{d}h_n}{\mathrm{d}t}(t) + \Lambda h_n(t) + F(h_n(t)) \ni g(t) & \text{for } t \in (0,T), \\ h_n(0) = h_{0,n} \end{cases} \tag{7.91}$$

is also a strong solution in $L^1(\partial\Omega)$ of Cauchy problem (7.91). Moreover, by Corollary 1.5 (in Chapter 1.11),

$$\lim_{n\to\infty} h_n = h \qquad \text{in } C([0,T];L^1(\partial\Omega)) \tag{7.92}$$

and h is the unique mild solution of Cauchy problem (7.88) in $L^1(\partial\Omega)$. Under the condition (1.80) on f, the functional φ_f defined by (1.79) can be extended continuously on $L^1(\partial\Omega)$ and so, we can apply h_n to the global estimate (1.81). Thus, the sequence

$$\left(\frac{\mathrm{d}h_n}{\mathrm{d}t}\right)_{n\geq 1} \qquad \text{is bounded in } L^2(0,T;L^2(\partial\Omega))$$

and so, there is a $\chi \in L^2(0,T;L^2(\partial\Omega))$ and a subsequence of $(h_n)_{n\geq 1}$, which, for simplicity, we denote again by $(h_n)_{n\geq 1}$ such that

$$\lim_{n\to\infty} \frac{\mathrm{d}h_n}{\mathrm{d}t} = \chi \qquad \text{weakly in } L^2(0,T;L^2(\partial\Omega)). \tag{7.93}$$

Let $\xi \in C_c^\infty(0,T)$ and $v \in L^2(\partial\Omega)$. Since $\frac{\mathrm{d}h_n}{\mathrm{d}t}$ is the weak derivative of h_n in $L^2(0,T;L^2(\partial\Omega))$, one has that

$$\int_0^T (\frac{\mathrm{d}h_n}{\mathrm{d}t}, v)_{L^2(\partial\Omega))}\xi(t)\,\mathrm{d}t = -\int_0^T (h_n, v)_{L^2(\partial\Omega))}\frac{\mathrm{d}}{\mathrm{d}t}\xi(t)\,\mathrm{d}t.$$

By (7.92) and (7.93), sending $n \to \infty$ in the last equation gives that

$$\int_0^T (\chi(t), v)_{L^2(\partial\Omega))}\xi(t)\,\mathrm{d}t = -\int_0^T (h, v)_{L^2(\partial\Omega))}\frac{\mathrm{d}}{\mathrm{d}t}\xi(t)\,\mathrm{d}t.$$

Since $\xi \in C_c^\infty(0,T)$ and $v \in L^2(\partial\Omega)$ were arbitrary, this proves that χ is the weak derivative of h in $L^2(0,T;L^2(\partial\Omega))$. Coming back to the global estimate (1.81) for $h = h_n$, if one takes the limit inferior in this inequality, and uses that φ_f is continuous on $L^1(\partial\Omega)$, then one sees that (1.81) also holds for initial data $h_0 \in L^1(\partial\Omega)$, completing the proof of statement (1) of this theorem. Statement (3) of Theorem 1.10 follows immediately from the theory developed in Chapter 5. This completes the proof of Theorem 1.10. □

7.4.4 Long-time stability

In this section, we give the proof of Theorem 1.11 on the long-time stability of the semigroup $\{e^{-t(\Lambda_{|L^q}+F)}\}_{t\geq 0}$ generated by $-(\Lambda_{|L^q(\partial\Omega)} + F)$ on $L^q(\partial\Omega)$.

We begin by the following proposition.

Proposition 7.20 *Let F be given by (1.75) with f satisfying (1.76) from Chapter 1.11, and φ be the functional given by (7.61). Then the following statements hold.*

1. *Let $\varphi_f : L^1(\partial\Omega) \to \mathbb{R}$ be the functional defined by*

$$\varphi_f(h) := \xi(h) + \int_{\partial\Omega} \int_0^{h(x)} f(x,r)\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^{d-1}, \qquad h \in L^1(\partial\Omega),$$

for every $h \in L^1(\partial\Omega)$*. Then, for every* $h_0 \in L^1(\partial\Omega)$*, the function*

$$t \mapsto \varphi_f(e^{-t(\Lambda+F)}h_0)$$

decreases monotonically along $(0,\infty)$*;*

2. *If* $F \equiv 0$*, then one has that*

$$\langle \frac{\mathrm{d}}{\mathrm{d}t} e^{-t\Lambda}h_0, e^{-t\Lambda}h_0 \rangle_{L^\infty,L^1} = -\varphi(e^{-t\Lambda}h_0) \tag{7.94}$$

for a.e. $t > 0$ *and every* $h_0 \in L^1(\partial\Omega)$*.*

3. *If* $F \equiv 0$*, then for every positive* $h_0 \in L^1(\partial\Omega)$*, one has that* $e^{-t\Lambda}h_0$ *is positive for every* $t \geq 0$ *and that*

$$\varphi(e^{-t\Lambda}h_0) \geq -\frac{1}{t}\|e^{-t\Lambda}h_0\|_2^2 \tag{7.95}$$

