Fractional Time Derivatives and Stochastic Processes

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List of Symbols

Spaces of functions

$\mathcal{X}, \mathcal{Y}$	normed space
$L^p(\Omega)$	L^p space on Ω
$W^{p,\beta}(\Omega)$	P.58
$W^{p,k}(\Omega)$	P.10
$C_b[0,\infty)$	bounded continuous functions
$C^m(\Omega)$	P.57
$C_c^m(\Omega)$	C^m with compact support
$C_c^\infty(\Omega)$	test functions
$C^\alpha(\Omega)$	P.57
$C^{\beta,p}(\Omega)$	P.56
$C_\infty^\beta(0,T)$	P.58
$AC(\Omega)$	absolutely continuous functions
$C_\infty(0,T]$	P.135
$\mathcal{BF}$	P.68
$\mathcal{CBF}$	P.68
$\mathcal{CM}$	P.68
R_0	slowly varying functions P.15
R_β	P.15
$C_\infty^\beta([0,b],w)$	P.57

General notation: probability

$\mathbb{P}, \mathbb{E}$	probability, expectation
$\mathcal{B}(\mathbb{R}^d)$	Borel σ algebra on $\mathbb{R}^d$
δ_x	Dirac measure at x
$\mathcal{F}_t$	filtration
a.s.	almost surely
δ	δ measure
PI	integration by parts

General notation: analysis

$\mathscr{L}$	Laplace transform P.13
$\mathscr{F}$	Fourier transform P.12
$\mathscr{M}$	Mellin transform P.14
Δ	Laplace operator
$\phi \asymp \psi$	P.113
$\phi * \psi$	convolution
$\phi \circledast \psi$	Mellin convolution P.15
$\mathcal{D}$	domain
${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{\cdot}$	P.37
${}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{\cdot}$	P.39
${}^{\mathrm{M}}\mathrm{D}^{\alpha}_{+}$	P.41
$(-\Delta)^\alpha$	P.78
${}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha}_{\cdot}$	P.36
${}^{\mathrm{Ce}}_{0}\mathrm{D}^{\alpha}_{\cdot}$	P.40
${}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{\cdot}$	P.73
${}^{\mathrm{C}}_{0}\mathrm{D}^{f}_{\cdot}$	P.73
${}^{\mathrm{Ce}}_{0}\mathrm{D}^{f}_{\cdot}$	P.110
${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot}$	P.74
$a \wedge b, a \vee b$	$\min\{a,b\}, \max\{a,b\}$

Stochastic processes

$\mathbb{P}^x, \mathbb{E}^x$	law, mean of a Markov process
$(L_t)_{t\geq 0}$	Lévy process
$\mathcal{A}$	generator
$\mathcal{A}^*$	adjoint of generator
$(R_\lambda)_{\lambda>0}$	resolvent
$(P_t)_{t\geq 0}$	semigroup
$(S_t)_{t\geq 0}$	subordinator

Introduction

In many applications, diffusion processes are known not to be optimal. Roughly speaking, a diffusion $X = (X_t)_{t\geq 0}$ moves with speed $\sqrt{t}$ (traditionally expressed by the variance), but in many applications we are either faster (superdiffusions) or slower (subdiffusions) and we usually do not have second moments. This has lead to the study of so–called **anomalous diffusions** which include **Lévy processes** (as models for superdiffusions) and (scaling limits of) **continuous–time random walks** (**CTRWs**) which are subdiffusive since they can be 'trapped' in a state for some time. While Lévy and Lévy-type processes are well-understood (see e.g. [11, 65]) the **mathematical** study of CTRWs started only in 2000; in physics they have been around since 1970, see [40] for a survey on methods and literature. The definition of a CTRW requires an independent, identically distribution sequence of pairs of random variables $(Y_i, J_i) \in \mathbb{R}^d \times [0, \infty)$, $i \in \mathbb{N}$; if, in addition, Y_1 and J_1 are independent, we speak of an **uncoupled CTRW**. A CTRW is defined as a continuous–time process X_t given by

$$X_0 := 0, \quad X_t := Y_1 + \cdots + Y_{N_t}, \qquad N_t := \max\{n : J_1 + \cdots + J_n \leq t\}.$$

An uncoupled CTRW with exponentially distributed J_1 is a **compound Poisson process (CPP)**; this is the only case which gives a Markov process X_t. In most applications, however, we want waiting times J_i which have no second moments, hence non–exponential J_i's. Consequently, we leave the familiar Markov setting.

A **Lévy process** is a stochastic process $L = (L_t)_{t\geq 0}$ in $\mathbb{R}^d$ with càdlàg (right–continuous, finite left limits) paths and independent and stationary increments. Our standard references are [36, 65]. Lévy processes are strong Markov, even Feller processes (see [11]) and they can be characterized through the **characteristic exponent** (or **symbol**):

$$\mathbb{E}^x \left(e^{i\xi\cdot(L_t - x)}\right) = \mathbb{E}^0 \left(e^{i\xi\cdot L_t}\right) = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d,$$

which is uniquely given by its **Lévy–Khintchine representation**

$$\psi(\xi) = ia\cdot\xi + \frac{1}{2}\xi\cdot Q\xi + \int_{x\neq 0} \left[1 - e^{ix\cdot\xi} - iy\cdot\xi\mathbb{1}_{(0,1)}(|x|)\right] \nu(dx),$$

here $a \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d\times d}$ is a positive semidefinite matrix and ν is a Radon measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{x\neq 0} \min\{|x|^2, 1\}\,\nu(dx) < \infty$. The triplet (a, Q, ν) has a good probabilistic meaning: a gives the drift part, Q the diffusion matrix and ν is the jump (or

Lévy) measure determining the jumps. These are also the differential semimartingale characteristics in the sense of [11], see He et al. [30] for the general picture. The set of Lévy processes is a larger subclasses of Markov processes which can be characterized in terms of a single deterministic function. In this thesis we focus on a special class of Lévy processes, the so–called subordinators.

A **subordinator** is a Lévy process S with $S_0 = 0$ and a.s. increasing paths. In this case one considers the Laplace transform (rather than the characteristic function), and one has

$$\mathbb{E}\left(e^{-\lambda S_t}\right) = e^{-tf(\lambda)}, \quad \lambda > 0, \tag{0.0.1}$$

where f is a **Bernstein function**. A Bernstein function can be expressed by

$$f(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda x})\, \mu(dx), \quad \lambda > 0, \tag{0.0.2}$$

where $b \geq 0$ and μ is a measure on $(0, \infty)$ such that $\int_0^\infty \min\{1, x\}\, \mu(dx) < \infty$. Since a subordinator is increasing, it cannot have a diffusion part, while (b, μ) corresponds to (a, ν) in the general setting. There is a deep connection between generators of subordinators and fractional derivatives. Let us recall some facts about fractional derivatives in general.

Fractional derivatives, in particular fractional time derivatives, have recently become important tools to model real-world phenomena. There are important applications in Physics, Chemistry and Biology (so–called "master equations" for "anomalous diffusions") as well as in Mathematics. A good introduction to applications is given in the monograph Klages el al [40]. One of the most frequently used extensions of the usual derivative is the **Riemann–Liouville fractional derivative** of order $\alpha \in (0, 1)$ on $[0, x]$

$${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\, \phi = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{\phi(s)}{(x-s)^\alpha}\, ds, \quad \alpha \in (0, 1).$$

It is defined in such a way that it "fills the gaps" between the zero-order "derivative" (i.e. $\phi = \left(\frac{d}{dx}\right)^0 \phi$) and the usual derivative (i.e. $\phi' = \left(\frac{d}{dx}\right)^1 \phi$) in a continuous way. Note that ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}$ is a nonlocal integral operator if $\alpha \notin \mathbb{N}_0$. Another version is the **fractional derivative in the sense of Caputo**

$${}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}\, \phi = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{\phi(s) - \phi(0)}{(x-s)^\alpha}\, ds, \quad t \geq 0,\ \alpha \in (0, 1).$$

It is closely related to the Riemann-Liouville derivative, but it is easier to handle, which comes at the price of higher regularity assumptions on ϕ. Let me point out that there are further types of derivatives, which are all called "fractional". Standard references are [40, 14] and [64]. Meerschaert et al. [51] consider a fractional Cauchy problem which replace the usual first-order time derivative by a fractional derivative involving Caputo fractional derivative. Hernández-Hernández and Kolokoltsov [34] provide well-posedness results and stochastic representations for the solutions to equations involving generalized operators of Caputo fractional derivative. The well-posedness and integral representations of the solutions to nonlinear equations involving generalized Caputo and Riemann–Liouville type fractional derivatives were studied in [33]. For a general background in fractional calculus and fractional differential equations, we refer to [14, 37, 56, 50]. When we solve differential equation involving a fractional derivative, for example Cauchy problems or resolvent equations, it is important to know the mapping properties of fractional derivatives and fractional integrals. Samko et.al [64] studied the mapping properties of fractional integral and derivatives in the L^p–scale which is based on the Hardy Littlewood Theorem, and Hölder spaces. Carbotti et al. [12] showed that the α-order fractional derivative maps the Sobolev space $W_0^{p,1}$ (see P.58 for the definition) to the fractional Sobolev-Slobodeckij space $W^{p,\alpha}$ for $\alpha \in (0, \min\{1/p, (p-1)/2p\})$, which implies that $\alpha < 1/2$. We show in Theorem 2.2.9. that $W_0^{p,s}$ is mapped to $W^{p,s-\alpha}$ for all $\alpha < s < 1$, which seems to be more natural and includes the above result, even extending it.

The probabilistic explanation of the Caputo and Riemann–Liouville fractional derivatives was explored in [43]. From this point of view, the α order Caputo and Riemann–Liouville fractional derivative can be treated as generators of α stable subordinator which are interrupted on crossing a boundary. In this thesis, we investigate that a **Bernstein censored fractional derivative** treated as generator of subordinator whose overshoots are suppressed on crossing a boundary. In the case of a smooth function that vanishes outside of $(0, \infty)$, it has been shown in [16] that the Riemann-Liouville fractional derivative is equal to the Caputo derivative. This equivalence is particularly significant as it corresponds to the generator of a killed Lévy process. This leads us to consider the fractional derivative as the generator of a Markov process. Our starting point is the so–called **Marchaud fractional derivative**, which is given by

$$ {}^{\mathrm{M}}\mathrm{D}_{+}^{\alpha}\, \phi(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^{\infty} \frac{\phi(x) - \phi(x-s)}{s^{1+\alpha}}\, ds. \tag{0.0.3} $$

If the function ϕ is only defined on the positive axis $(0, \infty)$ the formula (0.0.3) is not well defined unless we extend ϕ onto $(-\infty, \infty)$. Our main observation is that using the correct extension we obtain from (0.0.3) discussed fractional derivatives. For example, setting $\phi|_{(-\infty,0)} = 0$ (I call it **killing–type extension** ϕ^0), we get the Riemann–Liouville derivative, while the extension $\phi|_{(-\infty,0)} = \phi(0)$ (I call it **sticky–type extension** ϕ^σ) leads to the Caputo derivative. Other extensions are, of course, possible. In this thesis, we give a basic idea to get censored fractional derivative using Marchaud fractional derivative which involves to solve a convolution equation in Chapter 2, Section 1.

The advantage of the form (0.0.3) is that it relaxes on the regularity of ϕ and that it shows the rationale behind the derivative: it is a limit of weighted sums of the increments of ϕ, $\phi(x) - \phi(x-s)$ from various past values (i.e. in $(-\infty, x)$) up to the present value

t. This allows us to generalize fractional derivatives by using different kind of positive weights, using the theory of Bernstein functions [69]. Moreover this establishes in a natural way the connection to Lévy processes. Recently, censored fractional derivatives see below are studied in [16]. A **censored fractional derivative** is given by

$$ {}^{\mathrm{Ce}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^x (\phi(x) - \phi(x-s))\, s^{-\alpha-1}\, ds, \qquad (0.0.4) $$

where $\alpha \in (0,1)$. Considering (0.0.4) and (0.0.3), a natural question is whether it is possible to define censored fractional derivative using the extension method leading from the Marchaud derivative to the censored fractional derivative, as in the case of Riemann Louville fractional derivatives and Caputo derivatives. We will show that, in general, this is possible, but the extension is more involved than in the first two cases, see Chapter 2 for a detailed discussion.

Observing the expression of the Marchaud fractional derivative, we have an alternative choice to generalize this, starting from a Bernstein function (or the generator of a subordinator). To do so, it is useful to recall the following:

Since f from (0.0.1) is given by (0.0.2), the infinitesimal generator of the subordinator S is formally given by

$$ \mathcal{A}u(x) = b \cdot \frac{d}{dx}u(x) + \int_0^\infty (u(x) - u(x-t))\, \mu(dt), \quad u : \mathbb{R} \to \mathbb{R}. $$

and we can show that the formal adjoint A^* in $L^2(0,\infty)$ is given by

$$ \mathcal{A}^*u(x) = -b \cdot \frac{d}{dx}u(x) - \int_0^\infty (u(x) - u(x-t))\, \mu(dt). $$

Comparing this formula with the classical Marchaud derivative (0.0.3), the candidate for a general Marchaud fractional derivative, which we call **Bernstein Marchaud fractional derivative**, is

$$ {}^{\mathrm{M}}\mathrm{D}^{f}_{+}\, u(x) = b \cdot \frac{d}{dx}u(x) + \int_0^\infty (u(x) - u(x-t))\, \mu(dt). \qquad (0.0.5) $$

Observe that ${}^{\mathrm{M}}\mathrm{D}^{f}_{+}$ is connected to the Bernstein function f and its representing pair (b,μ). Moreover ${}^{\mathrm{M}}\mathrm{D}^{f}_{+}$ is the generator of an adjoint subordinator, but subject to a boundary condition that must reflect the type of the extension.

By using a similar idea, we obtain a **Bernstein Riemann–Liouville derivative** (0.0.6) using the killing type extension, and a **Bernstein Caputo derivative** (0.0.7) by the sticky type extension, from the Bernstein Marchaud fractional derivative (0.0.5).

$$ {}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\,\phi = b \cdot \frac{d}{dx}\phi^0(x) + \int_0^\infty \left(\phi^0(x) - \phi^0(x-s)\right) \mu(ds), \qquad (0.0.6) $$

$$ {}^{\mathrm{C}}_{0}\mathrm{D}^{f}_{x}\,\phi = b \cdot \frac{d}{dx}\phi^\sigma(x) + \int_0^\infty (\phi^\sigma(x) - \phi^\sigma(x-s))\, \mu(ds), \qquad (0.0.7) $$

where ϕ^0 and ϕ^σ are the killing–type extension and the sticky–type extension, respectively, that we mentioned in (0.0.3), see also P.41. Kochubei [42] develops a theory of relaxation and diffusion equations associated with operators in the time variable (0.0.7). Chen [13] defines a Bernstein Caputo derivative (0.0.7) and he studied the existence and uniqueness of solutions for general fractional time parabolic equations, and their probabilistic representations in terms of the corresponding inverse subordinators with or without drifts. Initial value problems of fractional differential equations are considered in Sin [72] where the fractional derivatives are defined as the Bernstein Caputo derivative (0.0.7). A probabilistic approach to solve linear equations involving Bernstein Caputo derivatives (0.0.7) and Riemann–Liouville (0.0.6) derivatives and using the probabilistic interpretation of these operators as the generators of interrupted Feller processes was proposed in [31]. Hernández-Hernández and Kolokoltsov [32] provide a probabilistic approach to solve resolvent equations involving Bernstein Caputo (0.0.7) and Bernstein Riemann-Liouville (0.0.6) derivatives. They establish, in particular, the well-posedness and integral representations of the solutions to nonlinear equations involving Bernstein Caputo (0.0.7) and Riemann–Liouville (0.0.6) fractional derivatives.

In this thesis, we introduce a general censored fractional derivative and study its properties. We discuss various aspects such as the resolvent equation and the construction of the corresponding Markov process. By investigating these topics, we aim to gain a deeper understanding of the characteristics and behavior of the censored fractional derivative. Examining the structure of (0.0.4), a good candidate for a generalized censored fractional derivative is

$$ {}^{\mathrm{Ce}}_{\;\,0}\mathrm{D}^{f}_{x}\,\phi(x) = \int_0^x (\phi(x) - \phi(x-s))\,\mu(ds). \tag{0.0.8}$$

Our starting point is that the operator given by (0.0.8) can be viewed as generator of a decreasing subordinator, where we allow only those jumps such that the generator lands inside $(0,\infty)$ and we suppress the jumps such that the subordinator would land outside of $(0,\infty)$. We will apply the Hille-Yosida theorem, which is a powerful tool to study the relation between the semigroup, the corresponding Markov process and its generator; for further information we refer the reader to [55]. In order to show that ${}^{\mathrm{Ce}}_{\;\,0}\mathrm{D}^{f}_{\cdot}$ generates a stochastic process we apply the Hille-Yosida theorem, which leads to the necessity to solve the resolvent equation in $C_{\bar{\mu}}[0,T]$ (see P.107 for the definition)

$$\begin{cases} {}^{\mathrm{Ce}}_{\;\,0}\mathrm{D}^{f}_{x}\,\phi(x) = \lambda\phi(x), \quad x>0, \quad \lambda\in\mathbb{R}, \\ \phi(0) = \phi_0. \end{cases} \tag{0.0.9}$$

For ϕ in $C_{\bar{\mu}}[0,T]$, a key step to solve (0.0.9) is to construct the right inverse ${}^{\mathrm{Ce}}_{\;\,0}\mathrm{I}^{f}_{\cdot}$ of ${}^{\mathrm{Ce}}_{\;\,0}\mathrm{D}^{f}_{\cdot}$. For this construction, we need so–called **Sonine pairs** and we refer the reader to [42, 62, 63, 29, 49] and P.85 in this thesis. There is a deep one–to–one connection between Sonine pairs and so–called special Bernstein functions – but this is a highly theoretical result, since we know not much about special Bernstein functions cf. [69]. Therefore, we use the more benign class of complete Bernstein functions. Thus, we will investigate the connection between complete Bernstein functions and Sonine pairs in Chapter 3. We can show that every complete Bernstein function corresponds to a completely monotone Sonine pairs. The properties of completely monotone Sonine pair will be reviewed and, furthermore, we provide several examples to illustrate different of Sonine pairs, see P.95.

In Chapter 4, we prove that (0.0.8) is the generator of the censored subordinator $S^c = (S^c_t)_{t\geq 0}$ in $(0, \infty)$ and show that under certain assumptions this process has the Feller property. We will construct S^c by repeatedly resurrecting the killed decreasing subordinator and using the piecing together procedure, for the construction of Markov processes by the piecing out method, developed in [54]. The starting point of the censored process is always assumed to be some fixed $x > 0$. This construction guarantees that we get a right continuous strong Markov process by using Theorem 1.1 in [54].

Chapter 5 is devoted to applications of the resolvent equation (0.0.9) in which the Sonine pair is assumed to be of regular variation. Here we will get more specific results due to the special properties of regularly varying functions. For further properties of regularly varying functions, we refer the reader to [9].

Chapter 1

Basics

The aim of this chapter is to briefly summarize definitions and results that we will frequently use in this thesis. First, we introduce some basic results and recall standard definitions from functional analysis. In Section 1.2, we gather some facts on the Fourier transform, Laplace transform, and Mellin transform. Section 1.3 is devoted to the class of regularly varying functions, which will play a crucial role in Chapter 5. After introducing Markov processes in Section 1.4, we define Lévy processes, subordinators, and discuss their most important properties in Section 1.5. Most of the results we present in this chapter are well–known; therefore, we do not include proofs for most of these results but merely provide references.

1.1 Some results in functional analysis

The results in this section are a digest of the material in Rudin [61] Chapter 2, Reed & Simon [60] Chapter 13 and Kreyszig [46] Chapter 5 and 8. The following definition is taken from the monograph [46].

Let $\mathcal{X}, \mathcal{Y}$ be two vector spaces over the same field $\mathbb{R}$ of scalars. A mapping $\Lambda : \mathcal{X} \to \mathcal{Y}$ is called **transformation** or **an operator**. An operator is said to be linear if $\Lambda(ax + by) = a\Lambda(x) + b\Lambda(y), \quad x, y \in \mathcal{X}, a, b \in \mathbb{R}$.

Definition 1.1.1. Let $\mathcal{X}, \mathcal{Y}$ be normed spaces and $\Lambda : \mathcal{D}(\Lambda) \to \mathcal{Y}$ a linear operator with domain $\mathcal{D}(\Lambda) \subset \mathcal{X}$. Then Λ is called a **closed linear operator** if its graph

$$\mathscr{G}(\Lambda) = \{(x, y) \mid x \in \mathcal{D}(\Lambda),\ y = \Lambda x\}$$

is closed in the normed space $\mathcal{X} \times \mathcal{Y}$, where the two algebraic operations of a vector space in $\mathcal{X} \times \mathcal{Y}$ are defined as usual, that is

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad a(x, y) = (ax, y)$$

(a a scalar) and the norm on $\mathcal{X} \times \mathcal{Y}$ is defined by

$$\|(x, y)\| = \|x\| + \|y\|.$$

We can obtain an equivalent and useful description of a closed operator from Definition 1.1.1: an operator Λ is said to be **closed** if, and only if, whenever a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Lambda)$ converges to x and $(\Lambda x_n)_{n\in\mathbb{N}}$ converges to y, it follows that $x \in \mathcal{D}(\Lambda)$ and $\Lambda x = y$. This property is often taken as a definition of closedness of a linear operator. Furthermore, A linear operator Λ on $\mathcal{X}$ is called **closable** if it has a closed linear extension. If Λ is closable, then the closure $\bar{\Lambda}$ of Λ is the minimal closed linear extension of Λ.

Here we provide an example to illustrate the concepts of closed operators and closable operators.

Example 1.1.2. Let $\Lambda = \frac{d}{dx}, \mathcal{X} = \mathcal{Y} = C([0,1])$ is the Banach space of all continuous functions on an interval $[0,1]$ with the supremum norm $\|\cdot\|_\infty$.

(1) If we take domain $\mathcal{D}(\Lambda) = C^1([0,1])$, then Λ is a closed operator.

(2) If we take domain $\mathcal{D}(\Lambda) = C^\infty([0,1])$, then Λ is not closed, but it will be closable.

Proof. (1) According to the above equivalent statement for closed operator, take a sequence u_n and assume $u_n \to u$ in $C^1([0,1])$, $\Lambda u_n = u_n' \to v$. We have

$$\int_0^x v(s)\,ds = \int_0^x \lim_{n\to\infty} u_n'(s)\,ds = \lim_{n\to\infty} \int_0^x u_n'(s)\,ds = u(x) - u(0).$$

Thus $u(x) = u(0) + \int_0^x v(s)\,ds$ and we get $u \in \mathcal{D}(T)$, and $u' = v$. Thus, Λ is a closed operator.

(2) If $\mathcal{D}(\Lambda) = C^\infty([0,1])$, The operator $\left(\frac{d}{dx}, C^\infty([0,1])\right)$ is not closed. Take a sequence $u_n \in C^\infty[0,1]$ as follows

$$u_n(x) = \frac{1}{2}(x-\frac{1}{2})\sqrt{(x-\frac{1}{2})^2 + 1/n} + \frac{1}{2n}\log\left((x-\frac{1}{2}) + \sqrt{(x-\frac{1}{2})^2 + 1/n}\right).$$

We can obtain $u_n'(x) = \sqrt{(x-\frac{1}{2})^2 + 1/n}$ by a direct calculation. Furthermore, we have $u_n(x) \to \frac{1}{2}(x-\frac{1}{2})|x-\frac{1}{2}|$ and $u_n'(x) \to |x-\frac{1}{2}|$ uniformly as n approaches ∞. However, $\frac{1}{2}(x-\frac{1}{2})|x-\frac{1}{2}|$ is not in $C^\infty[0,1]$. Hence, the operator $\left(\frac{d}{dx}, C^\infty([0,1])\right)$ is not closed. By (1), $\left(\frac{d}{dx}, C^1([0,1])\right)$ is a closed extension of $\left(\frac{d}{dx}, C^\infty([0,1])\right)$, hence it is closable. □

We will introduce the concept of bounded linear transformations, which provides an elegant definition of the Lebesgue integral, including the construction of the Riemann-Stieltjes integral. For a detailed proof and explanation, refer to Section 1 of Reed and Simon's book [60]. Later on, we will use this theorem to establish the mapping property in Chapter 3.

Theorem 1.1.3. *(Bounded linear transformation) Let Λ be a bounded linear operator on a Banach space $\mathcal{X}$ and $\mathcal{Y}$ be another Banach space. Let $D \subset \mathcal{X}$ be a dense subspace of $\mathcal{X}$. If Λ satisfies the following condition:*

$$\Lambda : D \to \mathcal{Y}, \quad \|\Lambda x\|_{\mathcal{Y}} \leq C\|x\|_{\mathcal{X}}, \quad \forall x \in D,$$

then there exists unique extension of $\tilde{\Lambda}$ to $\bar{D} = \mathcal{X}$ such that for the same C

$$\|\tilde{\Lambda} x\|_{\mathcal{Y}} \leq C\|x\|_{\mathcal{X}}, \quad \forall x \in \mathcal{X}.$$

In this case we write $\Lambda \in L(\mathcal{X}, \mathcal{Y})$ and set

$$\|\Lambda\|_{L(\mathcal{X},\mathcal{Y})} = \sup_{x \in \mathcal{X}\setminus\{0\}} \frac{\|\Lambda x\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}} = \sup_{\|x\|_{\mathcal{X}}=1} \|\Lambda x\|_{\mathcal{Y}}.$$

Now, we will provide examples of both bounded and unbounded operators to illustrate their differences and properties.

Example 1.1.4. Let $C_b[0,1]$ be a bounded continuous function space with supremum norm $\|\cdot\|_\infty$.

(1) If $\phi \in C_b[0,1]$ and $\phi : [0,1] \to \mathbb{R}$, then the integral $(\Lambda\phi)(t) = \int_0^t \phi(s)\,ds$, $\Lambda : C_b[0,1] \to C_b[0,1]$ defines a bounded linear transformation.

(2) If $g \in C_b[0,1]$ and $(\Lambda\phi)(t) = \int_0^t \phi(s)g(s)\,ds$, then $\Lambda : C_b[0,1] \to C_b[0,1]$ defines a bounded linear transformation.

(3) $\frac{d}{dx}$ is not a bounded operator.

Proof. For all of the above operators linearity is obvious. To see the boundedness note that:

(1) $\|(\Lambda\phi)(x)\|_\infty = \sup_{x\in[0,1]} |\int_0^x \phi(s)\,ds| \leq \int_0^1 \sup_{s\in[0,1]} |\phi(s)|\,ds = \|\phi\|_\infty$, this shows that $\int_0^x \phi(s)\,ds$ is a bounded operator.

(2) $\|(\Lambda\phi)(x)\|_\infty = \sup_{x\in[0,1]} |\int_0^x g(s)\phi(s)\,ds| \leq M \int_0^1 \sup_{s\in[0,1]} |\phi(s)|\,ds = M\|\phi\|_\infty$, due to $|g| \leq M$, this shows that $\int_0^t g(s)\phi(s)\,ds$ is a bounded operator.

(3) Take the sequence $u_n = \sin(nx)$ in $C_b([0,1])$, $\|\frac{d}{dx}u_n\|_\infty = \sup_{x\in[0,1]} |n\cos(nx)| = n$. We can conclude that Λ is not a bounded operator. □

It is often useful to form a non-pointwise product of two functions that smooth out irregularities of each them to produce a function better behaved local than either factor alone. Such a product is the **convolution** $\phi * u$ of two functions ϕ and u, defined by

$$\phi * u(x) = \int_{\mathbb{R}^d} \phi(x-y)u(y)\,dy. \tag{1.1.1}$$

A function j is called **mollifier** if it is a nonnegative, real valued function belonging to $C_c^\infty(\mathbb{R}^d)$ and having the following properties: $j(x) = 0$, $|x| \geq 1$ and $\int_{\mathbb{R}^d} j(x)\,dx = 1$. In general, mollifiers are smooth functions with special properties, used for example in distribution theory to create sequence of smooth functions approximating nonsmooth functions, via convolution. Let $\Omega \subset \mathbb{R}^n$ be open. Recall that:

$$W^{p,k}(\Omega) = \left\{\phi \in L^p(\Omega) \,:\, \nabla^\beta \phi \in L^p(\Omega), \text{for all } |\beta| \leq k\right\},$$

is the classical Sobolev space of order $k = 0, 1, 2, ...$ over $L^p(\Omega)$.

Theorem 1.1.5. *Let* $1 \leq p < \infty$ *and* $k \geq 0$. *Then the space* $C_c^\infty(\mathbb{R}^d)$ *is dense in* $W^{p,k}(\mathbb{R}^d)$.

Proof. Let $\phi \in C_c^\infty$ be the standard mollifier with the following properties: $\phi_n = n^d\phi(nx)$ and $\int_{\mathbb{R}^d} \phi_n\,dx = 1$. Take $g \in W^{p,k}(\mathbb{R}^d)$. Define $g_n(x) = \int_{\mathbb{R}^d} g(y)\phi_n(x-y)\,dy$. Since ϕ_n has compact support and $\nabla^j g_n \in L^p(\mathbb{R}), 0 \leq j \leq k$, then $g_n \in C^\infty(\mathbb{R}^d) \cap W^{p,k}(\mathbb{R}^d)$. Since $\nabla^j g_n$ is convergent to $\nabla^j g, 0 \leq j \leq k$ in L^p, we obtain by dominated convergence that g_n converges to g in $W^{p,k}(\mathbb{R}^d)$. Therefore $C^\infty(\mathbb{R}^d) \cap W^{p,k}(\mathbb{R}^d)$ is dense in $W^{p,k}(\mathbb{R}^d)$.

Now suppose that $g \in C^\infty(\mathbb{R}^d) \cap W^{p,k}(\mathbb{R}^d)$, and let $\psi \in C_c^\infty$ be a cut-off function such that

$$\psi(x) = \begin{cases} 1, & |x| \leq 1; \\ 0, & |x| \geq 2. \end{cases}$$

Define $\psi_n(x) = \psi(x/n)$ and $g_n(x) = g(x)\psi(x/n) \in C_c^\infty(\mathbb{R}^d)$. Then we obtain by Leibniz rule ,

$$\nabla g_n(x) = \nabla g(x)\psi(x/n) + \frac{1}{n} g(x)\nabla\psi(x/n)$$

The first term converges to $\nabla g(x)$ in L^p by dominated convergence, as ψ is uniformly bounded and ψ_n goes to 1 pointwise. As $\psi \in C_c^\infty(\mathbb{R}^d)$, $\nabla\psi$ is uniformly bounded and $g \in L^p$, we conclude that the second term goes to 0 if n goes to infinity. By a similar reasoning we can show $\nabla^j g_n$ converges to $\nabla^j g, 0 \leq j \leq k$ as n goes to ∞. So g_n converges to g in $W^{p,k}$ as n goes to infinity. It follows that $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{p,k}(\mathbb{R}^d)$. □

If Ω is a proper open subset of $\mathbb{R}^d$, then $C_c^\infty(\Omega)$ is not dense in $W^{p,k}(\Omega)$. Instead, its closure is the space of functions $W_0^{p,k}(\Omega)$ that vanish on the boundary $\partial\Omega$. The space $C^\infty(\Omega) \cap W^{p,k}(\Omega)$ is dense in $W^{p,k}(\Omega)$ for any open set cf. Meyers and Serrin [53].

Next we will introduce a important interpolation theorem which we will use in Chapter 2. For further information see Triebel [73, 74] , Bennett & Sharpeley [4] and Reed & Simon [60].

Definition 1.1.6. Let $\mathcal{X}$ be a complex vector space. Two norms $\|\cdot\|_{\mathcal{X}_0}, \|\cdot\|_{\mathcal{X}_1}$ on $\mathcal{X}$ are called **consistent** if any sequence $\{x_n\}$ that converges to zero in one norm, and which is Cauchy in other norm, converges to zero in both norms. If $\|\cdot\|_{\mathcal{X}_0}$ and $\|\cdot\|_{\mathcal{X}_1}$ are consistent, we define $\|x\|_{\mathcal{X}_0+\mathcal{X}_1} = \inf\{\|x_0\|_{\mathcal{X}_0} + \|x_1\|_{\mathcal{X}_1} \,:\, x = x_0 + x_1, x_j \in \mathcal{X}, j = 0, 1\}$

We denote by $\mathcal{X}_0$, $\mathcal{X}_1$ and $\mathcal{X}_0+\mathcal{X}_1$ the completion of $\mathcal{X}$ with respect to the norms $\|\cdot\|_{\mathcal{X}_0}$, $\|\cdot\|_{\mathcal{X}_1}$ and $\|\cdot\|_{\mathcal{X}_0+\mathcal{X}_1}$, respectively.

Assume $S=\{z\in\mathbb{C}: 0<\operatorname{Re} z<1\}$ be an open strip in the complex plane. Assume $\|\cdot\|_{\mathcal{X}_0}$ and $\|\cdot\|_{\mathcal{X}_1}$ be two consistent norms on a complex vector space $\mathcal{X}$. Denote $F[\mathcal{X}]$ to be the space of continuous functions $\phi(z)$, from $\overline{S}$ to $\mathcal{X}_0+\mathcal{X}_1$ which are analytic in S, with the following properties:

(1) $\sup_{z\in\overline{S}}\|\phi(z)\|_{\mathcal{X}_0+\mathcal{X}_1}<\infty$.

(2) $\phi(iy)\in\mathcal{X}_0$ and $\phi(1+iy)\in\mathcal{X}_1$ with $y\in\mathbb{R}$, are continuous in the respective norm $\|\cdot\|_{\mathcal{X}_0}$ and $\|\cdot\|_{\mathcal{X}_1}$

$$\|\phi\|_{F[\mathcal{X}]}=\sup_{y\in\mathbb{R}}\{\|\phi(iy)\|_{\mathcal{X}_0},\quad \|\phi(1+iy)\|_{\mathcal{X}_1}\}<\infty.$$

$F[\mathcal{X}]$ with norm $\|\cdot\|_{F[\mathcal{X}]}$ is a Banach space, and for each $t\in[0,1]$ the subspace $K_t=\{\phi\in F[\mathcal{X}]\,|\,\phi(t)=0\}$ is $\|\cdot\|_{F[\mathcal{X}]}$-closed. Define $\tilde{\mathcal{X}}_t=F[\mathcal{X}]/K_t, 0\le t\le 1$, and the quotient norm on $\tilde{\mathcal{X}}_t$ is $\|\cdot\|_{\tilde{\mathcal{X}}_t}$. Note that $\mathcal{X}$ may be identified with a subset of $\tilde{\mathcal{X}}_t$ which takes each $x\in\mathcal{X}$ into $[x]$, the equivalence class of the constant function whose value is x. Further, $\tilde{\mathcal{X}}_t$ may be identified with a subset of $\mathcal{X}_0+\mathcal{X}_1$ under the map which takes an equivalence class $[\phi]$ into the common value of its members at t. This map is clearly injective, and the following computation shows that it is continuous: Let $[\phi]\in\tilde{\mathcal{X}}_t$ and $x=\phi(t)$. $x\le\|\phi\|_{F[\mathcal{X}]}$. Thus

$$\|x\|_{\mathcal{X}_0+\mathcal{X}_1}\le\|[\phi]\|_{\tilde{\mathcal{X}}_t}=\inf\{\|\phi\|_{F[\mathcal{X}]}: \phi\in F[\mathcal{X}],\ \phi(t)=x\}.$$

We denote by $\mathcal{X}_t$ is the completion of $\mathcal{X}$ with respect to the norm $\|\cdot\|_{\tilde{\mathcal{X}}_t}$, as usual, we set

$$\|x\|_{\mathcal{X}_t}=\|[\phi]\|_{\tilde{\mathcal{X}}_t},$$

where $\phi\in F[\mathcal{X}]$ such that $\phi(t)=x$, see Reed & Simon [59].

The next theorem is the Calderón Lions Interpolation Theorem which is taken from Reed & Simon [59].

Theorem 1.1.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be complex vector spaces with given consistent norms $\|\cdot\|_{\mathcal{X}_0}, \|\cdot\|_{\mathcal{X}_1}$ on $\mathcal{X}$ and $\|\cdot\|_{\mathcal{Y}_0}, \|\cdot\|_{\mathcal{Y}_1}$ on $\mathcal{Y}$. Suppose $\Lambda(\cdot)$ is an analytic, uniformly bounded, continuous and $L(\mathcal{X}_0+\mathcal{X}_1,\mathcal{Y}_0+\mathcal{Y}_1)$-valued function on the strip $\overline{S}$ with the following properties:*

(1) $\Lambda(t):\mathcal{X}\to\mathcal{Y}$ for each $t\in(0,1)$,

(2) for all $y\in\mathbb{R}$, $\Lambda(iy):\mathcal{X}_0\to\mathcal{Y}_0$ and $M_0=\sup_{y\in\mathbb{R}}\|\Lambda(iy)\|_{L(\mathcal{X}_0,\mathcal{Y}_0)}<\infty$,

(3) for all $y\in\mathbb{R}$, $\Lambda(1+iy):\mathcal{X}_1\to\mathcal{Y}_1$ and $M_1=\sup_{y\in\mathbb{R}}\|\Lambda(1+iy)\|_{L(\mathcal{X}_1,\mathcal{Y}_1)}<\infty$,

then $T(t)[\mathcal{X}_t] \subset \mathcal{Y}_t$ *and* $\|T(t)\|_{L(\mathcal{X}_t,\mathcal{Y}_t)} \le M_0^{1-t} M_1^t$.

In the last part of this section, we will provide an important example as application of Theorem 1.1.7.

Example 1.1.8. If we take $\mathcal{X} = L^1(\Omega) \cap L^\infty(\Omega)$ and set $\|\cdot\|_{\mathcal{X}_0} := \|\cdot\|_{L^1(\Omega)}$ and $\|\cdot\|_{\mathcal{X}_1} := \|\cdot\|_{L^\infty(\Omega)}$, we obtain that $\mathcal{X}_t = L^{t^{-1}}(\Omega)$ for $t \in (0,1)$. For a detailed discussion see Reed and Simon [59, P.38].

1.2 Fourier, Laplace and Mellin transforms

In this section, we collect a few useful facts on the **Fourier transform** on $\mathbb{R}^d$, the (one-sided) **Laplace transform** and **Mellin transform** on $[0,\infty)$. Our standard reference for the Fourier transform, Laplace transform and Mellin transform are the books [27, 59, 70, 28].

Fourier transform. If ϕ is a function and μ a measure, the Fourier transform is defined as

$$\begin{aligned} \mathscr{F}\mu(\xi) = \mathscr{F}(\mu,\xi) &:= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\, \mu(dx) \\ \text{and} \quad \mathscr{F}\phi(\xi) = \mathscr{F}(\phi,\xi) &:= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} \phi(x)\, dx \end{aligned} \tag{1.2.1}$$

whenever the integrals are defined. The inverse Fourier transform $\mathscr{F}^{-1}$ coincides with the characteristic function from probability theory, i.e.

$$\mathscr{F}^{-1}(\nu,x) = \int_{\mathbb{R}^d} e^{i\xi\cdot x}\, \nu(dx) \quad \text{and} \quad \mathscr{F}^{-1}(\phi,x) = \int_{\mathbb{R}^d} e^{i\xi\cdot x}\, \phi(x)\, dx.$$

The following rules for the Fourier transform are useful:

$$\mu \mapsto \mathscr{F}\mu, \quad \phi \mapsto \mathscr{F}\phi \quad \text{is linear} \tag{1.2.2}$$

$$\mathscr{F}(\phi * \mu) = \mathscr{F}\phi \cdot \mathscr{F}\mu, \quad \mathscr{F}(\phi * u) = \mathscr{F}\phi \cdot \mathscr{F}u \tag{1.2.3}$$

$$\mathscr{F}(\nabla\phi,\xi) = i\xi\mathscr{F}\phi(\xi). \tag{1.2.4}$$

Recall that the space $L^2(\mathbb{R}^d)$ consist of all measurable functions $\phi : \mathbb{R}^d \to \mathbb{C}$ such that $\int_{\mathbb{R}^d} |\phi|^2 dx < \infty$.

Theorem 1.2.1 (Plancherel identity)**.** *The Fourier transform is an isometry of* $L^2(\mathbb{R}^d)$. *That is,* $\|\phi\|_{L^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{d/2}} \|\mathscr{F}\phi\|_{L^2(\mathbb{R}^d)}$ *i.e.*

$$\int_{\mathbb{R}^d} |\phi(x)|^2\, dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\mathscr{F}\phi(\xi)|^2\, d\xi$$

Laplace transform. If ϕ is a function and μ is a measure, we have

$$\begin{aligned} \mathscr{L}\mu(s) = \mathscr{L}(\mu; s) &:= \int_0^\infty e^{-sx}\,\mu(dx) \\ \text{and} \quad \mathscr{L}\phi(s) = \mathscr{L}(\phi; s) &:= \int_0^\infty e^{-sx}\phi(x)\,dx \end{aligned} \tag{1.2.5}$$

whenever the integrals are defined. The **Laplace convolution** of a function and a measure, or two functions, is defined as

$$\begin{aligned} \phi * \mu(x) &= \int_0^x \phi(x-t)\,\mu(dt) \\ \text{respectively} \quad \phi * u(x) &= \int_0^x \phi(x-t)u(t)\,dt = \int_0^x u(x-t)\phi(t)\,dt = u * \phi(x). \end{aligned} \tag{1.2.6}$$

The Laplace convolution is derived from the usual convolution if we agree that $\mu|_{(-\infty,0)} = 0$ and $\phi|_{(-\infty,0)} = 0$. The reason is as follows

$$\phi * \mu(x) = \int_{-\infty}^\infty \phi(x-t)\,\mu(dt) = \int_0^x \phi(t)\,\mu(dt) + \int_x^\infty \phi(x-t)\,\mu(dt) + \int_{-\infty}^0 \phi(x-t)\,\mu(dt).$$

We can obtain that the last two terms are 0.

The following rules for the Laplace transform are useful:

$$\mu \mapsto \mathscr{L}\mu, \quad \phi \mapsto \mathscr{L}\phi \quad \text{are linear} \tag{1.2.7}$$

$$\mathscr{L}(\phi * \mu) = \mathscr{L}\phi \cdot \mathscr{L}\mu, \quad \mathscr{L}(\phi * u) = \mathscr{L}\phi \cdot \mathscr{L}u \tag{1.2.8}$$

$$\mathscr{L}(\mu[0,x), s]) = \frac{1}{s}\mathscr{L}\mu(s), \quad \mathscr{L}\left(\int_0^x \phi(t)\,dt, s\right) = \frac{1}{s}\mathscr{L}\phi(s) \tag{1.2.9}$$

$$\mathscr{L}(\phi', s) = s\mathscr{L}\phi(s) - \phi(0+). \tag{1.2.10}$$

$$\mathscr{L}(\phi(x-b), s) = e^{-sb}\mathscr{L}[\phi(x), s]. \tag{1.2.11}$$

Only the first half of (1.2.9) looks a bit unusual. It follows from

$$\begin{aligned} \mathscr{L}(\mu; s) &= \int_0^\infty \int_0^x \mu(dt) e^{-sx}\,dx \\ &= \int_0^\infty \int_t^\infty e^{-sx}\,dx\,\mu(dt) \\ &= \int_0^\infty \left[-\frac{1}{s}e^{-sx}\right]_{x=t}^\infty \mu(dt) \\ &= \int_0^\infty \frac{1}{s}e^{-st}\,\mu(dt) = \frac{1}{s}\mathscr{L}\mu(s). \end{aligned}$$

An extremely useful Laplace transform is Euler's Gamma function. We have

$$\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}\,dx \;=\; \int_0^\infty x^r e^{-x}\,\frac{dx}{x}$$

$$\overset{x=s\,y}{=} \int_0^\infty s^r y^r e^{-sy} \frac{dy}{y}$$
$$= s^r \mathscr{L}(x^{r-1}, s)$$

and if we take $\alpha = 1 - r$ or $r - 1 = -\alpha$ we end up with

$$s^{1-\alpha} = \mathscr{L}\,(x^{-\alpha}/\Gamma(1-\alpha), s) \tag{1.2.12}$$

Moreover, we have **Lévy's continuity theorem** for Laplace transforms:

$$\mu_n \text{ converges vaguely} \iff \lim_n \mathscr{L}\mu_n(s) \text{ exists for all } s > 0 \text{ and is finite.}$$

In particular, the pointwise limit of the Laplace transforms is always the Laplace transform of a measure on $[0, \infty)$. A proof of this classical result can be found in [69, Corollary 1.7 and Theorem 1.4]. There is also a **sub-sequence version** which states that if the sequence $\mathscr{L}\mu_n(s)$ is bounded at $s = 0+$, then there is a subsequence such that $\mathscr{L}\mu_{n(k)}(s)$ converges for all $s > 0$, consequently the underlying measures converge vaguely, cf. [69, Corollary 1.8].

We will also need Euler's Beta function which is given by

$$B(x, y) \overset{\text{def}}{=} \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0. \tag{1.2.13}$$

In the last part of this section, we will introduce an important tool, namely the Tauberian theorem, cf. Feller [22, P.443]. The Tauberian theorem is a powerful tool that can be used to deal with asymptotics of measures and Laplace transforms.