Proof By taking $g \equiv 0$ in (7.90), and subsequently integrating over (s,t) for any $0 \leq s \leq t$, one sees that

$$\varphi_f(e^{-t(\Lambda+F)}h_0) \leq \varphi_f(e^{-s(\Lambda+F)}h_0),$$

showing that $t \mapsto \varphi_f(e^{-t(\Lambda+F)}h_0)$ is monotonically decreasing along $(0,\infty)$ provided the initial data $h_0 \in L^2(\partial\Omega)$. Now, let $h_0 \in L^1(\partial\Omega)$. Then, there is a sequence $(h_{0,n})_{n\geq 1}$ in $L^2(\partial\Omega)$ converging to h in $L^1(\partial\Omega)$. By Corollary 1.5 (in Chapter 1.11), $e^{-t(\Lambda+F)}h_{0,n} \to e^{-t(\Lambda+F)}h_0$ in $C([0,T];L^1(\partial\Omega))$ for every $T > 0$. Thus and by the continuity of φ_f on $L^1(\partial\Omega)$, for given $0 \leq s \leq t$, we can send $n \to \infty$ in

$$\varphi_f(e^{-t(\Lambda+F)}h_{0,n}) \leq \varphi_f(e^{-s(\Lambda+F)}h_{0,n})$$

and find that $t \mapsto \varphi_f(e^{-t(\Lambda+F)}h_0)$ is monotonically decreasing along $(0,\infty)$ also for every initial data $h_0 \in L^1(\partial\Omega)$. Next, we show that (7.94) holds. For this, we note that by Theorem 1.10, for every $h_0 \in L^1(\partial\Omega)$, $h(t) := e^{-t(\Lambda+F)}h_0$ is a strong solution of the homogeneous Cauchy problem (in $L^1(\partial\Omega)$)

$$\begin{cases} \frac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda h(t) \ni 0 & \text{for } t \in (0,T), \\ h(0) = h_0. & \text{on } \partial\Omega. \end{cases}$$

Hence, multiplying by $h(t)$ and using that φ is homogeneous of order one, yields (7.94). Finally, we show that for every positive $h_0 \in L^1(\partial\Omega)$, (7.95) holds. Note, that by Corollary 1.5, $e^{-t\Lambda}h_0$ is positive for every $t \geq 0$. Moreover, by (1.84) from Corollary 1.6, one has that

$$\frac{\mathrm{d}}{\mathrm{d}t_+} e^{-t\Lambda_1}h_0 \leq \frac{1}{t} e^{-t\Lambda_1}h_0 \qquad \text{for a.e. } t > 0.$$

Applying this to (7.94), one sees that (7.95) holds. □

For the rest of this section, we focus on the case $F \equiv 0$. Then, we have the following.

Proposition 7.21 *The semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$ generated by the negative Dirichlet-to-Neumann operator $-\Lambda$ on $L^1(\partial\Omega)$ conserves mass; in other words, one has that*

$$\int_{\partial\Omega} h_0 \,\mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\Omega} e^{-t\Lambda} h_0 \,\mathrm{d}\mathcal{H}^{d-1} \tag{7.96}$$

for all $t \geq 0$ and $h_0 \in L^1(\partial\Omega)$.

Proof Recall that by Theorem 1.10, for every $h_0 \in L^1(\partial\Omega)$, $h(t) := e^{-t\Lambda} h_0$ is a strong solution of the Cauchy problem (in $L^1(\partial\Omega)$)

$$\begin{cases} \dfrac{\mathrm{d}h}{\mathrm{d}t}(t) + \Lambda h(t) \ni 0 & \text{for } t \in (0,T), \\ h(0) = h_0. & \text{on } \partial\Omega. \end{cases} \tag{7.97}$$

Hence, for a.e. $t > 0$, there is a weak solution $u_{h(t)} \in BV(\Omega)$ of Dirichlet problem (7.30) and a vector field $\mathbf{z}_{h(t)} \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (7.36)-(7.39) with boundary data $h(t)$, and the generalized co-normal derivative

$$[\mathbf{z}_{h(t)}, \nu] = -\frac{\mathrm{d}h}{\mathrm{d}t}(t) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{7.98}$$

Let $\mathbb{1}_{\overline{\Omega}}$ denote the constant 1 function on $\overline{\Omega}$. Multiplying (7.98) by $\mathcal{T}r(\mathbb{1}_{\overline{\Omega}}) = \mathbb{1}_{\partial\Omega}$ with respect to the L^2-inner product and then, integrating by parts (Proposition 7.7) yields that

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega} h(t)\,\mathbb{1}\,\mathrm{d}\mathcal{H}^{d-1} &= -\int_{\partial\Omega} \frac{\mathrm{d}h}{\mathrm{d}t}(t)\,\mathbb{1}\,\mathrm{d}\mathcal{H}^{d-1} \\ &= \int_{\partial\Omega} [\mathbf{z}_{h(t)}, \nu]\,\mathbb{1}\,\mathrm{d}\mathcal{H}^{d-1} \\ &= \int_{\Omega} (\mathbf{z}_{h(t)}, D\mathbb{1}) = 0. \end{aligned}$$

Hence, integrating this equation over $(0,t)$ for any $t > 0$, shows that (7.96) holds. □

Next, we establish the long-time convergence in $L^q(\partial\Omega)$ of the semigroup $\{e^{-t\Lambda}\}_{t\geq 0}$.

Proposition 7.22 *Let $1 \leq q < \infty$, φ given by (7.61), and $h_0 \in L^q(\partial\Omega)$. Then, the following statements hold.*

1. *One has that*

$$\lim_{t\to\infty} e^{-t\Lambda} h_0 = \overline{h_0} := \tfrac{1}{\mathcal{H}^{d-1}(\partial\Omega)} \int_{\partial\Omega} h_0 \,\mathrm{d}\mathcal{H}^{d-1} \qquad \text{in } L^q(\partial\Omega); \tag{7.99}$$

2. *One has that*

$$\lim_{t\to\infty} \varphi(e^{-t\Lambda}h_0) = \varphi(\overline{h_0}) = 0; \tag{7.100}$$

3. *(Entropy-Transport inequalities) There is a $C > 0$ such that*

$$\|e^{-t\Lambda}h_0 - \overline{h_0}\|_1 \le C\,\varphi(e^{-t\Lambda}h_0) \qquad \textit{for all } t > 0;$$

Moreover, for every $1 < q < r \le \infty$ and $h_0 \in L^r(\partial\Omega)$, one has that

$$\|e^{-t\Lambda}h_0 - \overline{h_0}\|_q \le \|h_0 - \overline{h_0}\|_r^{\frac{(q-1)r}{q(r-1)}}\, C\left(\varphi(e^{-t\Lambda}h_0)\right)^{\frac{r-q}{q(r-1)}}$$

4. *For every $h_0 \in L^2(\partial\Omega)$, one has that*

$$\varphi(e^{-t\Lambda}h_0) \le 2\frac{\|h_0\|_2^2}{t} \qquad \textit{for all } t > 0. \tag{7.101}$$

Proof We first establish (7.99) for $h_0 \in L^2(\partial\Omega)$. Since the functional φ given by (7.61) is even, and since the Dirichlet-to-Neumann operator $\Lambda_{|L^2}$ is the subgradient of the restriction $\varphi_{|L^2}$ of φ on $L^2(\partial\Omega)$, the limit (7.99) follows from a classic result due to Bruck [43, Theorem 5] in the Hilbert space theory. Moreover, by the continuity of $\varphi_{|L^2}$ on $L^2(\partial\Omega)$, it follows that (7.100) holds.

Next, let $h_0 \in L^q(\partial\Omega)$ for a given $1 \le q \le \infty$. One can always construct a sequence $(h_{0,n})_{n\ge1}$ in $L^\infty(\partial\Omega)$ such that $h_{0,n} \to h_0$ in $L^q(\partial\Omega)$ and by the continuous embedding from $L^q(\partial\Omega)$ into $L^1(\partial\Omega)$, one also has that the mean-values $\overline{h_{0,n}} \to \overline{h_0}$ in $\mathbb{R}$ as $n \to \infty$. If $q = \infty$, then one simply choose the sequence $(h_{0,n})_{n\ge1}$ given by $h_{0,n} \equiv h_0$ for all $n \ge 1$. Then, for given $\varepsilon > 0$, there is a $n_0 = n_0(\varepsilon) \in \mathbb{N}$ large enough such that

$$\|h_{0,n} - h_0\|_q < \frac{\varepsilon}{3} \quad \text{and} \quad \|\overline{h_{0,\varepsilon}} - \overline{h_0}\|_q < \frac{\varepsilon}{3}.$$

Since each mapping $e^{-t\Lambda}$ is contractive in $L^q(\partial\Omega)$, one has that

$$\begin{aligned}
\|e^{-t\Lambda}h_0 - \overline{h_0}\|_q &\le \|e^{-t\Lambda}h_0 - e^{-t\Lambda}h_{0,n_0}\|_q + \|e^{-t\Lambda}h_{0,n_0} - \overline{h_{0,n_0}}\|_q \\
&\quad + \|\overline{h_{0,n_0}} - \overline{h_0}\|_q \\
&\le \|h_0 - h_{0,n_0}\|_q + \|e^{-t\Lambda}h_{0,n_0} - \overline{h_{0,n_0}}\|_q \\
&\quad + \|\overline{h_{0,n_0}} - \overline{h_0}\|_q \\
&\le 2\frac{\varepsilon}{3} + \|e^{-t\Lambda}h_{0,n_0} - \overline{h_{0,n_0}}\|_q.
\end{aligned}$$