Theorem 1.2.2. *Let m be a measure with Laplace transform $\mathscr{L}(m, \lambda) = \int_0^\infty e^{-\lambda x} m(dx)$ define for $\lambda > 0$. then the following two results are equivalent:*

(1) $\dfrac{\mathscr{L}(m, \lambda y)}{\mathscr{L}(m, y)} \to \frac{1}{\lambda^\rho}, \quad y \to 0$

(2) $\dfrac{m[0, xt)}{m[0, t)} \to x^\rho, \quad t \to \infty.$

Mellin transform. Let ϕ be a locally integrable function on $(0, \infty)$, i.e. $\int_K \phi(x)\,dx$ exists for all compact set K on $(0, \infty)$. The **Mellin transform** of ϕ is defined by

$$\mathscr{M}\phi(z) = \int_0^\infty x^{z-1}\phi(x)\,dx, \quad z \in \mathbb{C}, \tag{1.2.14}$$

when this integral converges. The domain of analyticity of $\mathscr{M}\phi$ is usually an infinite strip $a < \operatorname{Im} z < b$ with $a, b \in \mathbb{R}$ parallel to the imaginary axis. The inversion formula is given by

$$\phi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-z}\mathscr{M}\phi(z)\,dz, \tag{1.2.15}$$

with $a < c < b$.

One of the two convolution integrals associated with Mellin transform is of the form

$$g(x) = \int_0^\infty \psi(y)\phi(xy)\,dy, \quad x > 0, \tag{1.2.16}$$

where we denote $g(x) = \psi \circledast \phi(x)$ and $\mathscr{M}g(z) = \mathscr{M}\psi(1-z)\mathscr{M}\phi(z)$.

1.3 Regularly varying functions

The results in this section are mostly standard results from the theory of regularly varying functions, which we collect here for the reader's convenience. We collect some results on regularly varying functions. They have important applications in complex analysis, analytic number theory, and in particular, probability theory. First, we recall the definition and some properties of regular variation. We refer the reader to Bingham, Goldie & Teugels [9]. Then we will give some relationships between subaddtive functions, submultiplicative functions and regular variation. We need these concepts in Chapter 5.

Definition 1.3.1. Assume F is a measurable function, defined on $[a, \infty)$, $a \geq 0$. It is said to be of **regular variation**, if for all $\lambda \in (0, \infty)$,

$$\frac{F(\lambda x)}{F(x)} \to \phi(\lambda) \in (0, \infty), \quad x \to \infty, \quad \forall \lambda > 0. \tag{1.3.1}$$

F is called **slowly varying**, if $\phi \equiv 1$ in the above Definition 1.3.1. A function l is **slowly varying** if, and only if, it can be written in the form

$$l(x) = c(x) \exp\left(\int_1^{1\vee x} \frac{\varepsilon(u)}{u} du \right) \tag{1.3.2}$$

where $c(x)$ is measurable, nonnegative with $\lim_{x\to\infty} c(x) = c$, and $\varepsilon(u)$ is measurable and locally bounded with $\lim_{u\to\infty} \varepsilon(u) = 0$. This result is known as the **representation theorem**. For further information, we refer the reader to Bingham, Goldie and Teugels [9] Chapter 1. The simplest non-trivial examples of slowly varying functions are $\log x$, $\log\log x$ and $\log^\alpha(x)$, $(\log\log x)^\beta$.

The function ϕ in Definition 1.3.1 is necessarily of the form $\phi(\lambda) = \lambda^\beta$ for some $\beta \in \mathbb{R}$. We call F is **regularly varying** and β is the **index** of F. Let $\mathcal{R}_\beta$ be the class of locally bounded functions $F : [0, \infty) \to [0, \infty)$ that are regularly varying at infinity (similarly at 0) with index β. Thus, $F(x) = x^\beta l(x)$ with a slowly varying function l. If the representation in eq. (1.3.2) for l is chosen with $c(x) = 1$ for $x \in [0, 1]$, then l is called **normalized**.

$$l(x) \asymp \exp\left(\int_1^{1\vee x} \frac{\varepsilon(u)}{u} du \right), \quad x \to \infty, \tag{1.3.3}$$

Where $a \asymp b$ means that there exists c, C such that $ca \leq b \leq Ca$. The function appearing on the right hand side will be shown to be submulticative if ε is nonincreasing on $[1, \infty)$, see Theorem 1.3.10. Denote be $\mathcal{R}_0^*$ the class of all $l \in \mathcal{R}_0$ with nonincreasing ε.

Now we will introduce a crucial theorem, namely the **uniform convergence theorem** for regularly varying functions, see [9, P.22].

Theorem 1.3.2. (uniform convergence theorem for $\mathcal{R}_\beta$) *If F is regularly varying with index β, then $F(\lambda x)/F(x) \to \lambda^\beta$, uniformly for all λ from*

(a) on each $[a, b]$ $(0 < a \leq b < \infty)$ if $\beta = 0$,

(b) on each $(0, b]$ $(0 < b < \infty)$ if $\beta > 0$ and F is bounded on each interval $(0, M]$,

(c) on each $[a, \infty)$ $(0 < a < \infty)$ if $\beta < 0$.

We will use in Chapter 3 the uniform convergence theorem for slowly varying functions, i.e. $\beta = 0$. It states that under the aforementioned assumptions, the convergence is uniform on compact sets of λ values in the interval $(0, \infty)$.

Abelian theorems are results that use the asymptotic behavior of sequences and functions to deduce the asymptotic behavior of their generating functions and Laplace transforms. Conversely, results that deduce the asymptotic behavior of sequences and functions from the asymptotic behavior of their generating functions and Laplace transforms are called Tauberian theorems. In this context, we will introduce a Tauberian theorem for the Laplace transforms of distributions. For further information, we refer the reader to [9, 22]

Theorem 1.3.3. [9, P.37]. *Let ϕ be a non-decreasing right continuous function on $\mathbb{R}$ with $\phi(0) = 0$ for all $x < 0$. If l varies slowly and $c \geq 0, \beta \geq 0$, the following are equivalent:*

$$\phi(x) \sim \frac{cx^\beta l(x)}{\Gamma(1+\beta)}, \quad x \to \infty, \tag{1.3.4}$$

$$\mathscr{L}(\phi(x), \tau) \sim c\tau^{-\beta} l(1/\tau), \quad \tau \to 0. \tag{1.3.5}$$

If we denote $\Phi(x) = \int_0^x \phi(s)ds$ and $\Phi(t) \sim cx^\beta l(x), x \to \infty$, where $c, \beta \in \mathbb{R}, l \in \mathcal{R}_0$, and if ϕ is ultimately monotone, i.e. if ϕ is monotone from some $x > K$ onwards, then

$$\phi(x) \sim c\beta x^{\rho-1} l(x), \quad x \to \infty. \tag{1.3.6}$$

This is the **monotone density theorem**. If $c > 0$ our assumptions say that $\Phi \in \mathcal{R}_\beta$. However, (1.3.6) does not imply regular variation of ϕ, unless $c\beta > 0$.

Potter's bounds, as established by Potter [57], can be utilized to prove a significant result known as Breiman's Lemma, which concerns the tail behavior of products of independent random variables. Although Potter's results primarily focus on global bounds for $F(y)/F(x)$, the subsequent results extend Potter's findings, see [9, P.25].

Theorem 1.3.4. (Potter's bounds)

(a) If l is slowing varying then for any chosen constants $A > 1, \delta > 0$ there exists $M = M(A, \delta)$ such that

$$\frac{l(y)}{l(x)} \leq A \max\left\{ \left(\frac{y}{x}\right)^{\delta}, \left(\frac{y}{x}\right)^{-\delta} \right\}, \quad (x \geq M, y \geq M).$$

(b) If, further, l is bounded away from 0 and ∞ on every compact subset of $[0, \infty)$, then for every $\delta > 0$ there exists $C = C(\delta) > 1$ such that

$$\frac{l(y)}{l(x)} \leq C \max\left\{ \left(\frac{y}{x}\right)^{\delta}, \left(\frac{y}{x}\right)^{-\delta} \right\}, \quad (x > 0, y > 0).$$

(c) If ϕ is regularly varying of index β then for any chosen $a > 1, \delta > 0$ there exists $M = M(a, \delta)$ such that

$$\frac{\phi(y)}{\phi(x)} \leq a \max\left\{ \left(\frac{y}{x}\right)^{\beta+\delta}, \left(\frac{y}{x}\right)^{\beta-\delta} \right\}, \quad (x \geq M, y \geq M).$$

Next, we will discuss subadditive functions that are closely related to regularly varying functions. In this section, we will provide some definitions that serve as preliminaries for studying the index of regularly varying functions.

Definition 1.3.5. A function $F(x)$ defined on $\mathbb{R}$ or $(0, \infty)$ is said to be **subadditive** if for all x_1, x_2 we have

$$F(x_1 + x_2) \leq F(x_1) + F(x_2). \tag{1.3.7}$$

Lemma 1.3.6. *If ϕ is a concave function and $\phi \geq 0$, then ϕ is subadditive on $[0, \infty)$.*

Proof. According to the definition of concavity, for any $x, y \in [0, \infty)$ and $\alpha \in [0, 1]$,

$$(1 - \alpha)\phi(x) + \alpha\phi(y) \leq \phi((1 - \alpha)x + \alpha y).$$

Taking $x = 0$, we have $\alpha\phi(y) \leq \phi(\alpha y)$. Using this property, we have

$$\phi(x)+\phi(y) = \phi\left(\frac{x(x+y)}{x+y}\right)+\phi\left(\frac{y(x+y)}{x+y}\right) \geq \frac{x}{x+y}\phi(x+y)+\frac{y}{x+y}\phi(x+y) = \phi(x+y).$$

Thus, ϕ is subadditive. □

The next theorem provides the growth rate of additive functions and describes their behavior near the endpoints. This is an important result in the field.

Theorem 1.3.7. [45, P.52] *If $\phi(x)$ is subadditive and finite in $(-\infty, \infty)$, then*

$$\beta = \lim_{x\to\infty} \frac{\phi(x)}{x} = \inf_{x>0} \frac{\phi(x)}{x} < \infty. \tag{1.3.8}$$

$$\alpha = \lim_{x\to-\infty} \frac{\phi(x)}{x} = \sup_{x<0} \frac{\phi(x)}{x} < \infty. \tag{1.3.9}$$

Definition 1.3.8. [45, P.52]. A positive everywhere finite function $F(x)$ on $(0, \infty)$ is said to be **submultiplicative** if for all $x_1, x_2 \in (0, \infty)$ we have

$$F(x_1 x_2) \leq F(x_1)F(x_1) \tag{1.3.10}$$

This property does, in general, not hold for regularly varying functions.

If F is submultiplicative, positive and we introduce a new variable $x = \log y$, then the function $G(x) = \log F(e^x)$ is a finite subadditive function on $\mathbb{R}$. As an application of Theorem 1.3.7, we have

Lemma 1.3.9. *Let $\phi : (0, \infty) \to [0, \infty)$ be submultiplicative. There exist numbers α, β satisfying $-\infty < \alpha \leq \beta < \infty$ such that*

(1) for all $x > 1 : \phi(x) \geq x^{\beta}$;

(2) for all $x \in (0, 1) : \phi(x) \geq x^{\alpha}$;

(3) for all $\varepsilon > 0$, there exists $M_\varepsilon > 1$ for all $x > M_\varepsilon : \phi(x) \geq x^{\beta+\varepsilon}$;

(4) for all $\varepsilon > 0$, there exists $m_\varepsilon \in (0, 1)$ for all $x \in (0, m_\varepsilon) : \phi(t) \geq x^{\alpha-\varepsilon}$.

Moreover,

$$\alpha = \lim_{x\to 0} \frac{\log \phi(x)}{\log x} = \sup_{0<x<1} \frac{\log \phi(x)}{\log x}, \quad \beta = \lim_{x\to\infty} \frac{\log \phi(x)}{\log x} = \inf_{x>1} \frac{\log \phi(x)}{\log x}. \tag{1.3.11}$$

Proof. Using the above connection ϕ is submultiplicative on $(0, \infty)$ and $\phi \geq 0$, we have $\log \phi(e^x)$ is suadditive, $x \in \mathbb{R}$. By Theorem 1.3.7,

$$\alpha = \lim_{x\to 0} \frac{\log \phi(x)}{\log x} = \sup_{0<x<1} \frac{\log \phi(x)}{\log x}, \quad \beta = \lim_{x\to\infty} \frac{\log \phi(x)}{\log x} = \inf_{x>1} \frac{\log \phi(x)}{\log x}.$$

Further, for all $\varepsilon > 0$, there exists $m_\varepsilon \in (0, 1)$ such that $\alpha - \varepsilon \leq \frac{\log \phi(x)}{\log x} \leq \alpha$ and $\log x \leq 0$, since $x \in (0, 1)$. We obtain from here $(\alpha - \varepsilon)\log x \geq \log \phi(x) \geq \alpha \log x$. Thus, we get (2) and (4). Using a similar procedure, we will get (1) and (3). $\square$

The next theorem shows that slowly varying functions l are submultiplicative. For more information, refer to [1].

Theorem 1.3.10. [1, P.175]. *Let $l(x)$ be a slowly varying function, i.e. $l \in \mathcal{R}_0$, such that $l(x) = \exp\left(\int_1^{1\vee x} \frac{\varepsilon(u)}{u} du\right)$ for some nonincreasing $\varepsilon : [1, \infty) \to \mathbb{R}$ which is locally bounded and $\lim_{u\to 0} \varepsilon(u) = 0$. Then l is submultiplicative.*

The representation theorem tells us that $\phi \in \mathcal{R}_\alpha$ can be write as $\phi(x) = c(x)x^\alpha e^{\int_a^x \frac{\varepsilon(u)}{u}\,du}$ for $a > 0$, a measurable function $c(x)$ with $\lim_{x\to\infty} c(x) = c_0$ and $\lim_{x\to\infty} \varepsilon(x) = 0$. If we are interested in the **asymptotic** properties of $\phi \in \mathcal{R}_\alpha$ as $x \to \infty$, we can replace $\phi(x)$ by

$$\phi(x) \asymp c_0 x^\alpha e^{\int_a^x \frac{\varepsilon(u)}{u}\,du}.$$

and the function appearing on the right is differentiable. the following theorem follows from [9, P.44-45].

Theorem 1.3.11. *Let $\phi \in \mathcal{R}_\alpha$*

(a) If $\phi(x) = c_0 x^\alpha e^{\int_a^x \frac{\varepsilon(u)}{u}\,du}$, then $\lim_{x\to\infty} \frac{x\phi'(x)}{\phi(x)} = \alpha = \lim_{x\to\infty} \frac{\log\phi(x)}{\log x}$.

(b) If $\phi(x)$ is as in (a) and, in addition submultiplicative, then

$$\alpha = \lim_{x\to\infty} \frac{\log\phi(x)}{\log x} = \inf_{x>1} \frac{\log\phi(x)}{\log x}.$$

(c) If $\phi(x) = c(x)x^\alpha e^{\int_a^x \frac{\varepsilon(u)}{u}\,du}$, then

$$\alpha = \lim_{x\to\infty} \frac{\log\phi(x)}{\log x}.$$

Proof. (a) if $\phi \in \mathcal{R}_\alpha$, then $\phi(x) = x^\alpha l(x)$, where $l(x) = c_0 e^{\int_a^x \frac{\varepsilon(u)}{u}\,du}$. The function appearing on the right side is differentiable by [9, P.44, P.45]. By a direct calculation, we have

$$\phi'(x) = \alpha x^{\alpha-1} l(x) + x^\alpha l'(x),$$

and

$$\frac{x\phi'(x)}{\phi(x)} = \frac{x(\alpha x^{\alpha-1} l(x) + x^\alpha l'(x)}{x^\alpha l(x)} = \alpha + x\frac{l'(x)}{l(x)}.$$

Using the results [9, P.15], we obtain $xl'(x)/l(x) \to 0$ as x goes to ∞. Taking the limit on both sides of the above formula , we will get the result (a).

(b) Using the results of (a), we have $\alpha = \lim_{x\to\infty} \log\phi(x)/\log x$. In addition, ϕ is submultiplicative and satisfies the condition in Lemma 1.3.9, we get

$$\alpha = \lim_{x\to\infty} \frac{\log\phi(x)}{\log x} = \inf_{x>1} \frac{\log\phi(x)}{\log x}.$$

(c) $\phi(x) = c(x)x^\alpha e^{\int_a^x \frac{\varepsilon(u)}{u}\,du}$, then we have

$$\frac{\log\phi(x)}{\log x} = \alpha + \frac{c(x)}{\log x} + \frac{\int_a^x \frac{\varepsilon(u)}{u}\,du}{\log x},$$

when apply L'Hôpital's rule on the right side of the above equality and use representative theorem $\lim_{x\to\infty} c(x) = c_0$, $\lim_{x\to\infty} \varepsilon(x) = 0$, we have

$$\lim_{x\to\infty} \frac{\log\phi(x)}{\log x} = \alpha + 0 + \lim_{x\to\infty} \varepsilon(x) = \alpha.$$

□

We can extend the concept of regular variation to o-regular variation by using lim sup and lim inf instead of the limit. This extension is discussed in Dragan [15].

Definition 1.3.12. [9, P.65] The class of o-regularly varying functions is the set of positive measurable $F : [0, \infty) \to (0, \infty)$ with

$$0 < \liminf_{x\to\infty} \frac{F(\lambda x)}{F(x)} \leq \limsup_{x\to\infty} \frac{F(\lambda x)}{F(x)} < \infty, \forall \lambda \geq 1$$

Theorem 1.3.13. [2, P.27]. *If $F : (0, \infty) \to (0, \infty)$ is measurable and submultiplicative, then F is o-regularly varying.*

The above theorem provides a sufficient condition for a function to be o-regularly varying. For example, $\log(e + x)$ is a submultiplicative function but not regularly varying, while $\log(x)$ is regularly varying but not submultiplicative.

Here is the relation between regularly varying and o-regularly varying functions: every regularly varying function is also o-regularly varying. However, not every o-regularly varying function is regularly varying. In Chapter 3, we will construct a function that is o-regularly varying but not regularly varying.

1.4 Markov processes

In this section, we will provide a brief introduction to Markov processes and discuss some of their properties.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with some filtration $(\mathcal{F})_{t\geq 0}$. The next definition and properties are taken from the monographs [6, 24, 36].

Definition 1.4.1. [36, P.24] A (temporally homogeneous) **Markov transition function** is a measure kernel $p_t(x, A), t \geq 0, x \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d)$ such that

(a) $A \mapsto p_s(x, A)$ is a probability measure for every $s \geq 0$ and $x \in \mathbb{R}^d$;

(b) $(s, x) \mapsto p_s(x, A)$ is a Borel measurable function for every $A \in \mathcal{B}(\mathbb{R}^d)$;

(c) the Chapman-Kolmogorov equations hold

$$p_{s+t}(x, A) = \int p_t(y, A) p_s(x, dy) \quad \forall s, t \geq 0, \quad x \in \mathbb{R}, \quad A \in \mathcal{B}(\mathbb{R}^d). \qquad (1.4.1)$$

Definition 1.4.2. [36, P.24] A stochastic process $(X_t)_{t\geq 0}$ is called a **(temporally homogeneous) Markov process** with transition function if there exists a Markov transition function $p_t(x, A)$ such that

$$\mathbb{P}(X_t \in A | \mathcal{F}_s) = p_{t-s}(X_s, A) \ a.s. \quad \forall s \leq t, \quad A \in \mathcal{B}(\mathbb{R}^d). \qquad (1.4.2)$$

Here, we will review some definitions related to Feller semigroups and infinitesimal generators. For more detailed information, see [11, 36].

Definition 1.4.3. [36, P.26] A Feller semigroup is a family of linear operators P_t satisfying

(a) $P_t : \mathcal{B}(\mathbb{R}^d) \to \mathcal{B}(\mathbb{R}^d)$, (acting on $\mathcal{B}(\mathbb{R}^d)$).

(b) $P_{t+s} = P_t P_s = P_s P_t$ for all $s, t \geq 0$ and $P_0 = I$, (semigroup).

(c) $0 \leq \phi \leq 1 \Rightarrow 0 \leq P_t\phi \leq 1$, (sub-Markov).

(d) $\|P_t\phi\|_\infty \leq \|\phi\|_\infty$ for all $\phi \in \mathcal{B}(\mathbb{R}^d)$, (contractive).

(e) $P_t 1 = 1$, (conservative).

(f) $P_t : C_\infty(\mathbb{R}^d) \to C_\infty(\mathbb{R}^d)$, (Feller operator).

(g) $\lim_{t\to 0} \|P_t\phi - \phi\|_\infty = 0$ for all $\phi \in C_\infty(\mathbb{R}^d)$, (strong continuous).

$(P_t)_{t\geq 0}$ is called a **Markov semigroup** if (a)-(c) holds and it is called **Feller semigroup** if (a)-(d), (f), (g) hold. If (e) holds, P_t is said to be conservative.

Every strongly continuous contractive Markov semigroup has a generator and a resolvent, which plays a crucial role in probability theory. Let us recall the definition of the generator and its expression in terms of the Laplace transform of the semigroup.

Definition 1.4.4. Let $(P_t)_{t\geq 0}$ be a Feller semigroup. The **infinitesimal generator** a linear operator defined by

$$\mathcal{D}(\mathcal{A}) := \left\{ \phi \in C_\infty(\mathbb{R}^d) \quad | \quad \exists g \in C_\infty(\mathbb{R}^d) : \lim_{t\to 0} \left\| \frac{P_t\phi - \phi}{t} - g \right\|_\infty = 0 \right\} \quad (1.4.3)$$

$$\mathcal{A}\phi := \lim_{t\to 0} \frac{P_t\phi - \phi}{t}, \quad f \in \mathcal{D}(\mathcal{A}). \quad (1.4.4)$$

There is a one-to-one correspondence between Feller semigroups and generators. The following theorem serves as the key to establish this connection. For the proof, we refer to [55, 19], while a probabilistic approach can be found in [35].

Theorem 1.4.5. **(Hill-Yosida-Ray)** *A linear operator* $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ *on* $C_\infty(\mathbb{R}^d)$ *generates a Feller semigroup* $(P_t)_{t\geq 0}$ *if, and only if,*

(a) $\mathcal{D}(\mathcal{A}) \subset C_\infty(\mathbb{R}^d)$ dense.

(b) A is dissipative, i.e. $\|\lambda\phi - \mathcal{A}\phi\|_\infty \geq \lambda\|\phi\|_\infty$ for some (or all) $\lambda > 0$.

(c) $(\lambda - \mathcal{A})(\mathcal{D}(\mathcal{A})) = C_\infty(\mathbb{R}^d)$ for some (or all) $\lambda > 0$.

(d) $\mathcal{A}$ satisfies the **positive maximum principle**:

$$\phi \in \mathcal{D}(\mathcal{A}), \quad \phi(x_0) = \sup_{x\in\mathbb{R}^d} \phi(x) \geq 0 \implies \mathcal{A}\phi(x_0) \leq 0.$$

Assume that $(X_t, \mathcal{F}_t)_{t\geq 0}$ is a time homogeneous Markov process. From the Markov property it is easy to see that

$$P_t\phi(x) = \mathbb{E}^x\phi(X_t) := \int_{\mathbb{R}^d} \phi(y)\mathbb{P}^x(X_t \in dy), \quad \forall \phi \in \mathcal{B}(\mathbb{R}^d), \quad x \in \mathbb{R}^d. \tag{1.4.5}$$

defines a Markov semigroup.

Definition 1.4.6. A Feller process is a time homogeneous Markov process whose transition semigroup $P_t\phi(x) = \mathbb{E}^x\phi(X_t)$ is a Feller semigroup.

By the above definitions and theorem: There is a one to one correspondence between Feller semigroup and Feller processes $(X_t)_{t\geq 0}$.

Definition 1.4.7. Let $(P_t)_{t\geq 0}$ be a Feller semigroup. The **resolvent** is a family of linear operators given by

$$R_\lambda\phi(x) := \int_0^\infty e^{-\lambda t}P_t\phi(x)\,dt, \quad \phi \in \mathcal{B}(\mathbb{R}^d), \quad x \in \mathbb{R}^d, \quad \lambda > 0. \tag{1.4.6}$$

Let $\mathcal{A}$ be the generator and R_λ be the resolvent of a Feller semigroup. In view of Thereom 1.4.5 (c) one has, $R_\lambda = (\lambda - \mathcal{A})^{-1}$ for all $\lambda > 0$, that means $R_\lambda(C_\infty(\mathbb{R}^d)) \subset \mathcal{D}(\mathcal{A})$. From the above results, it is clear that there is a one-to-one relationship between the semigroup, generator, and resolvent. For further information and a detailed discussion of their properties, we refer the reader to [6, 36].

In the last part of this section, we will give a brief introduction to potential theory and some of its associated concepts. Now let us discuss some more specific definitions related to Markov processes: potential measure.

Definition 1.4.8. [6, P.31] Let $(X_t)_{t\geq 0}$ be a Markov process. The potential measure $U(x,\cdot)$ which corresponds to the 0-resolvent kernel is given by

$$U(x, A) = \int_0^\infty \mathbb{P}^x\left(X_t \in A\right)dt = \mathbb{E}^x\left(\int_0^\infty 1_{\{X_t\in A\}}dt\right) \tag{1.4.7}$$

Let $\mathbb{R}^d$ be the state space of a Markov process and $p_t(x, A)$ be the transition probability. By equality (1.4.7), $U(x, A) = \int_0^\infty \mathbb{P}^x(X_t \in A)dt = \int_0^\infty p_t(x, A)dt$, which represents the expected time for a particle starting at x to stay in the set A. Furthermore, the kernel U corresponds to the Green function in classical theory. Usually, the kernel U can be utilized in two dual ways: Assume ϕ is a function on $\mathbb{R}^d$, then $U\phi$ is defined by

$$(U\phi)(x) = \int_0^\infty \phi(y)U(x, dy) \tag{1.4.8}$$

and if m is a measure of subset of $\mathbb{R}^d$, the measure mU is given by the following way

$$(mU)(A) = \int_0^\infty U(x, A)m(dx). \tag{1.4.9}$$

In Chapter 3, we will study equation (1.4.8), exploring its connection with the concept of Sonine pairs. The measure m in equation (1.4.9) is commonly referred to as the speed measure. For more detailed information, we refer to Fukushima et al. [24]. Now we discuss the necessary and sufficient conditions for the transience of the Markov semigroup.

Theorem 1.4.9. [24, P.38] *A Markovian semigroup $(P_t)_{t\geq 0}$ is transient if, and only if, there exists a function $\phi \in L^1(\mathbb{R}; dx)$ such that ϕ is strictly positive $dx - a.e.$ on $\mathbb{R}$ and $U\phi(x) < \infty, dx.a.e.$.*

1.5 Lévy processes and subordinators

In this section, we will collect some results for an important subclass of Lévy processes: subordinators, their inverses and Bochner's subordination. These results are mostly standard and we collect them here for the reader's convenience cf. Bertoin [6, 7], Schilling et al [36].

Lévy process. A **Lévy process** $L = (L_t)_{t\geq 0}$ is a stochastic process on $\mathbb{R}^d$ with independent and stationary increments, càdlàg paths and $L_0 = 0$. Since L_t is infinitely divisible we can describe L_t, hence L, through

$$\mathbb{E}e^{i\xi\cdot L_t} = e^{-t\psi(\xi)} \tag{1.5.1}$$

where the characteristic exponent $\psi : \mathbb{R}^d \to \mathbb{C}$ is uniquely determined through the **Lévy–Khintchine formula**

$$\psi(\xi) = ia\cdot\xi + \frac{1}{2}Q\xi\cdot\xi + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{-i\xi\cdot x} - i\xi\cdot x\mathbb{1}_{(0,1)}(|x|)\right)\nu(dx) \tag{1.5.2}$$

where $a \in \mathbb{R}^d, Q \in \mathbb{R}^{d\times d}$ positive semidefinite and ν is the Lévy measure satisfying $\int_{\mathbb{R}^d\setminus\{0\}}(1\wedge|x|^2)\nu(dx) < \infty$. The triplet (a, Q, ν) is called **Lévy triplet** or characteristics of ψ.

Since L is a Markov process, there is a transition semigroup

$$P_t\phi(x) = \mathbb{E}^x\phi(L_t) = \mathbb{E}\phi(L_t + x) = \int \phi(x+y)\,p_t(dy), \quad p_t(dy) = \mathbb{P}(L_t \in dy). \tag{1.5.3}$$

Let us treat this semigroup as a Feller semigroup, i.e. on the space of continuous functions vanishing at infinity $C_\infty(\mathbb{R}^d) = \{\phi \in C(\mathbb{R}^d) : \lim_{|x|\to\infty}\phi(x) = 0\}$ which is, equipped with the supremum, a Banach space. The infinitesimal generator $\mathcal{A}$ is given by

$$\mathcal{A}\phi(x) = \lim_{t\to 0}\frac{1}{t}(P_t\phi(x) - \phi(x))$$

and the domain $\mathcal{D}(\mathcal{A})$ consists of all $\phi \in C_\infty(\mathbb{R}^d)$ such that the above limit exists pointwise (rather than uniformly!) [1] and defines a function in $C_\infty(\mathbb{R}^d)$. For $\phi \in C_\infty(\mathbb{R}^d)$, we take the Fourier transform on both sides of eq. (1.5.3). We have

$$\mathscr{F}\left[P_t\phi\right] = e^{-t\psi}\mathscr{F}\phi,$$

and we find

$$\mathscr{F}\left[\mathcal{A}\phi\right] = -\psi\mathscr{F}\phi \quad \text{or} \quad \mathcal{A}\phi = -\psi(D_x)\phi = \mathscr{F}^{-1}\left[-\psi\mathscr{F}\phi\right], \tag{1.5.4}$$

i.e. the generator is a pseudo-differential operator with symbol $-\psi$.

Subordinator. A **subordinator** $S = (S_t)_{t\geq 0}$ is a one-dimensional Lévy process with a.s. increasing sample paths. Since S is one-sided, we use the Laplace transform to characterize the one-dimensional transition functions, hence the process. Because of the independent and stationary increments we have

$$\mathbb{E}e^{-\lambda S_t} = e^{-tf(\lambda)}, \quad t, \lambda > 0 \tag{1.5.5}$$

where the characteristic exponent f is a **Bernstein function** (see [69]) with $f(0) = 0$, which is fully characterized by the Lévy–Khintchine formula

$$f(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda x})\,\mu(dx) \tag{1.5.6}$$

where $b \geq 0$ and μ is a measure on $(0, \infty)$ such that $\int_{(0,\infty)} 1 \wedge x\,\mu(dx) < \infty$. We call the pair (b, μ) the characteristics of the Bernstein function f. Let us treat this semigroup as a Feller semigroup, i.e. on the space of continuous functions vanishing at infinity

$$\phi \in C_\infty[0,\infty) \iff \phi \text{ continuous on } [0,\infty) \text{ and } \lim_{x\to\infty}\phi(x) = 0.$$

The infinitesimal generator $\mathcal{A}$ is given by

$$\mathcal{A}\phi(x) = \lim_{t\to 0}\frac{1}{t}(P_t\phi(x) - \phi(x))$$

and the domain $\mathcal{D}(\mathcal{A})$ consists of all $\phi \in C_\infty[0,\infty)$ such that the above limit exists pointwise (rather than uniformly!) and defines a function in $C_\infty[0,\infty)$. Using the Lévy–Khintchine representation one can show that the generator takes the following form

$$\mathcal{A}\phi(x) = b\phi'(x) + \int_0^\infty (\phi(x+y) - \phi(x))\,\mu(dy), \quad \phi \in C^1[0,\infty) \cap C_\infty[0,\infty). \tag{1.5.7}$$

In particular, $C^1_\infty[0,\infty) \subset \mathcal{D}(\mathcal{A})$. If $b = 0$, we have $C^1_b[0,\infty) \cap C_\infty[0,\infty) \subset \mathcal{D}(\mathcal{A})$. It is clear that the representation (1.5.7) extends to all functions $\phi \in C^1[0,\infty)$.

[1] That we can use pointwise limits to characterize the domain is a consequence of the positive maximum principle, see e.g. Sato [64, Lemma 31.7] or Schilling–Partzsch [67, Theorem 7.22].

Example 1.5.1 (Shift semigroup). Consider the deterministic subordinator $S_t = t$ which describes a movement to the right at unit speed. We have

$$T_t\phi(s) = \phi(s+t) \quad \text{and} \quad \mathcal{A}\phi(s) = \frac{d}{dt}\Big|_{t=0} T_t\phi(s) = \frac{d}{ds}\phi(s).$$

Example 1.5.2 (One-sided stable semigroup). Consider the one-sided α-stable ($0 < \alpha < 1$) subordinator S_t which is defined via the Laplace transform:

$$\mathbb{E}e^{-sS_t} = e^{-ts^\alpha} \quad \text{which means that} \quad f(s) = s^\alpha.$$

It is not hard to see that this Bernstein function has the following Lévy–Khintchine formula

$$s^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-sx}) \, \frac{dx}{x^{1+\alpha}}$$

and this means that the generator is of the following form

$$\mathcal{A}\phi(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{\phi(x+y) - \phi(y)}{y^{1+\alpha}} \, dy.$$

The Lévy measure of the Bernstein function $f(s) = s^\alpha$ is $\mu(dx) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-(1+\alpha)} \, dx$. It is easy to see that

$$\frac{\alpha}{\Gamma(1-\alpha)} x^{-(1+\alpha)} = \int_0^\infty e^{-yx} \, \sigma(dy),$$

with $\sigma(dy) = \frac{y^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \, dy$. This leads us to the definition of so–called **complete Bernstein functions** and $f(x) = x^\alpha$ is one of most prominent examples of this class.

If the represently measure μ in the Lévy–Khintchine formula (2.4.1) for the Bernstein function f has a density which is completely monotone, i.e.

$$\mu(dx) = m(x) \, dx \quad \text{and} \quad m(x) = \int_0^\infty e^{-xy} \, \hat{\sigma}(dy),$$

then f is a **complete Bernstein function**. After a short calculation we can show that we get the following representation

$$f(\lambda) = b\lambda + \int_0^\infty \frac{\lambda}{\lambda + y} \, \sigma(dy), \quad \lambda > 0. \tag{1.5.8}$$

Here b is as in eq. (2.4.1) and $\sigma(dy) = \hat{\sigma}(dy)/y$ such that $\int_{0+}^\infty (1+y)^{-1} \, \sigma(dy) < \infty$. The pair (b, σ) uniquely characterizes the complete Bernstein function f. Complete Bernstein functions have much better properties than general Bernstein functions, see [69].

Bochner's subordination. Subordination is a method to get new processes and semigroups from a given one. Let us begin with the functional analytic definition. We will denote by $(\mathcal{X}, \| \bullet \|)$ a Banach space.

Definition 1.5.3. Let S be a subordinator, f the corresponding Bernstein function, and denote by $\mu_t(dy) = \mathbb{P}(S_t \in dy)$ be the corresponding vaguely continuous convolution semigroup. For any strongly continuous semigroup $(P_t)_{t\geq 0}$ on $\mathcal{X}$ the following operators

$$P_t^f \phi := \int_0^\infty P_s \phi \, \mu_t(ds) \qquad \text{(Bochner integral)} \tag{1.5.9}$$

form again a strongly continuous semigroup. The semigroup $(P_t^f)_{t\geq 0}$ is *subordinate* (*in the sense of Bochner*) to $(P_t)_{t\geq 0}$.

The semigroup and strong continuity properties of $(P_t^f)_{t\geq 0}$ are obvious. Many other properties of $(P_t)_{t\geq 0}$ are inherited by the subordinate semigroup, for example (always with the proviso that the Banach space is such that the stated property makes sense!):

- Feller and strong Feller properties;
- positivity & Markov property;
- conservativeness;
- contraction property, and more general the type of the semigroup;
- analyticity, see Gomilko–Tomilov [26] and [25];

If $(P_t)_{t\geq 0}$ is the semigroup associated with a Markov process via $P_t\phi(x) = \mathbb{E}^x\phi(X_t)$, then $(P_t^f)_{t\geq 0}$ also belongs to a Markov process. Denote by $X^f = (X_t^f)_{t\geq 0}$ this Markov process. It is obtained by an independent stochastic time-change ("subordination") from the original process X. We assume that X, S are independent (otherwise we construct a new probability space such that the pair (X, S) is independent). Then

$$X_t^f(\omega) := X_{S_t(\omega)}(\omega), \quad \omega \in \Omega,\ t \geq 0. \tag{1.5.10}$$

It is not difficult to see that X^f is a Markov process. Moreover, by independence,

$$\begin{aligned}\mathbb{E}^x\phi(X_t^f) = \mathbb{E}^x\phi(X_{S_t}) &= \int_0^\infty \mathbb{E}^x\phi(X_s)\, \mathbb{P}(S_t \in ds) \\ &= \int_0^\infty P_s\phi(x)\, \mu_t(ds) = P_t^f \phi(x)\end{aligned}$$

in a pointwise sense, but this is enough to identify the Bochner integral. If X is a Lévy process with exponent ψ then X^f is a Lévy process with exponent $f\circ\psi$. The defining characteristics of a Lévy process are clearly preserved under subordination (here we use that S is both independent and itself Lévy), while the exponent is identified as follows:

$$\mathbb{E}e^{i\xi X_t^f} = \mathbb{E}e^{i\xi X_{S_t}} = \int_0^\infty \mathbb{E}e^{i\xi X_s}\, \mu_t(ds)$$

$$= \int_0^\infty e^{-s\psi(\xi)}\,\mu_t(ds) = \mathscr{L}[\mu_t;\psi(\xi)] = e^{-tf(\psi(\xi))}.$$

Since the subordinate semigroup is again strongly continuous, it makes sense to consider its infinitesimal generator. The following representation of the generator goes back to R.S. Phillips, cf. [69, p. 200 *et seq.*]

Theorem 1.5.4 (Phillips)**.** *Let $(P_t)_{t\geq 0}$ be a strongly continuous semigroup on X with generator $(\mathcal{A},\mathcal{D}(\mathcal{A}))$, and let f be a Bernstein function with characteristics (b,μ). Denote by $(P_t^f)_{t\geq 0}$ the subordinate semigroup. Its generator $(\mathcal{A}^f,\mathcal{D}(\mathcal{A}^f))$ has $\mathcal{D}(\mathcal{A})$ as an operator core and on $\mathcal{D}(\mathcal{A})$ it is of the form*

$$\mathcal{A}^f\phi = b\mathcal{A}\phi + \int_{0+}^\infty (P_s\phi-\phi)\,\mu(ds), \quad \phi\in\mathcal{D}(\mathcal{A}). \tag{1.5.11}$$

Example 1.5.5. Assume that P_t comes from a convolution semigroup (a Lévy process) i.e. $P_t\phi = \phi * \mu_t$, $\mu_t(dx) = \mathbb{P}(-X_t\in dt)$ where X_t is a Lévy process, then the following calculation establishes Theorem 3.4.(which holds for general C_0-semigroups).

Proof. According to the definition of a pseudo differential operator and the expression of the Fourier transform

$$P_t\phi(x) = \int \hat{\phi}(\xi)e^{i\xi x}e^{-t\psi(\xi)}\,d\xi$$

we have the following expression

$$\begin{aligned}
\mathcal{A}^f\phi = -f(\psi(D)) &= -\int \hat{\phi}(\xi)f(\psi(\xi))e^{ix\xi}\,d\xi\\
&= -b\int \hat{\phi}(\xi)e^{is\xi}\,d\xi - \int\hat{\phi}(\xi)\int_0^\infty(1-e^{-s\psi(\xi)})\mu(ds)e^{ix\xi}\,d\xi\\
&= b\mathcal{A}\phi - \int_0^\infty\int\hat{\phi}(\xi)(1-e^{-s\psi(\xi)})e^{ix\xi}\,d\xi\mu(ds)\\
&= b\mathcal{A}\phi - \int_0^\infty(\phi-P_s\phi)\,\mu(ds)\\
&= b\mathcal{A}\phi + \int_0^\infty(P_s\phi-\phi)\,\mu(ds),
\end{aligned}$$

where the fourth equality comes from the above expression of $P_t\phi$. □

If the Bernstein function f is a complete Bernstein function, we have the following result.

Theorem 1.5.6 (Hirsch; Berg Boyadzhiev deLaubenfels; Schilling)**.** *Let $(P_t)_{t\geq 0}$ be a strongly continuous semigroup on X with generator $(\mathcal{A},\mathcal{D}(\mathcal{A}))$ and resolvent $(R_\lambda)_{\lambda>0}$. Let f be a complete Bernstein function of the form* (1.5.8)*. Denote by $(P_t^f)_{t\geq 0}$ the subordinate* *semigroup. Its generator* $(\mathcal{A}^f, \mathcal{D}(\mathcal{A}^f))$ *has* $\mathcal{D}(\mathcal{A})$ *as an operator core and on* $\mathcal{D}(\mathcal{A})$ *it is of the form*

$$\mathcal{A}^f \phi = b\mathcal{A}\phi + \int_{0+}^{\infty} \mathcal{A}R_\lambda \phi \, \sigma(d\lambda), \quad \phi \in \mathcal{D}(\mathcal{A}). \tag{1.5.12}$$

Moreover, if $b = 0$, $\phi \in \mathcal{D}(\mathcal{A}^f)$ *if, and only if, the limit* $\lim_{k\to\infty} \int_{0+}^{k} \mathcal{A}R_\lambda \phi \, \sigma(d\lambda)$ *exists in the strong sense. If* $b \neq 0$, $\mathcal{D}(\mathcal{A}^f) = \mathcal{D}(\mathcal{A})$.

Proof. According to Theorem 1.5.4 (1.5.11) for all $\phi \in \mathcal{D}(\mathcal{A})$, we obtain,

$$\begin{aligned}
\mathcal{A}^f \phi &= b\mathcal{A}\phi + \int_{0+}^{\infty} (P_s\phi - \phi)\, \mu(ds) \\
&= b\mathcal{A}\phi + \int_{0+}^{\infty} (P_s\phi - \phi)\, \mathscr{L}[x\sigma(dx); s]\, ds \\
&= b\mathcal{A}\phi + \int_{0+}^{\infty} (P_s\phi - \phi) \int_{0}^{\infty} xe^{-sx}\, \sigma(dx)\, ds \\
&= b\mathcal{A}\phi + \int_{0}^{\infty} \int_{0+}^{\infty} (P_s\phi - \phi) xe^{-sx}\, ds\, \sigma(dx) \\
&= b\mathcal{A}\phi - \int_{0}^{\infty} \left[e^{-sx}(P_s\phi - \phi)|_0^\infty - \int_{0+}^{\infty} \frac{d(P_s\phi - \phi)}{ds} e^{-sx}\, ds \right] \sigma(dx) \\
&= b\mathcal{A}\phi + \int_{0}^{\infty} \int_{0+}^{\infty} \frac{d\int_0^s P_y \mathcal{A}\phi dy}{ds} e^{-sx} ds\, \sigma(dx) \\
&= b\mathcal{A}\phi + \int_{0}^{\infty} \int_{0+}^{\infty} \mathcal{A}P_s\phi e^{-sx}\, ds\, \sigma(dx) \\
&= b\mathcal{A}\phi + \int_{0}^{\infty} \mathcal{A}R_\lambda \phi\, \sigma(d\lambda),
\end{aligned}$$

where the second equality follows by using $\mu(ds) = \mathscr{L}[x\sigma(dx); s]\, ds$, using Fubini we will get the fourth equality and the last three equalities are obtained when we apply the following formula $P_s\phi - \phi = \int_0^s P_y \mathcal{A}\phi\, dy$. □

Next, we will discuss the definition of the potential measure of a Lévy process and a subordinator, as well as their associated properties that are connected to Chapter 3.

We begin with the definition of transience and recurrence for a Lévy process.

Definition 1.5.7. [6, P.32]

(i) A Lévy process is transient if the potential measures are Radon measures, that is, for every compact set K,

$$U(x, K) < \infty, \quad x \in \mathbb{R}^d.$$

(ii) A Lévy process is recurrent if $U(0, B) = \infty$ for every open ball B centered at the origin.

Theorem 1.5.8. [6, P.33]. *A Lévy process with characteristic exponent ψ is transient if, and only if, for some $r > 0$ small enough,*

$$\limsup_{\lambda \to 0} \int_{B_r} \operatorname{Re}\left(\frac{1}{\lambda + \psi(\xi)}\right) d\xi < \infty,$$

where B_r is the ball with radius $r > 0$ centered at the origin.

A subordinator is a transient Lévy process.

Definition 1.5.9. [7, P.10] The potential measure of a subordinator $U(dx)$ is called **renewal measure**. It is given by

$$\int_0^\infty \phi(x) U(dx) = \mathbb{E}\left(\int_0^\infty \phi(S_t) dt\right). \tag{1.5.13}$$

Remark 1.5.10. The distribution function of the renewal measure is given by

$$U(x) := U(0, x] = \mathbb{E}\left(\int_0^\infty 1_{\{S_t \leq x\}} dt\right). \tag{1.5.14}$$

Remark 1.5.11. By the Markov property the distribution function $U(x)$ of the potential measure is subadditive, that is

$$U(x + y) \leq U(x) + U(y), \quad \forall x, y \geq 0. \tag{1.5.15}$$

We denote by

$$E(x) = \inf\{t > 0 : S_t > x\}, \quad x \geq 0,$$

the generalized inverse of the subordinator S. In general $x \mapsto E(x)$ is only right continuous; If $t \mapsto S_t$ is almost surely strictly increasing (this is equivalent to S being not a compound Poisson process, i.e. the Lévy measure μ has infinite mass), then $x \mapsto E(x)$ is continuous. Moreover $\{E(x) \geq t\} = \{S_{t-} \leq x\}$.