Thus, in order to prove (7.99) in $L^q(\partial\Omega)$ for general $h_0 \in L^q(\partial\Omega)$, it is sufficient to establish (7.99) for $h_0 \in L^\infty(\partial\Omega)$. So, let $h_0 \in L^\infty(\partial\Omega)$. Since h_0 also belongs to $L^2(\partial\Omega)$, the first part of this proof implies that $e^{-t\Lambda}h_0 \to \overline{h_0}$ in $L^2(\partial\Omega)$ as $t \to \infty$. If $q < 2$, then by the continuous embedding of $L^2(\partial\Omega)$ into $L^q(\partial\Omega)$, one has that (7.99) needs to be true also in this case. Thus, let's focus now on the case $2 < q \le \infty$. Since $h_0 \in L^\infty(\partial\Omega)$, by the contractivity property of $e^{-t\Lambda}$ in $L^\infty(\partial\Omega)$, and by the fact that

$$e^{-t\Lambda}(c\mathbb{1}_{\partial\Omega}) = c\mathbb{1}_{\partial\Omega} \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \text{ for all } t \geq 0,$$

one sees that

$$\begin{aligned}\|e^{-t\Lambda}h_0 - \overline{h_0}\|_q &\leq \|e^{-t\Lambda}h_0 - \overline{h_0}\|_\infty^{\frac{q-2}{q}}\ \|e^{-t\Lambda}h_0 - \overline{h_0}\|_2^{\frac{2}{q}}\\ &\leq \|h_0 - \overline{h_0}\|_\infty^{\frac{q-2}{q}}\ \|e^{-t\Lambda}h_0 - \overline{h_0}\|_2^{\frac{2}{q}} \longrightarrow 0\end{aligned}$$

as $t \longrightarrow \infty$. This completes the proof of statement (1) and by using the continuity of the functional φ, it follows that (1) implies (2). To see that (3) holds, we recall that by Theorem 7.5,

$$\varphi(e^{-t\Lambda}h_0) = \int_\Omega |Du_{e^{-t\Lambda}h_0}| + \int_{\partial\Omega} |e^{-t\Lambda}h_0 - \mathcal{T}r(u_{e^{-t\Lambda}h_0})|\,\mathrm{d}\mathcal{H}^{d-1} \tag{7.102}$$

for every $t > 0$, where $Du_{e^{-t\Lambda}h_0} \in BV(\Omega)$ denotes a weak solution of Dirichlet problem (7.30) with boundary data $e^{-t\Lambda}h_0$. Further, by the Poincaré trace-inequality (7.3) for BV-functions, there is a constant $C_p > 0$ such that

$$\|\mathcal{T}r(u_{e^{-t\Lambda}h_0}) - \overline{\mathcal{T}r(u_{e^{-t\Lambda}h_0})}\|_1 \leq C_p \int_\Omega |Du_{e^{-t\Lambda}h_0}|. \tag{7.103}$$

Now, by using (7.102), (7.103), and (7.96), then one finds that

$$\begin{aligned}\int_{\partial\Omega} |e^{-t\Lambda}h_0 - \overline{h_0}|\,\mathrm{d}\mathcal{H}^{d-1} &\leq \int_{\partial\Omega} |e^{-t\Lambda}h_0 - \mathcal{T}r(u_{e^{-t\Lambda}h_0})|\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad + \int_{\partial\Omega} |\mathcal{T}r(u_{e^{-t\Lambda}h_0}) - \overline{\mathcal{T}r(u_{e^{-t\Lambda}h_0})}|\,\mathrm{d}\mathcal{H}^{d-1}\\ &\qquad + \int_{\partial\Omega} |\overline{\mathcal{T}r(u_{e^{-t\Lambda}h_0})} - \overline{h_0}|\,\mathrm{d}\mathcal{H}^{d-1}\\ &\leq \int_{\partial\Omega} |e^{-t\Lambda}h_0 - \mathcal{T}r(u_{e^{-t\Lambda}h_0})|\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad + C_p \int_\Omega |Du_{e^{-t\Lambda}h_0}|\\ &\qquad + \left|\int_{\partial\Omega} (\mathcal{T}r(u_{e^{-t\Lambda}h_0}) - h_0)\,\mathrm{d}\mathcal{H}^{d-1}\right|\\ &= \int_{\partial\Omega} |e^{-t\Lambda}h_0 - \mathcal{T}r(u_{e^{-t\Lambda}h_0})|\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad + C_p \int_\Omega |Du_{e^{-t\Lambda}h_0}|\\ &\qquad + \left|\int_{\partial\Omega} (\mathcal{T}r(u_{e^{-t\Lambda}h_0}) - e^{-t\Lambda}h_0)\,\mathrm{d}\mathcal{H}^{d-1}\right|\\ &\leq 2\int_{\partial\Omega} |e^{-t\Lambda}h_0 - \mathcal{T}r(u_{e^{-t\Lambda}h_0})|\,\mathrm{d}\mathcal{H}^{d-1}\end{aligned}$$