Theorem 1.5.12. *Let S be a subordinator with the distribution function $U(x)$ of the potential measure and generalized inverse E. Then $U(x) = \mathbb{E}(E(x))$.*

Proof. Using Remark 1.5.10, we have

$$\begin{aligned} U(x) &= \mathbb{E}\left(\int_0^\infty \mathbb{1}_{\{S_t \leq x\}} dt\right) \\ &= \mathbb{E}\left(\int_0^\infty \mathbb{1}_{\{S_{t-} \leq x\}} dt\right) \\ &= \mathbb{E}\left(\int_0^\infty \mathbb{1}_{\{E(x) \geq t\}} dt\right) \\ &= \mathbb{E}(E(x)), \end{aligned}$$

here we use $S_t = S_{t-}$ Lebesgue a.e. and $\{S_{t-} \leq x\} = \{E(x) \geq t\}$. □

Theorem 1.5.13. *Suppose that U is the distribution function of the potential measure of a subordinator S with Laplace exponent f. Then we have the following result:*

$$\mathscr{L}[U,\lambda] = \frac{1}{f(\lambda)}. \tag{1.5.16}$$

Proof. According to the definition of Laplace transform of a measure, we will obtain,

$$\begin{aligned}
\mathscr{L}[U,\lambda] &= \int_0^\infty e^{-\lambda t} U(dt) \\
&\overset{(1.5.13)}{=} \mathbb{E}\left[\int_0^\infty e^{-\lambda S_t} dt\right] \\
&= \int_0^\infty \mathbb{E}[e^{-\lambda S_t}] dt \\
&= \int_0^\infty e^{-tf(\lambda)} dt \\
&= \frac{1}{f(\lambda)},
\end{aligned}$$

where the third equality is obtained by using Fubini, and the last two equality follow from the definition of the Laplace exponent. □

The relation $\{E_s \geq t\} = \{S_{t-} \leq s\}$ allows us to calculate the Laplace transform of $\mathbb{P}(E_s \geq t)$ with respect to s:

$$\begin{aligned}
\mathscr{L}\left[\mathbb{P}(E_\bullet \geq t)\right](\lambda) &= \mathscr{L}\left[\mathbb{P}(S_t \leq \bullet)\right](\lambda) \\
&= \int_0^\infty e^{-\lambda s}\, \mathbb{P}(S_t \leq s)\, ds \\
&= \frac{1}{\lambda}\int_0^\infty e^{-\lambda s}\, \mathbb{P}(S_t \in ds) = \frac{1}{\lambda} e^{-tf(\lambda)}.
\end{aligned} \tag{1.5.17}$$

Differentiating in t yields

$$\mathscr{L}\left[\mathbb{P}(E_\bullet \in dt)\right](\lambda) = \frac{f(\lambda)}{\lambda} e^{-tf(\lambda)}\, dt \tag{1.5.18}$$

and we get for the Laplace-transform of $\mathbb{P}(E_s \in dt)$ in both s and t the following formula

$$\begin{aligned}
\int_0^\infty e^{-\lambda s}\mathbb{E}e^{-xE_s}\, ds = \int_0^\infty \int_0^\infty e^{-\lambda s} e^{-xt}\, \mathbb{P}(E_s \in dt)\, ds &= \int_0^\infty \frac{f(\lambda)}{\lambda} e^{-tf(\lambda)} e^{-xt}\, dt \\
&= \frac{1}{\lambda}\frac{f(\lambda)}{x+f(\lambda)}.
\end{aligned} \tag{1.5.19}$$

Remark 1.5.14. (a) If f is a complete Bernstein function, i.e. $f \in \mathcal{CBF}$, see Chapter 2 section 3, then $\frac{f(\lambda)}{x+f(\lambda)} \in \mathcal{CBF}$ and $\lambda \mapsto \frac{1}{\lambda}\frac{f(\lambda)}{x+f(\lambda)}$ is a Stieltjes function, i.e. a double-Laplace transform; consequently, $\mathbb{E}e^{-xL_s}$ can be written as $\mathscr{L}[\rho_s](x)$ for some s-dependent probability measure ρ_s; here we use $\mathbb{E}e^{-xL_0} = 1$.

(b) If f is a Bernstein function, i.e. $f \in \mathcal{BF}$, see Chapter 2 section 3, then $\frac{f(\lambda)}{\lambda}$ is a completely monotone function, see Chapter 2 section 3. Since $\frac{1}{x+f(\lambda)} \in \mathcal{CM}$, we find that $\lambda \mapsto \frac{1}{\lambda}\frac{f(\lambda)}{x+f(\lambda)}$ is a completely monotone function.

(c) We can also directly work out the Laplace transform of $\mathbb{E}e^{-xE_s}$:

$$\mathbb{E}e^{-xE_s} = \int_0^\infty \mathbb{P}(E_s \le u)\, xe^{-ux}\, du = \int_0^\infty \mathbb{P}(S_u \ge s)\, xe^{-ux}\, du.$$

The Blumenthal-Getoor index is a fascinating topic of study as it allows us to characterize various path properties, such as the Hausdorff dimension and the asymptotic behavior of $\sup_{s\le t} |X_s|$, as well as the asymptotic behavior of absolute moments.

In the case where the Lévy process is finitely active, meaning it has only a finite number of jumps in any finite time interval, the Blumenthal-Getoor index is zero.

The Blumenthal-Getoor index serves as a measure of the local Hölder regularity of Lévy processes. Now, let's introduce the concept of the process index. For the detailed proofs of these results, you can refer to the work of Blumenthal and Getoor [10].

Definition 1.5.15. [10, P498] The index of Lévy process X with characteristic exponent ψ is defined by

$$\beta = \inf\left\{\rho > 0 \,:\, x^{-\rho}\psi(x) \to 0, |x| \to \infty\right\}$$

If X is Lévy process X with characteristic exponent ψ and it is normalized, i.e. there is no Gaussian part that means $Q = 0$. Then

$$\beta = \inf\left\{\rho > 0 \,:\, \int_{|x|<1} |x|^\rho \nu(dx) < \infty\right\}. \tag{1.5.20}$$

Clearly $\beta \in [0, 2]$ and it is easy to see that $\beta = \alpha$ if X is an α stable process. By definition, The Blumenthal-Getoor index measures the intensity of small jumps. Roughly speaking: if X, Y are Lévy process with Blumenthal-Getoor index β_X, β_Y, respectively, if $\beta_X > \beta_Y$, then X has more small jumps than Y.

Next we give a special index η for subordinator whose Laplace exponent is

$$f(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda x})\, \mu(dx),$$

$$\eta = \sup\left\{\rho \le 1 \,:\, \int_1^\infty \frac{x^{\rho-1}}{f(x)} dx\right\}.$$

Theorem 1.5.16. [10, P.504]. *If $(S_t)_{t\ge0}$ is a subordinator whose Laplace exponent is $f(\lambda)$, then*

(1) $\beta = \inf\{\rho \ge 0 \,:\, f(\lambda)\lambda^{-\rho} \to 0, \lambda \to \infty\}$

(2) $\eta = \sup\{\rho \geq 0 : f(\lambda)\lambda^{-\rho} \to \infty, \lambda \to \infty\}$

(3) $\eta \geq \sup\{\rho \geq 0 : x^{\rho}\mu[x,\infty) \to \infty, x \to 0\}$.

$(S_t)_{t\geq 0}$ is a subordinator, so the support of μ is contained in the half line $x \geq 0$ and

$$\int_0^1 x\mu(dx) < \infty. \tag{1.5.21}$$

Thus the index β of a subordinator satisfies $0 \leq \beta \leq 1$.

Theorem 1.5.17. *Let $f(\lambda)$ be an exponent of a subordinator S_t. If $f(\lambda)$ is submultiplicative for sufficiently large λ, then the index β can be calculated in the following way*

$$\beta = \beta_\infty := \lim_{\lambda\to\infty} \frac{\log f(\lambda)}{\log \lambda}. \tag{1.5.22}$$

Proof. According to the Theorem 1.3.7, $\lim_{\lambda\to\infty} \frac{\log f(\lambda)}{\log \lambda}$ exists. Next we show $\beta_\infty \in [0,1]$. According to the definition of $f(\lambda) = b\lambda + \int_0^\infty (1-e^{-\lambda x})\,\mu(dx)$ and using the inequality $1 - e^{-\lambda x} \leq \lambda x$ and $f(\lambda) \geq 0$ we have

$$\begin{aligned} f(\lambda) &= b\lambda + \int_0^\infty (1-e^{-\lambda x})\,\mu(dx) \\ &= b\lambda + \int_0^1 \lambda x\mu(dx) + \int_1^\infty (1-e^{-\lambda x})\,\mu(dx) \\ &\leq M_1\lambda + M_2, \end{aligned}$$

where $M_1 = b + \int_0^1 x\,\mu(dx)$, $M_2 = \mu[1,\infty)$ are finite. Furthermore, there exists M_3 such that for sufficient large $\lambda > 0$

$$\frac{\log f(\lambda)}{\log \lambda} \leq \frac{M_3 + \log\lambda}{\log\lambda} \leq \frac{M_3}{\log\lambda} + 1.$$

From this it is clear that $\beta_\infty \in [0,1]$. Next we prove $\beta = \beta_\infty$. If we can show that for $\rho > \beta_\infty$, it follows that $\lim_{\lambda\to\infty} \frac{f(\lambda)}{\lambda^\rho} = 0$ and for $\rho < \beta_\infty$, we have $\lim_{\lambda\to\infty} \frac{f(\lambda)}{\lambda^\rho} = \infty$, then we are done. In fact, let $h \in \mathbb{R}$, then

$$\begin{aligned} \frac{f(\lambda)}{\lambda^{\beta_\infty + h}} &= \exp\left[\log f(\lambda) - \beta_\infty \log\lambda - h\log\lambda\right] \\ &= \exp\left[\left(\frac{\log f(\lambda)}{\log\lambda} - \beta_\infty\right)\log\lambda - h\log\lambda\right]. \end{aligned}$$

By definition, $\frac{\log f(\lambda)}{\log \lambda} - \beta_\infty \to 0$. If $h > 0$ there exists some λ_h we have

$$\frac{\log f(\lambda)}{\log\lambda} - \beta_\infty < \frac{h}{2},$$

and so

$$\exp\left[\left(\frac{\log f(\lambda)}{\log\lambda}-\beta_\infty\right)\log\lambda-h\log\lambda\right]\le\exp\left[-\frac{h}{2}\log\lambda\right]\to 0.$$

A similar reason applies to $h<0$ and we get for sufficient large $\lambda>0$

$$\exp\left[\left(\frac{\log f(\lambda)}{\log\lambda}-\beta_\infty\right)\log\lambda-h\log\lambda\right]\ge\exp\left[-\frac{h}{2}\log\lambda\right]\to\infty.$$

□

Chapter 2

Fractional derivatives and integrals

First, in Section 2.1, we establish some basic notation and recall the standard definition of classical fractional derivatives and integrals. Following the introduction of the definition of classical fractional derivatives and integrals, Section 2.2 delves into their most essential properties: mapping properties. We prove that the α order fractional derivative maps the Sobolev space $W_0^{p,s}$ (see P.58) to the fractional Sobolev-Slobodeckij space $W^{p,s-\alpha}$ for all $\alpha < s < 1$.

In Section 2.3, we revisit the standard definition of Bernstein functions. Section 2.4 is dedicated to defining general fractional derivatives and integrals using Bernstein functions. Here, we aim to provide a method to unify all fractional derivatives using an extension through Marchaud fractional derivatives.

The final two sections of this chapter will discuss the probabilistic interpretation of fractional derivatives and establish connections with the fractional Laplace operator.

2.1 Classical fractional integrals and derivatives

In this section, we will provide some tools to investigate classical integrals and derivatives, starting with the Cauchy formula. The Riemann-Liouville fractional integral is derived from the Cauchy formula. Assuming ϕ is integrable in (a, t) and denoting $\phi^{(-1)}$ as the integral of ϕ, we have:

$$\phi^{(-1)}(t) = \int_a^t \phi(s)\, ds.$$

We can extend this concept further by considering iterated integrals. For example, the two-fold integral is given by:

$$\phi^{(-2)}(t) = \int_a^t \int_a^{s_1} \phi(s_2)\, ds_2\, ds_1 = \int_a^t (t-s)\phi(s)\, ds.$$

By induction, we obtain the general form of the Cauchy formula:

$$\phi^{(-n)}(t) = \frac{1}{\Gamma(n)} \int_a^t (t-s)^{n-1} \phi(s)\, ds. \tag{2.1.1}$$

If we replace the integer n in the Cauchy formula (2.1.1) with a real number α, we obtain an integral of arbitrary order. This leads us to the definition of the Riemann-Liouville fractional integral, denoted by ${}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha}_{\cdot}$, which is discussed in detail in works by Podlubny and Samko et al. [56, 64].

Definition 2.1.1 (Riemann–Liouville fractional integral)**.** The **Riemann–Liouville fractional integral** of order $\alpha \in (0,1)$ defined on $[0,x]$ and $[x,\infty)$, respectively, is given by

$$
\begin{aligned}
{}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha}_{x}\,\phi &= \int_0^x \phi(t) \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\, dt, \\
{}^{\mathrm{R}}_{x}\mathrm{I}^{\alpha}_{\infty}\,\phi &= \int_x^\infty \phi(t) \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\, dt.
\end{aligned}
$$

The Riemann–Liouville fractional integral of order $\alpha > 0$ on $[x,0]$ and $(-\infty,x]$, respectively, is

$$
\begin{aligned}
{}^{\mathrm{R}}_{x}\mathrm{I}^{\alpha}_{0}\,\phi &= \int_x^0 \phi(t) \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\, dt, \\
{}^{\mathrm{R}}_{-\infty}\mathrm{I}^{\alpha}_{x}\,\phi &= \int_{-\infty}^x \phi(t) \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\, dt.
\end{aligned}
$$

Next, we will provide the definition of the Riemann-Liouville fractional derivative. The concept of fractional derivative is closely connected to Abel's integral equation.

$$\frac{1}{\Gamma(\alpha)} \int_0^x \frac{\phi(t)}{(x-t)^{1-\alpha}} dt = g(x). \tag{2.1.2}$$

From the above equation

$$
\begin{aligned}
\int_0^x \frac{g(t)}{(x-t)^{\alpha}}\, dt &= \frac{1}{\Gamma(\alpha)} \int_0^x \int_0^t \frac{\phi(s)}{(t-s)^{1-\alpha}}\, ds\, \frac{dt}{(x-t)^{\alpha}} \\
&= \frac{1}{\Gamma(\alpha)} \int_0^x \int_s^x \frac{\phi(s)}{(t-s)^{1-\alpha}} \frac{dt}{(x-t)^{\alpha}}\, ds \\
&= \Gamma(1-\alpha) \int_0^x \phi(s)\, ds.
\end{aligned}
$$

This definition is based on the Cauchy formula for fractional integrals which we have discussed earlier. The Riemann–Liouville fractional derivative allows us to generalize the notion of differentiation to non-integer orders. It has various applications in physics, engineering, and applied mathematics, particularly in the study of systems with memory and non-local behavior, see Kilbas et al [37], Diethelm [14].

The connection between fractional derivatives and Abel's integral equation arises from the fact that the Riemann–Liouville fractional derivative can be used to solve certain types of integral equations, including Abel's integral equation. The details of this connection and further properties of fractional derivatives can be found in relevant literature on the topic, we refer reader Samko et al [64] and Samko, Cardoso [63]. Next we will give the definition of Riemann–Liouville fractional derivative. In this thesis we restrict ourself to $\alpha \in (0,1)$ since this is the only case needed in stochastics. To define the Riemann–Liouville fractional derivative, let ϕ be a function defined on (a,b).

Definition 2.1.2. The Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$ on $[0,x]$ and $[x,\infty)$, respectively, is defined by

$$
\begin{aligned}
{}_0^{\mathrm{R}}\mathrm{D}_x^{\alpha}\,\phi &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x \frac{\phi(t)}{(x-t)^{\alpha}}\,dt = \frac{d}{dx}\,{}_0^{\mathrm{R}}\mathrm{I}_x^{1-\alpha}\,\phi,\\
{}_x^{\mathrm{R}}\mathrm{D}_\infty^{\alpha}\,\phi &= -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^\infty \frac{\phi(t)}{(t-x)^{\alpha}}\,dt = \frac{d}{dx}\,{}_x^{\mathrm{R}}\mathrm{I}_\infty^{1-\alpha}\,\phi.
\end{aligned}
$$

The Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$ on $[x,0]$ and $(-\infty,x]$, respectively, is given by

$$
\begin{aligned}
{}_x^{\mathrm{R}}\mathrm{D}_0^{\alpha}\,\phi &= -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^0 \frac{\phi(t)}{(t-x)^{\alpha}}\,dt = \frac{d}{dx}\,{}_x^{\mathrm{R}}\mathrm{I}_0^{1-\alpha}\,\phi,\\
{}_{-\infty}^{\mathrm{R}}\mathrm{D}_x^{\alpha}\,\phi &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^x \frac{\phi(t)}{(x-t)^{\alpha}}\,dt = \frac{d}{dx}\,{}_{-\infty}^{\mathrm{R}}\mathrm{I}_x^{1-\alpha}\,\phi.
\end{aligned}
$$

Higher-order fractional derivatives are then defined via

$$
{}_0^{\mathrm{R}}\mathrm{D}_x^{\alpha+n}\,\phi := \frac{d^n}{dx^n}\,{}_0^{\mathrm{R}}\mathrm{D}_x^{\alpha}\,\phi \quad \text{and} \quad {}_x^{\mathrm{R}}\mathrm{D}_\infty^{\alpha+n}\,\phi := (-1)^n\frac{d^n}{dx^n}\,{}_x^{\mathrm{R}}\mathrm{D}_\infty^{\alpha}\,\phi.
$$

For $\alpha \in (0,1)$ and $n \in \mathbb{N}$.

The fractional derivative is indeed the fractional anti integral. Define the following operator as

$$
I_x^n\phi = \int_0^x\int_0^{x_1}..\int_0^{x_{n-1}} \phi(s)\,ds\,dx_{n-1}...\,dx_1, \quad I_x^0\phi = \phi,
$$

for the classical derivative and integral, we have

$$
\frac{d^n}{dx^n}I_x^n\phi = \phi \quad \text{but} \quad I_x^n\frac{d^n}{dx^n}\phi = \phi + p, \quad p \in \operatorname{span}\left\{1, x, \dots, x^{n-1}\right\};
$$

Futher, one can easily show the following relation, see, for example, Samko–Kilbas–Marichev [64, Theorem 2.4, P.45].

$$
{}_0^{\mathrm{R}}\mathrm{D}_x^{\alpha}\,{}_0^{\mathrm{R}}\mathrm{I}_x^{\alpha}\,\phi = \phi \quad \text{but} \quad {}_0^{\mathrm{R}}\mathrm{I}_x^{\alpha}\,{}_0^{\mathrm{R}}\mathrm{D}_x^{\alpha}\,\phi = \phi + p, \quad p \in \operatorname{span}\{x^{\alpha-k}, 1 \le k \le \lfloor\alpha\rfloor + 1\},
$$

where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \le x\}$.
The first relation holds if ϕ is integrable, the second formula needs that ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi$ is integrable. Let us prove the second relation in the case $\alpha \in (0,1)$. Assume that ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi$ is integrable. Then

$$
\begin{aligned}
&{}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha+1}_{x}\ {}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi \\
&= \frac{1}{\Gamma(\alpha+1)} \int_0^x (x-t)^{\alpha}\, {}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{t}\,\phi\, dt \\
&= \frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \int_0^x (x-t)^{\alpha} \left(\frac{d}{dt} \int_0^t \frac{\phi(s)}{(t-s)^{\alpha}} \right) ds\, dt \\
&\overset{\mathrm{PI}}{=} \frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \left[(x-t)^{\alpha} \int_0^t \frac{\phi(s)}{(t-s)^{\alpha}}\, ds \Big|_{t=0}^{x} + \int_0^x \alpha (x-t)^{\alpha-1} \int_0^t \frac{\phi(s)}{(t-s)^{\alpha}}\, ds\, dt \right] \\
&= \frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \left[-x^{\alpha} \int_0^t \frac{\phi(s)}{(t-s)^{\alpha}}\, ds \Big|_{t=0} + \alpha \int_0^x \int_s^x (x-t)^{\alpha-1}(t-s)^{-\alpha}\, dt\, \phi(s)\, ds \right] \\
&= -\,{}^{\mathrm{R}}_{0}\mathrm{I}^{1-\alpha}_{x}\,\phi \Big|_{x=0+} \cdot \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \int_0^x \int_0^{x-s} u^{\alpha-1}(x-s-u)^{-\alpha}\, du\, \phi(s)\, ds \\
&= -\,{}^{\mathrm{R}}_{0}\mathrm{I}^{1-\alpha}_{x}\,\phi \Big|_{x=0+} \cdot \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \int_0^x \int_0^1 v^{\alpha-1}(1-v)^{-\alpha}\, dv\, \phi(s)\, ds
\end{aligned}
$$

in the last two lines we used the following variable changes: $t = x - u$ and $u = v(x-s)$

$$
\begin{aligned}
&= -\,{}^{\mathrm{R}}_{0}\mathrm{I}^{1-\alpha}_{x}\,\phi \Big|_{x=0+} \cdot \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha B(\alpha, 1-\alpha)}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \int_0^x \phi(s)\, ds \\
&= -\,{}^{\mathrm{R}}_{0}\mathrm{I}^{1-\alpha}_{x}\,\phi \Big|_{x=0+} \cdot \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \int_0^x \phi(s)\, ds.
\end{aligned}
$$

Since – without restrictions – $\frac{d}{dx}\,{}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha+1}_{x} = {}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha}_{x}$, we get from the previous calculation

$$
{}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha}_{x}\ {}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi = \phi(x) - {}^{\mathrm{R}}_{0}\mathrm{I}^{1-\alpha}_{x}\,\phi \Big|_{x=0+} \cdot \frac{\alpha x^{\alpha-1}}{\Gamma(\alpha+1)} = \phi(x) - {}^{\mathrm{R}}_{0}\mathrm{I}^{1-\alpha}_{x}\,\phi \Big|_{x=0+} \cdot \frac{x^{\alpha-1}}{\Gamma(\alpha)}.
$$

Note that the integrability assumption on ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi$ ensures the finiteness of the ${}^{\mathrm{R}}_{0}\mathrm{I}^{1-\alpha}_{x}\,\phi\Big|_{x=0+}$ and the Riemann–Liouville fractional integrals and derivatives are (essentially) Laplace-convolution operators. A good reference for this kind of calculation is the book [56, P.70–71] by Podlubny.

Next, we will introduce the Caputo–type fractional derivative. The Caputo fractional derivative is another approach to generalize the concept of differentiation to non–integer orders. The Caputo–type fractional derivative is particularly useful when dealing with initial value problems, as it takes into account the initial condition(s) of the function being differentiated. It is widely used in fractional calculus and has applications in various fields, including physics, engineering, and finance, we refer the reader to Meerschaert et al [51], Baeumer and Meerschaert [3] and Eidelman et al [18].

It is important to note that there are differences between the Riemann–Liouville fractional derivative and the Caputo–type fractional derivative. The main distinction lies in their treatment of the initial conditions. The Caputo–type derivative considers the initial conditions of the function, while the Riemann–Liouville derivative does not.

The relationship between these two derivatives is a subject of study in fractional calculus. Several theorems have been established to establish the connection and equal between the Riemann–Liouville and Caputo–type fractional derivatives. These theorems provide conditions under which the derivatives are equal or related to each other. Details and proofs of these theorems can be found in the relevant literature on fractional calculus, we refer the reader to Samko et al [64], Kochubei et al [41].

Definition 2.1.3. Let $\phi : [0, \infty) \to \mathbb{R}$ be differentiable and $\alpha \in (0, 1)$. The **Caputo fractional derivative** of order $\alpha \in (0, 1)$ is defined as

$$ {}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}\, \phi = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{\phi'(t)}{(x-t)^\alpha}\, dt. \tag{2.1.3} $$

Again we restrict our attention to the case $\alpha \in (0, 1)$.

Theorem 2.1.4. *Let $\phi : [0, \infty) \to \mathbb{R}$ be differentiable and $\alpha \in (0, 1)$. Then*

$$ {}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\, \phi = \frac{1}{\Gamma(1-\alpha)} \frac{\phi(0)}{x^\alpha} + \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{\phi'(t)}{(x-t)^\alpha}\, dt = \frac{1}{\Gamma(1-\alpha)} \frac{\phi(0)}{x^\alpha} + {}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}\, \phi \tag{2.1.4} $$

Moreover, if ϕ is absolutely continuous, then

$$ {}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha}_{x}\, {}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}\, \phi = \phi(x) - \phi(0) \tag{2.1.5} $$

Proof. This follows from the usual differentiation rules for Riemann integrals:

$$ \begin{aligned} \Gamma(1-\alpha)\, {}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\, \phi &= \frac{d}{dx} \int_0^x \frac{\phi(t)}{(x-t)^\alpha}\, dt \\ &= \frac{d}{dx} \int_0^x \frac{\phi(x-s)}{s^\alpha}\, ds \\ &= \frac{\phi(0)}{x^\alpha} + \int_0^x \frac{d}{dx} \frac{\phi(x-s)}{s^\alpha}\, ds \\ &= \frac{\phi(0)}{x^\alpha} + \int_0^x \frac{\phi'(t)}{(x-t)^\alpha}\, dt. \end{aligned} $$

The formula for the composition of the Riemann–Liouville integral with the Caputo derivative follows with a direct calculation:

$$ \begin{aligned} {}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha}_{x}\, {}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}\, \phi \quad &= \quad \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, {}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{t}\, \phi\, dt \\ &= \quad \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^x \int_0^t \phi'(s)(t-s)^{-\alpha}\, ds\, (x-t)^{\alpha-1}\, dt \end{aligned} $$

$$\begin{aligned}
&\overset{\text{Fubini}}{=} \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^x \int_s^x (t-s)^{-\alpha}(x-t)^{\alpha-1}\, dt\, \phi'(s)\, ds \\
&\overset{t=x-u}{=} \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^x \int_0^{x-s} (x-s-u)^{-\alpha}u^{\alpha-1}\, du\, \phi'(s)\, ds \\
&\overset{u=(x-s)v}{=} \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^x \int_0^1 (1-v)^{-\alpha}v^{\alpha-1}\, dv\, \phi'(s)\, ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^x B(1-\alpha,\alpha)\phi'(s)\, ds \\
&= \int_0^x \phi'(s)\, ds = \phi(x) - \phi(0).
\end{aligned}$$

Since ϕ is absolutely continuous, ϕ' exists a.e.; the use of Fubini's theorem can be justified with Young's inequality for convolutions. □

Lemma 2.1.5 (and Definition). *Let $\phi : [0,\infty) \to \mathbb{R}$ be differentiable and $\alpha \in (0,1)$, then*

$$ {}^{\mathrm{R}}_{x}\mathrm{D}^{\alpha}_{\infty}\, \phi = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^\infty \frac{\phi(t)}{(t-x)^\alpha}\, dt = \frac{-1}{\Gamma(1-\alpha)} \int_x^\infty \frac{\phi'(t)}{(t-x)^\alpha}\, dt \tag{2.1.6}$$

*with the **Caputo fractional derivative** ${}^{\mathrm{C}}_{x}\mathrm{D}^{\alpha}_{\infty}\, \phi$ of order $\alpha \in (0,1)$, which is defined as*

$$ {}^{\mathrm{C}}_{x}\mathrm{D}^{\alpha}_{\infty}\, \phi = \frac{-1}{\Gamma(1-\alpha)} \int_x^\infty \frac{\phi'(t)}{(t-x)^\alpha}\, dt.$$

Proof. A direct calculation based on the definition of the Riemann–Liouville fractional derivative yields

$$\begin{aligned}
{}^{\mathrm{R}}_{x}\mathrm{D}^{\alpha}_{\infty}\, \phi &= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^\infty \frac{\phi(t)}{(t-x)^\alpha}\, dt \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^\infty \frac{\phi(u+x)}{u^\alpha}\, du \\
&= \frac{-1}{\Gamma(1-\alpha)} \int_0^\infty \frac{d}{dx} \frac{\phi(u+x)}{u^\alpha}\, du \\
&= \frac{-1}{\Gamma(1-\alpha)} \int_x^\infty \frac{\phi'(t)}{(x-t)^\alpha}\, dt.
\end{aligned}$$

□

Definition 2.1.6. Let $\phi : [0,\infty) \to \mathbb{R}$ be a function, then the **censored fractional derivative** of order α, $\alpha \in (0,1)$, is defined as

$$ {}^{\mathrm{Ce}}_{0}\mathrm{D}^{\alpha}_{x}\, \phi(x) = {}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\, \phi - \phi(x)\frac{x^{-\alpha}}{\Gamma(1-\alpha)}. \tag{2.1.7}$$

We will now see how we can treat the various fractional derivatives in a unified way. The Marchaud derivatives are a priorily defined on the whole real axis and are another type of fractional derivative that extends the concept of differentiation to non-integer orders. It was introduced by Pierre Marchaud in the early 20th century and has found applications in various areas of mathematics and physics, we refer the reader to Samko et al [64]. The Marchaud derivative differs from the Riemann–Liouville and Caputo derivatives since it does not explicitly use derivatives. It has some distinct properties and behaviors, which make it suitable for certain applications, especially in probability, see Hernández-Hernández and Kolokoltsov [31] and Kolokoltsov [44].

Definition 2.1.7 (Marchaud fractional derivatives)**.** Let $u : \mathbb{R} \to \mathbb{R}$ be a function which is defined on the whole real axis and $\alpha \in (0,1)$. The **Marchaud fractional derivatives** are the following operators

$$ {}^{\mathrm{M}}\mathrm{D}_{+}^{\alpha} u(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^{\infty} \frac{u(x) - u(x-t)}{t^{1+\alpha}}\, dt, \tag{2.1.8} $$

$$ {}^{\mathrm{M}}\mathrm{D}_{-}^{\alpha} u(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^{\infty} \frac{u(x) - u(x+t)}{t^{1+\alpha}}\, dt. \tag{2.1.9} $$

If necessary, the integrals $\int_{0+}^{\infty}$ may be interpreted as improper Riemann integrals, i.e. $\int_{0+}^{\infty} := \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{\infty}$.

Remark 2.1.8. A simple change of variables shows that the Marchaud derivatives can be written in the following way which is close to the definitions of Riemann–Liouville and Caputo derivatives (here our notation)

$$ {}^{\mathrm{M}}\mathrm{D}_{+}^{\alpha} u(x) = {}_{-\infty}^{\quad\mathrm{M}}\mathrm{D}_{x}^{\alpha} u = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{u(x) - u(t)}{(x-t)^{1+\alpha}}\, dt, $$

$$ {}^{\mathrm{M}}\mathrm{D}_{-}^{\alpha} u(x) = {}_{x}^{\mathrm{M}}\mathrm{D}_{\infty}^{\alpha} u = \frac{\alpha}{\Gamma(1-\alpha)} \int_{x}^{\infty} \frac{u(x) - u(t)}{(t-x)^{1+\alpha}}\, dt. $$

Let us now investigate Marchaud derivatives for functions which are only defined on the half-axis $[0,\infty)$. When comparing Marchaud derivatives with fractional derivatives on $[0,\infty)$, we have to extend functions defined on the half-axis onto $(-\infty, 0]$.

Definition 2.1.9 (Extension)**.** Let $\phi : [0,\infty) \to \mathbb{R}$. Then we can extend ϕ in the following way to become a function on $\mathbb{R}$

- 'Killing type' extension: $\phi^{0}(x) = \phi(x)\mathbb{1}_{[0,\infty)}(x) + 0 \cdot \mathbb{1}_{(-\infty,0)}(x)$.
- 'Sticky type' extension: $\phi^{\sigma}(x) := \phi(x)\mathbb{1}_{[0,\infty)}(x) + \phi(0)\mathbb{1}_{(-\infty,0)}(x)$.
- 'Even' extension: $\phi^{e}(x) := \phi(x)\mathbb{1}_{[0,\infty)}(x) + \phi(-x)\mathbb{1}_{(-\infty,0)}(x)$.

The corresponding function spaces are denoted by

- $C_{\infty}^{[0]}[0,\infty) := \{u : \mathbb{R} \to \mathbb{R} : u|_{[0,\infty)} \in C_{\infty}[0,\infty) \text{ and } u|_{(-\infty,0)} \equiv 0\}$
- $C_{\infty}^{[\sigma]}[0,\infty) := \{u : \mathbb{R} \to \mathbb{R} : u|_{[0,\infty)} \in C_{\infty}[0,\infty) \text{ and } u|_{(-\infty,0)} \equiv u(0)\}$

Next we continue to explore further properties of the Caputo fractional derivative and the Riemann Liouville fractional derivative.

Lemma 2.1.10. *Let* $\phi : [0, \infty) \to \mathbb{R}$ *be differentiable and* $\alpha \in (0, 1)$. *Then*

$$ {}_{0}^{\mathrm{C}}\mathrm{D}_{x}^{\alpha}\, \phi \overset{\text{def}}{=} \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{\phi'(t)}{(x-t)^{\alpha}}\, dt = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{\phi(t)-\phi(0)}{(x-t)^{\alpha}} dt. \tag{2.1.10} $$

Proof. Let's start from the right hand side

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{\phi(t)-\phi(0)}{(x-t)^{\alpha}} dt \\
&\overset{\text{t=xu}}{=} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^1 x^{1-\alpha}(1-u)^{-\alpha}(\phi(xu)-\phi(0))du \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-\alpha)x^{-\alpha}(1-u)^{-\alpha}(\phi(xu)-\phi(0))du + \int_0^1 x^{1-\alpha}(1-u)^{-\alpha}u\phi'(xu)du \\
&\overset{\text{PI}}{=} -\frac{1}{\Gamma(1-\alpha)} \left[x^{-\alpha}(\phi(xu)-\phi(0))(1-u)^{1-\alpha}\Big|_{u=0}^{u=1} - \int_0^1 x^{-\alpha}\phi'(xu)x(1-u)^{1-\alpha}du \right] \\
&\quad + \frac{1}{\Gamma(1-\alpha)} \int_0^1 x^{1-\alpha}(1-u)^{-\alpha}u\phi'(xu)du \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\int_0^1 x^{1-\alpha}\phi'(xu)(1-u)^{1-\alpha}(1-u)du + \int_0^1 x^{1-\alpha}(1-u)^{-\alpha}u\phi'(xu)du \right] \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^1 x^{1-\alpha}\phi'(xu)(1-u)^{-\alpha}du \\
&\overset{\text{t=xu}}{=} \frac{1}{\Gamma(1-\alpha)} \int_0^x \phi'(u)(x-t)^{-\alpha}dt \\
&\overset{\text{def}}{=} {}_{0}^{\mathrm{C}}\mathrm{D}_{x}^{\alpha}\, \phi(x).
\end{aligned}
$$

From the definition of the Caputo derivative, eq. (2.1.3), we obtain the above equality eq. (2.1.10). □

The connection between Riemann–Liouville, Caputo and Marchaud derivatives is as follows:

Lemma 2.1.11. *Let* $\alpha \in (0, 1)$ *and* $\phi : [0, \infty) \to \mathbb{R}$ *be a function.*

$$
\begin{aligned}
{}_{0}^{\mathrm{R}}\mathrm{D}_{x}^{\alpha}\, \phi &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{\phi^0(x)-\phi^0(x-s)}{s^{\alpha+1}}\, ds = {}^{\mathrm{M}}\mathrm{D}_{+}^{\alpha}\, \phi^0(x), \\
{}_{0}^{\mathrm{C}}\mathrm{D}_{x}^{\alpha}\, \phi &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{\phi^{\sigma}(x)-\phi^{\sigma}(x-s)}{s^{\alpha+1}}\, ds = {}^{\mathrm{M}}\mathrm{D}_{+}^{\alpha}\, \phi^{\sigma}(x) \\
{}_{0}^{\mathrm{C}}\mathrm{D}_{x}^{\alpha}\, \phi + {}_{x}^{\mathrm{C}}\mathrm{D}_{\infty}^{\alpha}\, \phi &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{\phi^{e}(x)-\phi^{e}(x-s)}{s^{\alpha+1}}\, ds = {}^{\mathrm{M}}\mathrm{D}_{+}^{\alpha}\, \phi^{e}(x).
\end{aligned}
$$

whereas the formula

$$ {}_{x}^{\mathrm{C}}\mathrm{D}_{\infty}^{\alpha}\, \phi = {}_{x}^{\mathrm{R}}\mathrm{D}_{\infty}^{\alpha}\, \phi = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{\phi(x)-\phi(x+s)}{s^{\alpha+1}}\, ds = {}^{\mathrm{M}}\mathrm{D}_{-}^{\alpha}\, \phi(x) $$

is valid for any extension of ϕ (as we do not need to extend ϕ!).

Proof. We begin with the integral expression in the middle of the first displayed formula. As usual, we assume that ϕ is as regular as we need it for the calculation below. Using the definition of the killing-type extension we see

$$\begin{aligned}
&\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{\phi^0(x)-\phi^0(x-s)}{s^{\alpha+1}}\,ds\\
&\quad=\frac{\alpha}{\Gamma(1-\alpha)}\left\{\int_0^x \frac{\phi(x)-\phi(x-s)}{s^{\alpha+1}}\,ds+\int_x^\infty \frac{\phi(x)-0}{s^{\alpha+1}}\,ds\right\}\\
&\quad=\frac{1}{\Gamma(1-\alpha)}\left\{\alpha\int_0^x\int_0^s \phi'(x-t)\,dt\frac{ds}{s^{\alpha+1}}+(\phi(x)-0)x^{-\alpha}\right\}\\
&\quad=\frac{1}{\Gamma(1-\alpha)}\left\{\int_0^x\int_t^x \frac{\alpha\,ds}{s^{\alpha+1}}\,\phi'(x-t)\,dt+(\phi(x)-0)x^{-\alpha}\right\}\\
&\quad=\frac{1}{\Gamma(1-\alpha)}\left\{\int_0^x t^{-\alpha}\,\phi'(x-t)\,dt-x^{-\alpha}\int_0^x \phi'(x-t)\,dt+(\phi(x)-0)x^{-\alpha}\right\}\\
&\quad=\frac{1}{\Gamma(1-\alpha)}\left\{\int_0^x \frac{\phi'(x-t)}{t^\alpha}\,dt-\big(\phi(x)-\phi(0)\big)x^{-\alpha}+(\phi(x)-0)x^{-\alpha}\right\}\\
&\quad=\frac{1}{\Gamma(1-\alpha)}\left\{\int_0^x \frac{\phi'(x-t)}{t^\alpha}\,dt+\frac{\phi(0)-0}{x^\alpha}\right\}\\
&\quad={}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi
\end{aligned}$$

where we use the formula (2.1.4) in the last equality.

Note that the artificial "-0" should remind of the fact that we use the killing extension ϕ^0. If we use the sticky extension $\phi^0 \rightsquigarrow \phi^\sigma$ and if we change throughout the above calculation $-0 \rightsquigarrow -\phi(0)$, then we get the second claimed formula for the Caputo derivative.

Let us now consider the even extension ϕ^e. Note that

$$\begin{aligned}
\int_x^\infty \frac{\phi(x)-\phi(s-x)}{s^{\alpha+1}}\,ds &= -\int_x^\infty \left(\int_x^s \phi'(t-x)dt+(\phi(0)-\phi(x))\right)s^{-\alpha-1}\,ds\\
&=-\int_x^\infty\int_x^s \phi'(t-x)dt\,s^{-\alpha-1}\,ds-\frac{\phi(0)-\phi(x)}{\alpha x^\alpha}\\
&=-\int_x^\infty\int_t^\infty s^{-\alpha-1}\,ds\,\phi'(t-x)dt-\frac{\phi(0)-\phi(x)}{\alpha x^\alpha}\\
&=\frac{1}{\alpha}\left\{-\int_x^\infty \frac{\phi'(t-x)}{t^\alpha}dt+\frac{\phi(x)-\phi(0)}{x^\alpha}\right\}.
\end{aligned}$$

The first calculation of the present proof shows, in particular,

$$\frac{\alpha}{\Gamma(1-\alpha)}\int_0^x \frac{\phi(x)-\phi(x-s)}{s^{\alpha+1}}\,ds=\frac{1}{\Gamma(1-\alpha)}\left\{\int_0^x \frac{\phi'(x-t)}{t^\alpha}\,dt-\frac{\phi(x)-\phi(0)}{x^\alpha}\right\}.$$

Thus,

$$\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{\phi^e(x)-\phi^e(x-s)}{s^{\alpha+1}}\,ds$$

$$
\begin{aligned}
&= \frac{\alpha}{\Gamma(1-\alpha)}\left\{\int_0^x \frac{\phi(x)-\phi(x-s)}{s^{\alpha+1}}\,ds + \int_x^\infty \frac{\phi(x)-\phi(s-x)}{s^{\alpha+1}}\,ds\right\}\\
&= \frac{1}{\Gamma(1-\alpha)}\left\{\int_0^x \frac{\phi'(x-t)}{t^\alpha}\,dt - \int_x^\infty \frac{\phi'(t-x)}{t^\alpha}dt\right\}\\
&\underset{(2.1.6)}{\overset{(2.1.3)}{=}} {}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi + {}^{\mathrm{C}}_{x}\mathrm{D}^{\alpha}_{\infty}\,\phi.
\end{aligned}
$$

For the fourth formula we use either the very same calculation or we prove it directly:

$$
\begin{aligned}
{}^{\mathrm{C}}_{x}\mathrm{D}^{\alpha}_{\infty}\,\phi \overset{(2.1.6)}{=} {}^{\mathrm{R}}_{x}\mathrm{D}^{\alpha}_{\infty}\,\phi &= -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^\infty \frac{\phi(t)}{(t-x)^\alpha}\,dt\\
&= -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^\infty \frac{\phi(t+x)}{t^\alpha}\,dt\\
&= -\frac{1}{\Gamma(1-\alpha)}\int_0^\infty \frac{\phi'(t+x)}{t^\alpha}\,dt\\
&= -\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \phi'(t+x)\int_t^\infty \frac{ds}{s^{\alpha+1}}\,dt\\
&= -\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \int_0^s \phi'(t+x)\,dt\frac{ds}{s^{\alpha+1}}\\
&= -\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \big(\phi(s+x)-\phi(x)\big)\frac{ds}{s^{\alpha+1}}\\
&= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{\phi(x)-\phi(s+x)}{s^{\alpha+1}}\,ds\\
&= {}^{\mathrm{M}}\mathrm{D}^{\alpha}_{-}\,\phi.
\end{aligned}
$$

Observe that the last calculation also works in the first case. Indeed,

$$
\begin{aligned}
{}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x \frac{\phi(t)}{(x-t)^\alpha}\,dt\\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x \frac{\phi(x-t)}{t^\alpha}\,dt\\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^\infty \frac{\phi^0(x-t)}{t^\alpha}\,dt\\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^\infty \frac{(\phi^0)'(x-t)}{t^\alpha}\,dt\\
&= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \int_0^s (\phi^0)'(x-t)\,dt\frac{ds}{s^{\alpha+1}}\\
&= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \big(\phi^0(x)-\phi^0(x-s)\big)\frac{ds}{s^{\alpha+1}}\\
&= {}^{\mathrm{M}}\mathrm{D}^{\alpha}_{+}\,\phi^0.
\end{aligned}
$$

Since the reference meansure ds has no atom at $s=0$, there is no problem when considering ϕ' resp. $(\phi^0)'$ at $t=0$. □

The first part of the proof gives, in fact, the following alternative form of the Riemann–Liouville, Caputo and censored derivatives.

Corollary 2.1.12. *Let* $\alpha \in (0,1)$. *Then*

$$ {}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^x \frac{\phi(x)-\phi(x-s)}{s^{\alpha+1}}\,ds + \frac{\phi(x)}{\Gamma(1-\alpha)x^{\alpha}}; \tag{2.1.11}$$

$$ {}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^x \frac{\phi(x)-\phi(x-s)}{s^{\alpha+1}}\,ds + \frac{\phi(x)-\phi(0)}{\Gamma(1-\alpha)x^{\alpha}}; \tag{2.1.12}$$

$$ {}^{\mathrm{Ce}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^x \frac{\phi(x)-\phi(x-s)}{s^{\alpha+1}}\,ds. \tag{2.1.13}$$

We note that there are other natural possibilities, e.g. odd extensions etc. Now, let us consider a non–trivial example.

Example 2.1.13. For a function $u : [0.\infty) \to \mathbb{R}$, the so–called censored derivative, defined in Definition 2.1.6 and Corollary 2.1.12, is

$$\frac{\alpha}{\Gamma(1-\alpha)}\int_{0+}^x \frac{u(x)-u(x-t)}{t^{1+\alpha}}\,dt.$$

If we compare it with the Marchaud derivative, we need an extension

$$\tilde{u}(x) = \begin{cases} u(x), & x \geq 0, \\ u(0) = v(0), & \text{else}, \\ v(x), & x < 0. \end{cases} \tag{2.1.14}$$

such that

$$\frac{\alpha}{\Gamma(1-\alpha)}\int_x^\infty \frac{\tilde{u}(x)-\tilde{u}(x-t)}{t^{1+\alpha}}\,dt = \frac{\alpha}{\Gamma(1-\alpha)}\int_x^\infty \frac{u(x)-v(x-t)}{t^{1+\alpha}}\,dt = 0$$

for all $x > 0$. By using the change of variables $-y = x - t$, we obtain

$$\frac{u(x)}{\alpha x^{\alpha}} = \int_0^\infty \frac{v(y)}{y^{\alpha}}\left(\frac{1}{(\frac{x}{y}+1)^{1+\alpha}}\right)\frac{dy}{y}.$$

By multiplying both sides with αx^{α}, we get a Mellin convolution (defined by $\circledast$)

$$u(x) = \alpha\int_0^\infty v(y)\left(\frac{(\frac{x}{y})^{\alpha}}{(\frac{x}{y}+1)^{1+\alpha}}\right)\frac{dy}{y} =: \alpha v \circledast g(x), \tag{2.1.15}$$

where $g(r) := r^{\alpha}(1+r)^{-\alpha-1}$.

Recall that the **Mellin transform** of a function $\phi : [0,\infty) \to \mathbb{R}$ is defined by

$$\mathscr{M}(\phi; s) = \int_0^\infty x^{s-1}\phi(x)\,dx, \quad s \in \mathbb{C}.$$

If $x^{c-1}\phi(x) \in L^1(0,\infty)$ for all $c \in (a,b)$ and some $a, b \in \mathbb{R}$, then $\mathscr{M}(\phi; s)$ exists and it is analytic in the strip $a < \operatorname{Re} s < b$. If ϕ is continuous and $t \mapsto \mathscr{M}(\phi; c+it)$ is integrable in $(-\infty,\infty)$, then we have an inverse transform,

$$\phi(x) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} x^{-z}\mathscr{M}(\phi; z)\, dz.$$

Taking the Mellin transform on both sides of eq. (2.1.15) we have

$$\mathscr{M}u(z) = \alpha\mathscr{M}v(z)\mathscr{M}g(z),$$

for all $z \in \mathbb{C}$, where the Mellin transforms of v and g are well-defined. Calculating the Mellin transform of g, we obtain that

$$\mathscr{M}g(z) = B(1-z, \alpha+z) \text{ for all } -\alpha < \operatorname{Re}(z) < 1.$$

Formally, we obtain that v is given by

$$v(x) = \mathscr{M}^{-1}\left(\frac{\mathscr{M}u}{\alpha\mathscr{M}g}\right)(x) = \mathscr{M}^{-1}\left(\frac{\mathscr{M}u}{\alpha B(1-\cdot, \alpha+\cdot)}\right)(x).$$

If we assume for some $x \in (-\alpha, 1)$, that

$$y \mapsto \frac{\mathscr{M}u(x+iy)}{\alpha B(1-x-iy, \alpha+x+iy)} \in L^1(\mathbb{R}),$$

then the above inverse Mellin transform is well defined i.e.

$$v(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-z}\frac{\mathscr{M}u(z)}{\alpha B(1-z, \alpha+z)}\, dz < \infty \tag{2.1.16}$$

with $-\alpha < c < 1$. This gives us a idea to extend Marchaud fractional derivative to get furthermore other fractional derivatives.

The Fourier transform gives us a unique way of viewing a function. Here we define the Laplace transform and the Fourier transform of the fractional integral and derivative. Denote that S is the space of rapidly decreasing function on $\mathbb{R}$ and S' the space of tempered distribution on $\mathbb{R}$, see e.g. [61].

Theorem 2.1.14. *Let $\phi \in L^1(\mathbb{R})$ and $\alpha \in (0,1)$. The Fourier transform of the fractional integral*

$$\mathscr{F}\left({}_{-\infty}^{\;\;\mathrm{R}}\mathrm{I}_x^{\alpha}\,\phi; \xi\right) = \frac{\mathscr{F}\phi(\xi)}{(i\xi)^{\alpha}} \tag{2.1.17}$$

and

$$\mathscr{F}\left({}_{x}^{\mathrm{R}}\mathrm{I}_{\infty}^{\alpha}\,\phi; \xi\right) = \frac{\mathscr{F}\phi(\xi)}{(-i\xi)^{\alpha}} \tag{2.1.18}$$

Proof. Let us use distributions in the sense of L.Schwartz. We have

$$t_+^{\alpha-1} = \begin{cases} t^{\alpha-1}, & t > 0; \\ 0, & t \le 0. \end{cases} \tag{2.1.19}$$

And in the sense of S'

$$\mathscr{F}(t_+^{\alpha-1}; \xi) = \frac{\Gamma(\alpha)}{(i\xi)^\alpha}.$$

Now we have for $\alpha \in (0, 1)$,

$$\begin{aligned} {}_{-\infty}^{\ \ \mathrm{R}}\mathrm{I}_x^\alpha \phi &= \int_{-\infty}^x \phi(t) \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\, dt \\ &= \int_0^\infty \phi(x-t) \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}\, dt \\ &= \int_{-\infty}^\infty \phi(x-t) \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}\, dt \\ &= \left(\frac{t_+^{\alpha-1}}{\Gamma(\alpha)} * \phi \right)(\xi). \end{aligned}$$

then we have

$$\mathscr{F}\left({}_{-\infty}^{\ \ \mathrm{R}}\mathrm{I}_x^\alpha \phi; \xi \right) = \mathscr{F}\left(\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}; \xi \right) \mathscr{F}(\phi, \xi); \quad \mathscr{F}(\phi; \xi) = \frac{\mathscr{F}(\phi; \xi)}{(i\xi)^\alpha}.$$

Define

$$t_-^{\alpha-1} = \begin{cases} 0, & t \ge 0; \\ (-t)^{\alpha-1}, & t < 0. \end{cases} \tag{2.1.20}$$

In the sense of S'

$$\mathscr{F}(t_-^{\alpha-1}; \xi) = \frac{\Gamma(\alpha)}{(-i\xi)^\alpha}.$$

$$\begin{aligned} {}_x^{\mathrm{R}}\mathrm{I}_\infty^\alpha \phi &= \int_x^\infty \phi(t) \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\, dt \\ &\overset{t=x+u}{=} \int_0^\infty \phi(x+u) \frac{u^{\alpha-1}}{\Gamma(\alpha)}\, du \\ &\overset{u=-t}{=} \int_{-\infty}^0 \phi(x-u) \frac{(-t)^{\alpha-1}}{\Gamma(\alpha)}\, dt \\ &= \int_{-\infty}^\infty \phi(x-u) \frac{t_-^{\alpha-1}}{\Gamma(\alpha)}\, dt \\ &= \left(\frac{t_-^{\alpha-1}}{\Gamma(\alpha)} * \phi \right)(\xi). \end{aligned}$$

then we have

$$\mathscr{F}({}_x^{\mathrm{R}}\mathrm{I}_\infty^\alpha \phi; \xi) = \mathscr{F}(\frac{t_-^{\alpha-1}}{\Gamma(\alpha)}; \xi)\mathscr{F}(\phi; \xi); \quad \mathscr{F}({}_x^{\mathrm{R}}\mathrm{I}_\infty^\alpha \phi; \xi) = \frac{\mathscr{F}(\phi; \xi)}{(-i\xi)^\alpha}.$$ □

Theorem 2.1.15. *The Fourier transforms of fractional derivatives are given by*

$$\mathscr{F}({}_{-\infty}^{\mathrm{R}}\mathrm{D}_x^{\alpha}\, \phi; \xi) = \mathscr{F}\phi(\xi)(i\xi)^{\alpha} \tag{2.1.21}$$

and

$$\mathscr{F}({}_{x}^{\mathrm{R}}\mathrm{D}_{\infty}^{\alpha}\, \phi; \xi) = \mathscr{F}\phi(\xi)(-i\xi)^{\alpha} \tag{2.1.22}$$

Proof. Let us check eq. (2.1.21).

$$\begin{aligned} \mathscr{F}({}_{-\infty}^{\mathrm{R}}\mathrm{D}_x^{\alpha}\, \phi; \xi) &= \frac{1}{\Gamma(1-\alpha)}\mathscr{F}(\frac{d}{dx}\, {}_{-\infty}^{\mathrm{R}}\mathrm{I}_x^{1-\alpha}\, \phi; \xi) \\ &= -\frac{i\xi}{\Gamma(1-\alpha)}\mathscr{F}({}_{-\infty}^{\mathrm{R}}\mathrm{I}_x^{1-\alpha}\, \phi; \xi) \\ &\overset{(2.1.17)}{=} -\frac{i\xi}{\Gamma(1-\alpha)}\frac{\Gamma(1-\alpha)\mathscr{F}\phi(\xi)}{(i\xi)^{1-\alpha}} \\ &= \mathscr{F}\phi(\xi)(i\xi)^{\alpha} \end{aligned}$$

By using the same argument, we get eq. (2.1.22). □

Theorem 2.1.16 (Laplace transform of the fractional derivative & integral)**.** *Let* $\alpha \in (0,1)$, $\phi : [0,\infty) \to \mathbb{R}$ *and* $\phi \in L^1[0,\infty)$. *We have*

$$\begin{aligned} \mathscr{L}({}_{0}^{\mathrm{R}}\mathrm{D}_x^{\alpha}\, \phi; s) &= s^{\alpha}\mathscr{L}\phi(s), \\ \mathscr{L}({}_{0}^{\mathrm{C}}\mathrm{D}_x^{\alpha}\, \phi; s) &= s^{\alpha}\mathscr{L}\phi(s) - \phi(0)s^{\alpha-1}, \\ \mathscr{L}({}_{0}^{\mathrm{R}}\mathrm{I}_x^{\alpha}\, \phi; s) &= s^{-\alpha}\mathscr{L}\phi(s), \end{aligned}$$

Proof. We observe that ${}_{0}^{\mathrm{R}}\mathrm{D}_x^{\alpha}\, \phi = \frac{d}{dx}\left(\phi * \frac{(\bullet)^{-\alpha}}{\Gamma(1-\alpha)}\right)$ is (the derivative of) a Laplace convolution. With the rules for the Laplace transform from Section 1.2 we get

$$\begin{aligned} \mathscr{L}({}_{0}^{\mathrm{R}}\mathrm{D}_x^{\alpha}\, \phi; s) &= \mathscr{L}\left(\frac{d}{dx}\left(\phi * \frac{(\bullet)^{-\alpha}}{\Gamma(1-\alpha)}\right); s\right) \\ &\overset{(1.2.10)}{=} s\mathscr{L}\left(\phi * \frac{(\bullet)^{-\alpha}}{\Gamma(1-\alpha)}; s\right) \\ &\overset{(1.2.8)}{=} s\mathscr{L}(\phi; s)\frac{1}{\Gamma(1-\alpha)}\mathscr{L}\left[x^{-\alpha}; s\right) \\ &\overset{(1.2.12)}{=} s\mathscr{L}(\phi; s)s^{\alpha-1} = s^{\alpha}\mathscr{L}\phi(s). \end{aligned}$$

For the Caputo derivative we find in a similar way

$$\begin{aligned} \mathscr{L}({}_{0}^{\mathrm{C}}\mathrm{D}_x^{\alpha}\, \phi; s) &\overset{(2.1.3)}{=} \mathscr{L}\left(\phi' * \frac{(\bullet)^{-\alpha}}{\Gamma(1-\alpha)}; s\right) \\ &\overset{(1.2.8)}{=} \mathscr{L}(\phi'; s)\frac{1}{\Gamma(1-\alpha)}\mathscr{L}\left(x^{-\alpha}; s\right) \end{aligned}$$

$$
\begin{aligned}
&\overset{(1.2.10)}{\underset{(1.2.12)}{=}} (s\mathscr{L}(\phi;s) - \phi(0))\, s^{\alpha-1} \\
&= \quad s^{\alpha}\mathscr{L}\phi(s) - \phi(0)s^{\alpha-1}.
\end{aligned}
$$

Finally we calculate the Laplace transfrom of the fractional integral if $\alpha \in (0,1)$:

$$
\begin{aligned}
\mathscr{L}({}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha}_{x}\,\phi; s) &= \mathscr{L}\left(\left(\phi * \frac{(\bullet)^{\alpha-1}}{\Gamma(r)}\right); s\right) \\
&\overset{(1.2.8)}{=} \mathscr{L}(\phi; s)\frac{1}{\Gamma(r)}\mathscr{L}\left[x^{\alpha-1}; s\right) \\
&\overset{(1.2.12)}{=} \mathscr{L}(\phi; s)\, s^{-\alpha}.
\end{aligned}
$$

□

Remark 2.1.17. The result from the previous lemma shows that

$$
\mathscr{L}({}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi; s) \underset{\alpha\uparrow 1}{\longrightarrow} s\mathscr{L}\phi(s) = \mathscr{L}(\phi'; s) + \phi(0)
$$

and

$$
\mathscr{L}({}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi; s) \underset{\alpha\uparrow 1}{\longrightarrow} s\mathscr{L}\phi(s) - \phi(0) = \mathscr{L}(\phi'; s).
$$

This is a further indication that the Riemann–Liouville derivative is – up to a boundary term at $s = 0$ – the generalization of the usual derivative. The Caputo derivative has the boundary term included.

Here is a calculation which indicates why the last remark on the Caputo derivative should be correct. We assume that $\phi \in C^2_b[0,\infty)$. Consider

$$
{}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi - c_\alpha\phi' + c_\alpha\phi' - \phi',
$$

with c_α defined as

$$
\begin{aligned}
c_\alpha^{-1} &= \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{1}{(x-t)^\alpha}\,dt \overset{\text{t=ux}}{\underset{\text{dt = x du}}{=}} \frac{1}{\Gamma(1-\alpha)}\int_0^1 \frac{x}{x^\alpha(1-u)^\alpha}\,du \\
&= \frac{1}{x^{\alpha-1}}\frac{1}{\Gamma(1-\alpha)}\int_0^1 v^{-\alpha}\,dv = \frac{1}{x^{\alpha-1}}\frac{1}{\Gamma(1-\alpha)}\left[\frac{v^{1-\alpha}}{1-\alpha}\right]_{v=0}^1 \\
&= \frac{1}{x^{\alpha-1}}\frac{1}{(1-\alpha)\Gamma(1-\alpha)} = \frac{1}{x^{\alpha-1}}\frac{1}{\Gamma(2-\alpha)} \underset{\alpha\uparrow 1}{\longrightarrow} 1.
\end{aligned}
$$

Our choice of c_α shows that it is enough to look at

$$
\begin{aligned}
&\left|\frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{\phi'(t)-\phi'(x)}{(x-t)^\alpha}\,dt\right| \\
&\overset{\text{Taylor}}{\leq} \frac{1}{\Gamma(1-\alpha)}\|\phi''\|_\infty \int_0^x \frac{|x-t|}{(x-t)^\alpha}\,dt \\
&= \frac{1}{\Gamma(1-\alpha)}\|\phi''\|_\infty \int_0^x (x-t)^{1-\alpha}\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)}\|\phi''\|_\infty \left[-\frac{(x-t)^{2-\alpha}}{2-\alpha}\right]_{t=0}^{t=x} \\
&= \frac{1}{\Gamma(1-\alpha)}\|\phi''\|_\infty \frac{x^{2-\alpha}}{2-\alpha}\frac{1-\alpha}{1-\alpha} \\
&= (1-\alpha)\frac{1}{(1-\alpha)\Gamma(1-\alpha)}\|\phi''\|_\infty \frac{x^{2-\alpha}}{2-\alpha} \\
&= (1-\alpha)\frac{1}{\Gamma(2-\alpha)}\|\phi''\|_\infty \frac{x^{2-\alpha}}{2-\alpha} \xrightarrow[\alpha\uparrow 1]{} 0.
\end{aligned}
$$

Remark 2.1.18. Here are some interesting explicit fractional derivatives and integrals. As usual, $\alpha \in (0,1)$:

1. ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\mathbb{1} = \dfrac{x^{-\alpha}}{\Gamma(1-\alpha)}$ and ${}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}\,\mathbb{1} = 0$.

2. ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}[x^p] = {}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}[x^p] = \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}x^{p-\alpha}$ with $p>0$, cf. [56, P.72].

3. $$
\begin{aligned}
{}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}\,e^{-sx} &= \sum_{p=0}^{\infty} {}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}[x^p]\frac{(-s)^p}{\Gamma(p+1)} \\
&= \sum_{p=1}^{\infty} \frac{(-s)^p x^{p-\alpha}}{\Gamma(p+1-\alpha)} \\
&= x^{-\alpha}\sum_{p=1}^{\infty} \frac{(-sx)^p}{\Gamma(p+1-\alpha)}
\end{aligned}
$$

   Since the term for $p=0$ is treated differently by the RL-derivative, we see with essentially the same calculation that

   $$ {}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,e^{-sx} = x^{-\alpha}\sum_{p=0}^{\infty} \frac{(-sx)^p}{\Gamma(p+1-\alpha)}. $$

4. Eigenfunctions of the fractional derivative (Meerschaert–Sikorskii [52, (2.28)]). Define the Mittag–Leffler function E_β

   $$ E_\beta(y) := \sum_{p=0}^{\infty} \frac{y^p}{\Gamma(1+\beta p)} \quad \text{and} \quad \phi_\beta(x) := E_\beta(sx^\beta). $$

   Then we have for $\beta \in (0,1)$ and $s>0$

   $$ {}^{\mathrm{C}}_{0}\mathrm{D}^{\beta}_{x}\,\phi_\beta = s\phi_\beta(x). $$

5. ${}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha}_{x}[x^{-p}] = \dfrac{\Gamma(1-p)}{\Gamma(\alpha+1-p)}x^{-p+\alpha}$. Indeed, we find by a direct calculation

   $$ {}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha}_{x}[x^{-p}] = \frac{1}{\Gamma(\alpha)}\int_0^x t^{-p}(x-t)^{\alpha-1}\,dt \underset{dt\,=\,x\,du}{\overset{t\,=\,ux}{=}} \frac{1}{\Gamma(\alpha)}\int_0^1 (xu)^{-p}(x-xu)^{\alpha-1}\,x\,du $$

$$
\begin{aligned}
&= \frac{x^{-p+\alpha}}{\Gamma(\alpha)} \int_0^1 u^{-p}(1-u)^{\alpha-1}\, du \overset{(1.2.13)}{=} \frac{B(1-p,\alpha)}{\Gamma(\alpha)} x^{-p+\alpha} \\
&= \frac{\Gamma(1-p)}{\Gamma(\alpha+1-p)} x^{-p+\alpha}
\end{aligned}
$$

with $p < 1$. In particular for $p = \alpha$, we have ${}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha}_{x}[x^{-\alpha}] = \Gamma(1-\alpha)$.

As usual, denote by A^* the formal L^2 adjoint (conjugate) operator; here we work in the space $L^2([0,\infty), dx)$.

Theorem 2.1.19 (formal adjoint; integration by parts)**.** *Let* $\phi \in L^2[0,\infty)$. *We have for* $\alpha \in (0,1)$

$$
({}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x})^* \phi = {}^{\mathrm{R}}_{x}\mathrm{D}^{\alpha}_{\infty}\, \phi. \tag{2.1.23}
$$

Proof. Assume that ϕ, g are smooth, integrable and with compact support, so that all calculations below are justified. Since such functions are dense in $L^2[0,\infty)$, this is no restriction at all. We have

$$
\begin{aligned}
&\int_0^\infty {}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\, \phi \cdot g(x)\, dx \\
&\quad= \int_0^\infty \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \phi(x-t)\, t^{-\alpha}\, dt\, g(x)\, dx \\
&\quad\overset{\mathrm{PI}}{=} \frac{1}{\Gamma(1-\alpha)} \left[\int_0^x \phi(x-t)\, t^{-\alpha}\, dt\, g(x) \Big|_{x=0}^{\infty} - \int_0^\infty \int_0^x \phi(x-t)\, t^{-\alpha} \cdot g'(x)\, dt\, dx \right] \\
&\quad= -\frac{1}{\Gamma(1-\alpha)} \int_0^\infty \int_0^x \phi(x-t)\, t^{-\alpha} \cdot g'(x)\, dt\, dx \\
&\quad\overset{\mathrm{Fubini}}{=} -\frac{1}{\Gamma(1-\alpha)} \int_0^\infty \int_t^\infty \phi(x-t) g'(x)\, dx\, t^{-\alpha}\, dt \\
&\quad= -\frac{1}{\Gamma(1-\alpha)} \int_0^\infty \int_0^\infty \phi(x) g'(x+t)\, dx\, t^{-\alpha}\, dt \\
&\quad\overset{\mathrm{Fubini}}{=} -\frac{1}{\Gamma(1-\alpha)} \int_0^\infty \phi(x) \int_0^\infty g'(x+t)\, t^{-\alpha}\, dt\, dx \\
&\quad= -\frac{1}{\Gamma(1-\alpha)} \int_0^\infty \phi(x) \frac{d}{dx} \int_0^\infty g(x+t)\, t^{-\alpha}\, dt\, dx \\
&\quad= -\frac{1}{\Gamma(1-\alpha)} \int_0^\infty \phi(x) \frac{d}{dx} \int_x^\infty g(s)(s-x)^{-\alpha}\, ds\, dx \\
&\quad= \int_0^\infty \phi(x)\, {}^{\mathrm{R}}_{x}\mathrm{D}^{\alpha}_{\infty}\, g(x)\, dx.
\end{aligned}
$$

□

If $\alpha \in (0,1)$, then we have in the space $L^2([0,\infty), dx)$

$$
{}^{\mathrm{C}}_{0}\mathrm{D}^{\alpha}_{x}\, \phi = ({}^{\mathrm{R}}_{x}\mathrm{D}^{\alpha}_{\infty})^* \phi(x) - \frac{1}{\Gamma(1-\alpha)} \frac{\phi(0)}{x^\alpha}. \tag{2.1.24}
$$

Hardy and Littlewood made significant contributions to the study of mapping properties for the Riemann-Liouville fractional integral ${}^{\mathrm{R}}_{0}\mathrm{I}^{\alpha}_{\cdot}$. Their results primarily focus on the integrability properties of functions and the continuity properties of functions. The following two statements are known as the **Hardy-Littlewood theorem**. We refer the reader to [64, P.66, P.103]. The first case considers functions given on the whole axis: If $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, the fractional integration operator ${}^{\mathrm{R}}_{a}\mathrm{I}^{\alpha}_{\cdot} : L^p \to L^q$ with $q = \frac{p}{1-\alpha p}$. The second case considers functions on the half axis or the axis and gives the following: Let $1 \le p \le \infty$, $1 \le q \le \infty$, $\alpha > 0$. The operators ${}^{\mathrm{R}}_{-\infty}\mathrm{I}^{\alpha}_{\cdot} : L^p(\mathbb{R}) \to L^q(\mathbb{R})$ and ${}^{\mathrm{R}}_{\cdot}\mathrm{I}^{\alpha}_{\infty} : L^p(\mathbb{R}) \to L^q(\mathbb{R})$ if, and only if, $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $q = \frac{p}{1-\alpha p}$.

Theorem 2.1.20. *Let $-\infty \le a < b \le \infty$. If $\phi(x) \in L^p([a,b])$, $g(x) \in L^q([a,b])$ with $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$, $p, q \ge 1$ and $q, p \ne 1$, for $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$, then the following formula holds*

$$\int_a^b \phi(x)\, {}^{\mathrm{R}}_{a}\mathrm{I}^{\alpha}_{x}\, g(x)\, dx = \int_a^b g(x)\, {}^{\mathrm{R}}_{x}\mathrm{I}^{\alpha}_{b}\, \phi(x)\, dx.$$

Proof. The situation $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$ can be reduced to $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$. The reason is that $L^r[a,b] \hookrightarrow L^s[a,b]$ if $r > s$, for the detail, see [64, P.67]. Without loss of generality, let $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$, we look at the left hand side of the above formula $\int_a^b \phi(x)\, {}^{\mathrm{R}}_{a}\mathrm{I}^{\alpha}_{x}\, g(x)\, dx$, and we use the assumptions $\phi \in L^p[a,b]$, $g \in L^q[a,b]$. Using the results of the Hardy-Littlewood theorem, we have ${}^{\mathrm{R}}_{a}\mathrm{I}^{\alpha}_{x}\, g \in L^{\frac{q}{1-\alpha q}}[a,b]$. Now use Hölder with $r, s \ge 1$ where we pick $r = p$, $s = \frac{q}{1-\alpha q}$ to get $\frac{1}{r} + \frac{1}{s} = 1$, so r and s are conjugate. We get

$$\int_a^b \phi(x)\, {}^{\mathrm{R}}_{a}\mathrm{I}^{\alpha}_{x}\, g(x)\, dx \le \|\phi\|_{L^r} \|\, {}^{\mathrm{R}}_{a}\mathrm{I}^{\alpha}_{\cdot}\, g\|_{L^s}.$$

We are allowed to do the following calculation,

$$\begin{aligned}
\int_a^b \phi(x)\, {}^{\mathrm{R}}_{a}\mathrm{I}^{\alpha}_{x}\, g(x)\, dx &= \frac{1}{\Gamma(\alpha)} \int_a^b \phi(x) \int_a^x (x-y)^{\alpha-1} g(y)\, dy\, dx \\
&\overset{\text{Fubini}}{=} \frac{1}{\Gamma(\alpha)} \int_a^b g(y) \int_y^b (x-y)^{\alpha-1} \phi(x)\, dx\, dy \\
&= \int_a^b g(x)\, {}^{\mathrm{R}}_{x}\mathrm{I}^{\alpha}_{b}\, \phi(x)\, dx.
\end{aligned}$$

□

Theorem 2.1.21. *Let S_t be a subordinator with characteristics (b, μ) and the generator of S_t is given by*

$$\mathcal{A}\phi(x) = b\phi'(x) + \int_0^\infty [\phi(x+y) - \phi(x)]\, \mu(dy)$$

for all $\phi \in C_c^\infty(0,\infty)$ and with $\phi(x) = 0, x \in (-\infty, 0]$. Denote by $\mathcal{A}^*$ the adjoint of $\mathcal{A}$ in $L^2((0,\infty), dy)$. It is given by

$$\mathcal{A}^*\phi(x) = -b\phi'(x) + \int_0^x [\phi(x-y) - \phi(x)]\, \mu(dy) - \mu[x,\infty)\phi(x).$$

Proof. Denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2([0,\infty), dy)$. We show first that $\mathcal{A}$ and $\mathcal{A}^*$ map $C_c^\infty(0,\infty)$ into $L^2([0,\infty), dy)$. Using a Taylor expansion we get for $\phi \in C_c^\infty(0,\infty)$

$$\mathcal{A}\phi(x) = b\phi'(x) + \int_0^1 \int_0^1 \phi'(x+ty)\, y\, dt\, \mu(dy) + \int_1^\infty [\phi(x+y) - \phi(x)]\, \mu(dy).$$

Now

$$\|\mathcal{A}\phi\|_{L^2} \leq b\|\phi'(x)\|_{L^2} + \int_0^1 \int_0^1 \|\phi'(\cdot + ty)\|_{L^2}\, dt\, y\, \mu(dy) + \int_1^\infty \left(\|\phi(\cdot + y)\|_{L^2} + \|\phi\|_{L^2} \right) \mu(dy).$$

Here we use the fact that we extend $\phi \in C_c^\infty$ continuous by 0 onto $\mathbb{R}$ and that

$$\left\| \int_0^\infty h(x,y)\, \nu(dy) \right\|_{L^2(dx)} \leq \int_0^\infty \|h(x,y)\|_{L^2}\, \nu(dy).$$

So

$$\|\mathcal{A}\phi\|_{L^2} \leq b\|\phi'(x)\|_{L^2} + \|\phi'\|_{L^2} \int_0^1 y\, \mu(dy) + 2\|\phi\|_{L^2} \mu[1,\infty).$$

Let us now estimate $\mathcal{A}^* g$ for $g \in C_c^\infty(0,\infty)$

$$\begin{aligned}
\mathcal{A}^* g(x) &= -bg'(x) + \int_0^x [g(x-y) - g(x)]\, \mu(dy) - \mu[x,\infty) g(x) \\
&= -bg'(x) + \int_0^{x \wedge 1} [g(x-y) - g(x)]\, \mu(dy) + \int_{x \wedge 1}^x [g(x-y) - g(x)]\, \mu(dy) \\
&\quad - \mu[x,\infty) g(x),
\end{aligned}$$

So

$$\begin{aligned}
\|\mathcal{A}^* g\|_{L^2} &\leq b\|g'\|_{L^2(dx)} + \left\| \int_0^{x\wedge 1} [g(x-y) - g(x)]\, \mu(dy) \right\|_{L^2(dx)} \\
&\quad + \left\| \int_{x \wedge 1}^x [g(x-y) - g(x)]\, \mu(dy) \right\|_{L^2(dx)} + \|\mu[x,\infty) g\|_{L^2(dx)} \\
&\leq b\|g'\|_{L^2(dx)} + \left\| \int_0^1 [g(x-y) - g(x)]\, \mu(dy) \right\|_{L^2(dx)} \\
&\quad + \left\| \int_1^\infty [g(x-y) - g(x)]\, \mu(dy) \right\|_{L^2(dx)} + \|\mu[x,\infty) g\|_{L^2(dx)} \\
&\leq b\|g'\|_{L^2(dx)} + \|g'\|_{L^2(dx)} \int_0^1 y\, \mu(dy) + 2\|g\|_{L^2} \mu[1,\infty) + \|\mu[x,\infty) g\|_{L^2(dx)}.
\end{aligned}$$

The last inequality follows similarly as in the estimate for $\mathcal{A}$. We have to assume now $g(0) = 0$ and this is clear since $g \in C_c^\infty(0,\infty)$. This gives for the last term

$$\begin{aligned}
\|\mu[x,\infty) g\|_{L^2(dx)} &= \|\mu[x,\infty)(g(x) - g(0))\|_{L^2(dx)} \\
&= \left\| x\mu[x,\infty) \int_0^1 g'(tx)\, dt \right\|_{L^2(dx)}
\end{aligned}$$

$$
\begin{aligned}
&\leq c_\mu \left\| \int_0^1 \left(\int_0^\infty (g'(tx))^2 \, dx \right)^{1/2} dt \right\|_{L^2(dx)} \\
&= c_\mu \|g'\|_{L^2(dx)} \int_0^1 \frac{1}{\sqrt{t}} \, dt \\
&= C_\mu \|g'\|_{L^2(dx)};
\end{aligned}
$$

in the first inequality we use Jensen and $x\mu[x,\infty) \leq c_\mu$ for all $x \in (0,1)$. This follows from

$$
\begin{aligned}
\infty = \int_0^1 x \, \mu(dx) = -x\mu[x,\infty)\Big|_{x=0+}^{x=1} + \int_0^1 \mu[y,\infty) \, dy \\
\geq \lim_{x\to 0} x\mu[x,\infty) - \mu[x,\infty).
\end{aligned}
$$

So $\sup_{x\in(0,1)} x\mu[x,\infty) \leq c_\mu < \infty$. Now, we have for $\phi, g \in C_c^\infty(0,\infty)$,

$$
\begin{aligned}
&\langle \mathcal{A}\phi(x), g(x) \rangle \\
&= \int_0^\infty \mathcal{A}\phi(x) g(x) \, dx \\
&= \int_0^\infty \left[b\phi'(x) + \int_{(0,\infty)} [\phi(x+y) - \phi(x)] \mu(dy) \right] g(x) \, dx \\
&= \int_0^\infty b\phi'(x) g(x) \, dx + \int_0^\infty \int_0^\infty [\phi(x+y) - \phi(x)] \; \mu(dy) g(x) \, dx \\
&\overset{\text{Fubini}}{=} -b \int_0^\infty \phi(x) g'(x) \, dx + \int_0^\infty \int_0^\infty [\phi(x+y) - \phi(x)] \; \mu(dy) g(x) \, dx \\
&= -b \int_0^\infty \phi(x) g'(x) \, dx + \lim_{\varepsilon\to 0} \left[\int_0^\infty \int_\varepsilon^\infty \phi(x+y) \, \mu(dy) g(x) \, dx \right. \\
&\quad \left. - \int_0^\infty \int_\varepsilon^\infty \phi(x) \, \mu(dy) g(x) \, dx \right] \\
&\overset{\text{x+y=u}}{=} -b \int_0^\infty \phi(x) g'(x) \, dx + \lim_{\varepsilon\to 0} \left[\int_\varepsilon^\infty \int_0^\infty \phi(u) g(u-y) \, du \, \mu(dy) \right. \\
&\quad \left. - \int_\varepsilon^\infty \int_0^\infty \phi(x) g(x) \, dx \, \mu(dy) \right] \\
&= -b \int_0^\infty \phi(x) g'(x) \, dx + \lim_{\varepsilon\to 0} \left[\int_\varepsilon^\infty \int_y^\infty \phi(x) g(x-y) \, dx \, \mu(dy) \right. \\
&\quad \left. - \int_\varepsilon^\infty \int_y^\infty \phi(x) g(x) \, dx \, \mu(dy) - \int_\varepsilon^\infty \int_0^y \phi(x) g(x) \, dx \, \mu(dy) \right] \\
&= -b \int_0^\infty \phi(x) g'(x) \, dx + \lim_{\varepsilon\to 0} \left[\int_\varepsilon^\infty \int_y^\infty \phi(x) \left(g(x-y) - g(x) \right) \, dx \, \mu(dy) \right. \\
&\quad \left. - \int_\varepsilon^\infty \int_0^y \phi(x) g(x) \, dx \, \mu(dy) \right] \\
&= -b \int_0^\infty \phi(x) g'(x) \, dx + \lim_{\varepsilon\to 0} \left[\int_\varepsilon^\infty \int_\varepsilon^x \left(g(x-y) - g(x) \right) \; \mu(dy) \phi(x) \, dx \right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \int_\varepsilon^\infty \int_0^y \phi(x)g(x)\,dx\,\mu(dy)\Bigg] \\
&= -b\int_0^\infty \phi(x)g'(x)\,dx + \lim_{\varepsilon\to 0}\left[\int_\varepsilon^\infty \int_\varepsilon^x (g(x-y)-g(x))\ \mu(dy)\phi(x)\,dx\right] \\
&\quad - \int_0^\infty \int_x^\infty \mu(dy)\phi(x)g(x)\,dx \\
&= \langle \phi(x), \mathcal{A}^* g(x)\rangle.
\end{aligned}
$$

Note that all limits exist because of our estimates on $\mathcal{A}\phi$ and $\mathcal{A}^*g$. If we use the "killing" extension, we see that

$$
\mathcal{A}^* g(x) = -bg'(x) + \int_0^\infty [g^{(0)}(x-y) - g^{(0)}(x)]\mu(dy)
$$

where $g^{(0)}(x) = g(x)$ when $x \geq 0$ and $g^{(0)}(x) = 0$ when $x < 0$. □

Lemma 2.1.22. *In the space $L^2(\mathbb{R})$ we have $({}^{\mathrm{M}}\mathrm{D}_+^\alpha)^* = {}^{\mathrm{M}}\mathrm{D}_-^\alpha$.*

Proof. Let $u, v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be as regular as we need for the following calculation.

$$
\begin{aligned}
&\int_{-\infty}^\infty {}^{\mathrm{M}}\mathrm{D}_+^\alpha u(x)\cdot v(x)\,dx \\
&= \int_{-\infty}^\infty \frac{\alpha}{\Gamma(1-\alpha)}\int_{0+}^\infty \frac{u(x)-u(x-t)}{t^{1+\alpha}}\,dt\cdot v(x)\,dx \\
&= \frac{\alpha}{\Gamma(1-\alpha)}\lim_{\epsilon\downarrow 0}\int_{-\infty}^\infty\int_\epsilon^\infty \frac{u(x)-u(x-t)}{t^{1+\alpha}}\,dt\cdot v(x)\,dx \\
&\overset{\text{Fubini}}{=} \frac{\alpha}{\Gamma(1-\alpha)}\lim_{\epsilon\downarrow 0}\left[\int_{-\infty}^\infty\int_\epsilon^\infty \frac{u(x)v(x)}{t^{1+\alpha}}\,dt\,dx - \int_\epsilon^\infty\int_{-\infty}^\infty \frac{u(x-t)v(x)}{t^{1+\alpha}}\,dx\,dt\right] \\
&\underset{\text{in 2nd } \int}{\overset{x \rightsquigarrow x+t}{=}} \frac{\alpha}{\Gamma(1-\alpha)}\lim_{\epsilon\downarrow 0}\left[\int_{-\infty}^\infty\int_\epsilon^\infty \frac{u(x)v(x)}{t^{1+\alpha}}\,dt\,dx - \int_\epsilon^\infty\int_{-\infty}^\infty \frac{u(x)v(x+t)}{t^{1+\alpha}}\,dx\,dt\right] \\
&\overset{\text{Fubini}}{=} \frac{\alpha}{\Gamma(1-\alpha)}\lim_{\epsilon\downarrow 0}\left[\int_{-\infty}^\infty\int_\epsilon^\infty \frac{u(x)v(x)}{t^{1+\alpha}}\,dt\,dx - \int_{-\infty}^\infty\int_\epsilon^\infty \frac{u(x)v(x+t)}{t^{1+\alpha}}\,dt\,dx\right] \\
&= \lim_{\epsilon\downarrow 0}\int_{-\infty}^\infty u(x)\frac{\alpha}{\Gamma(1-\alpha)}\int_\epsilon^\infty \frac{v(x)-v(x+t)}{t^{1+\alpha}}\,dt\,dx \\
&= \int_{-\infty}^\infty u(x)\frac{\alpha}{\Gamma(1-\alpha)}\int_{0+}^\infty \frac{v(x)-v(x+t)}{t^{1+\alpha}}\,dt\,dx \\
&= \int_{-\infty}^\infty u(x)\,{}^{\mathrm{M}}\mathrm{D}_-^\alpha v(x)\,dx.
\end{aligned}
$$

□

Meerschaert–Sikorskii call the Marchaud representation the **generator form** of the fractional derivative. We have, in fact, straight from the definition of the Marchaud derivative:

Corollary 2.1.23. *Let S be an α-stable subordinator – cf. Example 1.5.2 – with $0 < \alpha < 1$. Then its generator is $-{}^{\mathrm{M}}\mathrm{D}_-^\alpha$.*

Recall the alternative notation ${}^{\mathrm{M}}\mathrm{D}_{+}^{\alpha}\, u = {}_{-\infty}^{\ \ \mathrm{M}}\mathrm{D}_{x}^{\alpha}\, u$, which we introduced in the connection of different fractional derivatives.

Corollary 2.1.24. *Let u be differentiable with killing extension and $\alpha \in (0, 1)$. By Lemma 2.1.5, Definition 2.1.7 and Lemma 2.1.11 , then*

$$\begin{aligned} {}_{-\infty}^{\ \ \mathrm{M}}\mathrm{D}_{x}^{\alpha}\, u &= {}_{-\infty}^{\ \ \mathrm{R}}\mathrm{D}_{x}^{\alpha}\, u = {}_{-\infty}^{\ \ \mathrm{C}}\mathrm{D}_{x}^{\alpha}\, u = {}_{0}^{\mathrm{R}}\mathrm{D}_{x}^{\alpha}\, u \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^x \frac{u(x) - u(x-t)}{t^{1+\alpha}}\, dt + u(x) \frac{x^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned} \tag{2.1.25}$$

2.2 Mapping properties of fractional integrals and derivatives

The mapping properties of various forms of fractional integration operators in known function spaces is of great interest. Fractional integration shares some properties with classical integrals, enhancing the properties of functions in certain ways. On the other hand, fractional differentiation often deteriorates the properties of functions. When we discuss mapping properties, we refer to the following question: given a specific function space $\mathcal{X}$, can we characterize ${}_{0}^{\mathrm{R}}\mathrm{I}_{\cdot}^{\alpha}\, \mathcal{X} = \{ {}_{0}^{\mathrm{R}}\mathrm{I}_{\cdot}^{\alpha}\, \phi, \phi \in \mathcal{X}\}$ as a subspace of another function space $\mathcal{Y}$.

It is important to understand the increased regularity provided by fractional integrals ${}_{0}^{\mathrm{R}}\mathrm{I}_{\cdot}^{\alpha}\, \phi$ compared to the original function $\phi \in L^p$. The following statements highlight this relationship:

Theorem 2.2.1. [5, 12, P.940, P.25]. *Let* $0 < \alpha < 1$. *Then*

(i) $1 \le p < \infty$, ${}_{a}^{\mathrm{R}}\mathrm{I}_{\cdot}^{\alpha} : L^p(a, b) \to L^p(a, b)$.

(ii) $p = \infty$, ${}_{a}^{\mathrm{R}}\mathrm{I}_{\cdot}^{\alpha} : L^\infty(a, b) \to C^\alpha[a, b]$.

These statements demonstrate that the fractional integrals enhance the regularity of functions compared to their original space in L^p or in the space of continuous functions. Based on the previous theorem, we observe that fractional integrals preserve the space $L^p(a, b)$. If we consider the case when $p = 1$, we can derive additional information: Let ${}_{0}^{\mathrm{R}}\mathrm{I}_{\cdot}^{\alpha}(L^1(0, b))$, where $0 < \alpha < 1$, denote the range of fractional integrals. Then, for a function ϕ, we have $\phi \in {}_{0}^{\mathrm{R}}\mathrm{I}_{\cdot}^{\alpha}(L^1)$ if and only if ${}_{0}^{\mathrm{R}}\mathrm{I}_{\cdot}^{1-\alpha}\, \phi \in AC((0, b])$, where $AC((0, b])$ represents the space of absolutely continuous functions on the interval $(0, b]$.

This statement establishes a connection between fractional integrals in L^1 and the absolute continuity of the corresponding functions. Here we denote that $\beta = m + \alpha$, $0 < \alpha < 1$, $p \ge 0$ and $M > 0$ is a constant.

$$C^{\beta,p}(\Omega) = \left\{ \phi \in C^m(\Omega) : \left| \phi^{(m)}(x+y) - \phi^{(m)}(x) \right| \le M\, |y|^\alpha \left(\log \frac{1}{|y|} \right)^p, |y| \le 0.5 \right\},$$

and its norm

$$\|\phi\|_{C^{\beta,p}} = \|\phi\|_{C^m} + \sup_{x,x+y\in\Omega,|y|\le 0.5} \frac{\phi^{(m)}(x+y)-\phi^{(m)}(x)}{|y|^\alpha \left(\log \frac{1}{|y|}\right)^p}.$$

Theorem 2.2.2. *Let $\alpha > 0, p > \frac{1}{\alpha}$, then the fractional integral operator satisfies:*

$${}_{a}^{\mathrm{R}}\mathrm{I}_{\cdot}^{\alpha} : L^p(a,b) \to C^{\alpha-\frac{1}{p}}(a,b), \quad \alpha - \frac{1}{p} \notin \mathbb{Z}_+,$$

$${}_{a}^{\mathrm{R}}\mathrm{I}_{\cdot}^{\alpha} : L^p(a,b) \to C^{\alpha-\frac{1}{p},\frac{1}{p'}}(a,b), \quad \alpha - \frac{1}{p} \in \mathbb{Z}_+, \quad \frac{1}{p} + \frac{1}{p'} = 1;$$

and

$${}_{a}^{\mathrm{R}}\mathrm{I}_{\cdot}^{\alpha}\, \phi = o((x-a)^{\alpha-\frac{1}{p}}), \quad x \to a. \tag{2.2.1}$$

Proof. We get the above results (2.2.1) by the Hölder inequality: Let $p, p' \ge 1$ such that $1/p + 1/p' = 1$, then there exists a constant M, such that

$$\begin{aligned}
|\,{}_{a}^{\mathrm{R}}\mathrm{I}_{x}^{\alpha}\, \phi| &= \left|\frac{1}{\Gamma(\alpha)} \int_a^x \phi(t)(x-t)^{\alpha-1}\, dt\right| \\
&\le \frac{1}{\Gamma(\alpha)} \left(\int_a^x |\phi(t)|^p\, dt\right)^{\frac{1}{p}} \left(\int_a^x (x-t)^{(\alpha-p)p'}\, dt\right)^{\frac{1}{p'}} \\
&\le M(x-a)^{\alpha-\frac{1}{p}} \left(\int_a^x |\phi(t)|^p\, dt\right)^{\frac{1}{p}}.
\end{aligned}$$

For the other two estimates cf. Theorem 3.6 [64, P67]. □

Recall Hölder space $\beta = m + r,\ 0 < r < 1,\ M > 0$ is a constant and $C^m(\Omega)$ is m times continuous differentiable functions on Ω.

$$C^\beta(\Omega) = \left\{\phi \in C^m(\Omega) : \left|\phi^{(m)}(x+y) - \phi^{(m)}(x)\right| \le M\,|y|^r\right\},$$

and its norm

$$\|\phi\|_{C^\beta} = \|\phi\|_{C^m} + \sup_{x,x+y\in\Omega} \frac{\phi^{(m)}(x+y)-\phi^{(m)}(x)}{|y|^r}.$$

The next results focuses on the mapping properties of fractional integrals in Hölder spaces. Specifically, the fractional integration of order α enhances the Hölder exponent β by α units. cf [64, P.53].