$$
\begin{aligned}
&+C_p \int_\Omega |Du_{e^{-t\Lambda}h_0}| \\
&\leq (2+C_p)\,\varphi(e^{-t\Lambda}h_0)
\end{aligned}
$$

for all $t \geq 0$, proving (3). Finally, to see that (7.101) holds, one simply applies (1.83) to

$$
[\mathbf{z}_{e^{-t\Lambda}h_0}, \nu] = -\frac{\mathrm{d}h}{\mathrm{d}t_+}(t) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega,
$$

where the vector field $\mathbf{z}_{e^{-t\Lambda}h_0} \in L^\infty(\Omega;\mathbb{R}^d)$ is such that

$$
[\mathbf{z}_{e^{-t\Lambda}h_0}, \nu] = \Lambda^\circ(e^{-t\Lambda}h_0).
$$

Then, one finds that

$$
\begin{aligned}
\varphi(e^{-t\Lambda}h_0) &= \int_{\partial\Omega} [\mathbf{z}_{e^{-t\Lambda}h_0}, \nu]\, e^{-t\Lambda}h_0 \,\mathrm{d}\mathcal{H}^{d-1} \\
&\leq \int_{\partial\Omega} |[\mathbf{z}_{e^{-t\Lambda}h_0}, \nu]|\, |e^{-t\Lambda}h_0| \,\mathrm{d}\mathcal{H}^{d-1} \\
&\leq 2 \int_{\partial\Omega} \frac{|e^{-t\Lambda}h_0|^2}{t} \,\mathrm{d}\mathcal{H}^{d-1}
\end{aligned}
$$

for all $t > 0$. This completes the proof of this proposition. □

Appendix A
Weighted Sobolev Spaces

In this first chapter of the appendix, we provide a brief review about weighted Sobolev spaces $H^{1,p}\mu(\Omega)$, which is sufficient to understand well the content of this thesis. Here, we mainly follow the book [89] by Heinonen, Kilpeläinen, and Martio.

A.1 p-admissible weights

Throughout this section, let Ω be an open subset of the Euclidean space $\mathbb{R}^d$ for dimension $d \geq 2$.

For a given function $\omega \in L^1_{\text{loc}}(\mathbb{R}^d)$, we canonically associate a Radon measure μ with Radon-Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}x} = \omega$ by

$$\mu(O) = \int_O \omega(x)\mathrm{d}x \tag{A.1}$$

for every open subset O of $\mathbb{R}^d$. Since the family of open subsets O of $\mathbb{R}^d$ generate the Borel σ-Algebra on $\mathbb{R}^d$, the Radon measure μ is uniquely defined by (A.1).

Definition A.1 A function $\omega \in L^1_{\text{loc}}(\mathbb{R}^d)$ is called *p-admissible* if $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^d$ and the associated measure μ satisfies the following four conditions:

1. (*Doubling condition*) There is a constant $C > 0$ such that $\mu(2B) \leq C\,\mu(B)$ for every ball B in $\mathbb{R}^d$.
2. (*Well-definedness of the weak gradient* ∇u) Let Ω be an open subset of $\mathbb{R}^d$ and $1 \leq p < \infty$. If for $\mathbf{v} \in L^p_\mu(\Omega; \mathbb{R}^d)$, there is a sequence $(\varphi_n)_{n\geq 1}$ of functions $\varphi \in C^\infty(\Omega)$ satisfying

$$\lim_{n\to\infty} \int_\Omega |\varphi_n|^p \,\mathrm{d}\mu = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_\Omega |\nabla\varphi_n - \mathbf{v}|^p \,\mathrm{d}\mu = 0,$$

   then one has that $\mathbf{v} \equiv 0$.