Theorem 2.2.3. *Let $\phi(x) \in C^\beta[0,b]$, $0 \le \beta \le 1$, $0 < \alpha < 1$. Then the fractional integral ${}_{0}^{\mathrm{R}}\mathrm{I}_{\cdot}^{\alpha}\, \phi$ has the form*

$${}_{0}^{\mathrm{R}}\mathrm{I}_{x}^{\alpha}\, \phi = \frac{\phi(0)}{\Gamma(1+\alpha)} x^\alpha + \psi(x), \tag{2.2.2}$$

where $\psi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\phi(t)-\phi(0)}{(x-t)^{1-\alpha}} dt$, $\psi \in C^{\beta+\alpha}[0,b]$ *if* $\beta + \alpha \neq 1$; *and* $\psi \in C^{\beta+\alpha,1}[0,b]$ *if* $\beta + \alpha = 1$. *Moreover the estimate*

$$| \psi(x) | \leq C x^{\beta+\alpha}, \tag{2.2.3}$$

holds.

Let $\beta, \alpha \in (0,1)$. The operator

$$\phi \mapsto \Psi\phi, \quad \Psi\phi(x) = \psi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\phi(t) - \phi(0)}{(x-t)^{1-\alpha}} dt$$

is a bounded as operator $\Psi : C^{\beta}[0,b] \to C^{\beta+\alpha}[0,b]$, if $\beta + \alpha \neq 1$; and if $\beta + \alpha = 1$, $\Psi : C^{\beta}[0,b] \to C^{\beta+\alpha,1}[0,b]$.

If we consider the weighted space

$$C^{\beta}_{\infty}([0,b], w) = \left\{\phi : w(x)\phi(x) \in C^{\beta}[0,b] \quad \text{and} \quad w(0)\phi(0) = w(b)\phi(b) = 0\right\},$$

where $w : [0,b] \to [0,\infty)$ is a weight function. Usually, we consider $w(x)$ to be a power function. Consider the operator ${}^{R}_{0}\mathrm{I}^{\alpha}_{\cdot} : C^{\beta}([0,b], w) \to C^{\beta+\alpha}([0,b], w)$, where $\beta + \alpha < 1$. If we have $w(x) = x^p$ with $p < \beta + 1$, then the operator is well-defined. cf [64, P.57]

Next, we will discuss the regularizing properties of the fractional integral when it acts on functions in the Sobolev space $W^{p,k}(0,T)$, where $1 \leq p < \infty$ and $1 \leq k < \infty$. Let $1 \leq p < \infty$ and $\alpha \in (0,1)$ such that $\alpha p < 1$. Carbotti et al [12, P.46] show that ${}^{R}_{0}\mathrm{I}^{1-\alpha}_{\cdot} : W^{p,1}(0,T) \to W^{p,1}(0,T)$. If $\alpha \in (0,1)$ and $1 \leq p \leq 1/\alpha$. If $0 < \alpha < 1$ and $1 \leq p \leq \frac{1}{\alpha}$, the fractional derivative ${}^{R}_{0}\mathrm{D}^{\alpha}_{\cdot} : W^{p,1}(0,T) \to L^p(0,T)$. These results indicate that the fractional integral can improve the regularity of functions in the Sobolev space, while the fractional derivative preserves the regularity but may reduce the space to a lower regularity.

In general, when we take the derivative of a function, we may lose regularity. However, there are similar results for the Riemann-Liouville derivative that indicate certain improvements in regularity. For instance, if our function is in the α-order Hölder space, we can obtain a continuous function. Let $C^{\beta}_{\infty}(0,T] = \{\phi \in C^{\beta}(0,T] : \phi(0+) = \phi(T) = 0\}$, which represents the space of functions in $C^{\beta}(0,T)$ with zero boundary conditions. The following theorem provides results in this context; see [64, P.240] for a proof.

Theorem 2.2.4. *Let* $\beta > 0$, $\phi \in C^{\beta}_{\infty}(0,T]$, $0 < \alpha < \beta \leq 1$. *Then* ${}^{R}_{0}\mathrm{D}^{\alpha}_{\cdot}\, \phi \in C^{\beta-\alpha}(0,T)$.

Definition 2.2.5. The fractional Sobolev sapce $W^{p,\beta}(0,T)$, $p \in [1,\infty)$, $\beta \in (0,1]$, $T > 0$ consists of all $\phi \in L^p(0,T)$ such that the norm

$$\|\phi\|^p_{W^{p,\beta}} = \|\phi\|^p_{L^p} + \int_0^T \int_0^T \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{p\beta+1}} dx\, dy,$$

is finite. If $p=\infty$, we have

$$\|\phi\|_{W^{\infty,\beta}} = \|\phi\|_\infty + \operatorname*{esssup}_{(x,y)\in(0,T)\times(0,T)\setminus\{x=y\}} \frac{|\phi(x)-\phi(y)|}{|x-y|^\beta}.$$

Denote by $W_0^{p,\beta}(0,T) := \overline{C_c^\infty(0,T)}^{\|\cdot\|_{W^{p,\beta}}} = \{\phi \in W^{p,\beta}(0,T) : \phi(0+) = \phi(T-) = 0\}$.

Remark 2.2.6. If $\phi \in W^{\infty,\beta}(0,T)$, there exists a representative $\bar{\phi}$ that is Hölder continuous, i.e. $\bar{\phi} \in C^\beta(0,T)$, see [48, P.13].

We recall that the denseness of smooth compactly supported functions in fractional Sobolev space $W^{p,\beta}$ is ensured only in some cases cf. [12, P.23] and Triebel [73, P.209].

Theorem 2.2.7. *Let $\beta \in (0,1)$ and $1 \le p < \infty, \beta p < 1$. Then the following result $\overline{C_c^\infty(0,T)}^{\|\cdot\|_{W^{p,\beta}}} = W^{p,\beta}(0,T)$ holds; that is , $C_c^\infty(0,T)$ is dense in $W^{p,\beta}(0,T) = W_0^{p,\beta}(0,T)$.*

We can prove that, when we apply the fractional integral ${}^{\mathrm{R}}_{0}\mathrm{I}^{1-\alpha}_{\cdot}\,\phi$ to a function in $W^{1,p}(0,T)$ for some $p > 1$, we gain more smoothness. This implies that the function ${}^{\mathrm{R}}_{0}\mathrm{I}^{1-\alpha}_{\cdot}\,\phi$ belongs to a higher order fractional Sobolev space. According to [12, P.47], we have the following results: Let $p > 1$ and $\alpha \in (0, \min\{\frac{1}{p}, \frac{p-1}{2p}\})$. For all $\phi \in W_0^{p,1}(0,T)$, it holds that ${}^{\mathrm{R}}_{0}\mathrm{I}^{1-\alpha}_{x}\,\phi \in W^{p,\alpha+1}(0,T)$ and ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi \in W^{p,\alpha}(0,T)$.

This result indicates that, by applying the fractional integral ${}^{\mathrm{R}}_{0}\mathrm{I}^{1-\alpha}_{\cdot}$ to a function in $W^{1,p}(0,T)$, we obtain a function that belongs to the fractional Sobolev space $W^{p,\alpha+1}(0,T)$. Similarly, applying the fractional derivative ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}$ to a function in $W^{1,p}(0,T)$ gives us a function in the fractional Sobolev space $W^{p,\alpha}(0,T)$. For a detailed proof and further information, we refer to [12, P.47].

Theorem 2.2.8. *Let $1 \le p < \infty$ and $\alpha \in (0,1]$. ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{\cdot} : W^{p,k}(0,T) \to W^{p,k-1}(0,T)$.*

Proof. Step 1. if $k=1, 0<\alpha<1$, we need to show ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi \in L^p(0,T)$. If $\alpha p \le 1$ and $\phi(0+)$ exists, we have

$$\begin{aligned}
&\|\,{}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi\|^p_{L^p}\\
&= \frac{1}{\Gamma(1-\alpha)^p}\int_0^T \left|\frac{d}{dx}\int_0^x \frac{\phi(t)}{(x-t)^\alpha}dt\right|^p dx\\
&= \frac{1}{\Gamma(1-\alpha)^p}\int_0^T \left|\frac{d}{dx}\int_0^x \frac{\phi(x-t)}{t^\alpha}dt\right|^p dx\\
&= \frac{1}{\Gamma(1-\alpha)^p}\int_0^T \left|\frac{\phi(0)}{x^\alpha} + \int_0^x \frac{\phi'(x-t)}{t^\alpha}dt\right|^p dx\\
&\le \frac{C}{\Gamma(1-\alpha)^p}\int_0^T \left|\frac{\phi(0)}{x^\alpha}\right|^p dx + \frac{C}{\Gamma(1-\alpha)^p}\int_0^T \left|\int_0^x \frac{\phi'(x-t)}{t^\alpha}dt\right|^p dx\\
&= \mathrm{I} + \mathrm{II}
\end{aligned}$$

For Part I, if $\alpha p < 1$, then we can get $\displaystyle\int_0^T \left|\frac{\phi(0)}{x^\alpha}\right|^p dx < \infty$.

For Part II, we have

$$\begin{aligned}
&\int_0^T \left|\int_0^x \frac{\phi'(x-t)}{t^\alpha} dt\right|^p dx \\
=& \int_0^T \left|\int_0^x x\frac{\phi'(x-t)}{t^\alpha}\frac{dt}{x}\right|^p dx \\
=& \int_0^T x^p \left|\int_0^x \frac{\phi'(x-t)}{t^\alpha}\frac{dt}{x}\right|^p dx \\
\overset{\text{Jensen}}{\leq}& \int_0^T x^p \int_0^x \frac{|\phi'(x-t)|^p}{t^{p\alpha}}\frac{dt}{x} dx \\
\overset{\text{Fubini}}{=}& \int_0^T \int_t^T x^{p-1}|\phi'(x-t)|^p \, dx \, t^{-p\alpha} \, dt \\
\leq& \; T^{p-1}\int_0^T \int_t^T |\phi'(x-t)|^p \, dx \, t^{-p\alpha} \, dt \\
\leq& \; T^{p-1}\|\phi'\|_{L^p}\int_0^T t^{-p\alpha} \, dt \\
\leq& \; CT^{p-1}\|\phi'\|_{L^p}
\end{aligned}$$

From here we have $\| {}_0^{\mathrm{R}}\mathrm{D}_x^\alpha \phi\|_{L^p} \leq C_T\|\phi\|_{W^{p,1}}$.

Step 2. if $k > 1, 0 < \alpha \leq 1, \alpha p \leq 1$ let $\phi \in W^{p,k}(0,T)$, $\phi^{(j)}(0+) = 0$, $0 \leq j \leq k$, let us calculate the norm of ${}_0^{\mathrm{R}}\mathrm{D}_x^\alpha \phi$ in $W^{p,k-1}(0,T)$, for all m, $0 \leq m \leq k-1$,

$$\begin{aligned}
&\left\|\frac{d^m}{dx^m}\, {}_0^{\mathrm{R}}\mathrm{D}_x^\alpha \phi\right\|_{L^p}^p \\
\overset{(2.1.11)}{=}& \frac{1}{\Gamma(1-\alpha)^p}\int_0^T \left|\frac{d^m}{dx^m}\frac{d}{dx}\int_0^x \frac{\phi(t)}{(x-t)^\alpha} dt\right|^p dx \\
=& \frac{1}{\Gamma(1-\alpha)^p}\int_0^T \left|\frac{d^m}{dx^m}\frac{d}{dx}\int_0^x \frac{\phi(x-t)}{t^\alpha} dt\right|^p dx \\
=& \frac{1}{\Gamma(1-\alpha)^p}\int_0^T \left|\frac{d^m}{dx^m}\int_0^x \frac{\phi'(x-t)}{t^\alpha} dt\right|^p dx \\
=& \frac{1}{\Gamma(1-\alpha)^p}\int_0^T \left|\int_0^x \frac{\phi^{(m+1)}(x-t)}{t^\alpha} dt\right|^p dx
\end{aligned}$$

A similar reasoning as in Step 1 gives $\|\frac{d^m}{dx^m}\, {}_0^{\mathrm{R}}\mathrm{D}_x^\alpha \phi\|_{L^p}^p \leq C_T\|\phi^{(m+1)}\|_{L^p}$. Thus we have

$$\| {}_0^{\mathrm{R}}\mathrm{D}_x^\alpha \phi\|_{W^{p,k-1}}^p \leq C_T\sum_{m=0}^{k-1}\|\frac{d^m}{dx^m}\, {}_0^{\mathrm{R}}\mathrm{D}_x^\alpha \phi\|_{L^p}^p \leq C_T\sum_{m=0}^{k-1}\|\phi^{(m+1)}\|_{L^p}^p \leq \|\phi\|_{W^{p,k}}. \qquad \square$$

For $p = 1$, we obtain the following result for $W^{1,s}(0,T)$, $s \in (0,1)$. α order derivative we will lose α order smoothness.

Theorem 2.2.9. *Let $\phi \in W^{1,s}(0,T), 0<\alpha<s<1$, then we have ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{\cdot}\,\phi \in W^{1,s-\alpha}(0,T)$.*

Proof. According to the definition of $W^{1,s-\alpha}(0,T)$, we have the following expression

$$\begin{aligned}
\|{}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi\|_{W^{1,s-\alpha}} &= \|{}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi\|_{L^1} + \left(\int_0^T \int_0^T \frac{|{}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi - {}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{y}\,\phi|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy\right) \\
&= \mathrm{I} + \mathrm{II}
\end{aligned}$$

We use the Marchaud representation formula of the Riemann Liouville fractional derivative, see Corollary 2.1.12:

$$\begin{aligned}
\|{}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{x}\,\phi\|_{L^1} &= \left|\int_0^T \left[\frac{\phi(x)}{\Gamma(1-\alpha)x^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^x \frac{\phi(x)-\phi(t)}{(x-t)^{\alpha+1}}\,dt\right] dx\right| \\
&\le \left|\int_0^T \frac{\phi(x)}{\Gamma(1-\alpha)x^{\alpha}}\,dx\right| + \left|\int_0^T \frac{\alpha}{\Gamma(1-\alpha)}\int_0^x \frac{\phi(x)-\phi(t)}{(x-t)^{\alpha+1}}\,dt\,dx\right| \\
&\le C_{\alpha}\left[\int_0^T \frac{|\phi(x)|}{x^{\alpha}}\,dx + \int_0^T\int_0^x \frac{|\phi(x)-\phi(t)|}{|x-t|^{\beta+1}}\frac{|x-t|^{\beta}}{|x-t|^{\alpha}}\,dt\,dx\right] \\
&\le C_{\alpha,T}\left[\|\phi\|_{W^{1,s}} + \int_0^T\int_0^T \frac{|\phi(x)-\phi(t)|}{|x-t|^{\beta+1}}\,dt\,dx\right] \\
&= 2C_{\alpha,T}\|\phi\|_{W^{1,s}} \\
&< \infty
\end{aligned}$$

For the last two inequalities we can use Hardy's inequality. Indeed, we assume that $\phi \in C_c^{\infty}(\mathbb{R})$ such that $\mathrm{supp}(\phi) \subset (0,T)$ and we have

$$|\phi(x)| \le |\phi(x)-\phi(y)| + |\phi(y)| \tag{2.2.4}$$

we divide (2.2.4) by $ax^{\alpha+1}$, and integrate over $x>0$ and $ax<y<2ax$. Here, the constant $a>0$ will be chosen later. Using Fubini's theorem, we find that

$$\begin{aligned}
&\int_0^T \left|\frac{\phi(x)}{x^{\alpha}}\right| dx \\
&= \int_0^T \int_{ax}^{2ax} \left|\frac{\phi(x)}{ax^{\alpha+1}}\right| dy\,dx \\
&\le \int_0^{\infty}\int_{ax}^{2ax} \left|\frac{\phi(y)}{ax^{\alpha+1}}\right| dy\,dx + \int_0^T\int_{ax}^{2ax} \frac{|\phi(x)-\phi(y)|}{ax^{\alpha+1}}\,dy\,dx \\
&\le \int_0^{\infty}\int_{y/2a}^{y/a} \left|\frac{\phi(y)}{ax^{\alpha+1}}\right| dx\,dy + \int_0^T\int_{ax}^{2ax} \frac{|\phi(x)-\phi(y)|}{ax^{\alpha+1}}\,dy\,dx \\
&= \frac{a^{\alpha-1}(2^{\alpha}-1)}{\alpha}\int_0^T \left|\frac{\phi(y)}{y^{\alpha}}\right| dy + \int_0^T\int_{ax}^{2ax\wedge T} \frac{|\phi(x)-\phi(y)|}{|x-y|^{\alpha+1}}\frac{|x-y|^{\alpha+1}}{ax^{\alpha+1}}\,dy\,dx \\
&\quad + \int_0^T\int_{2ax\wedge T}^{2ax} \frac{|\phi(x)-\phi(y)|}{ax^{\alpha+1}}\,dy\,dx.
\end{aligned}$$

Denote $C_{\alpha,a} := \sup_{0<ax<y<2ax} \frac{|x-y|^{\alpha+1}}{ax^{\alpha+1}} = \frac{(|a-1| \vee |2a-1|)^{1+\alpha}}{a} < \infty$.

$$\begin{aligned}
&\int_0^T \left|\frac{\phi(x)}{x^\alpha}\right| dx \\
&\leq \frac{a^{\alpha-1}(2^\alpha-1)}{\alpha} \int_0^T \left|\frac{\phi(y)}{y^\alpha}\right| dy + C_{\alpha,a} \int_0^T \int_0^T \frac{|\phi(x)-\phi(y)|}{|x-y|^{\alpha+1}}\, dy\, dx \\
&\quad + \int_0^T \int_{2ax\wedge T}^{2ax} \frac{|\phi(x)-\phi(y)|}{ax^{\alpha+1}}\, dy\, dx \\
&\leq \frac{a^{\alpha-1}(2^\alpha-1)}{\alpha} \int_0^T \left|\frac{\phi(y)}{y^\alpha}\right| dy + C_{\alpha,a} \int_0^T \int_0^T \frac{|\phi(x)-\phi(y)|}{|x-y|^{\alpha+1}}\, dy\, dx \\
&\quad + \int_{T/(2a)}^T \int_T^{2ax} \frac{|\phi(x)-\phi(y)|}{ax^{\alpha+1}}\, dy\, dx \\
&= \frac{a^{\alpha-1}(2^\alpha-1)}{\alpha} \int_0^T \left|\frac{\phi(y)}{y^\alpha}\right| dy + C_{\alpha,a} \int_0^T \int_0^T \frac{|\phi(x)-\phi(y)|}{|x-y|^{\alpha+1}}\, dy\, dx \\
&\quad + \int_{T/(2a)}^T \int_T^{2ax} \frac{|\phi(x)|}{ax^{\alpha+1}}\, dy\, dx \qquad \left(\text{Since } \operatorname{supp}\phi \subset (0,T)\right) \\
&\leq \frac{a^{\alpha-1}(2^\alpha-1)}{\alpha} \int_0^T \left|\frac{\phi(y)}{y^\alpha}\right| dy + C_{\alpha,a} \int_0^T \int_0^T \frac{|\phi(x)-\phi(y)|}{|x-y|^{\alpha+1}}\, dy\, dx \\
&\quad + 2\int_{T/(2a)}^T \frac{|\phi(x)|}{x^\alpha}\, dx \\
&\leq \frac{a^{\alpha-1}(2^\alpha-1)}{\alpha} \int_0^T \left|\frac{\phi(y)}{y^\alpha}\right| dy + C_{\alpha,a} \int_0^T \int_0^T \frac{|\phi(x)-\phi(y)|}{|x-y|^{\alpha+1}}\, dy\, dx \\
&\quad + \left(\frac{1}{T^\alpha} \wedge \frac{(2a)^\alpha}{T^\alpha}\right) 2\int_{T/(2a)}^T |\phi(x)|\, dx \\
&\leq \frac{a^{\alpha-1}(2^\alpha-1)}{\alpha} \int_0^T \left|\frac{\phi(y)}{y^\alpha}\right| dy + C_{\alpha,a} \int_0^T \int_0^T \frac{|\phi(x)-\phi(y)|}{|x-y|^{\alpha+1}}\, dy\, dx \\
&\quad + \frac{2^{\alpha+1}a^\alpha}{T^\alpha} \int_0^T |\phi(x)|\, dx
\end{aligned}$$

Now we pick a such that $\frac{a^{\alpha-1}(2^\alpha-1)}{\alpha} < 1$ and denote $C_{\alpha,a,T} = \frac{2^{\alpha+1}a^\alpha}{T^\alpha} \vee C_{\alpha,a}$, then we have

$$\begin{aligned}
&\int_0^T \left|\frac{\phi(x)}{x^\alpha}\right| dx \\
&\leq \frac{a^{\alpha-1}(2^\alpha-1)}{\alpha} \int_0^T \left|\frac{\phi(y)}{y^\alpha}\right| dy + C_{\alpha,a,T} \left[\int_0^T \int_0^T \frac{|\phi(x)-\phi(y)|}{|x-y|^{\alpha+1}}\, dy\, dx + \int_0^T |\phi(x)|\, dx\right].
\end{aligned}$$

That implies

$$\begin{aligned}
&\left(1 - \frac{a^{\alpha-1}(2^\alpha-1)}{\alpha}\right) \int_0^T \left|\frac{\phi(x)}{x^\alpha}\right| dx \\
&\leq C_{\alpha,a,T} \left[\int_0^T \int_0^T \frac{|\phi(x)-\phi(y)|}{|x-y|^{\alpha+1}}\, dy\, dx + \int_0^T |\phi(x)|\, dx\right]
\end{aligned}$$

$$\leq C_{\alpha,a,T}\|\phi\|_{W^{1,\alpha}(0,T)}.$$

Now let $\phi \in W^{1,\alpha}(0,T)$. By Theorem 2.2.7, there exists a sequence $\phi_n \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}(\phi_n) \subset (0,T)$ such that ϕ_n converges to ϕ in $W^{1,\alpha}(0,T)$. There exists a subsequence ϕ_{n_k} converging pointwise to ϕ a.e., therefore

$$\begin{aligned}
\int_0^T \frac{|\phi(x)|}{x^\alpha}\,dx &= \int_0^T \liminf_{k\to\infty} \frac{|\phi_{n_k}(x)|}{x^\alpha}\,dx \\
&\leq \liminf_{k\to\infty} \int_0^T \frac{|\phi_{n_k}(x)|}{x^\alpha}\,dx \\
&\leq \liminf_{k\to\infty} C'_{\alpha,a,T}\left[\int_0^T\int_0^T \frac{|\phi_{n_k}(x)-\phi_{n_k}(y)|}{|x-y|^{\alpha+1}}\,dy\,dx + \int_0^T |\phi_{n_k}(x)|\,dx\right]. \\
&\leq \liminf_{k\to\infty} C'_{\alpha,a,T}\|\phi_{n_k}\|_{W^{1,\alpha}} \\
&= C'_{\alpha,a,T}\|\phi\|_{W^{1,\alpha}}.
\end{aligned}$$

We obtain $\int_0^T \left|\frac{\phi(x)}{x^\alpha}\right| dx \leq C'_{\alpha,a,T}\|\phi\|_{W^{1,\alpha}} \leq C''_{\alpha,a,T}\|\phi\|_{W^{1,s}}$ since $\alpha < s$.
For part II, we have

$$\begin{aligned}
&|{}^{\mathrm{R}}_0\mathrm{D}^\alpha_x\phi - {}^{\mathrm{R}}_0\mathrm{D}^\alpha_y\phi| \\
&= \left|\frac{\phi(x)}{\Gamma(1-\alpha)x^\alpha} - \frac{\phi(y)}{\Gamma(1-\alpha)y^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\left[\int_0^x \frac{\phi(x)-\phi(t)}{(x-t)^{\alpha+1}}\,dt - \int_0^y \frac{\phi(y)-\phi(t)}{(y-t)^{\alpha+1}}\,dt\right]\right| \\
&\leq \left|\frac{\phi(x)}{\Gamma(1-\alpha)x^\alpha} - \frac{\phi(y)}{\Gamma(1-\alpha)y^\alpha}\right| + \left|\frac{\alpha}{\Gamma(1-\alpha)}\left[\int_0^x \frac{\phi(x)-\phi(t)}{(x-t)^{1+\alpha}}\,dt - \int_0^y \frac{\phi(y)-\phi(t)}{(y-t)^{\alpha+1}}\,dt\right]\right| \\
&= \mathrm{III} + \mathrm{IV}
\end{aligned}$$

For the double integral are Part III, using the definition of the norm and the symmetry of the integral, we have

$$\begin{aligned}
&\int_0^T\int_0^T \frac{|\phi(x)x^{-\alpha} - \phi(y)y^{-\alpha}|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy \\
&= 2\int_0^T\int_0^x \frac{|\phi(x)x^{-\alpha} - \phi(y)y^{-\alpha}|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy \\
&\leq 2\int_0^T\int_0^x \frac{|\phi(x)-\phi(y)|x^{-\alpha} + |\phi(y)|(x^{-\alpha}-y^{-\alpha})|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy \\
&= 2\int_0^T\int_0^x \frac{|\phi(x)-\phi(y)|x^{-\alpha}}{|x-y|^{(s-\alpha)+1}}\,dx\,dy + 2\int_0^T\int_0^x \frac{|\phi(y)|(x^{-\alpha}-y^{-\alpha})|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy \\
&\leq 2\int_0^T\int_0^x \frac{|\phi(x)-\phi(y)|}{|x-y|^{s+1}}\frac{|x-y|^\alpha}{x^\alpha}\,dx\,dy + 2\int_0^T\int_0^x \frac{|\phi(y)|(x^{-\alpha}-y^{-\alpha})|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy \\
&\leq C\|\phi\|_{W^{1,s}} + 2\int_0^T\int_0^x \frac{|\phi(y)|(x^{-\alpha}-y^{-\alpha})|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy
\end{aligned}$$

In order to bound the integral expression, we use the following estimate. After a change of variables we can get,

$$\begin{aligned}
&\int_0^T \int_0^x \frac{|\phi(y)|(x^{-\alpha} - y^{-\alpha})|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy \\
&= \int_0^T \int_0^1 \frac{|\phi(xu)||x^{-\alpha}(1-\frac{1}{u^\alpha})|}{|x(1-u)|^{(s-\alpha)+1}} x\,du\,dx \\
&= \int_0^1 \int_0^T \frac{|\phi(xu)|}{|xu|^s}\,dx\,\frac{|u^s(1-\frac{1}{u^\alpha})|}{|1-u|^{(s-\alpha)+1}}\,du \\
&= \int_0^1 \int_0^{Tu} \frac{|\phi(y)|}{|y|^s}\,dx\,\frac{|u^{s-1}(1-\frac{1}{u^\alpha})|}{|1-u|^{(s-\alpha)+1}}\,du \\
&\le \int_0^T \frac{|\phi(y)|}{|y|^s}\,dx \int_0^1 \frac{|u^{s-1}(1-\frac{1}{u^\alpha})|}{|1-u|^{(s-\alpha)+1}}\,du \\
&\le C\|\phi\|_{W^{1,s}}
\end{aligned}$$

Where we used the same argument as for Part I. Without loss of generality, we assume $x \ge y$ and use the symmetry. For part IV, we have the following estimate,

$$\begin{aligned}
&\int_0^T \int_0^T \frac{\left|\int_0^x \frac{\phi(x)-\phi(t)}{(x-t)^{1+\alpha}}\,dt - \int_0^y \frac{\phi(y)-\phi(t)}{(y-t)^{\alpha+1}}\,dt\right|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy \\
&= 2\int_0^T \int_0^x \frac{\left|\int_0^x \frac{\phi(x)-\phi(t)}{(x-t)^{1+\alpha}}\,dt - \int_0^y \frac{\phi(y)-\phi(t)}{(y-t)^{\alpha+1}}\,dt\right|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy \\
&\le 2\int_0^T \int_0^x \frac{\left|\int_0^y \frac{\phi(x)-\phi(t)}{(x-t)^{1+\alpha}}\,dt - \int_0^y \frac{\phi(y)-\phi(t)}{(y-t)^{\alpha+1}}\,dt\right|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy \\
&\quad + 2\int_0^T \int_0^x \frac{\left|\int_y^x \frac{\phi(x)-\phi(t)}{(x-t)^{1+\alpha}}\,dt\right|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy \\
&= 2(\text{V}+\text{VI})
\end{aligned}$$

For part VI, we we can just use Fubini's theorem. In fact,

$$\begin{aligned}
&\int_0^T \int_0^x \frac{\left|\int_y^x \frac{\phi(x)-\phi(t)}{(x-t)^{1+\alpha}}\,dt\right|}{|x-y|^{(s-\alpha)+1}}\,dx\,dy \\
&\le \int_0^T \int_0^x \frac{\int_y^x \frac{|\phi(x)-\phi(t)|}{|x-t|^{1+\alpha}}\,dt}{|x-y|^{(s-\alpha)+1}}\,dx\,dy \\
&= \int_0^T \int_0^x \int_0^t \frac{|\phi(x)-\phi(t)||x-t|^{-1-\alpha}}{|x-y|^{(s-\alpha)+1}}\,dy\,dx\,dt \\
&\le 2\int_0^T \int_0^x \frac{|\phi(x)-\phi(t)|}{|x-t|^{1+s}}\,dx\,dt
\end{aligned}$$

$$\leq \|\phi\|_{W^{1,s}}$$

In the penultimate estimate we calculate the integral exactly and use $|x|^{\alpha-s} \leq |x-t|^{\alpha-s}$ to get the above result.

For part V, we will get

$$\begin{aligned}
&\int_0^T \int_0^x \frac{\left|\int_0^y \frac{\phi(x)-\phi(t)}{(x-t)^{1+\alpha}}\,dt - \int_0^y \frac{\phi(y)-\phi(t)}{(y-t)^{\alpha+1}}\,dt\right|}{|x-y|^{(s-\alpha)+1}}\,dy\,dx \\
&\leq \int_0^T \int_0^x \frac{|\phi(x)-\phi(y)| \int_0^y (x-t)^{-1-\alpha}\,dt}{|x-y|^{(s-\alpha)+1}}\,dy\,dx \\
&\quad + \int_0^T \int_0^x \frac{\int_0^y |\phi(y)-\phi(t)||(x-t)^{-\alpha-1}-(y-t)^{-\alpha-1}|\,dt}{|x-y|^{(s-\alpha)+1}}\,dy\,dx \\
&\leq 2\int_0^T \int_0^x \frac{|\phi(x)-\phi(y)|}{|x-y|^{s+1}}\,dy\,dx \\
&\quad + \int_0^T \int_0^x \frac{\int_0^y |\phi(y)-\phi(t)||(x-t)^{-\alpha-1}-(y-t)^{-\alpha-1}|\,dt}{|x-y|^{(s-\alpha)+1}}\,dy\,dx \\
&\leq 2\|\phi\|_{W^{1,s}} + \int_0^T \int_0^x \frac{\int_0^y |\phi(y)-\phi(t)||(x-t)^{-\alpha-1}-(y-t)^{-\alpha-1}|\,dt}{|x-y|^{(s-\alpha)+1}}\,dy\,dx.
\end{aligned}$$

Next we deal with the the right hand side, by using the monotonicity of x^α to get

$$\begin{aligned}
&\int_0^T \int_0^x \frac{\int_0^y |\phi(y)-\phi(t)||(x-t)^{-\alpha-1}-(y-t)^{-\alpha-1}|\,dt}{|x-y|^{(s-\alpha)+1}}\,dy\,dx \\
&\leq C_\alpha \int_0^T \int_0^x \frac{\int_0^y |\phi(y)-\phi(t)| \frac{(x-t)^\alpha |x-y|}{(x-t)^{\alpha+1}(y-t)^{\alpha+1}}|\,dt}{|x-y|^{(s-\alpha)+1}}\,dy\,dx \\
&= C_\alpha \int_0^T \int_0^x \int_0^y \frac{|\phi(y)-\phi(t)|}{|y-t|^{s+1}} \frac{|y-t|^{s+1}}{|x-y|^{s-\alpha}|x-t||y-t|^{\alpha+1}}\,dt\,dy\,dx \\
&= C_\alpha \int_0^T \int_t^T \int_y^T \frac{|\phi(y)-\phi(t)|}{|y-t|^{s+1}} \frac{|y-t|^{s+1}}{|x-y|^{s-\alpha}|x-t||y-t|^{\alpha+1}}\,dx\,dy\,dt
\end{aligned}$$

Finally we show that $\int_y^T \frac{|y-t|^{s+1}}{|x-y|^{s-\alpha}|x-t||y-t|^{\alpha+1}}\,dx$ is bounded. In fact, after a change of variable we have the following estimate,

$$\begin{aligned}
&\int_y^T \frac{|y-t|^{s+1}}{|x-y|^{s-\alpha}|x-t||y-t|^{\alpha+1}}\,dx \\
&= \int_0^{T-y} \frac{1}{|\frac{u}{y-t}|^{s-\alpha}} \frac{1}{|1+\frac{u}{y-t}|} \frac{du}{|y-t|} \\
&= \int_0^{\frac{T-y}{y-t}} \frac{1}{|v|^{s-\alpha}} \frac{1}{|1+v|}dv \\
&\leq \int_0^\infty \frac{1}{|v|^{s-\alpha}} \frac{1}{|1+v|}dv
\end{aligned}$$

$$< M$$

Since $M < \infty$, so we have $\| {}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{\cdot}\, \phi\|_{W^{1,s-\alpha}} \leq C\|\phi\|_{W^{1,s}}$. □

Now, we use the interpolation theorem to show: When $p \geq 1$, the α order Riemann-Liouville fractional derivative maps $W^{p,s}(0,T)$ to $W^{p,s-\alpha}(0,T)$ under certain kind of conditions.

Theorem 2.2.10. *Let $\phi \in W^{p,s}(0,T)$, $0 < \alpha < s < 1 \leq p$, then we have ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{\cdot}\, \phi \in W^{p,s-\alpha}(0,T)$.*

Proof. We want to use the Calderón–Lions interpolation theorem, cf. Theorem 1.1.7. Using the notation from Theorem 1.1.7, we have

$$\mathcal{X} = C_c^{\infty}(0,T), \quad \mathcal{Y} = W^{1,s-\alpha}(0,T) \cap W^{\infty,s-\alpha}(0,T)$$

and

$$\|\cdot\|_{\mathcal{X}_1} = \|\cdot\|_{W^{1,s}}, \quad \|\cdot\|_{\mathcal{X}_0} = \|\cdot\|_{C^s},$$

$$\|\cdot\|_{\mathcal{Y}_1} = \|\cdot\|_{W^{1,s-\alpha}}, \quad \|\cdot\|_{\mathcal{Y}_0} = \|\cdot\|_{W^{\infty,s-\alpha}}.$$

Denote $C^s_{(0)}(0,T) = \overline{\mathcal{X}}^{\|\cdot\|_{C^s}}$. Moreover $T(t) = {}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{\cdot}$ (independent of t).

We know ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{\cdot} : C^s_{(0)}(0,T) \to C^{s-\alpha}(0,T)$, as well as ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{\cdot} : W^{1,s}(0,T) \to W^{1,s-\alpha}(0,T)$ see Theorem 2.2.4 and 2.2.9, respectively. We also have $W^{1,s}(0,T) = W^{1,s}_0(0,T)$ using Theorem 2.2.7, and by the definition of the respective norm, see Definition 2.2.5, $C^{s-\alpha}(0,T) = W^{\infty,s-\alpha}(0,T)$ and $C^s_{(0)}(0,T) = W^{\infty,s}_0(0,T)$, where Therefore, all three conditions of Theorem 1.1.7 are satisfied, and we get that ${}^{\mathrm{R}}_{0}\mathrm{D}^{\alpha}_{\cdot} : \mathcal{X}_t \to \mathcal{Y}_t$, $t \in [0,1]$. It remains to identify the interpolation spaces. We will see that

$$\mathcal{X}_t = W^{1/t,s}_0(0,T), \quad \text{and} \quad \mathcal{Y}_t = W^{1/t,s-\alpha}(0,T).$$

In order to identify $\mathcal{X}_t$ we remark that we can identify $W^{1/t,s}_0(0,T)$ with a closed subspace of

$$L^{1/t}((0,T),dx) \times L^{1/t}\left((0,T)\times(0,T) \setminus \{x=y\}, \quad \frac{dx\,dy}{|x-y|}\right),$$

which is equipped with the norm

$$\|(g,h)\|^t_{1/t} = \|g\|^t_{L^{1/t}(dx)} + \|h\|^t_{L^{1/t}(dm)},$$

where $dm = dx\,dy/|x-y|$.

Therefore, it will be enough to show that the norms $\|\cdot\|_{\mathcal{X}_t}$ and $\|\cdot,\cdot\|_{1/t}$ are equivalent on the space

$$\hat{\mathcal{X}} = \left\{ \left(\phi(u), \frac{\phi(x)-\phi(y)}{|x-y|^s}\right) : \phi \in C_c^{\infty}(0,T) \right\}.$$

Proof of $W_0^{1/t,s} \subset \mathcal{X}_t$. Recall from the discussion preceding Theorem 1.1.7 that

$$\|\phi\|_{\mathcal{X}_t} = \inf\{\|\tilde{\phi}\|_{F[\mathcal{X}]} : \tilde{\phi} \in F[\mathcal{X}], \quad \tilde{\phi}(t) = \phi\},$$

where $F[\mathcal{X}]$ was the family of analytic maps connecting $\mathcal{X}_0$ and $\mathcal{X}_1$, see Theorem 1.1.7.

Pick any $\phi \in \mathcal{X} = C_c^\infty(0,T)$ such that $\|\phi\|_{W^{1/t,s}} = 1$ and define for $z \in \bar{S}$, where $S = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}$ an analytic function

$$g(z)(u,x,y) = \begin{pmatrix} |\phi(u)|^{z/t} \exp(i \arg \phi(u)) \\ \left|\frac{\phi(x)-\phi(y)}{|x-y|^s}\right|^{z/t} \exp\left(i \arg\left(\frac{\phi(x)-\phi(y)}{|x-y|^s}\right)\right) \end{pmatrix}$$

Note that we can identity $\mathcal{X}$ and $\hat{\mathcal{X}}$. Clearly, $g(z) \in \hat{\mathcal{X}}$ and for all $v \in (0,1)$

$$\|g(iv)\|_\infty = 2,$$

and

$$\begin{aligned} \|g(1+iv)\|_1 &= \int_0^T ||\phi(u)|^{(1+iv)/t}|\, du + \int_0^T \int_0^T \left|\left|\frac{\phi(x)-\phi(y)}{|x-y|^s}\right|^{(1+iv)/t}\right| \frac{dx\,dy}{|x-y|} \\ &= \|\phi\|_{W^{1/t,s}} \\ &= 1. \end{aligned}$$

Thus, by the three lines theorem $\|g\|_{F[\mathcal{X}]} \le 2 = 2\|\phi\|_{W^{1/t,s}}$ and, using the definition of $\|\cdot\|_{\mathcal{X}_t}$ we get $\|\phi\|_{\mathcal{X}_t} \le 2\|\phi\|_{W^{1/t,s}}$ for all $\phi \in \mathcal{X} = C_c^\infty(0,T)$. This proves that $W_0^{1/t,s}(0,T)$ is a subset of $\mathcal{X}_t$, i.e. $W_0^{1/t,s}(0,T) \subset \mathcal{X}_t$.

In oder to see the converse in conclusion, $W_0^{1/t,s}(0,T) \supset \mathcal{X}_t$, we take any map

$$\psi(z)(u,x,y) = \begin{pmatrix} \psi_1(z)(u) \\ \psi_2(z)(x,y)) \end{pmatrix} \in \hat{\mathcal{X}}$$

with $\psi(z) \in F[\mathcal{X}]$ and we fix $\phi \in C_c^\infty(0,T)$ and $l(\cdot,\cdot) \in C_c^\infty((0,T)\times(0,T)\setminus\{x=y\})$ such that $\|(\phi,l)\|_{1/(1-t)} = 1$. Note that $\|\cdot\|_{\frac{1}{1-t}}$ and $\|\cdot\|_{\frac{1}{t}}$ are norms in duality as $1/\frac{1}{t} + 1/\frac{1}{1-t} = 1$. Define

$$h(z)(u,x,y) = \begin{pmatrix} h_1(z)(u) \\ h_2(z)(x,y) \end{pmatrix} = \begin{pmatrix} |\phi(u)|^{(1-z)/(1-t)} \exp(i \arg \phi(u)) \\ |l(x,y)|^{(1-z)/(1-t)} \exp(i \arg l(x,y)) \end{pmatrix}$$

and

$$\begin{aligned} H(z) = &\int_0^T \psi_1(z)(x)|\phi(x)|^{(1-z)/(1-t)} \exp(i \arg \phi(x))\, dx \\ &+ \int_0^T \int_0^T \psi_2(z)(x,y)|l(x,y)|^{(1-z)/(1-t)} \exp(i \arg l(x,y))\, \frac{dx\,dy}{|x-y|}. \end{aligned}$$

Then we get

$$H(t) = \int_0^T \psi_1(t)(x)\phi(x)\, dx + \int_0^T \int_0^T \psi_2(t)(x,y)l(x,y)\, \frac{dx\,dy}{|x-y|}.$$

Since $H(z)$ is analytic on S, we can apply the three line theorem to get

$$
\begin{aligned}
&|H(t)| \\
&\leq \sup_{v\in\mathbb{R}} \{|H(iv)|, \quad |H(1+iv)|\} \\
&\leq \sup_{v\in\mathbb{R}} \Big\{ \|\psi_1(iv)h_1(iv)\|_{L^1(dx)} + \|h_2(iv)\psi_2(iv)\|_{L^1(dm)}, \\
&\qquad \|\psi_1(1+iv)h_1(1+iv)\|_{L^1(dx)} + \|h_2(1+iv)\psi_2(1+iv)\|_{L^1(dm)} \Big\} \\
&\leq \sup_{v\in\mathbb{R}} \Big\{ \|\psi_1(iv)\|_{L^\infty(dx)}\|h_1(iv)\|_{L^1(dx)} + \|\psi_2(iv)\|_{L^\infty(dm)}\|h_2(iv)\|_{L^1(dm)}, \\
&\qquad \|\psi_1(1+iv)\|_{L^1(dx)}\|h_1(1+iv)\|_{L^\infty(dx)} + \|\psi_2(1+iv)\|_{L^1(dm)}\|h_2(1+iv)\|_{L^\infty(dm)} \Big\} \\
&\leq \sup_{v\in\mathbb{R}} \Big\{ \left(\|\psi_1(iv)\|_{L^\infty(dx)} + \|\psi_2(1+iv)\|_{L^\infty(dm)}\right)\left(\|h_1(iv)\|_{L^1(dx)} + \|h_2(iv)\|_{L^1(dm)}\right), \\
&\qquad \left(\|\psi_1(1+iv)\|_{L^1(dx)} + \|\psi_2(1+iv)\|_{L^1(dm)}\right)\left(\|h_1(1+iv)\|_{L^\infty(dx)} + \|h_2(1+iv)\|_{L^\infty(dm)}\right) \Big\} \\
&= \sup_{v\in\mathbb{R}} \Big\{ \|\psi_1(iv)\|_{L^\infty(dx)} + \|\psi_2(iv)\|_{L^\infty(dm)}, \|\psi_1(1+iv)\|_{L^1(dx)} + \|\psi_2(1+iv)\|_{L^1(dm)} \Big\} \\
&\qquad \times \sup_{v\in\mathbb{R}} \Big\{ \|h_1(iv)\|_{L^1(dx)} + \|h_2(iv)\|_{L^1(dm)}, \|h_1(1+iv)\|_{L^\infty(dx)} + \|h_2(1+iv)\|_{L^\infty(dm)} \Big\} \\
&= \|\psi\|_{F[\mathcal{X}]}\|(\phi, l)\|_{\frac{1}{1-t}} \\
&= \|\phi\|_{\mathcal{X}_t}\|(\phi, l)\|_{\frac{1}{1-t}}
\end{aligned}
$$

since $\sup(ab, AB) \leq \sup(a, A)\sup(b, B)$ we can get last two inequalities. Since $H(t)$ is the scalar product (duality relation) for the pair $(L^{1/t}, L^{1/1-t})$, we finally see

$$
\|\phi\|_{W^{1/t,s}} = \sup_{\|(\phi,l)\|_{1/1-t}=1} |H(t)| \leq \|\phi\|_{\mathcal{X}_t},
$$

for all $\phi \in C_c^\infty(0, T)$ finishing this part of the proof.
The identification of $\mathcal{Y}_t = W^{1/t,s-\alpha}(0, T)$ goes along the same lines. □

2.3 Bernstein functions

The following definitions and properties are, essentially, taken from the book [69] by Schilling et al. Here we collect some results which we collect for the reader's convenience.

Definition 2.3.1. A function $f : (0, \infty) \to \mathbb{R}$ is a **Bernstein function** if f is of class C^∞, $f(\lambda) \geq 0$ for all $\lambda > 0$ and

$$
(-1)^{n-1} f^{(n)}(\lambda) \geq 0 \quad \text{for all} \quad n \in \mathbb{N} \text{ and } \lambda > 0 \tag{2.3.1}
$$

The set of all Bernstein functions will be denoted by $\mathcal{BF}$.

Theorem 2.3.2. *A function $f : (0,\infty) \to \mathbb{R}$ is a Bernstein function if, and only if, it admits the representation*

$$f(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(dt) \tag{2.3.2}$$

where $a, b \geq 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\mu(dt) < \infty$.

Definition 2.3.3. A function $f : (0,\infty) \to \mathbb{R}$ is a **completely monotone function** if f is of class C^∞ and

$$(-1)^{n-1} f^{(n)}(\lambda) \geq 0 \qquad \text{for all} \quad n \in \mathbb{N} \ \text{ and } \ \lambda > 0 \tag{2.3.3}$$

The family of all completely monotone functions is denote by $\mathcal{CM}$.

It is well known (Bochner's theorem) that every $f \in \mathcal{CM}$ is the Laplace transform $\mathscr{L}(\mu, \lambda)$ of some measure μ on $[0,\infty)$, and vice versa.

Definition 2.3.4. A Bernstein function f is said to be a **complete Bernstein function**, if its Lévy measure μ has a completely monotone density $m(t)$ with respect to the Lebesgue measure, so (2.3.2) takes the following form

$$f(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t}) m(t)\, dt \tag{2.3.4}$$

where m satisfies $\int_{(0,\infty)} (1 \wedge t) m(t)\, dt < \infty$. The family of all complete Bernstein functions is denote by $\mathcal{CBF}$.

Definition 2.3.5. A (non-negative) Stielties function is a function $f : (0,\infty) \to [0,\infty)$ which can be written in the form

$$f(\lambda) = \frac{a}{\lambda} + b + \int_0^\infty \frac{1}{\lambda + t} \sigma(dt) \tag{2.3.5}$$

where $a, b \geq 0$ are non-negative constants and σ is a measure on $(0,\infty)$ such that $\int_0^\infty \frac{1}{1+t}\sigma(dt) < \infty$. The family of all Stielties functions is denote by $\mathcal{S}$.

There are several ways to define complete Bernstein functions, given by the following theorem.

Theorem 2.3.6. *If $f : (0,\infty) \to [0,\infty)$, then the following conditions are equivalent*

(1) $f \in \mathcal{CBF}$.

(2) $\frac{1}{f(\lambda)} \in \mathcal{S}$.

(3) $\frac{\lambda}{f(\lambda)} \in \mathcal{CBF}$.

(4) $\frac{f(\lambda)}{\lambda} \in \mathcal{S}$.

Let us recall a few properties of Bernstein functions.

Theorem 2.3.7. *(i) The set $\mathcal{BF}$ is a convex cone: $sf_1 + tf_2$ for all $f_1, f_2 \in \mathcal{BF}$ and $s, t \geq 0$.*

(ii) The set $\mathcal{BF}$ is closed under pointwise limits: if $(f_n)_{n\in\mathbb{N}} \subset \mathcal{BF}$, and if the limit $\lim_{n\to\infty} f_n(\lambda) = f(\lambda)$ exists for every $\lambda > 0$, then $f \in \mathcal{BF}$.

(iii) The set $\mathcal{BF}$ is closed under composition: if $f_1, f_2 \in \mathcal{BF}$, then $f_1 \circ f_2 \in \mathcal{BF}$. In particular, $\lambda \mapsto f_1(c\lambda)$ is in $\mathcal{BF}$ for any $c > 0$.

(iv) $f \in \mathcal{BF}$ is bounded if, and only if, in (2.3.2) we have $b = 0$ and $\mu(0, \infty) < \infty$.

(v) Let $f_1, f_2 \in \mathcal{BF}$ and $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta \leq 1$. Then $\lambda \mapsto f_1(\lambda^\alpha) f_2(\lambda^\beta)$ is again a Bernstein function.

(vi) Let $f \in \mathcal{BF}$. For every $\kappa > 0$ the function $f_\kappa(\lambda) = f(\lambda) + f(\kappa) - f(\lambda + \kappa)$, $\lambda > 0$, is again a Bernstein function. In particular, $f \in \mathcal{BF}$ is subadditive, i.e.$f(\lambda + \kappa) \leq f(\lambda) + f(\kappa)$ for all $\kappa, \lambda > 0$.

(vii) For any Bernstein function, there exists $c > 0$ such that $f(\lambda) \leq c\lambda$ for all $\lambda > 1$.

Definition 2.3.8. A **vaguely continuous convolution semigroup** of sub-probability measures on $[0, \infty)$ is a family of measures $(\nu_t)_{t\geq 0}$ satisfying the following prperties:

(i) $\nu_t[0, \infty) \leq 1$ for all $t \geq 0$.

(ii) $\nu_{t+s} = \nu_t \star \nu_s$.

(iii) vague-$\lim_{t\to 0} \nu_t = \delta_0$.

Definition 2.3.9. A function $f \in \mathcal{BF}$ is said to be a **special Bernstein function** if the function $f^\star(\lambda) = \frac{\lambda}{f(\lambda)}$ is again a Bernstein function, i.e. $f^\star \in \mathcal{BF}$. The family of all special Bernstein functions is denoted by $\mathcal{SBF}$.

A subordinator S whose Laplace exponent f belongs to $\mathcal{SBF}$ will be called a **special subordinator**.

Definition 2.3.10. A function $f : (0, \infty) \to (0, \infty)$ is said to be a **potential** if $\frac{1}{g}$ where $g \in \mathcal{BF}^* = \mathcal{BF} \setminus \{0\}$. The set of all potentials will be denoted by $\mathcal{P}$.

Suppose $f \in \mathcal{BF}$. Then $\frac{\lambda}{f(\lambda)} \in \mathcal{BF}$ if and only if $\frac{f(\lambda)}{\lambda} \in \mathcal{P}$. Hence $f \in \mathcal{SBF}$ if and only if $f \in \mathcal{BF}$ and $\frac{f(\lambda)}{\lambda} \in \mathcal{P}$.

Theorem 2.3.11. *Let S be a subordinator with potential measure U. Then S is special if, and only if,*

$$U(dt) = a_0\delta_0(dt) + u(t)dt, \tag{2.3.6}$$

for some $a_0 \geq 0$ and some decreasing function $u : (0,\infty) \to (0,\infty)$ satisfying

$$\int_0^1 u(t)dt < \infty.$$

Theorem 2.3.12. *Let f be a special Bernstein function with representation* (2.3.2) *satisfying $b > 0$ or $\mu(0,\infty) = \infty$. Then*

$$bu(t) + \int_0^t u(s)v(t-s)ds = bu(t) + \int_0^t v(s)u(t-s)ds = 1, t > 0. \tag{2.3.7}$$

As an application of Theorem 2.3.11, we can get the following results.

Theorem 2.3.13. *Assume that*

$$U(dt) = b_1\delta_0(dt) + u(t)dt, \quad V(dt) = b_2\delta_0(dt) + v(t)dt,$$

where $b_1, b_2 \geq 0$ and $u(t), v(t)$ are positive, decreasing and satisfy

$$\int_0^1 u(t)\,dt < \infty, \quad \int_0^1 v(t)\,dt < \infty.$$

If $U * V(dt) = 1$, then $b_1 b_2 = 0$ and $\lambda\mathscr{L}[U(dt),\lambda]$ and $\lambda\mathscr{L}[V(dt),\lambda]$ are special Bernstein functions.

Definition 2.3.14. (i) A function $f : (0,\infty) \to \infty$ is said to be **logarithmically convex** if $\log f$ is convex.

(ii) A sequence $(a_n)_{n\geq 0}$ of non-negative real numbers is **logarithmically convex** if $a_n^2 \leq a_{n-1}a_{n+1}$ for all $n \geq 1$.

Definition 2.3.15. We call a function $f \in \mathcal{CM}$ a **Hirsch function** if for all $\lambda > 0$ the sequence

$$\left(\frac{(-1)^n}{n!} f^{(n)}(\lambda)\right)_{n\geq 0}$$

is logarithmically convex in the sense of Definition 2.3.14. The set of all Hirsch functions will be denoted by $\mathcal{H}$.

Theorem 2.3.16. *Suppose that $u : (0,\infty) \to (0,\infty)$ is decreasing logarithmically convex and satisfies $\int_0^1 u(t)dt < \infty$, and let $b \geq 0$. Define a measure U by*

$$U(dt) = b\delta_0(dt) + u(t)dt \tag{2.3.8}$$

Then $\mathscr{L}U \in \mathcal{H}$. Coneversely, every $f \in \mathcal{H}$ is the Laplace transfrom of a measure U of the form (2.3.8).

Corollary 2.3.17. *(i)* $\frac{1}{\mathcal{H}^*} \subset \mathcal{SBF}, \mathcal{H}^* \subset \mathcal{P}$*, where* $\mathcal{H}^* = \mathcal{H} \setminus \{0\}$.

(ii) If $g \in \mathcal{H}$*, then* $\lambda g(\lambda) \in \mathcal{SBF}$.

(iii) If $f \in \mathcal{BF}$ *and* $\bar{\mu}(x) = \mu(x, \infty)$ *is logarithmically convex, then* $\frac{f(\lambda)}{\lambda} \in \mathcal{H}$.

(iv) $\mathcal{S} \subset \mathcal{H}$.

Definition 2.3.18. A Bernstein function f is **self-similar** with respect to $\alpha \in (0, 1)$ and $\lambda > 1$ if it adnits the discrete scale invariance

$$f(\lambda^{\frac{1}{\alpha}} x) = \lambda f(x) \quad \forall x > 0. \tag{2.3.9}$$

2.4 Fractional derivatives based on Bernstein Functions

Recall that a Bernstein function with $f(0) = 0$ is represented by

$$f(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda x})\, \mu(dx) \tag{2.4.1}$$

where $b \geq 0$ and μ is a measure on $(0, \infty)$ such that $\int_0^\infty 1 \wedge x\, \mu(dx) < \infty$. We will now define the Marchaud fractional derivative and Riemann–Liouville fractional derivative induced by a Bernstein function f.

Definition 2.4.1. [generalized Marchaud fractional derivative] Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function which is defined on the whole real axis and let f be a Bernstein function given by (2.4.1). The **Bernsterin Marchaud fractional derivatives for the Bernstein function** f are the following operators

$$^{\mathrm{M}}\mathrm{D}_+^f\, \phi(x) := b \cdot \frac{d}{dx}\phi(x) + \int_0^\infty (\phi(x) - \phi(x - t))\, \mu(dt) \tag{2.4.2}$$

$$^{\mathrm{M}}\mathrm{D}_-^f\, \phi(x) := -b \cdot \frac{d}{dx}\phi(x) + \int_0^\infty (\phi(x) - \phi(x + t))\, \mu(dt) \tag{2.4.3}$$

If necessary, the integrals $\int_{0+}^\infty$ may be interpreted as improper Riemann integrals, i.e. $\int_{0+}^\infty := \lim_{\epsilon \downarrow 0} \int_\epsilon^\infty$.

Remark 2.4.2. Let us briefly discuss when the integral expressions (2.4.2) and (2.4.3) make sense. Let us consider (2.4.2) in an $L^2(\mathbb{R}, dx)$-context. Our argument is similar to the one used in [66, Lemma 3.4]. We have for any test function ϕ

$$\begin{aligned} &\| ^{\mathrm{M}}\mathrm{D}_+^f\, \phi\|_{L^2} \\ &\leq b\|\phi'\|_{L^2} + \left\| \int_0^1 (\phi(x) - \phi(x - t))\, \mu(dt) \right\|_{L^2(dx)} + \left\| \int_1^\infty (\phi(x) - \phi(x - t))\, \mu(dt) \right\|_{L^2(dx)} \end{aligned}$$

$$= b\|\phi'\|_{L^2} + \left\| \int_0^1 \int_0^t (-\phi')(x-\theta)\, d\theta\, \mu(dt) \right\|_{L^2(dx)} + \left\| \int_1^\infty (\phi(x) - \phi(x-t))\, \mu(dt) \right\|_{L^2(dx)}$$
$$\le b\|\phi'\|_{L^2} + \int_0^1 \int_0^t \|\phi'(x-\theta)\|_{L^2(dx)}\, d\theta\, \mu(dt) + \int_1^\infty \left(\|\phi(x)\|_{L^2(dx)} + \|\phi(x-t)\|_{L^2(dx)} \right) \mu(dt)$$
$$= b\|\phi'\|_{L^2} + \int_0^1 \|\phi'\|_{L^2}\, t\, \mu(dt) + 2\|\phi\|_{L^2} \int_1^\infty \mu(dt)$$
$$\le C \left(\|\phi\|_{L^2} + \|\phi'\|_{L^2} \right)$$

with the constant $C = b + 2\int_0^\infty t \wedge 1\, \mu(dt)$. Now we use a classical boundedness argument and extend this's estimate to all functions u for which the right-hand side gives a finite expression.

A similar calculation goes through for any Banach-space norm (we have only used that $\| \int \ldots d\mu \| \le \int \| \ldots \|\, d\mu$), i.e. the Marchaud derivatives are actually defined for all spaces $W^{1,p}(\mathbb{R})$, $p \ge 1$, (classical L^p-style Sobolev spaces) and $C^1(\mathbb{R})$. It is clear that for $b \ne 0$ these domains are 'maximal' while for $b = 0$ there are more functions u for which this holds. Note that the Sobolev-Slobodeckij spaces of fractional order seem to be the right 'domain' for classical fractional Marchaud derivatives (i.e. where $f(\lambda) = \lambda^\alpha$, $0 < \alpha < 1$).[1]

Recall the extensions ϕ^0, ϕ^σ of ϕ on $\mathbb{R}$ from Definition 2.1.9. In analogy to the classical case, we define – based on the various extensions – the following fractional derivatives for the Bernstein function f.

Definition 2.4.3. Let ϕ be a function on $[0, \infty)$ and f any Bernstein function with characteristics (b, μ). The (**backwards**) **Bernstein Riemann Liouville**, (**backwards**) **Bernstein Caputo fractional derivatives**are defined by

$$ {}^{\mathrm{R}}_{0}\mathrm{D}^f_x\, \phi := {}^{\mathrm{M}}\mathrm{D}^f_+\, \phi^0(x) = b \cdot \frac{d}{dx}\phi^0(x) + \int_0^\infty \left(\phi^0(x) - \phi^0(x-s) \right) \mu(ds), \tag{2.4.4}$$
$$ {}^{\mathrm{C}}_{0}\mathrm{D}^f_x\, \phi := {}^{\mathrm{M}}\mathrm{D}^f_+\, \phi^\sigma(x) = b \cdot \frac{d}{dx}\phi^\sigma(x) + \int_0^\infty (\phi^\sigma(x) - \phi^\sigma(x-s))\, \mu(ds) \tag{2.4.5}$$

The corresponding forward fractional derivatives are given by

$$ {}^{\mathrm{R}}_{x}\mathrm{D}^f_\infty\, \phi = {}^{\mathrm{C}}_{x}\mathrm{D}^f_\infty\, \phi := -b \cdot \frac{d}{dx}\phi(x) + \int_0^\infty (\phi(x) - \phi(x+s))\, \mu(ds) = {}^{\mathrm{M}}\mathrm{D}^f_-\, \phi(x). \tag{2.4.6}$$

[1] The standard reference for such spaces are the books by Triebel [74, 75]. If $\alpha \in (0,1)$, the fractional Sobolev-Slobodeckij norm is given by

$$\|\phi\|^p_{W^{p,\alpha}(\mathbb{R}^d)} \asymp \|\phi\|^p_{L^p(\mathbb{R}^d)} + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{d+\alpha p}}\, dx\, dy, \quad p \in [1, \infty).$$

It arises by real interpolation between $L^p(\mathbb{R}^d, dx)$ and $W^{p,1}(\mathbb{R}^d, dx) = \{\phi \in L^p(\mathbb{R}^d, dx) : \nabla\phi \in L^p(\mathbb{R}^d, dx)\}$.

By the definition of the extensions ϕ^0 and ϕ^σ, respectively, we note that

$$ {}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\,\phi(x) - {}^{\mathrm{C}}_{0}\mathrm{D}^{f}_{x}\,\phi(x) = \int_x^\infty \phi(0)\mu(ds) = \phi(0)\bar{\mu}(x) \tag{2.4.7} $$

where $\bar{\mu}(x) := \mu(x, \infty)$.

Definition 2.4.4. [gernalized Riemann–Liouville fractional integral] Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function which is defined on the whole real axis and let f be a Bernstein function given by (2.4.1). The **Riemann–Liouville fractional integral for the Bernstein function** f are the following operators

$$ {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi(x) = \int_0^x \phi(z)k(x-z)dz \tag{2.4.8} $$

where k is defined by the following Laplace transfrom $\mathscr{L}[k;\lambda] = 1/f(\lambda)$.

Lemma 2.4.5. *Let ϕ be a function on $[0,\infty)$ and f be a Bernstein function with characteristics (b,μ). The backward Bernstein Riemann–Liouville, Bernstein Caputo fractional derivatives are*

$$ {}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\,\phi = b\cdot\frac{d}{dx}\phi(x) + \frac{d}{dx}\int_0^x \phi(t)\,\bar{\mu}(x-t)\,dt, \tag{2.4.9} $$

$$ {}^{\mathrm{C}}_{0}\mathrm{D}^{f}_{x}\,\phi = b\cdot\frac{d}{dx}\phi(x) + \int_0^x \phi'(t)\bar{\mu}(x-t)\,dt, \tag{2.4.10} $$

where $\bar{\mu}(s) := \mu(s,\infty)$. For the corresponding forward fractional derivative one has

$$ {}^{\mathrm{R}}_{x}\mathrm{D}^{f}_{\infty}\,\phi = {}^{\mathrm{C}}_{x}\mathrm{D}^{f}_{\infty}\,\phi = -b\cdot\frac{d}{dx}\phi(x) - \frac{d}{dx}\int_x^\infty \phi(t)\,\bar{\mu}(t-x)\,dt. \tag{2.4.11} $$

Proof. The proof is very similar to the proof of Lemma 2.1.11. We assume that ϕ is as regular as we need it for the calculation below. The definition of the sticky-type extension implies that

$$ \begin{aligned} &{}^{\mathrm{C}}_{0}\mathrm{D}^{f}_{x}\,\phi - b\cdot\frac{d}{dx}\phi(x) \\ &\quad= \int_0^\infty (\phi^\sigma(x) - \phi^\sigma(x-s))\,\mu(ds) \\ &\quad= \int_0^x (\phi(x) - \phi(x-s)\,\mu(ds) + \int_x^\infty (\phi(x) - \phi(0))\,\mu(ds) \\ &\quad= \int_0^x \int_0^s \phi'(x-t)\,dt\,\mu(ds) + (\phi(x)-\phi(0))\bar{\mu}(x) \\ &\quad\overset{\text{Fubini}}{=} \int_0^x \int_t^x \mu(ds)\,\phi'(x-t)\,dt + (\phi(x)-\phi(0))\bar{\mu}(x) \\ &\quad= \int_0^x (\bar{\mu}(t) - \bar{\mu}(x))\,\phi'(x-t)\,dt + (\phi(x)-\phi(0))\bar{\mu}(x) \end{aligned} $$

$$
\begin{aligned}
&= \int_0^x \phi'(x-t)\bar{\mu}(t)\,dt - \int_0^x \phi'(x-t)\,dt\bar{\mu}(x) + (\phi(x)-\phi(0))\bar{\mu}(x) \\
&= \int_0^x \phi'(t)\bar{\mu}(x-t)\,dt - \big(\phi(x)-\phi(0)\big)\bar{\mu}(x) + (\phi(x)-\phi(0))\bar{\mu}(x) \\
&= \int_0^x \phi'(t)\bar{\mu}(x-t)\,dt.
\end{aligned}
$$

Note that

$$
\frac{d}{dx}\int_0^x \phi(t)\bar{\mu}(x-t)\,dt = \frac{d}{dx}\int_0^x \phi(x-t)\bar{\mu}(t)\,dt = \phi(0)\bar{\mu}(x) + \int_0^x \phi'(x-t)\bar{\mu}(t)\,dt.
$$

Combining the formula (2.4.7) and the above observations, we get

$$
\begin{aligned}
{}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\,\phi &= {}^{\mathrm{C}}_{0}\mathrm{D}^{f}_{x}\,\phi + \phi(0)\bar{\mu}(x) \\
&= b\cdot\frac{d}{dx}\phi(x) + \int_0^x \phi'(t)\bar{\mu}(x-t)\,dt + \phi(0)\bar{\mu}(x) \\
&= b\cdot\frac{d}{dx}\phi(x) + \frac{d}{dx}\int_0^x \phi(t)\bar{\mu}(x-t)\,dt.
\end{aligned}
$$

For the third formula we use either the very same calculation or we make a direct attack like this:

$$
\begin{aligned}
\frac{d}{dx}\int_x^\infty \phi(t)\bar{\mu}(t-x)\,dt &= \frac{d}{dx}\int_0^\infty \phi(t+x)\bar{\mu}(t)\,dt \\
&= \int_0^\infty \phi'(t+x)\bar{\mu}(t)\,dt \\
&= \int_0^\infty \phi'(t+x)\int_t^\infty \mu(ds)\,dt \\
&= \int_0^\infty \int_0^s \phi'(t+x)\,dt\mu(ds) \\
&= -\int_0^\infty \big(\phi(x)-\phi(s+x)\big)\mu(ds).
\end{aligned}
$$

□

The following Corollary 2.4.6 gives a different representation of the Bernstein-Caputo derivative.

Corollary 2.4.6. *If $\phi \in C^1[0,\infty)$ we have the following equality*

$$
{}^{\mathrm{C}}_{0}\mathrm{D}^{f}_{x}\,\phi = b\cdot\frac{d}{dx}\phi(x) + \frac{d}{dx}\int_0^x (\phi(t)-\phi(0))\bar{\mu}(x-t)\,dt \tag{2.4.12}
$$

Proof. If $\phi \in C^1[0,\infty)$ we have following equalities

$$
\begin{aligned}
{}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\,\phi &= b\cdot\frac{d}{dx}\phi(x) + \frac{d}{dx}\int_0^x \phi(t)\,\bar{\mu}(x-t)\,dt \\
&= b\cdot\frac{d}{dx}\phi(x) + \frac{d}{dx}\int_0^x \left[\phi(0) + \int_0^t \phi'(s)\,ds\right]\bar{\mu}(x-t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= b\cdot \frac{d}{dx}\phi(x) + \frac{d}{dx}\int_0^x \phi(0)\bar{\mu}(x-t)\,dt + \frac{d}{dx}\int_0^x\int_0^t \phi'(s)\,ds\,\bar{\mu}(x-t)\,ds\,dt \\
&\overset{\text{Fubini}}{=} b\cdot \frac{d}{dx}\phi(x) + \frac{d}{dx}\int_0^x \phi(0)\bar{\mu}(x-t)\,dt + \frac{d}{dx}\int_0^x\int_s^x \phi'(s)\,ds\,\bar{\mu}(x-t)\,dt\,ds \\
&\overset{\text{x-t=u}}{=} b\cdot \frac{d}{dx}\phi(x) + \frac{d}{dx}\int_0^x \phi(0)\bar{\mu}(u)\,du + \frac{d}{dx}\int_0^x\int_0^{x-s} \bar{\mu}(u)\,du\,\phi'(s)\,ds \\
&= b\cdot \frac{d}{dx}\phi(x) + \phi(0)\bar{\mu}(x) + \int_0^x \frac{d}{dx}\int_0^{x-s} \bar{\mu}(u)\,du\,\phi'(s)\,ds \\
&= b\cdot \frac{d}{dx}\phi(x) + \phi(0)\bar{\mu}(x) + \int_0^x \bar{\mu}(x-s)\phi'(s)\,ds \\
&= \phi(0)\bar{\mu}(x) + {}_0^{\mathrm{C}}\mathrm{D}_x^f\,\phi
\end{aligned}
$$

This shows

$$
b\cdot \frac{d}{dx}\phi(x) + \frac{d}{dx}\int_0^x (\phi(t)-\phi(0))\bar{\mu}(x-t)\,dt = {}_0^{\mathrm{R}}\mathrm{D}_x^f\,\phi - \phi(0)\bar{\mu}(x) = {}_0^{\mathrm{C}}\mathrm{D}_x^f\,\phi. \qquad \square
$$

Lemma 2.4.7 (Laplace transform of Bernstein fractional derivatives)**.** *We have*

$$
\begin{aligned}
\mathscr{L}({}_0^{\mathrm{R}}\mathrm{D}_x^f\,\phi; s) &= f(s)\mathscr{L}\phi(s) - b\phi(0), \\
\mathscr{L}({}_0^{\mathrm{C}}\mathrm{D}_x^f\,\phi; s) &= f(s)\mathscr{L}\phi(s) - s^{-1}f(s)\phi(0),
\end{aligned}
$$

Proof. For $\bar{\mu}(s) = \mu(s,\infty)$, we observe that by Fubini's theorem

$$
\mathscr{L}\bar{\mu}(s) = \int_0^\infty e^{-sx}\int_s^\infty \mu(dt)\,dx = \frac{1}{s}\int_0^\infty (1-e^{-st})\,\mu(dt),
$$

and therefore

$$
b + \mathscr{L}\bar{\mu}(s) = s^{-1}f(s). \tag{2.4.13}
$$

Note that ${}_0^{\mathrm{R}}\mathrm{D}_x^f\,\phi = b\phi'(x) + (\phi * \bar{\mu})'(x)$ is (the derivative of) a Laplace convolution. With the rules for the Laplace transform from Section 1.2 we get

$$
\begin{aligned}
\mathscr{L}({}_0^{\mathrm{R}}\mathrm{D}_x^f\,\phi; s) &\overset{(2.4.9)}{=} b\mathscr{L}\left(\phi'; s\right) + \mathscr{L}\left(\frac{d}{dx}(\phi * \bar{\mu}); s\right) \\
&\overset{(1.2.10)}{=} b\left(s\mathscr{L}\phi; s) - \phi(0)\right) + s\mathscr{L}\left(\phi * \bar{\mu}; s\right) \\
&\overset{(1.2.8)}{=} s\mathscr{L}(\phi; s)(b + \mathscr{L}\bar{\mu}) - b\phi(0) \\
&\overset{(2.4.13)}{=} f(s)\mathscr{L}\phi(s) - b\phi(0).
\end{aligned}
$$

For the Caputo derivative, we get in a similar way

$$
\begin{aligned}
\mathscr{L}({}_0^{\mathrm{C}}\mathrm{D}_x^f\,\phi; s) &\overset{(2.4.10)}{=} b\mathscr{L}\left(\phi'; s\right) + \mathscr{L}\left(\phi' * \bar{\mu}; s\right) \\
&\overset{(1.2.8)}{=} \mathscr{L}(\phi'; s)(b + \mathscr{L}(\bar{\mu}; s))
\end{aligned}
$$

$$\overset{(1.2.10)}{=} (s\mathscr{L}(\phi; s) - \phi(0))\, s^{-1}(b + \mathscr{L}\,(\bar{\mu}; s))$$

$$\overset{(2.4.13)}{=} f(s)\mathscr{L}\phi(s) - s^{-1} f(s)\phi(0). \qquad \square$$

Remark 2.4.2 shows a sufficient condition for $\phi \in L^2(\mathbb{R})$ so that ${}^{\mathrm{M}}\mathrm{D}^f_{\pm} \in L^2(\mathbb{R})$. This we will need now.

Lemma 2.4.8. *Let $\phi, \psi \in L^2(\mathbb{R})$ be such that ${}^{\mathrm{M}}\mathrm{D}^f_{+}\,\phi$, ${}^{\mathrm{M}}\mathrm{D}^f_{-}\,\psi \in L^2(\mathbb{R})$. Then,*

$$\langle\, {}^{\mathrm{M}}\mathrm{D}^f_{+}\,\phi, \psi\rangle = \langle \phi,\, {}^{\mathrm{M}}\mathrm{D}^f_{-}\,\psi\rangle.$$

Proof. The proof is similar to the proof of Lemma 2.1.22 with $\mu(dt) = \alpha(\Gamma(1-\alpha)t^{1+\alpha})^{-1}dt$. Let $\phi, \psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be as regular as we need it for the following calculation.

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{0+}^{\infty} (\phi(x) - \phi(x-t))\;\mu(dt)\cdot\psi(x)\,dx \\
&= \lim_{\epsilon\downarrow 0}\int_{-\infty}^{\infty}\int_{\epsilon}^{\infty} (\phi(x) - \phi(x-t))\;\mu(dt)\cdot\psi(x)\,dx \\
&\overset{\text{Fubini}}{=} \lim_{\epsilon\downarrow 0}\left[\int_{-\infty}^{\infty}\int_{\epsilon}^{\infty} \phi(x)\psi(x)\,\mu(dt)\,dx - \int_{\epsilon}^{\infty}\int_{-\infty}^{\infty} \phi(x-t)\psi(x)\,dx\,\mu(dt)\right] \\
&\underset{\text{in 2nd }\int}{\overset{x \rightsquigarrow x+t}{=}} \lim_{\epsilon\downarrow 0}\left[\int_{-\infty}^{\infty}\int_{\epsilon}^{\infty} \phi(x)\psi(x)\,\mu(dt)\,dx - \int_{\epsilon}^{\infty}\int_{-\infty}^{\infty} \phi(x)\psi(x+t)\,dx\,\mu(dt)\right] \\
&\overset{\text{Fubini}}{=} \lim_{\epsilon\downarrow 0}\left[\int_{-\infty}^{\infty}\int_{\epsilon}^{\infty} \phi(x)\psi(x)\,\mu(dt)\,dx - \int_{-\infty}^{\infty}\int_{\epsilon}^{\infty} \phi(x)\psi(x+t)\,\mu(dt)\,dx\right] \\
&= \lim_{\epsilon\downarrow 0}\int_{-\infty}^{\infty}\phi(x)\int_{\epsilon}^{\infty} (\psi(x) - \psi(x+t))\;\mu(dt)\,dx \\
&= \int_{-\infty}^{\infty}\phi(x)\int_{0+}^{\infty} (\psi(x) - \psi(x+t))\;\mu(dt)\,dx
\end{aligned}$$

Since $\int_{-\infty}^{\infty} b\phi'(x)\cdot\psi(x)dx = -\int_{-\infty}^{\infty} b\phi(x)\cdot\psi'(x)dx$, we get the conclusions. $\square$

2.5 Probabilistic interpretation of fractional derivatives

Let us consider a simple stochastic jump process and its corresponding generator in this section.

Take a Markov chain $P = \{p_{i,j}\}_{i,j\in\Omega}$ with state space Ω and write out its generator

$$\mathcal{G}\phi(x) := (P - I)\phi(x) = \sum_{y\in\Omega}(\phi(y) - \phi(x))p_{x,y},\, x \in \Omega \qquad (2.5.1)$$

Here the intuition is as follows: the jump from x to y is assigned the probability $p_{x,y}$. The operator $\mathcal{G}$ is non-local. If we modify the process (impose boundary conditions), say by forcing the process to absorbed at $a \in \Omega$ once it tries to jump to a state $y \notin \Omega$, we obtain a new generator

$$\mathcal{G}\phi^{\sigma}(x) := (P - I)\phi^{\sigma}(x) = \sum_{y\in\Omega}(\phi(y) - \phi(x))p_{x,y} + (\phi(a) - \phi(x)) \sum_{y\notin\Omega} p_{x,y}, x \in \Omega \quad (2.5.2)$$

If we kill it, the new generator will be

$$\mathcal{G}\phi^{0}(x) := (P - I)f^{0}(x) = \sum_{y\in\Omega}(\phi(y) - \phi(x))p_{x,y} - \phi(x) \sum_{y\notin\Omega} p_{x,y}, x \in \Omega \quad (2.5.3)$$

If we shift it, the new generator will be

$$\mathcal{G}\phi^{s}(x) := (P - I)\phi^{s}(x) = \sum_{y\in\Omega}(\phi(y) - \phi(x))p_{x,y}, x \in \Omega \quad (2.5.4)$$

As mentioned in a comment above, the boundary conditions are reflected in the representation of the operator away from the boundary due to the non-locality of $\mathcal{G}$.

Let $(S_t)_{t\geq 0}$ be a subordinator with characteristic pair (b, μ). Assume that S_β is a β stable subordinator with generator $\mu_\beta(s) = \frac{\beta}{\Gamma(1-\beta)} s^{1+\beta}$.

$$\mathcal{A}_\beta\phi(t) = \int_0^\infty (\phi(t + s) - \phi(t))\ \mu_\beta(ds). \quad (2.5.5)$$

S^*_β is the adjoint of β stable subordinator with generator

$$\mathcal{A}^*_\beta\phi(t) = \int_0^\infty (\phi(t - s) - \phi(t))\, \mu_\beta(ds). \quad (2.5.6)$$

then $-{}^{\mathrm{M}}\mathrm{D}^\beta_+ = \mathcal{A}_\beta, -{}^{\mathrm{M}}\mathrm{D}^\beta_- = \mathcal{A}^*_\beta$.

Furthermore, ${}^{\mathrm{R}}_{0}\mathrm{D}^\beta_\cdot\, u = {}^{\mathrm{M}}\mathrm{D}^\beta_+\, u^0(x) = -\mathcal{A}^*_\beta u^0$, see Corollary 2.1.23. ${}^{\mathrm{R}}_{0}\mathrm{D}^\beta_x$ is the adjoint of the generator of a β stable subordinator killed on the first attempt to jump inside $\Omega = (-\infty, 0)$.

${}^{\mathrm{M}}\mathrm{D}^\beta_+\, u^\sigma(x) = {}^{\mathrm{C}}_{0}\mathrm{D}^\beta_x\, u = -\mathcal{A}^*_\beta u^\sigma$. Thus, ${}^{\mathrm{C}}_{0}\mathrm{D}^\beta_\cdot$ is the adjoint of the generator of a β stable subordinator absored at $\{0\}$ on the first attempt to jump into $\Omega = (-\infty, 0)$.

2.6 Fractional Laplace operator

In recent years, there has been significant interest in utilizing the fractional Laplacian as a mathematical tool for modeling various physical phenomena, including anomalous diffusion, turbulence, and water waves. The fractional Laplacian finds applications in probability theory and finance as well. Notably, it can be viewed as the infinitesimal generator of a stable Lévy process and plays a crucial role in describing anomalous diffusions. Moreover, it is closely connected to the concept of fractional derivatives. In this context, we present several commonly used definitions that have been frequently referenced in the literature which taken from Kwasnicki [47].

Definition 2.6.1. Let $B_r(x) = \{y : |x-y| < r\}$ and

(1) $\mathcal{D}((-\Delta)^\alpha, x) := \{\phi \mid \forall x \in \mathbb{R}^d$ (2.6.1) exists pointwise$\}$.

(2) $\mathcal{D}((-\Delta)^\alpha_s, L^p) := \{\phi \in L^p \mid$ (2.6.4) exists in $L^p, p \in [1,\infty)\}$.

(3) $\mathcal{D}((-\Delta)^\alpha, L^p) := \{\phi \in L^p \mid$ (2.6.1) exists in $L^p, p \in [1,\infty)\}$.

(4) $\mathcal{D}((-\Delta)^\alpha_v, x) := \{\phi \mid \forall x \in \mathbb{R}^d, \nabla f(x)$ exists and (2.6.2) converges absolutely$\}$.

(5) $\mathcal{D}((-\Delta)^\alpha_{va}, x) := \{\phi \mid \forall x \in \mathbb{R}^d$, (2.6.3) converges absolutely$\}$.

(6) $\mathcal{D}((-\Delta)^\alpha_{fc}, x) := \{\phi \in L^p \mid \lim_{n\to\infty} \int_0^n (-\Delta) R_\lambda (-\Delta)\phi \lambda^{1-\alpha}\, d\lambda$ exists in L^p sense$\}$.

There are several (equivalent) ways to define the fractional Laplacian:

(a) singular integral definition:

$$(-\Delta)^\alpha \phi(x) = c_{d,\alpha} \int_{\mathbb{R}^d} \frac{\phi(x) - \phi(y)}{|x-y|^{d+2\alpha}} dy, \tag{2.6.1}$$

where $c_{d,\alpha} = \frac{4^\alpha \Gamma(\frac{d}{2}+\alpha)}{\pi^{\frac{d}{2}} |\Gamma(-\alpha)|}$,

1) if $\phi \in C_c^\infty$

i) $2\alpha < 1$ then (2.6.1) is convergent and it is a classical integral.

ii) $2\alpha \geq 1$ then (2.6.1) is given by the Cauchy principal value integral that is

$$(-\Delta)^\alpha \phi(x) = c_{d,\alpha} \lim_{r\to 0} \int_{B_r^c} \frac{\phi(x) - \phi(y)}{|x-y|^{d+2\alpha}} dy,$$

This limit can be understood in the sense of pointwise convergence, namely for a given $x \in \mathbb{R}^d$ the limit exists.

2) finally, we can extend $(-\Delta)^\alpha$ to more general functions (Lipschitz, Hölder or L^p...).

(b) variant of the singular integral definition:

$$(-\Delta)^\alpha_v \phi(x) = c_{d,\alpha} \int_{\mathbb{R}^d} \frac{\phi(x+z) - \phi(x) - z\nabla\phi(x) 1_{B_r(z)}}{|z|^{d+2\alpha}} dz, \tag{2.6.2}$$

where $c_{d,\alpha} = \frac{4^\alpha \Gamma(\frac{d}{2}+\alpha)}{\pi^{\frac{d}{2}} |\Gamma(-\alpha)|}$.

(c) another variant of the singular integral definition:

$$(-\Delta)^\alpha_{va} \phi(x) = \frac{1}{2} c_{d,\alpha} \int_{\mathbb{R}^d} \frac{\phi(x+z) + \phi(x-z) - 2\phi(x)}{|z|^{d+2\alpha}} dz, \tag{2.6.3}$$

where $c_{d,\alpha} = \frac{4^\alpha \Gamma(\frac{d}{2}+\alpha)}{\pi^{\frac{d}{2}} |\Gamma(-\alpha)|}$.

(d) semigroup definition:

$$(-\Delta)^\alpha_s \phi = L^p - \lim_{t\to 0} \frac{P_t\phi - \phi}{t}, \tag{2.6.4}$$

Here $\mathcal{F}p_t(\xi) = e^{-t|\xi|^\alpha}$, $P_t\phi = \phi * p_t$. The operators P_t form a strongly continuous contraction semigroup in $L^p, p \in [1, \infty)$.

(e) Bernstein functional calculus definition (see [69] and [68], [20]) Denote $(-\Delta)$ the Laplace operator on $L^p(\mathbb{R}^d)$ with domain $\mathcal{D}(-\Delta) = W^{p,2}(\mathbb{R}^d)$ and write $(R_\lambda(-\Delta))_{\lambda>0}$ for its resolvent

$$(-\Delta)^\alpha_{fc}\phi(x) = c_\alpha \int_0^\infty (-\Delta)R_\lambda(-\Delta)\phi(x)\lambda^{\alpha-1}\, d\lambda,$$

with $c_\alpha = \sin(\alpha\pi)/\alpha$.

(f) Fourier definition:
$\mathcal{S}$ denotes the set of Schwartz functions. If $\phi \in \mathcal{S}$, then

$$\mathscr{F}[(-\Delta)^\alpha\phi(x), \xi] = |\xi|^{2\alpha}\mathscr{F}\phi(\xi). \tag{2.6.5}$$

Next, we will give the relationship between pointwise definitions involved in Definition 2.6.1. The following two results are standard; we refer to Kwasnicki [47].

Remark 2.6.2. For a given x, if $\phi \in \mathcal{D}((-\Delta)^\alpha_v, x)$ then

$$\phi \in \mathcal{D}((-\Delta)^\alpha, x),\ (-\Delta)^\alpha_v\phi(x) = (-\Delta)^\alpha\phi(x);$$

If $\phi \in \mathcal{D}((-\Delta)^\alpha_{va}, x)$, then

$$\phi \in \mathcal{D}((-\Delta)^\alpha, x),\ (-\Delta)^\alpha_{va}\phi(x) = (-\Delta)^\alpha\phi(x).$$

Furthermore the pointwise results can be re-used for the corresponding statements for norm convergence in any of the spaces $L^p(\mathbb{R}^d)$. $\phi \in \mathcal{D}((-\Delta)^\alpha, L^p(\mathbb{R}^d))$ if and only if $\phi \in \mathcal{D}((-\Delta)^\alpha_s, L^p(\mathbb{R}^d))$.

Example 2.6.3. Take $\mathcal{A} = (-\Delta)^\alpha, \mathcal{D}(\mathcal{A}) = \mathcal{D}((-\Delta)^\alpha, L^p(\mathbb{R}^d))$. According to the Hille–Yosida Theorem 1.4.5 and Definition 2.6.1, we know that $\big((-\Delta)^\alpha, \mathcal{D}((-\Delta)^\alpha, L^p(\mathbb{R}^d))\big)$ is closed.

An application of the BLT Theorem 1.1.3 gives the following Corollary.

Corollary 2.6.4. $(-\Delta)^\alpha$ *defined by (2.6.1), extends by continuity to* $W^{p,2}(\mathbb{R}^d)$. *In fact, for all* $\phi \in W^{p,2}(\mathbb{R}^d)$ *we have*

$$\|(-\Delta)^\alpha\phi\|_{L^p} \le C(\|(-\Delta)\phi\|_{L^p} + \|\phi\|_{L^p}) \tag{2.6.6}$$

Proof. We take $D = C_c^2(\mathbb{R}^d), \mathcal{X} = W^{p,2}(\mathbb{R}^d), A = (-\Delta)^\alpha, \mathcal{Y} = L^p(\mathbb{R}^d)$ in Theorem 1.1.3. According to Theorem 1.1.5, $D = C_c^2(\mathbb{R}^d)$ is a dense subspace in $W^{p,2}(\mathbb{R}^d)$. Using Farkas, Jacob and Schilling [20] Theorem 1.4.8, we have the following estimate

$$\begin{aligned}
&|(-\Delta)^\alpha)\phi(x)| \\
&= C_\alpha \left| \int_0^\infty (-\Delta) R_\lambda (-\Delta)\phi(x)\lambda^{\alpha-1}\, d\lambda \right| \\
&\le C_\alpha \left| \int_0^1 (-\Delta) R_\lambda (-\Delta)\phi(x)\lambda^{\alpha-1}\, d\lambda \right| + C_\alpha \left| \int_1^\infty (-\Delta) R_\lambda (-\Delta)\phi(x)\lambda^{\alpha-1}\, d\lambda \right| \\
&\le C_\alpha \left| \int_0^1 \phi(x)\lambda^{\alpha-1}\, d\lambda \right| + C_\alpha \left| \int_0^1 \lambda R_\lambda (-\Delta)\phi(x)\lambda^{\alpha-1}\, d\lambda \right| \\
&\quad + C_\alpha \left| \int_1^\infty (-\Delta) R_\lambda (-\Delta)\phi(x)\lambda^{\alpha-1}\, d\lambda \right| \\
&\le 2C_\alpha \|f\|_{L^p} + C_\alpha \|(-\Delta)\phi\|_{L^p} \\
&< \infty
\end{aligned}$$

Then we have $\|(-\Delta)^\alpha \phi\|_{L^p} \le C(\|(-\Delta)\phi\|_{L^p} + \|\phi\|_{L^p}) \le C\|\phi\|_{W^{p,2}}$. Applying the bounded linear transform Theorem, we can extend $(-\Delta)^\alpha$ to $W^{p,2}(\mathbb{R}^d)$ continuously. □

The Marchaud derivatives are known see [23] for their connection to the fractional Laplacian. Note that the Marchaud derivatives are equivalent to the left and right Riemann–Liouville derivatives under sufficiently good assumptions on the behavior of the function. The relationship between the Marchaud derivatives and the 1-dimensional fractional Laplacian is given in the following theorem. Furthermore, under some suitable conditions, we can get the relationship between general fractional Marchaud derivatives and generalized Laplace operators defined by Bernstein function.

Theorem 2.6.5. *Let $\alpha \in (0,1)$ and $\phi \in \mathcal{D}((-\Delta)_{va}^{\alpha/2})$, see Definition 2.1.7 and Definition 2.6.1. Then we have*

$$ {}^{\mathrm{M}}\mathrm{D}_+^\alpha \phi(x) + {}^{\mathrm{M}}\mathrm{D}_-^\alpha \phi(x) = -\frac{\alpha}{c_{1,\alpha}\Gamma(1-\alpha)}(-\Delta)^{\frac{\alpha}{2}}\phi(x), \tag{2.6.7}$$

where $c_{1,\alpha} = 4^\alpha \Gamma(\frac{1}{2}+\alpha)/(\pi^{\frac{1}{2}}|\Gamma(-\alpha)|)$.