3. (*Weighted Sobolev inequality*) There are $\kappa > 1$ and $C_S > 0$ such that

$$\left[\fint_{B_r} |\varphi|^{\kappa p} \, \mathrm{d}\mu \right]^{\frac{1}{\kappa p}} \le C_S \, r \left[\fint_{B_r} |\nabla \varphi|^p \, \mathrm{d}\mu \right]^{\frac{1}{p}} \tag{A.2}$$

for every open ball $B_r := \{x \in \mathbb{R}^d \mid |x - x_0| < r\}$ of radius $r > 0$ (and $x_0 \in \mathbb{R}^d$), and every $\varphi \in C_0^\infty(B_r)$, where the *integral average* $\fint_B f \, \mathrm{d}\mu := \int_B f \, \mathrm{d}\mu / \mu(B)$, and $1 \le p < \infty$.
4. (*Weighted Poincaré inequality*) There is a $C_P > 0$ such that

$$\int_{B_r} |\varphi - \varphi_{B_r}|^p \, \mathrm{d}\mu \le C_P \, r^p \int_{B_r} |\nabla \varphi|^p \, \mathrm{d}\mu$$

for every open ball B_r in $\mathbb{R}^d$ of radius $r > 0$, and every $\varphi \in C^\infty(B_r)$, where $\varphi_{B_r} := \fint_B \varphi \, \mathrm{d}\mu$, and $1 \le p < \infty$.

There are many examples of p-admissible weights (see, for example, the list of literature on weighted Sobolev and Poincaré inequalities in [89, Notes to Chapter 1]. We mention here only a few important ones.

Example A.1 It is quite obvious that the constant function $\omega \equiv 1$ admits the classical Lebesgue measure as canonically associated measure μ. By using the rsults from the classical theory of Sobolev Spaces (for example, see [41, 77] or [99]), then one easily verifies that the Lebesgue measure satisfies all four conditions from Definition A.1. Here, the parameter κ in the weighted local Sobolev inequality (A.2) is given by $\kappa = d/(d - p)$ if $1 \le p < d$ and for $p \ge d$, κ can be chosen $\kappa = 2$.

Example A.2 Another important class of p-admissible weights is given by the so-called A_p *Mouckenhoupt class*. For $1 < p < \infty$, the class A_p consists of all weights $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying $\omega(x) > 0$ a.e. on $\mathbb{R}^d$ and

$$\sup \left[\fint_B \omega \, \mathrm{d}x \right] \left[\fint_{B_r} \omega^{\frac{1}{1-p}} \, \mathrm{d}x \right]^{p-1}$$

is finite, where the supremum is taken over all open balls B in $\mathbb{R}^d$. Every weight $\omega \in A_p$ is p-admissible, where the emphasize is one the same p. More specifically, the weight $\omega(x) = |x|^\delta$ belongs to A_p if and only if $-d < \delta < d(p - 1)$.

Further, a weight $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to the *Mouckenhoupt class* A_1 if $\omega(x) > 0$ a.e. on $\mathbb{R}^d$ and there is a constant $C > 0$ such that

$$\fint_B \omega \, \mathrm{d}x \le C \operatorname{ess\,inf} \omega,$$

where the essential infimum is taken over all open balls B in $\mathbb{R}^d$. One has the inclusion $A_1 \subseteq A_p$ for all $p > 1$. Thus, every A_1-weight is p-admissible.

Appendix B
Mean spaces by Lions and Peetre

B.1 The connection between mean spaces and L^p spaces

The first part of the following theorem has been proved in [105, Théorème 1.1 of Chapter IV] by using so-called *discrete mean spaces* (cf. [105, Chapter II]). Here, we improve this result by showing that both spaces are isometrically isomorphic. This result serves us in the proof of Theorem 4.10 and Theorem 4.11 to determine the convergence of the constants in inequality (5.34) as $m \to \infty$. This part of the appendix has been taken from the monograph [**?**]

Theorem B.1 *Let* (Σ, μ) *be a* σ*-finite measure space,* (X_0, X_1) *be an interpolation couple,* $1 \le p_0$*,* $p_1 \le \infty$ *and* $0 < \theta < 1$*. Then for* $1 \le p \le \infty$ *given by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{B.1}$$

one has that

$$(L^{p_0}(\Sigma, X_0; \mu), L^{p_1}(\Sigma, X_1; \mu))_{\theta, p_0, p_1} = L^p(\Sigma, (X_0, X_1)_{\theta, p_0, p_1}; \mu) \tag{B.2}$$

with equal norms.

Proof (of Theorem B.1) We only outline the proof for $1 \le p_0 < \infty$ and $1 \le p_1 < \infty$ since the other cases are shown similarly.

First, let u be an element of $(L^{p_0}(\Sigma, X_0; \mu), L^{p_1}(\Sigma, X_1; \mu))_{\theta, p_0, p_1}$. By definition, there are measurable functions $v_i : (0, \infty) \to L^{p_i}(\Sigma, \mu)$ for $i = 0, 1$ such that $t^{-\theta} v_0 \in L^{p_0}_*(L^{p_0}(\Sigma, X_0; \mu))$, $t^{1-\theta} v_1 \in L^{p_1}(\Sigma, X_1; \mu)$ and

$$u(x) = v_0(t, x) + v_1(t, x)$$

for a.e. $(t, x) \in (0, \infty) \times \Sigma$. Since (Σ, μ) and $(\mathbb{R}_+, \frac{dt}{t})$ are both σ-finite measure spaces, Fubini's theorem implies that