Proof. Using Remark 2.6.2 we have we have $(-\Delta)_{va}^{\alpha/2} = (-\Delta)^{\alpha/2}$. Using the definition 2.1.7, we obtain

$$\begin{aligned}
&{}^{\mathrm{M}}\mathrm{D}_+^\alpha \phi(x) + {}^{\mathrm{M}}\mathrm{D}_-^\alpha \phi(x) \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^\infty \frac{\phi(x)-\phi(x-t)}{t^{1+\alpha}}\, dt + \frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^\infty \frac{\phi(x)-\phi(x+t)}{t^{1+\alpha}}\, dt \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_{0+}^\infty \frac{2\phi(x)-\phi(x+t)-\phi(x-t)}{t^{1+\alpha}}\, dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{2\Gamma(1-\alpha)} \int_{-\infty}^{\infty} \frac{2\phi(x) - \phi(x+t) - \phi(x-t)}{|t|^{1+\alpha}}\, dt \\
&= -\frac{\alpha}{c_{1,\alpha}\Gamma(1-\alpha)} \frac{c_{1,\alpha}}{2} \int_{-\infty}^{\infty} \frac{\phi(x+t) + \phi(x-t) - 2\phi(x)}{|t|^{1+\alpha}}\, dt \\
&= -\frac{\alpha}{c_{1,\alpha}\Gamma(1-\alpha)} (-\Delta)^{\frac{\alpha}{2}} \phi(x).
\end{aligned}
$$ □

Theorem 2.6.6. *Let f_1, f_2 be Bernstein functions given by*

$$
f_1(\lambda) = \int_0^{\infty} (1 - e^{-\lambda x})\, \mu_1(dx), \quad \text{and} \quad f_2(\lambda) = b_2 \lambda + \int_0^{\infty} (1 - e^{-\lambda x})\, \mu_2(dx),
$$

respectively where $b_2 \geq 0$ and μ_1, μ_2 are Lévy measures satisfying $\int_0^1 \sqrt{x}\, \mu_1(dx) < \infty$, and

$$
\mu_2(dt) = \int_0^{\infty} k_s(t)\, \mu_1(ds) dt, \quad k_s(t) = (4\pi s)^{-\frac{1}{2}} e^{\frac{-t^2}{4s}},
$$

then we have for all ϕ in $\mathcal{D}((-\Delta)^{f_1})$ [2] *with $\phi \in \mathcal{D}({}^{\mathrm{M}}\mathrm{D}_{\pm}^{f_2})$,*

$$
{}^{\mathrm{M}}\mathrm{D}_{+}^{f_2}\, \phi(x) + {}^{\mathrm{M}}\mathrm{D}_{-}^{f_2}\, \phi(x) = -(-\Delta)^{f_1} \phi(x). \tag{2.6.8}
$$

Proof. By the definition we have $\int_0^1 t\mu_1(dt) < \infty$. Let us show the following inequality, $\int_0^1 t\mu_2(dt) < \infty$. By the definition of $k_s(t)$, we have

$$
\begin{aligned}
\int_0^1 t\mu_2(dt) &= \int_0^1 t \int_0^{\infty} (4\pi s)^{-\frac{1}{2}} e^{\frac{-t^2}{4s}}\, \mu_1(ds)\, dt \\
&\overset{\text{Tonelli}}{=} \int_0^{\infty} (4\pi s)^{-\frac{1}{2}} \int_0^1 t e^{\frac{-t^2}{4s}}\, dt\, \mu_1(ds) \\
&= \pi^{-\frac{1}{2}} \int_0^{\infty} s^{\frac{1}{2}} (1 - e^{-\frac{1}{4s}})\, \mu_1(ds).
\end{aligned}
$$

Let us show that the following integral is finite:

$$
\begin{aligned}
&\frac{1}{\sqrt{\pi}} \int_0^{\infty} \sqrt{s}(1 - e^{-\frac{1}{4s}})\, \mu_1(ds) \\
&\leq \frac{1}{\sqrt{\pi}} \int_0^1 \sqrt{s}\, \mu_1(ds) + \frac{1}{\sqrt{\pi}} \int_1^{\infty} \frac{\sqrt{s}}{4s}\, \mu_1(ds) \\
&\leq \frac{1}{\sqrt{\pi}} \int_0^1 \sqrt{s}\, \mu_1(ds) + \frac{1}{\sqrt{\pi}} \int_1^{\infty} \mu_1(ds) \\
&< \infty,
\end{aligned}
$$

where the last two we use the elementary estimate $1 - e^{-x} \leq x$ if $x \in (0, 1)$.

[2] $\mathcal{D}((-\Delta)^{f_1})$ is the domain of the subordinate generator, see [69, Chapter 13] and [68, 21].

According to Theorem 1.5.4, we can get the expression of the subordinate Laplace operator. Using symmetry of $k_s(z)$, we have

$$\begin{aligned}(-\Delta)^{f_1}u(x) &= \int_0^\infty (e^{s\Delta}\phi(x)-\phi(x))\,\mu_1(ds)\\ &= \int_0^\infty (\phi * k_s(x)-\phi(x))\,\mu_1(ds)\\ &= \int_0^\infty \int_{\mathbb{R}} (\phi(x+z)-\phi(x))k_s(z)\,dz\,\mu_1(ds)\\ &\overset{\text{Fubini}}{=} \int_{\mathbb{R}} (\phi(x+z)-\phi(x)) \int_0^\infty k_s(z)\,\mu_1(ds)\,dz.\end{aligned}$$

By the same process, we have due to the even symmetry of $z \mapsto k_s(z)$

$$(-\Delta)^{f_1}\phi(x) = \int_{\mathbb{R}} (\phi(x-z)-\phi(x)) \int_0^\infty k_s(z)\,\mu_1(ds)\,dz.$$

Recall that $f_2(\lambda) = b_2\lambda + \int_0^\infty (1-e^{-\lambda x})\mu_2(dx),$ and $\mu_2(dz) = \int_0^\infty k_s(z)\mu_1(ds)dz$, then from Definition 2.4.1, we have

$$\begin{aligned}{}^{\mathrm{M}}\mathrm{D}_+^{f_2}\,\phi(x) &:= b_2\cdot\frac{d}{dx}\phi(x) + \int_0^\infty (\phi(x)-\phi(x-t))\ \mu_2(dt).\\ {}^{\mathrm{M}}\mathrm{D}_-^{f_2}\,\phi(x) &:= -\,b_2\cdot\frac{d}{dx}\phi(x) + \int_0^\infty (\phi(x)-\phi(x+t))\ \mu_2(dt).\end{aligned}$$

Compared with the above results, we have the following connection,

$$\begin{aligned}{}^{\mathrm{M}}\mathrm{D}_+^{f_2}\,\phi(x) + {}^{\mathrm{M}}\mathrm{D}_-^{f_2}\,\phi(x) &= \int_0^\infty (\phi(x)-\phi(x-t))\ \mu_2(dt) + \int_0^\infty (\phi(x)-\phi(x+t))\ \mu_2(dt)\\ &= \int_0^\infty (2\phi(x)-\phi(x-t)-\phi(x+t))\ \mu_2(dt)\\ &= \frac{1}{2}\int_{\mathbb{R}} (2\phi(x)-\phi(x-t)-\phi(x+t))\ \mu_2(dt)\\ &= \frac{1}{2}\int_{\mathbb{R}} (2\phi(x)-\phi(x-t)-\phi(x+t)) \int_0^\infty k_s(t)\,\mu_1(ds)\,dt\\ &= -(-\Delta)^{f_1}u(x).\end{aligned}$$

□

Chapter 3

Censored Bernstein fractional derivative and integral

In this chapter, we focus on studying the censored Bernstein fractional derivative and integral. Initially, we gather some results and examples concerning Sonine pairs in the first two sections. Sonine pairs play a key role in determining whether a given pair of functions can serve as kernels for generalized fractional derivatives and their associated generalized fractional integrals. Upon introducing Sonine pairs, we define the corresponding function space in Section 3.3 and further investigate the mapping properties of fractional derivatives. Section 3.4 is dedicated to defining the censored Bernstein fractional derivative and integrals. Lastly, in Section 3.5, we examine the existence and uniqueness of a linear censored initial equation.

3.1 Sonine pairs

In this section, we introduce a special first kind of Volterra equations, known as the Sonine pairs (Sonine equation), given by equation (3.1.1). The formal solution to such equations was discovered a long time ago and has been extensively studied, see Samko, Kilbas, and Marichev [64]. The research on such kernels dates back to the 19th century.

Definition 3.1.1. Let $h, g \in L^1_{\text{loc}}(0, \infty)$. We call h, g a **positive Sonine pair** if

$$h * g(x) = 1, \quad \forall x > 0. \tag{3.1.1}$$

where $h, g \geq 0$ are either both functions or by abuse of notation one of them is a positive function and the other is a positive measure.

A generalized Riemann-Liouville fractional derivative defined by Lemma 2.4.5, denoted by ${}^{\text{R}}_{0}\text{D}^{f}_{\cdot}$, and an associated generalized fractional integral, denoted by ${}^{\text{R}}_{0}\text{I}^{f}_{\cdot}$ with corresponding kernels $\bar{\mu}$ and k, are shown to be a Sonine pair, as given by,

$$(k * \bar{\mu})(x) = \int_0^x k(x-s)\bar{\mu}(s)ds = 1, x > 0. \tag{3.1.2}$$

If we consider $k(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}$ and $\bar{\mu}(x) = \frac{1}{\Gamma(1-\alpha)x^{\alpha}}$, then the α-order fractional integral and derivative can be defined, and their convolution is equal to 1, as in eq. (3.1.1).

The connection between the complete Bernstein functions and Sonine pairs have been noticed by [42], and it has significant applications due to the "nonlinear" properties of complete Bernstein functions.

If $f \in \mathcal{BF}$, then $\frac{f(\lambda)}{\lambda} \in \mathcal{CM}$ and $\frac{1}{f(\lambda)} \in \mathcal{CM}$, i.e. there exist measures ν and κ on $[0, \infty)$ with

$$\frac{1}{\lambda} = \frac{f(\lambda)}{\lambda}\frac{1}{f(\lambda)} = \mathscr{L}(\nu; \lambda)\mathscr{L}(\kappa, \lambda) = \mathscr{L}(\nu * \kappa, \lambda).$$

Thus, $\nu * \kappa(dx) = dx$ i.e ν, κ are a Sonine pair if one (or both) of the measures are absolutely continuous w.r.t. Lebesgue measure. From Theorem 1.5.12 we know that κ is the potential or renewal measure of the subordinator with Laplace exponent $f \in \mathcal{BF}$, the following renewal theorem gives some first information.

Theorem 3.1.2. *Let $f \in \mathcal{BF}$ with triplet (a, b, μ) and assume that $b = \lim_{\lambda\to\infty} \frac{f(\lambda)}{\lambda} > 0$. Then $\kappa(dx) = k(x)\,dx$ and $k(x) > 0$ and $x \mapsto k(x)$ is continuous.*

Proof. A proof can be fund in Bertoin [8] preposition 1.7 but we prefer to give a simple argument in the spirit of this thesis. We write $f(\lambda) = b\lambda + f_0(\lambda)$, i.e. $f_0 \in \mathcal{BF}$ with triplet $(a, 0, \mu)$. Clearly, $f_0(\lambda)/\lambda \in \mathcal{CM}$ and

$$\frac{f_0(\lambda)}{\lambda} = \frac{a}{\lambda} + \frac{1}{\lambda}\int_0^\infty (1 - e^{-\lambda x})\,\mu(dx) = \mathscr{L}(a\delta_0 + \bar{\mu}, \lambda) = \mathscr{L}(\nu_0, \lambda),$$

for the measure $\nu_0 = a\delta_0 + \bar{\mu}$, and $\bar{\mu}(x) = \mu[x, \infty)$.

$$\begin{aligned}\frac{1}{f(\lambda)} &= \frac{1}{b\lambda + f_0(\lambda)} = \frac{1}{b\lambda}\frac{1}{1 + \frac{f_0(\lambda)}{b\lambda}} \\ &= \frac{1}{b\lambda}\sum_{i=0}^{\infty}(-1)^i \left(\frac{f_0(\lambda)}{b\lambda}\right)^i.\end{aligned}$$

Notice that this series converges for λ large enough since $\lim_{\lambda\to\infty} \frac{f_0(\lambda)}{b\lambda} = 0$. Now

$$\begin{aligned}\mathscr{L}(\kappa, \lambda) = \frac{1}{f(\lambda)} &= \sum_{i=0}^{\infty}(-1)^i \frac{1}{b^{i+1}}\mathscr{L}(1; \lambda)\mathscr{L}(\nu_0; \lambda)^i \\ &= \mathscr{L}\left(\sum_{i=0}^{\infty}(-1)^i \frac{1}{b^{i+1}} 1 * \nu_0^{*i}, \lambda\right) \\ &= \mathscr{L}\left(1 * \sum_{i=0}^{\infty}(-1)^i \frac{1}{b^{i+1}} \nu_0^{*i}, \lambda\right).\end{aligned}$$

This proves our claim since, by the uniqueness of the Laplace transform,

$$\kappa = 1 * \rho, \quad \rho = \sum_{i=0}^{\infty} (-1)^i \frac{1}{b^{i+1}} \nu_0^{*i}.$$

Here ρ is a (possibly signed) non-trivial measure. Note that

$$(a\delta_0 + \bar{\mu})^{*n} = \sum_{i=0}^{n-1} \binom{n}{i} a^i \bar{\mu}^{*(n-i)} + a^n \delta_0,$$

i.e.

$$\begin{aligned}
\rho &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{b^{n+1}} \sum_{i=0}^{n} \binom{n}{i} a^i \bar{\mu}^{*(n-i)} \\
&= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} (-1)^n \frac{1}{b^{n+1}} a^i \bar{\mu}^{*(n-i)} \\
&= \sum_{i=0}^{\infty} (-1)^i \frac{1}{b} \frac{a^i}{b^i} \left(\delta_0 + \sum_{n=1}^{\infty} (-1)^n \binom{n+i}{i} \bar{\mu}^{*n} \right) \\
&= \frac{\delta_0}{a+b} + \sum_{i=0}^{\infty} (-1)^i \frac{1}{b} \frac{a^i}{b^i} \left(\sum_{n=1}^{\infty} (-1)^n \binom{n+i}{i} \bar{\mu}^{*n} \right).
\end{aligned}$$

These calculations are justified as

$$\sum_{i=0}^{\infty} \left| (-1)^i \frac{1}{b^{i+1}} \frac{1}{\lambda} \left(\frac{f_0(\lambda)}{\lambda} \right)^i \right| = \frac{1}{b\lambda} \sum_{i=0}^{\infty} \left(\frac{f_0(\lambda)}{\lambda b} \right)^i < \infty,$$

for λ large enough. Thus,

$$\kappa = \frac{1}{a+b} 1 * \delta_0 + 1 * \sum_{i=0}^{\infty} (-1)^i \frac{1}{b} \frac{a^i}{b^i} \left(\sum_{n=1}^{\infty} (-1)^n \binom{n+i}{i} \bar{\mu}^{*n} \right),$$

i.e. $\kappa(dx) = k(x)\,dx$ with the continuous function

$$k(x) = \frac{1}{a+b} + \int_0^x \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{i+n} \frac{1}{b} \frac{a^i}{b^i} \binom{n+i}{i} \bar{\mu}^{*n}(x)\,dx.$$

This finishes this proof. □

We will now discuss futher examples of Sonine pairs by specializing our choice of Bernstein functions and present here several results concerning the relationship between complete Bernstein functions and Sonine pairs.

Let $f \in \mathcal{BF}$ be given by (2.3.5) and assume that $f^\star(\lambda) = \frac{\lambda}{f(\lambda)}$. Then

$$f^\star(\lambda) = \frac{\lambda}{f(\lambda)} = a^\star + b^\star \lambda + \int_0^\infty (1 - e^{-\lambda t}) \mu^\star(dt) \tag{3.1.3}$$

Theorem 3.1.3. *Let $f \in \mathcal{CBF}$ be a complete Bernstein function with triplet (a, b, μ), $\bar{\mu}(x) = \mu[x, \infty)$ and $f^\star = \lambda/f(\lambda)$ with triplet $(a^\star, b^\star, \mu^\star)$, $k(x) = \mu^\star[x, \infty)$ as in* (3.1.3)*. Then*

$$\frac{f(\lambda)}{\lambda} = \mathscr{L}(a + \bar{\mu} + b\delta_0; \lambda);$$

$$\frac{1}{f(\lambda)} = \mathscr{L}(a^\star + k + b^\star\delta_0; \lambda).$$

are Stieltjes functions and $a + \bar{\mu}$ and $a^\star + k$ are completely monotone functions. Moreover, using the convention that $1/\infty = 0$. One has

$$f^\star(0) = a^\star = \lim_{\lambda\to 0}\frac{\lambda}{f(\lambda)} = \begin{cases} 0, & a > 0, \\ \dfrac{1}{b + \int_0^\infty t\mu(dt)}, & a = 0. \end{cases} \tag{3.1.4}$$

$$b^\star = \lim_{\lambda\to\infty}\frac{f^\star(\lambda)}{\lambda} = \lim_{\lambda\to\infty}\frac{1}{f(\lambda)} = \begin{cases} 0, & b > 0, \\ \dfrac{1}{a + \int_0^\infty \mu(dt)}, & b = 0. \end{cases} \tag{3.1.5}$$

and $(a + \bar{\mu}(x)) + b\delta_0$ and $(a^\star + k) + b^\star\delta_0$ is a Sonine pair. If $b = b^\star = 0$ see Table 3.1 or (3.1.4) *and* (3.1.5)*, then $a + \bar{\mu}(x)$ and $a^\star + k$ is a Sonine pair of completely monotone functions.*

Proof. We have $f(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \mu(dt)$ and so

$$\begin{aligned} \frac{f(\lambda)}{\lambda} &= \frac{a}{\lambda} + b + \int_0^\infty \frac{1 - e^{-\lambda t}}{\lambda}\, \mu(dt) \\ &= \frac{a}{\lambda} + b + \int_0^\infty \int_0^t e^{-\lambda t}\, dt\, \mu(dt) \\ &\overset{\text{Fubini}}{=} \frac{a}{\lambda} + b + \int_0^\infty \int_x^\infty \mu(dt) e^{-\lambda t}\, dt \\ &= \frac{a}{\lambda} + b + \int_0^\infty \bar{\mu}(x) e^{-\lambda t}\, dt \\ &= \mathscr{L}(a + b\delta_0 + \bar{\mu}(x); \lambda). \end{aligned}$$

denote by $f^\star = \frac{\lambda}{f(\lambda)}$ the conjugate Bernstein function. Since $f \in \mathcal{CBF}$, so is $f^\star$ and using $(a^\star, b^\star, \mu^\star)$ for its triplet, we get

$$\frac{1}{f(\lambda)} = \frac{f^\star(\lambda)}{\lambda} = \mathscr{L}(a^\star + b^\star\delta_0 + k(x); \lambda).$$

Since $\frac{1}{\lambda} = \frac{1}{f(\lambda)} \frac{f(\lambda)}{\lambda}$. It is obvious that $a + b\delta_0 + \bar{\mu}(x)$ and $a^\star + b^\star\delta_0 + k(x)$ is a Sonine pair (note that $bb^\star = 0$, i.e. at least one Sonine factor is a function) and the relation (3.1.4) and (3.1.5) follows from [69, Chapter 11] or Theorem 2.3.11.

Note that (see Table 3.1) $b = b^\star = 0$ if and only if we are in the case (1), (2), (4) as shown in Table 3.1. Since f is completely monotone function, $\frac{f(\lambda)}{\lambda}$, $\frac{1}{f(\lambda)}$ are Stieltjes functions, hence $a + \bar{\mu}$ and $a^\star + k$ are completely monotone functions. □

Table 3.1

Nr.	a	b	$m_0 = \int_0^\infty \mu(dt)$	$m_1 = \int_0^\infty t\,\mu(dt)$	$a^\star$	$b^\star$
(1)	0	0	– (no condition)	∞	0	0
(2)	0	0	∞	$< \infty$	$1/m_1$	0
(3)	0	0	$< \infty$	$< \infty$	$1/m_1$	$1/m_0$
(4)	> 0	0	∞	–	0	0
(5)	> 0	0	$< \infty$	–	0	$1/(a+m_0)$
(6)	0	> 0	–	∞	0	0
(7)	0	> 0	–	$< \infty$	$1/(b+m_1)$	0
(8)	> 0	> 0	–	–	0	0

Corollary 3.1.4. *Suppose $S = (S_t)_{t\geq 0}$ is a special subordinator with Laplace exponent given by*

$$f(\lambda) = a + \int_0^\infty (1 - e^{-\lambda t})\mu(dt)$$

where μ satisfies $\mu(0, \infty) = \infty$. Then

$$f^\star(\lambda) = \frac{\lambda}{f(\lambda)} = a^\star + \int_0^\infty (1 - e^{-\lambda t})\mu^\star(dt) \tag{3.1.6}$$

where the Lévy measure $\mu^\star$ satisfies $\mu^\star(0, \infty) = \infty$.

Based on the proof above and assume the measure μ satisfying the following condition: $\mu(0, \infty) = \infty$ and $\int_0^\infty t\mu(dt) = \infty$, we can obtain $b^\star = 0$ and $a^\star = 0$, see Table 3.1. From this we can conclude that every $f \in \mathcal{CBF}$ with (a, b, μ) as in rows (1), (2), (4) of Table 3.1 gives a completely monotone Sonine pair. However, not all Sonine pairs belong to $\mathcal{CM}$. For instance, $(\pi t)^{-1/2}\cos(2t^{1/2})$ and $(\pi t)^{-1/2}\cosh(2t^{1/2})$ form a Sonine pair but the first function changes sign, so it is not $\mathcal{CM}$. In Section 3.2, we will provide some figures to illustrate this point.

The following theorems give a characterization of Sonine pairs and its relationship with special Bernstein functions and potential measures.

Theorem 3.1.5. *Let $h, g \geq 0$ be a positive Sonine pair. Then $1/\mathscr{L}[g,\lambda] \in \mathcal{SBF}$ if and only if $1/\mathscr{L}[h,\lambda] \in \mathcal{SBF}$.*

Proof. By using the definition of Sonine pair (3.1.1) and taking Laplace transform, we will obtain $\mathscr{L}[h,\lambda]\mathscr{L}[g,\lambda] = 1/\lambda$. If we assume that $1/\mathscr{L}[g,\lambda]$ belongs to the class of special Bernstein functions denoted by $\mathcal{SBF}$. Denote the potentials as $\mathcal{P}$, then we have the following equivalence:

$$\begin{aligned}
&\frac{1}{\mathscr{L}[g,\lambda]} \in \mathcal{SBF} \\
&\Leftrightarrow \lambda\mathscr{L}[h,\lambda] \in \mathcal{SBF} \\
&\Leftrightarrow \lambda\mathscr{L}[h,\lambda] \in \mathcal{BF}, \mathscr{L}[h,\lambda] \in \mathcal{P} \\
&\Leftrightarrow \lambda\mathscr{L}[h,\lambda] \in \mathcal{BF}, \mathscr{L}[h,\lambda] \in \mathcal{CM}, \frac{1}{\mathscr{L}[h,\lambda]} \in \mathcal{BF} \\
&\Leftrightarrow \frac{1}{\mathscr{L}[h,\lambda]} \in \mathcal{SBF},
\end{aligned}$$

where the first equality is derived from the relation of Laplace transform, the second equality is obtained using Definition 2.3.10, the third equality follows from Schilling et al [69] Definition 5.16, and the last equality is obtained using the Definition 2.3.9. □

Theorem 3.1.6. *Let $h, g \geq 0$ be positive Sonine pair and $h = a\delta_0 + u, g = b\delta_0 + v$, where u, v are decreasing and $ab = 0$. If one of them is a potential density, then we have $1/\mathscr{L}[g,\lambda]$ and $1/\mathscr{L}[h,\lambda]$ are both special Bernstein functions, denoted by $\mathcal{SBF}$.*

Proof. Assume $h = a\delta_0 + u$ is a potential density, we can conclude that $\mathscr{L}[h,\lambda]$ is a potential function, based on the definition of a potential. Using the similar argument in Theorem 3.1.5, we can obtain that $1/\mathscr{L}[g,\lambda]$ and $1/\mathscr{L}[h,\lambda]$ are both special Bernstein function. □

Complete Bernstein functions are a subset of the special Bernstein functions i.e. $\mathcal{CBF} \subset \mathcal{SBF}$. If a Bernstein function f is a complete Bernstein function, then the corresponding Sonine pairs are completely monotone functions. On the other hand, if a Bernstein function f is a special Bernstein function, then Sonine pairs are related to the potential densities.

Remark 3.1.7. Let f be a special Bernstein function with conjugate function $f^\star \in \mathcal{SBF}$. We write (a, b, μ) and $(a^\star, b^\star, \mu^\star)$ for the corresponding triplets. Since

$$a = \lim_{\lambda\to 0} f(\lambda), \quad a^\star = \lim_{\lambda\to 0} f^\star(\lambda); \tag{3.1.7}$$

$$b = \lim_{\lambda\to 0} \frac{f(\lambda)}{\lambda}, \quad b^\star = \lim_{\lambda\to 0} \frac{f^\star(\lambda)}{\lambda} \tag{3.1.8}$$

and since $1/f(\lambda) = f^\star(\lambda)/\lambda$, it is easy to see that the conditions

$$\frac{1}{f(\lambda)} \to \infty, \quad \lambda \to 0, \quad \frac{1}{f(\lambda)} \to 0, \quad \lambda \to \infty; \tag{3.1.9}$$

$$\frac{\lambda}{f(\lambda)} \to 0, \quad \lambda \to 0, \quad \frac{\lambda}{f(\lambda)} \to \infty, \quad \lambda \to \infty. \tag{3.1.10}$$

are equivalent to seeing that

$$a = b = a^\star = b^\star = 0 \tag{3.1.11}$$

Theorem 3.1.8. *Assume f is a special Bernstein function with triplet (a, b, μ) and (3.1.9), (3.1.10) holds, then there exists a decreasing Sonine pair u and v.*

Conversely, assuming u and v are a decreasing Sonine pair and one of u, v is a potential density if, and only if $1/\mathscr{L}[v, \lambda]$ and $1/\mathscr{L}[u, \lambda]$ are special Bernstein function.

Proof. According to the conditions of f is a special Bernstein function with triplet (a, b, μ). Let

$$f(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \mu(dt), \tag{3.1.12}$$

$$f^\star(\lambda) = a^\star + b^\star\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \mu^\star(dt). \tag{3.1.13}$$

be representations of f and its conjugate $f^\star(\lambda)$. Applying the proof of Theorem 3.1.3 we can obtain (3.1.4) and (3.1.5). Using (3.1.9), (3.1.10), we have $a = b = 0$ and $a^\star = b^\star = 0$. We will get there exists u and v is a decreasing Sonine pair by Theorem 10.3, Corollary 10.8 and Theorem 10.9 in Schilling et al [69].

Conversely, if u and v are a Sonine pair, we have $u * v = 1$, and taking the Laplace transform on both sides gives us $\mathscr{L}[u, \lambda]\mathscr{L}[v, \lambda] = 1/\lambda$. Under the condition that u is decreasing, and assuming u is a potential density, it follows from Remark 10.26 in Schilling [69] that $1/\mathscr{L}[u, \lambda]$ is a special Bernstein function. By utilizing Theorem 3.1.5, we can deduce that $1/\mathscr{L}[v, \lambda]$ is a special Bernstein function. □

It can be easily to shown that a Sonine pair is not bounded on any neighborhood of 0 under an additional assumption.

Definition 3.1.9. A function $\phi : (a, b) \to \mathbb{R}$ has an integrable singularly at x_0 if

(a) for all $\varepsilon > 0$: $\int_{(a,b)\setminus(x_0-\varepsilon,x_0+\varepsilon)} |\phi(x)|\, dx < \infty$.

(b) $\lim_{\varepsilon\to 0} \int_{(a,b)\setminus(x_0-\varepsilon,x_0+\varepsilon)} \phi(x)\, dx$ exists.

If $x_0 = a$ or $x_0 = b$, then the above definition is modified in the usual way.

The following results is from [29, P.214] .

Theorem 3.1.10. *If $\bar{\mu}, k \in L^1_{loc}(0, \infty)$ are Sonine pairs, then $\bar{\mu}, k$ have an integrable singularity at 0.*

Not every function is Sonine function, for example, e^{-t}. In fact, if we assume that there exits a function g such that e^{-t} and g satisfy the Sonine equation i.e. $\int_0^x e^{-(x-r)}g(r)\,dr = 1$, then we obtain $\int_0^x e^r g(r)\,dr = e^x$. Differentiating this equation yields $g \equiv 1$. This is impossible since it leads to the contradiction $\int_0^x e^r\,dr = 1$ for all $x > 0$. The following result is from [29, P214].

Theorem 3.1.11. *(a) A locally integrable completely monotone function is a factor in a Sonine pair if and only if it is singular at 0.*

(b) The associated Sonine factor of an singular locally integrable completely monotone function is an singular locally integrable completely monotone function.

Based on the following assumptions. One may derive many special properties of a function to be Sonine kernel as its behavior at $0, \infty$ e.g. [62, P.3616]. Define **monotonicity near the origin**: There exists a neighborhood $0 < x < \varepsilon_0$ where

$$k(x) \geq 0, \quad \bar{\mu}(x) \geq 0, \quad k(x) \downarrow, \quad \bar{\mu} \downarrow, \quad 0 < x < \varepsilon_0. \tag{3.1.14}$$

In Samko et.al [62, P.3617], it is proved that:

Theorem 3.1.12. *A locally integrable Sonine pair $(\bar{\mu}, k)$ satisfying assumption (3.1.14) has the following properties:*

$$xk(x)\bar{\mu}(x) \leq 1, \quad 0 < x < \varepsilon_0, \tag{3.1.15}$$

$$k(x)\int_0^x \bar{\mu}(s)ds \leq 1, \quad \bar{\mu}(x)\int_0^x k(s)ds \leq 1, \quad 0 < x < \varepsilon_0, \tag{3.1.16}$$

$$k(x)\int_0^x \bar{\mu}(s)ds + \bar{\mu}(x)\int_0^x k(s)ds \geq 1, \quad 0 < x < \varepsilon_0, \tag{3.1.17}$$

$$\int_0^x k(s)ds \int_0^x \bar{\mu}(s)ds \geq x, \quad 0 < x < \varepsilon_0, \tag{3.1.18}$$

$$\lim_{x\to 0} k(x) = \lim_{x\to 0} \bar{\mu}(x) = \infty, \tag{3.1.19}$$

$$\lim_{x\to 0} xk(x) = \lim_{x\to 0} x\bar{\mu}(x) = 0, \tag{3.1.20}$$

$$\sup_{0<x<\varepsilon_0}\left[\int_0^x |k'(s)| \int_{x-s}^x \bar{\mu}(t)dtds\right] \leq 1. \tag{3.1.21}$$

Next we will explore a special class of Sonine pairs which are regularly varying cf [29, P216] and [58]. Suppose that there have two locally integrable functions $\bar{\mu}, k$ on $(0, \infty)$ which are regularly varying at 0 with index α, β, respectively and satisfy the Sonine equation (3.1.1). If $0 < \alpha < 1$, then $\beta = 1 - \alpha$ and we obtain the following expression:

$$\bar{\mu}(x) = \frac{l_1(x)x^{-\alpha}}{\Gamma(1-\alpha)}, \quad k(x) = \frac{l_2(x)x^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \alpha < 1, \tag{3.1.22}$$

where the functions l_1, l_2 are of slow variation at 0.

Theorem 3.1.13. *Assume that $\bar{\mu}$, k are regularly varying at 0 with index $-\alpha$, $\alpha - 1$, respectively and which are given by* (3.1.22). *If $\bar{\mu}, k$ satisfy the Sonine equation* (3.1.1), *then we have* $\lim_{x\to 0} l_1(x)l_2(x) = 1$.

Proof. Using the Sonine equation (3.1.1) and the expression (3.1.22) , we obtain

$$\begin{aligned}
1 &= \lim_{x\to 0}\int_0^x \bar{\mu}(r)k(x-r)\,dr \\
&= \lim_{x\to 0}\int_0^x \frac{l_1(r)r^{-\alpha}}{\Gamma(1-\alpha)}\frac{l_2(x-r)(x-r)^{\alpha-1}}{\Gamma(\alpha)}\,dr \\
&= \lim_{x\to 0}\int_0^1 \frac{l_1(xu)(xu)^{-\alpha}}{\Gamma(1-\alpha)}\frac{l_2(x-xu)(x-xu)^{\alpha-1}}{\Gamma(\alpha)}x\,du \\
&= \lim_{x\to 0}\int_0^1 l_1(x)l_2(x)\frac{l_1(xu)}{l_1(x)}\frac{u^{-\alpha}}{\Gamma(\alpha)}\frac{l_2(x-xu)}{l_2(x)}\frac{(1-u)^{\alpha-1}}{\Gamma(1-\alpha)}\,du \\
&= \lim_{x\to 0} l_1(x)l_2(x)\underbrace{\int_0^1 \frac{l_1(xu)}{l_1(x)}\frac{u^{-\alpha}}{\Gamma(\alpha)}\frac{l_2(x-xu)}{l_2(x)}\frac{(1-u)^{\alpha-1}}{\Gamma(1-\alpha)}\,du}_{\text{I}},
\end{aligned}$$

where the third equality comes from a change of variables. Now let us calculate Part I. We have:

$$\begin{aligned}
&\lim_{x\to 0}\int_0^1 \frac{l_1(xu)}{l_1(x)}\frac{u^{-\alpha}}{\Gamma(\alpha)}\frac{l_2(x-xu)}{l_2(x)}\frac{(1-u)^{\alpha-1}}{\Gamma(1-\alpha)}\,du \\
&= \lim_{x\to 0}\lim_{\varepsilon\to 0}\left[\int_0^\varepsilon + \int_\varepsilon^{1-\varepsilon} + \int_{1-\varepsilon}^1\right]\frac{l_1(xu)}{l_1(x)}\frac{u^{-\alpha}}{\Gamma(\alpha)}\frac{l_2(x-xu)}{l_2(x)}\frac{(1-u)^{\alpha-1}}{\Gamma(1-\alpha)}\,du
\end{aligned}$$

We consider the three integrals separately. Because slow variation

$$\lim_{x\to 0}\frac{l_1(xu)}{l_1(x)} = \lim_{x\to 0}\frac{l_2(x-xu)}{l_2(x)} = 0,$$

uniformly for $u \in [\varepsilon, 1-\varepsilon]$. Therefore we can exchange the order of the limits and calculate

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{x\to 0}\int_\varepsilon^{1-\varepsilon} \frac{l_1(xu)}{l_1(x)}\frac{u^{-\alpha}}{\Gamma(\alpha)}\frac{l_2(x-xu)}{l_2(x)}\frac{(1-u)^{\alpha-1}}{\Gamma(1-\alpha)}\,du \\
&= \lim_{\varepsilon\to 0}\int_\varepsilon^{1-\varepsilon}\frac{u^{-\alpha}}{\Gamma(\alpha)}\frac{(1-u)^{\alpha-1}}{\Gamma(1-\alpha)}\,du \\
&= \int_0^1 \frac{u^{-\alpha}}{\Gamma(\alpha)}\frac{(1-u)^{\alpha-1}}{\Gamma(1-\alpha)}\,du \\
&= 1.
\end{aligned}$$

In this calculation we used the continuity theorem for parameter–dependent integrals. For the remaining two terms we use Potter's bound Theorem 1.3.4, to get

$$\left[\int_0^\varepsilon + \int_{1-\varepsilon}^1\right]\frac{l_1(xu)}{l_1(x)}\frac{u^{-\alpha}}{\Gamma(\alpha)}\frac{l_2(x-xu)}{l_2(x)}\frac{(1-u)^{\alpha-1}}{\Gamma(1-\alpha)}\,du$$

$$\leq C\int_0^{\varepsilon} u^{-\delta}\frac{u^{-\alpha}}{\Gamma(\alpha)}\frac{(1-u)^{\alpha-1}}{\Gamma(1-\alpha)}\,du + \int_{1-\varepsilon}^{1}\frac{u^{-\alpha}}{\Gamma(\alpha)}(1-u)^{-\delta}\frac{(1-u)^{\alpha-1}}{\Gamma(1-\alpha)}\,du$$
$$\to 0, \quad \text{as} \quad \varepsilon \to 0.$$

$\lim_{x\to 0} l_1(x)l_2(x) = 1$ follows from Part I has limit 1. This finishes our proof. □

Example 3.1.14. Here is an example of a Sonine pair $(\bar{\mu}, k)$ consisting of completely monotone and regularly varying functions.

Let $\beta \geq 0, \quad 0 \leq \alpha \leq \beta \leq 1$ and denote by

$$E_{\alpha,\beta}(x) := \sum_{i=0}^{\infty}\frac{x^i}{\Gamma(\alpha i + \beta)},$$

the (generalized) Mittag–Leffler functions. Set

$$k(x) = x^{\beta-1}E_{\alpha,\beta}(-x^{\alpha}),$$

and

$$\bar{\mu}(x) = \frac{x^{-\beta}}{\Gamma(1-\beta)} + \frac{x^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}.$$

Clearly

$$\mathscr{L}(k,\lambda) = \int_0^{\infty} e^{-\lambda x}x^{\beta-1}E_{\alpha,\beta}(-x^{\alpha})\,dx = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}+1},$$

$$\mathscr{L}(\bar{\mu},\lambda) = \int_0^{\infty} e^{-\lambda x}\left(\frac{x^{-\beta}}{\Gamma(1-\beta)} + \frac{x^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)dx = \lambda^{\beta-1} + \lambda^{\beta-\alpha-1},$$

i.e. $\mathscr{L}(k,\lambda)\mathscr{L}(\bar{\mu},\lambda) = 1/\lambda$ proving that $k, \bar{\mu}$ is a Sonine pair. From this it is easy to see that

$$l_1(x) = \frac{x^{\alpha}\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)} + 1, \quad l_2(x) = \Gamma(\beta)E_{\alpha,\beta}(-x^{\alpha}).$$

Take $\alpha = 0.25, \beta = 0.5$, we have the following figure for $l_1 l_2$.

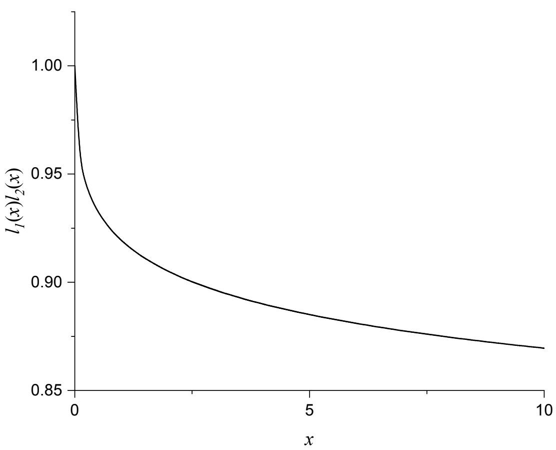

Figure 3.1

In the following, we will present several equivalent conditions to identify when a Sonine pair is regularly varying and determine its index.

Theorem 3.1.15. *Let f be a complete Bernstein function and $\bar{\mu}, k$ be the corresponding Sonine pair and $f(\lambda)/\lambda = \mathscr{L}\bar{\mu}(x,\lambda), \mathscr{L}k(\lambda) = 1/f(\lambda)$. Then the following assertions are equivalent:*

(a) f is regularly varying at ∞ with index ρ.

(b) $\bar{\mu}$ is regularly varying at 0 with index $-\rho$.

(c) k is regularly varying at 0 with index $\rho - 1$.

Proof. $(a) \iff (b)$: According to the definition that f is of regular variation at ∞, there exists ρ such that $\frac{f(\lambda x)}{f(x)} \to \lambda^\rho$, as $x \to \infty$. Using the relation between f and $\bar{\mu}$ and Theorem 1.2.2 we have $\frac{\mathscr{L}\bar{\mu}(\lambda x)}{\mathscr{L}\bar{\mu}(x)} = \frac{f(\lambda x)}{\lambda x} \Big/ \frac{f(x)}{x} \to \lambda^{\rho-1}$, as $x \to \infty$ if, and only if, $\frac{\bar{\mu}(tx)}{\bar{\mu}(x)} \to t^{-\rho}$ as $x \to 0$.

$(a) \iff (c)$: Using a similar argument we have $\frac{\mathscr{L}k(\lambda x)}{\mathscr{L}k(x)} = \frac{1}{f(\lambda x)} \Big/ \frac{1}{f(x)} \to \lambda^{-\rho}$ as $x \to \infty$ if and only if $\frac{k(tx)}{k(x)} \to t^{\rho-1}$ as $x \to 0$. □

Example 3.1.16. If $f(\lambda) = \lambda^\beta = \frac{\lambda}{\Gamma(1-\beta)} \int_0^\infty e^{-\lambda x} x^{-\beta}\, dx$. Using the above theorem, f has index $\rho = \beta$, $\bar{\mu}$ has index $-\beta$. For $\frac{1}{f(\lambda)} = \lambda^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\lambda x} x^{\beta-1}\, dx$. By a similar reasoning, we obtain k has index $\beta - 1$.

3.2 Examples of Sonine pairs

Here we collect some examples for Sonine pairs. We use the notation introduced above, i.e. the kernels $\bar{\mu}$ and k are such that $\bar{\mu} * k = 1$ and their Laplace transforms are $\frac{f(s)}{s} = \mathscr{L}[\bar{\mu}(x), s]$, and $\frac{1}{f(s)} = \mathscr{L}[k(x), s]$ so that $\frac{f(s)}{s}\frac{1}{f(s)} = \frac{1}{s}$.

(1) Stable kernels

$$\bar{\mu}(x) = \frac{x^{-\beta}}{\Gamma(1-\beta)}, \quad k(x) = \frac{x^{\beta-1}}{\Gamma(\beta)}$$

where $0 < \beta < 1$ and

$$\frac{f(s)}{s} = \mathscr{L}[\bar{\mu}(x), s] = s^{\beta-1}, \quad \frac{1}{f(s)} = \mathscr{L}[k(x), s] = s^{-\beta}.$$

(2) Regularly varying kernel

$$\bar{\mu}(x) = \frac{x^{-\beta}}{\Gamma(1-\beta)} + \frac{x^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}, \quad k(x) = x^{\beta-1} E_{\alpha,\beta}(-x^\alpha)$$

where $E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}$ is the Mittag-Leffler function and $0 < \alpha \le \beta < 1$ and

$$\frac{f(s)}{s} = \mathscr{L}[\bar{\mu}(x), s] = s^{-1+\beta} + s^{-1-a+\beta}, \quad \frac{1}{f(s)} = \mathscr{L}[k(x), s] = \frac{s^{-\beta}}{1+s^{-\alpha}}.$$

(3) Analytic function kernel

$$\bar{\mu}(x) = \frac{x^{-\beta}}{\Gamma(1-\beta)}\mathrm{M}(-\alpha, 1-\beta, -\lambda x), \quad k(x) = \frac{x^{\beta-1}}{\Gamma(\beta)}\mathrm{M}(\alpha, \beta, -\lambda x)$$

where $0 < \beta < 1$ and $\mathrm{M}(\alpha, \beta, z) = \sum_{j=0}^{\infty} \frac{(\alpha)_j}{(\beta)_j} \frac{z^j}{j!}$ is the Kummer function where $\alpha_0 = 1, (\alpha)_1 = \alpha, (\alpha)_j = \alpha(\alpha+1)...(\alpha+j-1) = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)}$. These functions form a Sonine pair.

Proof. We use Laplace transforms to show that the analytic function kernels are a Sonine pair.

$$\begin{aligned}
\mathscr{L}_{x\to s}[\bar{\mu}, s] &= \mathscr{L}_{x\to s}\left[\frac{x^{-\beta}}{\Gamma(1-\beta)}\mathrm{M}(-\alpha, 1-\beta, -\lambda x), s\right] \\
&= \mathscr{L}_{x\to s}\left[\frac{x^{-\beta}}{\Gamma(1-\beta)} \sum_{j=0}^{\infty} \frac{(-\alpha)_j}{(1-\beta)_j} \frac{(-\lambda x)^j}{j!}, s\right] \\
&= \sum_{j=0}^{\infty} \frac{(-\alpha)_j}{(1-\beta)_j} \frac{(-\lambda)^j \Gamma(j-\beta+1)}{j!\Gamma(1-\beta) s^{j-\beta+1}} \\
&= \sum_{j=0}^{\infty} \frac{(-\alpha)_j (-\lambda)^j}{j!} \frac{1}{s^{j-\beta+1}},
\end{aligned}$$

In the last equality, we use the definition of $\frac{\Gamma(j-\beta+1)}{\Gamma(1-\beta)} = (1-\beta)_j$. By a similar calculation, we have

$$\begin{aligned}
\mathscr{L}_{x\to s}[k, s] &= \mathscr{L}_{x\to s}\left[\frac{x^{\beta-1}}{\Gamma(\beta)}\mathrm{M}(\alpha, \beta, -\lambda x), s\right] \\
&= \mathscr{L}_{x\to s}\left[\frac{x^{\beta-1}}{\Gamma(\beta)} \sum_{i=0}^{\infty} \frac{(\alpha)_i}{(\beta)_i} \frac{(-\lambda x)^i}{i!}, s\right] \\
&= \sum_{i=0}^{\infty} \frac{(\alpha)_i}{(\beta)_i} \frac{(-\lambda)^i \Gamma(i+\beta)}{i!\Gamma(\beta) s^{i+\beta}} \\
&= \sum_{i=0}^{\infty} \frac{(\alpha)_i}{i!} \frac{(-\lambda)^i}{s^{i+\beta}}.
\end{aligned}$$

This allows us to calculate the product of the Laplace transforms:

$$\begin{aligned}
\mathscr{L}_{x\to s}[\bar{\mu}, s]\mathscr{L}_{x\to s}[k, s] &= \sum_{j=0}^{\infty} \frac{(-\alpha)_j (-\lambda)^j}{j!} \frac{1}{s^{j-\beta+1}} \sum_{i=0}^{\infty} \frac{(\alpha)_i}{i!} \frac{(-\lambda)^i}{s^{i+\beta}} \\
&= \frac{1}{s} \sum_{j=0}^{\infty} \frac{(-\alpha)_j (-\lambda)^j}{j!} \frac{1}{s^j} \sum_{i=0}^{\infty} \frac{(\alpha)_i}{i!} \frac{(-\lambda)^i}{s^i}
\end{aligned}$$

$$= \frac{1}{s} \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{(-\alpha)_i (-\lambda)^i}{i!} \frac{1}{s^i} \frac{(\alpha)_{j-i} (-\lambda)^{j-i}}{(j-i)!} \frac{1}{s^{j-i}}$$
$$= \frac{1}{s} \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{s^j} \sum_{i=0}^{j} \frac{(-\alpha)_i}{i!} \frac{(\alpha)_{j-i}}{(j-i)!}.$$

By a direct calculation we obtain for $j = 0$ that $\sum_{i=0}^{j} \frac{(-\alpha)_i}{i!} \frac{(\alpha)_{j-i}}{(j-i)!} = 1$. For $j \geq 1$, we use induction to show $\sum_{i=0}^{j} \frac{(-\alpha)_i}{i!} \frac{(\alpha)_{j-i}}{(j-i)!} = 0$.