$$t^{-\theta} v_0(\cdot, x) \in L^{p_0}_*(X_0) \quad \text{and} \quad t^{1-\theta} v_1(\cdot, x) \in L^{p_1}(\Sigma, \mu)$$

for a.e. $x \in \Sigma$. Thus by definition of the mean space and by (4.40), one has $a(x) \in (X_0, X_1)_{\theta,p_0,p_1}$ for a.e. $x \in \Sigma$ and

$$\|u(x)\|_{(X_0,X_1)_{\theta,p_0,p_1}} \le \|t^{-\theta} v_0(\cdot,x)\|^{1-\theta}_{L^{p_0}_*(X_0)} \, \|t^{1-\theta} v_1(\cdot,x)\|^{\theta}_{L^{p_1}_*(X_1)}.$$

Integrating the latter inequality over Σ, taking pth root, applying Hölder's inequality (where one uses (B.1)) and Fubini's theorem, we see that

$$\begin{aligned}
&\|u\|_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)} \\
&\quad \le \left[\int_\Sigma \|t^{-\theta} v_0(\cdot,x)\|^{p_0}_{L^{p_0}_*(X_0)} \,\mathrm{d}\mu\right]^{\frac{1-\theta}{p_0}} \left[\int_\Sigma \|t^{1-\theta} v_1(\cdot,x)\|^{p_1}_{L^{p_1}_*(X_1)} \,\mathrm{d}\mu\right]^{\frac{\theta}{p_1}} \\
&\quad = \|t^{-\theta} v_0\|^{1-\theta}_{L^{p_0}_*(L^{p_0}(\Sigma,X_0;\mu))} \, \|t^{1-\theta} v_1\|^{\theta}_{L^{p_1}_*(L^{p_1}(\Sigma,X_1;\mu))}.
\end{aligned}$$

Taking in this inequality the infimum over all representation pairs (v_0, v_1) of u and applying (4.40) yields

$$\|u\|_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)} \le \|u\|_{(L^{p_0}(\Sigma,X_0;\mu),L^{p_1}(\Sigma,X_1;\mu))_{\theta,p_0,p_1}}.$$

Now, let $u \in L^p(\Sigma, (X_0, X_1)_{\theta,p_0,p_1}; \mu)$ be a step function given by

$$u(x) = \sum_{\nu=1}^{m} a_\nu \mathbb{1}_{B_\nu}(x)$$

for finitely many different values $a_\nu \in (X_0, X_1)_{\theta,p_0,p_1}$ attained on pairwise disjoint measurable subsets B_ν of Σ. Let $\varepsilon > 0$. By the definition of $(X_0, X_1)_{\theta,p_0,p_1}$ and the infimum, for every $\nu = 1, \ldots, m$, there are measurable functions $v_{i\nu} : (0,\infty) \to X_i$ for $i = 0, 1$ satisfying

$$a_\nu = v_{0\nu}(t) + v_{1\nu}(t) \tag{B.3}$$

for a.e. $t \in (0, \infty)$ and

$$\max\left\{\|t^{-\theta} v_{0\nu}\|_{L^{p_0}_*(X_0)}, \|t^{1-\theta} v_{1\nu}\|_{L^{p_1}_*(X_1)}\right\} \le \|a_\nu\|_{(X_0,X_1)_{\theta,p_0,p_1}} (1+\varepsilon). \tag{B.4}$$

Set $\lambda = (p_0 - p)/\theta\, p_0$ and for every $\nu = 1, \ldots, m$ and $i = 0, 1$ define

$$w_{i\nu}(t) = v_{i\nu}(\|a_\nu\|^{\lambda}_{(X_0,X_1)_{\theta,p_0,p_1}} t).$$

Then applying the substitution $s = \|a_\nu\|^{\lambda}_{(X_0,X_1)_{\theta,p_0,p_1}} t$ together with (B.4) yields

$$\begin{aligned}
\|t^{-\theta} w_{0\nu}\|^{p_0}_{L^{p_0}_*(X_0)} &= \|a_\nu\|^{-\lambda\,\theta\, p_0}_{(X_0,X_1)_{\theta,p_0,p_1}} \int_0^\infty \|s^{-\theta} v_{0\nu}(s)\|^{p_0}_{X_0} \frac{\mathrm{d}s}{s} \\
&\le (1+\varepsilon)^{p_0} \|a_\nu\|^{p_0-\lambda\,\theta\, p_0}_{(X_0,X_1)_{\theta,p_0,p_1}} \\
&= (1+\varepsilon)^{p_0} \, \|a_\nu\|^{p}_{(X_0,X_1)_{\theta,p_0,p_1}}
\end{aligned} \tag{B.5}$$

for every $\nu = 1,\dots,m$. By the relation (B.1), one sees that the same λ satisfies $p_1 + \lambda(1-\theta)p_1 = p$. Thus the same arguments gives

$$\|t^{1-\theta} w_{1\nu}\|^{p_1}_{L^{p_1}_*(X_1)} \leq (1+\varepsilon)^{p_1} \, \|a_\nu\|^p_{(X_0,X_1)_{\theta,p_0,p_1}}. \tag{B.6}$$

For $i = 0,1$ and every $t \in (0,\infty)$, we define the step functions

$$w_i(t,x) = \sum_{\nu=1}^m w_{i\nu}(t)\, \mathbb{1}_{B_\nu}(x)$$

for a.e. $x \in \Sigma$. Then by (B.5) and (B.6) as well as by Fubini's theorem,

$$\begin{aligned}\int_0^\infty \|t^{-\theta} w_0(t,\cdot)\|^{p_0}_{L^{p_0}(\Sigma,X_0;\mu)} \frac{\mathrm{d}t}{t} &= \int_\Sigma \|t^{-\theta} w_0(\cdot,x)\|^{p_0}_{L^{p_0}_*(X_0)} \,\mathrm{d}\mu \\ &\leq (1+\varepsilon)^{p_0} \sum_{i=1}^m \|a_\nu\|^p_{(X_0,X_1)_{\theta,p_0,p_1}} \mu(B_\nu) \\ &= (1+\varepsilon)^{p_0} \, \|u\|^p_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)}\end{aligned} \tag{B.7}$$

and similarly,

$$\int_0^\infty \|t^{1-\theta} w_1(t,\cdot)\|^{p_1}_{L^{p_1}(\Sigma,X_1;\mu)} \frac{\mathrm{d}t}{t} \leq (1+\varepsilon)^{p_1} \, \|u\|^p_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)}. \tag{B.8}$$

Therefore, for $i = 0,1$, the functions $w_i : (0,\infty) \to L^{p_i}(\Sigma, X_i;\mu)$ are well defined step functions and so strongly measurable. In addition, by (B.3),

$$\begin{aligned} w_0(t,x) + w_1(t,x) &= \sum_{\nu=1}^m \big(v_{0\nu}(\|a_\nu\|^\lambda_{(X_0,X_1)_{\theta,p_0,p_1}} t) + v_{1\nu}(\|a_\nu\|^\lambda_{(X_0,X_1)_{\theta,p_0,p_1}} t)\big)\, \mathbb{1}_{B_\nu}(x) \\ &= \sum_{\nu=1}^m a_\nu \, \mathbb{1}_{B_\nu}(x) \\ &= u(x) \end{aligned}$$

for a.e. $x \in \Sigma$. Thus $u \in (L^{p_0}(\Sigma, X_0;\mu), L^{p_1}(\Sigma, X_1;\mu))_{\theta,p_0,p_1}$ and by (4.40), (B.7), (B.8), and (B.1),

$$\begin{aligned} \|u\|_{(L^{p_0}(\Sigma,X_0;\mu),L^{p_1}(\Sigma,X_1;\mu))_{\theta,p_0,p_1}} &\leq \|t^{-\theta} w_0\|^{1-\theta}_{L^{p_0}_*(L^{p_0}(\Sigma,X_0;\mu))} \, \|t^{1-\theta} w_1\|^{\theta}_{L^{p_1}_*(L^{p_1}(\Sigma,X_1;\mu))} \\ &\leq (1+\varepsilon) \, \|u\|^{\frac{(1-\theta)p}{p_0}+\frac{\theta p}{p_1}}_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)} \\ &= (1+\varepsilon) \, \|u\|_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)}. \end{aligned}$$

Sending $\varepsilon \to 0+$ shows that inequality

$$\|u\|_{(L^{p_0}(\Sigma,X_0;\mu),L^{p_1}(\Sigma,X_1;\mu))_{\theta,p_0,p_1}} \leq \|u\|_{L^p(\Sigma,(X_0,X_1)_{\theta,p_0,p_1};\mu)}.$$

holds for step functions. Since the set of step functions is dense in the space $L^p(\Sigma, (X_0, X_1)_{\theta,p_0,p_1}; \mu)$ the claim of this theorem holds. □

As an immediate consequence of Theorem B.1, we obtain the following corollary improving the statement in [105, Corollaire 1.1 of Chapter IV].

Corollary B.1 *Let* (Σ, μ) *be a* σ*-finite measure space,* $1 \le p_0$*,* $p_1 \le \infty$ *and* $0 < \theta < 1$*. Then for* $1 \le p \le \infty$ *satisfying the relation* (B.1)*, one has that*

$$(L^{p_0}(\Sigma, \mu), L^{p_1}(\Sigma, \mu))_{\theta,p_0,p_1} = L^p(\Sigma, \mu)$$

with equal norms.

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Symbols