Indeed, $j = 1$, we have $\frac{(-\alpha)_0}{0!} \frac{(\alpha)_1}{1!} + \frac{(-\alpha)_1}{1!} \frac{(\alpha)_0}{0!} = 0$. Assume the above results holds for $j \in \mathbb{N}$. For $j \rightsquigarrow j+1$, we obtain

$$\sum_{i=0}^{j+1} \frac{(-\alpha)_i}{i!} \frac{(\alpha)_{j+1-i}}{(j+1-i)!}$$
$$= \sum_{i=0}^{j+1} \frac{(-\alpha)_i (\alpha)_{j+1-i}}{(j+1)!} \frac{(j+1)!}{(j+1-i)! i!}$$
$$= \frac{1}{(j+1)!} \sum_{i=0}^{j+1} \binom{i}{j+1} (-\alpha)_i (\alpha)_{j+1-i}$$
$$= \frac{1}{(j+1)!} \left[(-\alpha)_{j+1} + (\alpha)_{j+1} + \sum_{i=1}^{j} \binom{i}{j+1} (-\alpha)_i (\alpha)_{j+1-i} \right]$$
$$= \frac{1}{(j+1)!} \left[(-\alpha)_{j+1} + (\alpha)_{j+1} + \sum_{i=1}^{j} \left(\binom{i-1}{j} + \binom{i}{j} \right) (-\alpha)_i (\alpha)_{j+1-i} \right]$$
$$= \frac{1}{(j+1)!} \left[(-\alpha)_{j+1} + (\alpha)_{j+1} + \sum_{i=1}^{j} \binom{i-1}{j} (-\alpha)_i (\alpha)_{j+1-i} + \sum_{i=1}^{j} \binom{i}{j} (-\alpha)_i (\alpha)_{j+1-i} \right]$$
$$= \frac{1}{(j+1)!} \left[\sum_{i=0}^{j} \binom{i}{j} (-\alpha)_{i+1} (\alpha)_{j-i} + \sum_{i=0}^{j} \binom{i}{j} (-\alpha)_i (\alpha)_{j+1-i} \right]$$
$$= \frac{1}{(j+1)!} \left[\sum_{i=0}^{j} \binom{i}{j} (-\alpha)_i (\alpha)_{j-i} (-\alpha + i) + \sum_{i=0}^{j} \binom{i}{j} (-\alpha)_i (\alpha)_{j-i} (\alpha + j - i) \right]$$
$$= \frac{1}{(j+1)!} \left[\sum_{i=0}^{j} \binom{i}{j} (-\alpha)_i (\alpha)_{j-i} (-\alpha + i + \alpha + j - i) \right]$$
$$= \frac{j}{(j+1)!} \left[\sum_{i=0}^{j} \binom{i}{j} (-\alpha)_i (\alpha)_{j-i} \right]$$
$$= 0,$$

the last equality holds because of the induction assumption. This finishes the proof of induction. Combining the above results we get $\mathscr{L}[\bar{\mu}, s] \mathscr{L}[k, s] = 1/s$. □

(4) Power-exponential function and the incomplete gamma function

$$\frac{f(s)}{s} = \bar{\mu}(x) = \lambda^{\beta} + \frac{\beta}{\Gamma(1-\beta)} \int_x^{\infty} \frac{e^{-\lambda t}}{t^{1+\beta}}\, dt, \quad k(x) = \frac{e^{-\lambda x}}{\Gamma(\beta) x^{1-\beta}}$$

where $0 < \beta < 1$ and $\lambda > 0$ and

$$\mathscr{L}[\bar{\mu}(x), s] = \frac{(\lambda+\beta)^{\alpha}}{s}, \quad \frac{1}{f(s)} = \mathscr{L}[k(x), s] = \frac{1}{(\lambda+\beta)^{\alpha}}.$$

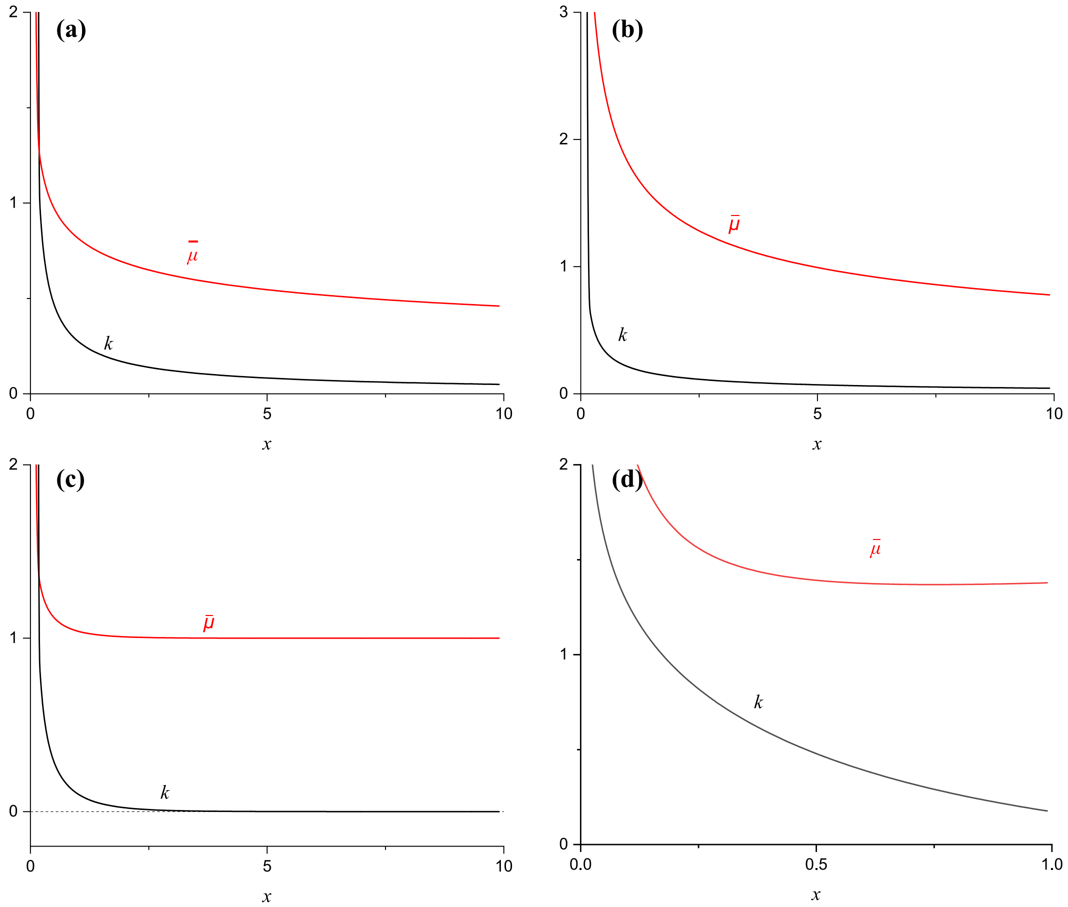

Figure 3.2: Examples for the evolution of k (black line) and $\bar{\mu}$ (red line) as a function of x under various conditions. (a) Stable kernels with $\beta = 0.25$. (b) Regularly varying kernel with $\alpha = 0.25, \beta = 0.5$. (c) Power-exponential function and the incomplete gamma function with $\lambda = 1, \beta = 0.25$. (d) Analytic function kernel with $\alpha = 1, \beta = 0.25, \lambda = 1$.

The following examples of Sonine pairs are composed of non-positive kernels, i.e. they do not fit into the present context, but they are nevertheless worth recording.

(5) Special non-analytic function kernel

$$\frac{f(s)}{s} = \bar{\mu}(x) = b + \frac{1}{\Gamma(0.5)\sqrt{x}}, \quad k(x) = \frac{1}{\sqrt{\pi x}} - b e^{b^2 x} \mathrm{erfc}(b\sqrt{x}), b = \lambda\Gamma(0.5)$$

where $\lambda > 0$ and $\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} = \int_0^z e^{-r^2}\, dr$ and

$$\frac{f(s)}{s} = \mathscr{L}[\bar{\mu}(x), s] = \frac{b}{s} + \frac{1}{\sqrt{s}}, \quad \frac{1}{f(s)} = \mathscr{L}[k(x), s] = \frac{1}{\sqrt{s}} - \frac{b(\sqrt{s} - b)}{(s - b^2)\sqrt{s}}.$$

(6) Bessel-type functions

$$\bar{\mu}(x) = (\sqrt{x})^{-\beta} J_{-\beta}(2\sqrt{x}), \quad k(x) = (\sqrt{x})^{\beta-1} I_{\beta-1}(2\sqrt{x})$$

where $J_\beta(x) = \sum_{j=0}^{\infty} \frac{(-1)^j (0.5x)^{2j+\beta}}{j!\Gamma(j+\beta+1)}$ and $I_\beta(x) = \sum_{j=0}^{\infty} \frac{(0.5x)^{2j+\beta}}{j!\Gamma(j+\beta+1)}$ are the Bessel and modified Bessel functions, respectively, $0 < \beta < 1$ and

$$\frac{f(s)}{s} = \mathscr{L}[\bar{\mu}(x), s] = \frac{e^{-1/s}}{s^{1-\beta}}, \quad \mathscr{L}[k(x), s] = e^{1/s} s^{-\beta}.$$

(7) Special Bessel-type functions

$$\bar{\mu}(x) = \frac{\cos(2\sqrt{x})}{\sqrt{\pi x}}, \quad \frac{1}{f(s)} = k(x) = \frac{\cosh(2\sqrt{x})}{\sqrt{\pi x}}$$

and

$$\frac{f(s)}{s} = \mathscr{L}[\bar{\mu}(x), s] = \frac{e^{-1/s}}{\sqrt{s}}, \quad \frac{1}{f(s)} = \mathscr{L}[k(x), s] = \frac{e^{1/s}}{\sqrt{s}}.$$

(8) Non-analytic function kernel

$$\bar{\mu}(x) = 1 - \frac{\lambda}{x^{1-\beta}}, \quad k(x) = bx^{-\beta} E_{1-\beta,1-\beta}(bx^{1-\beta})$$

where $\lambda > 0$, and $b = \lambda\Gamma(\beta)$, $0 < \beta < 1$ and

$$\frac{f(s)}{s} = \mathscr{L}[\bar{\mu}(x), s] = \frac{1}{s} - \lambda s^{-\beta}\Gamma(\beta), \quad \frac{1}{f(s)} = \mathscr{L}[k(x), s] = \frac{\lambda s^{-1+\beta}}{1 - \lambda s^{-1+\beta}\Gamma(\beta)}.$$

(9) Power-logarithmic type

$$\bar{\mu}(x) = \int_0^\infty \frac{x^{t-\beta} e^{bt}}{\Gamma(t+1-\beta)}\, dt, \quad k(x) = \frac{\log(1/x) + a}{\Gamma(\beta)x^{1-\beta}}$$

where $b = \frac{\Gamma(\beta)'}{\Gamma(\beta)} - a$ and

$$\frac{f(s)}{s} = \mathscr{L}[\bar{\mu}(x), s] = \frac{s^{\beta-1}}{\log(s) - b}, \quad \frac{1}{f(s)} = \mathscr{L}[k(x), s] = \frac{\log(s) - b}{s^\beta}.$$

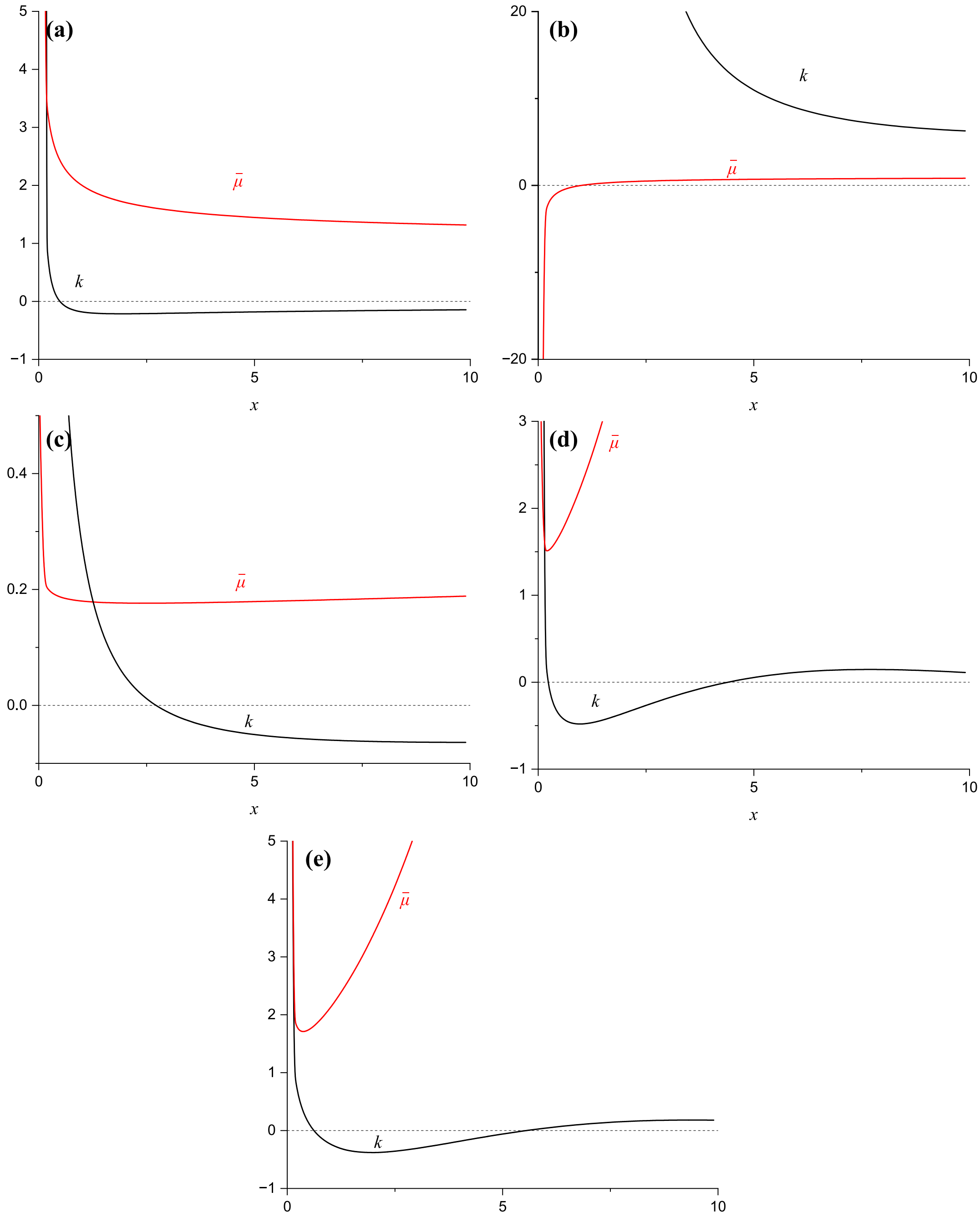

Figure 3.3: Examples for the evolution of k (black line) and $\bar{\mu}$ (red line) as a function of x under various conditions. (a) Special non–analytic kernel with $b = 1$ and $\lambda = 1$. (b) Power–logarithmic type kernel with $\beta = 0.25$ and $a = 1$. (c) Non–analytic kernel with $\beta = 0.25$ and $\lambda = 1$. (d) Bessel–type kernels with $\beta = 0.8$. (e) Special Bessel–type kernels.

3.3 Mapping properties of general fractional derivatives

We will now study the mapping behavior of general Bernstein functional derivatives. Throughout this section f will be a Bernstein function given by

$$f(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\,\mu(dx).$$

Consequently, $f(\lambda)/\lambda = \mathscr{L}(\bar{\mu}, \lambda)$ with $\bar{\mu}(x) = \mu[x, \infty)$ so that $\bar{\mu}, k$ is a Sonine pair with (possibly measure–valued) k given by $\mathscr{L}(k, \lambda) = 1/f(\lambda)$. For simplicity, we assume that also k is a function.

Recall $C^\alpha[0,T]$ (resp. $C^\alpha_\infty[0,T]$) are the α Hölder functions (with $\phi(0) = 0$) and

$$C^\alpha(\bar{\mu}^{-1}, [0,T]) = \left\{ \phi \in C^\alpha[0,T] : \quad \frac{|\phi(x)|}{\bar{\mu}(x)} \leq x^\alpha \right\}.$$

$\phi \in C^\alpha_\infty(\bar{\mu}^{-1}, [0,T]) \iff \phi \in C^\alpha(\bar{\mu}^{-1}, [0,T])$ with $\phi(0) = 0$.

Lemma 3.3.1. *If* $\phi \in C^\alpha_\infty\left[\bar{\mu}^{-1}, [0,T]\right]$, *then* $|{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot}\,\phi| \leq x^\alpha$.

Proof. Let $\phi \in C^\alpha\left[\bar{\mu}^{-1}, [0,T]\right]$. We have

$$\begin{aligned}
{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi &= \int_0^x \phi(s)k(x-s)\,ds \\
&= \int_0^x (\phi(s) - \phi(0))k(x-s)\,ds \\
&= \int_0^x (\phi(s) - \phi(0))\bar{\mu}(s)^{-1}\bar{\mu}(s)k(x-s)\,ds \\
&\leq \int_0^x s^\alpha \bar{\mu}(s)k(x-s)\,ds \\
&\leq x^\alpha
\end{aligned}$$

□

Theorem 3.3.2. *Let* $0 < \alpha < 1$, *the operator* ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot} : C^\alpha_\infty[0,T] \to C^\alpha[0,T]$ *and* ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot} : C_\infty[0,T] \to C[0,T]$.

Proof. We have ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi = \int_0^x \phi(s)k(x-s)\,ds$. Let $\phi \in C^\alpha[0,T]$ and $h \geq 0$ such that $0 \leq x \leq x+h \leq T$,

$$\begin{aligned}
&|{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi - {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x+h}\,\phi| \\
&= \left| \int_0^x \phi(s)k(x-s)\,ds - \int_0^{x+h} \phi(s)k(x+h-s)\,ds \right| \\
&= \left| \int_0^x \phi(s)k(x-s)\,ds - \int_{-h}^{x} \phi(s+h)k(x-s)\,ds \right| \\
&\leq \int_0^x |\phi(s) - \phi(s+h)|k(x-s)\,ds + \left| \int_{-h}^{0} \phi(s+h)k(x-s)\,ds \right|.
\end{aligned}$$

We note that up to now we have not used $\alpha > 0$. In fact our argument up to now shows that ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot}$ maps $C_\infty[0,T]$ into $C[0,T]$. We assume now $\alpha > 0$ and $\phi(0) = 0$ to continue

$$\begin{aligned}
&\int_0^x |\phi(s) - \phi(s+h)| k(x-s)\, ds + \left| \int_{-h}^0 (\phi(s+h) - \phi(0)) k(x-s)\, ds \right| \\
&\leq C \int_0^x h^\alpha k(x-s)\, ds + Ch^\alpha \int_0^h k(x+s)\, ds \\
&\leq Ch^\alpha \left[\int_0^x k(s)\, ds + \int_0^h k(x+s)\, ds \right],
\end{aligned}$$

where $\int_0^x k(s)\, ds < \infty, x \in [0,T]$ since it is Sonine factor. □

Next, we assume that the Sonine pair is regularly varying and l_1, l_2 are slowing varying at 0,

$$\bar{\mu}(x) = \frac{x^{-r}}{\Gamma(1-r)} l_2(x), \quad k(x) = \frac{x^{r-1}}{\Gamma(r)} l_2(x).$$

Theorem 3.3.3. *Let $\phi \in C^\alpha_\infty[0,T]$ and $0 \leq \alpha \leq 1$, $0 < \alpha + r < 1$. Then we have ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot} : C^\alpha_\infty[0,T] \to C^{\alpha+r-\delta}[0,T]$.*

Proof. Let $\phi \in C^\alpha[0,T]$ with $0 \leq \alpha \leq 1$ and $\phi(0) = 0$. We have

$$\begin{aligned}
{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \phi &= \int_0^x \phi(s) k(x-s)\, ds \\
&= \int_0^x (\phi(s) - \phi(0)) k(x-s)\, ds.
\end{aligned}$$

Let $h > 0$ be such that $0 \leq x \leq x + h \leq T$,

$$\begin{aligned}
&{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x+h}\, \phi - {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \phi \\
&= \int_0^{x+h} \phi(s) k(x+h-s)\, ds - \int_0^x \phi(s) k(x-s)\, ds \\
&= \int_{-h}^x \phi(x-s) k(h+s)\, ds - \int_0^x \phi(x-s) k(s)\, ds \\
&= \phi(x) \int_{x-h}^x k(h+s)\, ds + \int_{-h}^0 (\phi(x-s) - \phi(x)) k(h+s)\, ds \\
&\quad + \int_0^x (k(h+s) - k(s))(\phi(x-s) - \phi(x))\, ds \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}$$

We estimate I, II, III separately, I: If $h \geq x$, then as $\phi(0) = 0$

$$\begin{aligned}
|\mathrm{I}| &= \int_{x-h}^x k(h+s)\, ds\, |\phi(x) - \phi(0)| \\
&\leq Cx^\alpha \int_{x-h}^x (h+s)^{r-1-\delta}\, ds,
\end{aligned}$$

for some $\delta > 0$ by Potter's bounds. Thus

$$\begin{aligned} |\mathrm{I}| &\le C_1 h^\alpha |(h+x)^{r-\delta} - x^{r-\delta}| \\ &\le C_2 h^{\alpha+r-\delta}. \end{aligned}$$

If $h < x$, we use the estimate $(1+t)^\beta - 1 \le \beta t$ to get

$$\begin{aligned} |\mathrm{I}| &\le C_1 x^\alpha x^{r-\delta} \left|(1+h/x)^{r-\delta} - 1\right| \\ &\le C_2 h x^{\alpha+r-\delta-1} \\ &\le C_3 h^{\alpha+r-\delta}, \end{aligned}$$

where we use that $\alpha + r \le 1$.

II: By Hölder continuity and Potter's bounds

$$\begin{aligned} |\mathrm{II}| &\le \int_{-h}^{0} |\phi(x-s) - \phi(x)| k(h+s)\, ds \\ &\le C \int_{-h}^{0} |s|^\alpha (h+s)^{1+r-\delta}\, ds \\ &\le C_1 h^{r+\alpha-\delta}, \end{aligned}$$

for a suitable (any) $\delta > 0$ with some constant $C_1 = C_\delta$.

III: We have

$$\begin{aligned} |\mathrm{III}| &\le C \int_{0}^{x} s^\alpha |k(h+s) - k(s)|\, ds \\ &= C \int_{0}^{x} s^\alpha (k(h+s) - k(s))\, ds, \end{aligned}$$

since k is decreasing. If $x \le h$ we get with Potter's bounds

$$\begin{aligned} |\mathrm{III}| &\le C \int_{0}^{x} s^\alpha k(s)\, ds \\ &\le C_\delta \int_{0}^{x} s^\alpha s^{r-1-\delta}\, ds \\ &\le C_\delta^1 h^{\alpha-r-\delta}. \end{aligned}$$

If $x \ge h$, we use the previous part to get

$$\begin{aligned} |\mathrm{III}| &\le C \int_{0}^{x} s^\alpha (k(h+s) - k(s))\, ds \\ &\le C_\delta^2 h^{\alpha-r-\delta} + C \int_{h}^{x} s^\alpha (k(h+s) - k(s))\, ds. \end{aligned}$$

Since $k \in \mathcal{CM} \cap \mathcal{R}_{r-1}$, $k' \in \mathcal{R}_{r-2}$ (see Bingham et.al. [9]) and $|k'|$ is decreasing. Thus, by Potter's bounds and the mean value theorem

$$
\begin{aligned}
\text{III} \;&\leq\; C_\delta^2 h^{\alpha-r-\delta} + C \int_h^x s^\alpha |k'(s)| h \, ds \\
&\overset{\text{t=sh}}{\leq}\; C_\delta^2 h^{\alpha-r-\delta} + h^\alpha \int_1^{x/h} t^\alpha |k'(t/h)| \, dt \\
&\leq\; C_\delta^2 h^{\alpha-r-\delta} + h^\alpha \int_1^{x/h} t^\alpha (t/h)^{r+\delta-2} \, dt \\
&\leq\; C_\delta^2 h^{\alpha-r-\delta} + h^{\alpha-r-\delta+2} \int_1^\infty t^{\alpha+r+\delta-2} \, dt \\
&\leq C_\delta^3 h^{\alpha-r-\delta},
\end{aligned}
$$

If $\alpha + r + \delta < 1$. Combing all previous estimates proves the Theorem. here we note that δ must be chosen such a way that $r + \alpha < 1$ and $r + \alpha + \delta < 1$, which is always possible. □

Let us now turn to the generalized fractional derivative.

Lemma 3.3.4. *Let $\phi \in C_\infty^\alpha[0, T)$. If $\alpha - r > 0$, then ${}_0^{\mathrm{R}}\mathrm{D}_x^f \, \phi \in C_\infty^{\alpha-r-\delta}[0, T)$ for any $\delta > 0$ with $\alpha - r - \delta > 0$.*

Proof. We have

$$
\begin{aligned}
\psi(x) := {}_0^{\mathrm{R}}\mathrm{D}_x^f \, \phi &= \int_0^x (\phi(x) - \phi(x-t)) \; \mu(dt) \\
&= \int_0^x (\phi(x) - \phi(x-t)) \, m(t) \, dt,
\end{aligned}
$$

with $m \in \mathcal{CM} \cap \mathcal{R}_{-r-1}$. If $\phi \in C_\infty^\alpha[0, T)$, we get

$$
\begin{aligned}
\psi(x+h) - \psi(x) &= \int_0^x (\phi(x) - \phi(x-t)) \, (m(t+h) - m(t)) \; dt \\
&\quad + \int_{-h}^0 (\phi(x+h) - \phi(x-t)) \, m(t+h) \, dt \\
&\quad + \int_0^x (\phi(x+h) - \phi(x)) \, m(t+h) \, dt \\
&= \text{I+II+III}.
\end{aligned}
$$

We estimate these terms seperately, if $x \leq h$, then by Potto's bounds

$$
\begin{aligned}
|\text{I}| &\leq C \int_0^x t^\alpha \, (m(t) - m(t+h)) \; dt \\
&\leq C \int_0^x t^\alpha m(t) \, dt
\end{aligned}
$$

$$
\begin{aligned}
&\le C \int_0^x t^\alpha t^{-1-r-\delta}\, dt \\
&\le C_1 x^{\alpha-r-\delta} \\
&\le C_1 h^{\alpha-r-\delta}.
\end{aligned}
$$

If $x > h$, then by the previous calculation

$$
\begin{aligned}
|\mathrm{I}| \quad &\le \quad C\left(\int_0^h + \int_h^x\right) t^\alpha \,(m(t) - m(t+h))\, dt \\
&\le \quad C_1 h^{\alpha-r-\delta} + \int_h^x t^\alpha \,(m(t) - m(t+h))\, dt \\
&\overset{\text{mean value theorem}}{\le} \quad C_1 h^{\alpha-r-\delta} + \int_h^x t^\alpha |m'(t+\theta h)|\, dt \\
&\overset{\text{hs=t}}{=} \quad C_1 h^{\alpha-r-\delta} + h^\alpha \int_1^{x/h} h^2 |m'((s+\theta)h)|\, dt \\
&\le \quad C_1 h^{\alpha-r-\delta} + h^{\alpha+2} \int_1^\infty (sh)^{-2-r+\delta}\, dt \\
&\le \quad C_1 h^{\alpha-r-\delta} + C_2 h^{\alpha-r+\delta} \\
&\le \quad C_3 h^{\alpha-r-\delta},
\end{aligned}
$$

here $\theta \in (0,1)$ in the third inequality and the last third inequality by using Potter's bounds and $m' \in \mathcal{R}_{-2-r}$.

II: Again by Potter's bounds,

$$
\begin{aligned}
|\mathrm{II}| &\le C \int_{-h}^0 (t+h)^\alpha m(t+h)\, dt \\
&\le C \int_0^h s^\alpha s^{-r-1-\delta}\, ds \\
&\le C_1 h^{\alpha-r-\delta}.
\end{aligned}
$$

III: We have similar to case I:

$$
\begin{aligned}
|\mathrm{III}| &\le C \int_0^x h^\alpha m(t+h)\, dt \\
&\le C h^{\alpha-r-\delta}.
\end{aligned}
$$

□

Corollary 3.3.5. *Theorem 3.3.3 is still valid if* $\alpha + r \in (1,2)$.

Proof. Since $\alpha \in [0,1]$ and $r \in (0,1)$ the bound $\alpha + r < 2$ is best possible. Note that if f corresponds to the Sonine pair $(\bar{\mu}, k)$, then $f^\star = \lambda / f(\lambda)$ corresponds to the Sonine pair $(k, \bar{\mu})$. Let $\phi \in C^\alpha_\infty[0,T]$ and $\psi = {}^{\mathrm{R}}_0\mathrm{I}^f_x\, \phi$, we want to show for $\alpha + r > 1$ and $\alpha + r - \delta > 1$, $\psi \in C^{\alpha+r-\delta}[0,T]$. This is equivalent to

$$
\psi' = \frac{d}{dx}\, {}^{\mathrm{R}}_0\mathrm{I}^f_x\, \phi \in C^{\alpha+r-\delta-1}[0,T],
$$

$$\text{i.e.} \quad \psi' = \frac{d}{dx}\phi * k(x) \in C^{\alpha+r-\delta-1}[0,T],$$
$$\text{or} \quad \psi' = {}^{\mathrm{R}}_{0}\mathrm{D}^{f^\star}_{x}\,\phi \in C^{\alpha+r-\delta-1}[0,T] = C^{\alpha-\delta-(1-r)}[0,T],$$

but the latter is ensured by the previous Theorem 3.1.3 applied to $f^\star$ and Sonine pair $(k, \bar{\mu})$. □

3.4 Censored Bernstein fractional derivative and integral

Let $\{S_t, t \geq 0\}$ be a subordinator with Laplace transform [7]

$$\mathbb{E}(e^{-\lambda S_t}) = e^{-tf(\lambda)}$$

Here $f(\lambda)$ is the Laplace exponent given by

$$f(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda x})\mu(dx). \tag{3.4.1}$$

We make the following assumptions regarding to the Laplace transform (3.4.3).

Assumption 1. (1) f is a complete Bernstein function, i.e. the Lévy measure μ has a completely monotone density m with respect to the Lebesgue measure, so that (3.4.1) takes the form (2.3.4).

(2) (3.1.9) and (3.1.10) from Remark 3.1.7 hold.

Lemma 3.4.1. *If Assumption 1 is satisfied, $\bar{\mu}$ and k is a Sonine pair consisting of completely monotone functions and* (3.1.9)*,* (3.1.10) *hold.*

Proof. From (3.1.9) and (3.1.10), we get $a = b = 0$ and

$$\int_0^\infty m(t)\,dt = \int_0^\infty tm(t)\,dt = \infty. \tag{3.4.2}$$

$a^\star = b^\star = 0$ follows from the proof of Theorem 3.1.3 and (3.4.2). □

If $\int_0^\infty m(t)\,dt = \infty$, this allows to exclude the case of compound Poisson process. Now we have the following Laplace transform:

$$\frac{f(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda x}\bar{\mu}(x)\,dx \tag{3.4.3}$$

$$\frac{1}{f(\lambda)} = \int_0^\infty e^{-\lambda x}k(x)\,dx. \tag{3.4.4}$$

k is a potential density. If $U(dx) = k(x)\,dx$, we then see that $U(x) = \mathbb{E}(E(x))$ by Theorem 1.5.12.

Lemma 3.4.2. *If ϕ is a positive and decreasing function, then the following statements are equivalent:*

(1) $\mathscr{L}_{x\to\lambda}[\phi,\lambda] < \infty$ *for all* $0 < \lambda < \infty$.

(2) $\mathscr{L}_{x\to\lambda_0}[\phi,\lambda_0] < \infty$ *for one* λ_0.

(3) $\phi \in L^1_{\text{loc}}[0,\infty)$.

Proof. (1) $\Longrightarrow$ (2) is trivial. Now we show (2) $\Longrightarrow$ (3): we have the following inequality by ϕ is positive and we choose a compact set $[d_1,d_2] \subset [0,\infty)$ where,

$$\begin{aligned}\infty &> \mathscr{L}_{x\to\lambda_0}(\phi,\lambda_0)\\ &= \int_0^\infty e^{-\lambda_0 x}\phi(x)\,dx\\ &\geq \int_{d_1}^{d_2} e^{-\lambda_0 x}\phi(x)\,dx\\ &\geq e^{-\lambda_0 d_2}\int_{d_1}^{d_2}\phi(x)\,dx.\end{aligned}$$

(3) $\Longrightarrow$ (1): since ϕ is decreasing and positive,

$$\begin{aligned}\mathscr{L}_{x\to\lambda}(\phi,\lambda) &= \int_0^\infty e^{-\lambda x}\phi(x)\,dx\\ &= \int_0^1 e^{-\lambda x}\phi(x)\,dx + \int_1^\infty e^{-\lambda x}\phi(x)\,dx\\ &\leq \int_0^1 \phi(x)\,dx + \phi(1)\int_1^\infty e^{-\lambda x}\,dx\\ &< \infty.\end{aligned}$$

This shows that the above three statements are equivalent. □

Combing the above Assumption 1 and Lemma 3.4.2, we conclude that $\bar{\mu}$ and k are locally integrable on $[0,\infty)$. Multiplying (3.4.3) and (3.4.4), we conclude that $\bar{\mu}$ and k are a Sonine pair. Now let us define the following function spaces:

$$C_{\bar{\mu}}(0,T] = \left\{\varphi(x) \in C(0,T] \cap L^1(0,T] : (\bar{\mu} * \varphi)(x) \in C^1(0,T]\right\},$$

$$C_{\bar{\mu}}[0,T] = C[0,T] \cap C_{\bar{\mu}}(0,T].$$

Here if we take the kernel $\bar{\mu}(x) = \frac{x^{-\beta}}{\Gamma(1-\beta)}$ the above space is the same as $C_\beta(0,T]$ in Du, Toniazzi and Xu [16].

Let us define the following Bernstein Riemann-Liouville fractional derivative for $\phi \in C_{\bar{\mu}}(0,T]$ and Bernstein Riemann-Liouville integral $\varphi \in C(0,T] \cap L^1(0,T]$,

$$ {}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\,\phi = \frac{d}{dx}\int_0^x \phi(s)\,\bar{\mu}(x-s)\,ds, \tag{3.4.5} $$

$$ {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\varphi = \int_0^x \phi(s)k(x-s)\,ds. \tag{3.4.6} $$

The following theorem is due to [42, P.589]. We will give an independent proof.

Theorem 3.4.3. *If ϕ is a locally bounded measurable function on $[0,\infty)$ i.e. $\phi \in L^\infty_{\mathrm{loc}}[0,\infty)$, then ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi \to 0$ when $x \to 0, x \in (0,T]$.*

Proof. Since k is completely monotone, there exists a measure η on $[0,\infty)$ and assume that $k(x) = \mathscr{L}(\eta, x)$. Since $\mathscr{L}(k,\lambda) = 1/f(\lambda)$, we see that

$$ \begin{aligned} \mathscr{L}\left(\mathscr{L}(\eta,x),\lambda\right) &= \int_0^\infty e^{-\lambda x}\int_0^\infty e^{-xz}\,\eta(dz)\,dx \\ &\overset{\text{Fubini}}{=} \int_0^\infty \int_0^\infty e^{-\lambda x}e^{-xz}\,dx\,\eta(dz) \\ &= \int_0^\infty \frac{1}{\lambda+z}\,\eta(dz). \end{aligned} $$

In particular, $\int_0^\infty \frac{1}{\lambda+z}\,\eta(dz) < \infty$.

According to (3.4.6), we have

$$ \begin{aligned} |\,{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi| &= \left|\int_0^x \phi(s)k(x-s)\,ds\right| \\ &= \left|\int_0^x \phi(s)\int_0^\infty e^{-(x-s)z}\,\eta(dz)\,ds\right| \\ &\overset{\text{Fubini}}{=} \left|\int_0^\infty \int_0^x \phi(s)e^{-(x-s)z}\,ds\,\eta(dz)\right|. \end{aligned} $$

Since ϕ is locally bounded in $[0,\infty)$, we get

$$ \begin{aligned} |\,{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi| &\le C\int_0^\infty \int_0^x e^{-(x-s)z}\,ds\,\eta(dz) \\ &= C\int_0^\infty \frac{1-e^{-xz}}{z}d\eta(z) \\ &\le C\int_0^1 \frac{xz}{z}\,\eta(dz) + C\int_1^\infty \frac{1-e^{-xz}}{z}\,\eta(dz). \end{aligned} $$

Here we use the elementary estimate $1-e^{-y} \le y$. Since $1/z \le z/(1+z)$ for $z \ge 1$. We have

$$ |\,{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi| \le Cx\int_0^1 \eta(dz) + 2C\int_1^\infty \frac{1-e^{-xz}}{z}\,\eta(dz), $$

and using dominated convergence we get $\lim_{x\to 0} |\,{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi| = 0$. □

From now on we always assume Assumption 1.

Theorem 3.4.4. *If $\phi \in C(0,T] \cap L^1(0,T]$, then ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi \in C_{\bar{\mu}}(0,T]$ and ${}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{\cdot}$ is the left inverse of ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot}$.*

Proof. For the first claim, we need to show ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi \in C(0,T] \cap L^1(0,T]$ and $(\bar{\mu} * {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot})(x) \in C^1(0,T]$. Firstly we show that ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi \in C(0,T] \cap L^1(0,T]$. Since ϕ and k are in $C(0,T] \cap L^1(0,T]$, ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi$ is well defined and finite. Let $\varepsilon \in (0,x)$

$$ {}^{\mathrm{R}}_{0}\mathrm{J}^{f}_{x-\varepsilon}\,\phi = \int_0^{x-\varepsilon} \phi(s)k(x-s)\,ds, \tag{3.4.7} $$

note that ${}^{\mathrm{R}}_{0}\mathrm{J}^{f}_{x-\varepsilon} = {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x-\varepsilon}$. Given $T_0 \in (0,T]$, for all $x \in [T_0,T]$ and $\varepsilon \in (0,T_0)$, we have

$$ \begin{aligned} \left| {}^{\mathrm{R}}_{0}\mathrm{J}^{f}_{x-\varepsilon}\,\phi - {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi \right| &= \left| \int_{x-\varepsilon}^{x} \phi(s)k(x-s)\,ds \right| \\ &\le \int_{x-\varepsilon}^{x} |\phi(s)k(r-s)|\,ds \\ &\le \|\phi\|_{C[T_0-\varepsilon,T]} \int_{x-\varepsilon}^{x} k(r-s)\,ds, \end{aligned} $$

Using the local integrability of k, we can conclude that the above limit goes to 0 uniformly, when ε goes to 0. Applying dominated convergence theorem, ${}^{\mathrm{R}}_{0}\mathrm{J}^{f}_{x-\varepsilon}\,\phi$ is continuous on $[T_0,T]$. So, being a uniform limit of continuous function, ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi$ is continuous on $[T_0,T]$, and thus on $(0,T]$. The integrability of ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi$ follows by

$$ \begin{aligned} \int_0^T \left| {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi \right| dr &\le \int_0^T \int_0^x |\phi(r)|\,k(x-r)\,dr\,dx \\ &= \int_0^T |\phi(r)| \int_r^T k(x-r)\,dx\,dr \\ &\le \int_0^T k(r)\,dr \int_0^T |\phi(r)|\,dr \\ &< \infty. \end{aligned} $$

${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot} \in C(0,T] \cap L^1(0,T]$, and $\frac{d}{dx}\bar{\mu} * {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi$ is well defined on $(0,T]$. For $x \in (0,T]$,

$$ \begin{aligned} \bar{\mu} * {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\,\phi &= \int_0^x \bar{\mu}(x-r)\,{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{r}\,\phi(r)\,dr \\ &= \int_0^x \bar{\mu}(x-r) \int_0^r \phi(s)k(r-s)\,ds\,dr \\ &\overset{\text{Fubini}}{=} \int_0^x \phi(s) \int_s^x \bar{\mu}(x-r)k(r-s)\,dr\,ds \qquad (3.4.8) \\ &\overset{\text{r=s+t}}{=} \int_0^x \phi(s) \int_0^{x-s} \bar{\mu}(x-s-t)k(t)\,dt\,ds \\ &= \int_0^x \phi(s)\,ds \end{aligned} $$

then we get $(\bar{\mu} * {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot})\phi \in C^1(0,T]$ and Using the definition of Bernstein Riemann–Liouville fractional derivative Lemma 2.4.5 and noting that in this section we assume $b = 0$, we get ${}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\, {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \phi = \frac{d}{dx}\left(\bar{\mu} * {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \phi\right) = \phi$, when we take derivative on both sides of (3.4.8). □

Lemma 3.4.5. *(i) If $\psi \in C(0,T] \cap L^1(0,T]$. Then $\phi = {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \psi$ if and only if $\phi \in C_{\bar{\mu}}(0,T]$, ${}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\, \phi = \psi$ and $\lim_{x\to 0}(\bar{\mu} * \phi)(x) = 0$.*

(ii) If $\phi \in C_{\bar{\mu}}[0,T]$, ${}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\, \phi = 0$, then $\phi = 0$.

Proof. (i)$'' \Longrightarrow''$ Using the proof of Theorem 3.4.4, we have $\phi \in C_{\bar{\mu}}(0,T]$ and ${}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\, \phi = \psi$. By assumptions we can easily get $\lim_{x\to 0}(\bar{\mu} * \phi)(x) = 0$.

$'' \Longleftarrow''$ Using the results of Theorem 3.4.4, ${}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{\cdot}$ is the left inverse of ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot}$. Then we have ${}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\, \phi = \psi = {}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\, {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \psi$. Consequently ${}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}(\phi - {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \psi) = 0$ and $\bar{\mu} * (\phi - {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \psi)$ is a constant. By (3.4.8), we know $\lim_{x\to 0} \bar{\mu} * {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \psi = 0$, and $\lim_{x\to 0}(\bar{\mu} * \phi)(x) = 0$ by assumption. Therefore, $\bar{\mu} * (\phi - {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \psi)$ must be 0. Using the results of Theorem 3.4.4, ${}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{\cdot}$ is the left inverse of ${}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot}$. We can conclude that

$$
\begin{aligned}
\phi - {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \psi &= {}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\, {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}[\phi - {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \psi] \\
&= \frac{d}{dx}\left[\bar{\mu} * ({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot}(\phi - {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot}\, \psi))\right] \\
&= \frac{d}{dx}\left[\bar{\mu} * (k * (\phi - {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot}\, \psi))\right] \\
&= \frac{d}{dx}\left[k * (\bar{\mu} * (\phi - {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot}\, \psi))\right] \\
&= \frac{d}{dx}[k * 0] \\
&= 0,
\end{aligned}
$$

where the last third equality, we use the associativity of convolution. Thus, We can obtain $\phi - {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, \psi = 0$.

(ii) For $\phi \in C_{\bar{\mu}}[0,T]$, $(\bar{\mu} * \phi)(x) \le \|\phi\|_\infty \int_0^x \bar{\mu}(s)ds \to 0, \quad x \to 0$. Using the results of (i), $\phi = {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, 0 = 0$. □

Definition 3.4.6. We define the **censored Bernstein fractional derivative** of any $\varphi \in C_{\bar{\mu}}(0,T]$ as

$$
{}^{\mathrm{Ce}}_{0}\mathrm{D}^{f}_{x}\, \varphi(x) = {}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\, \varphi(x) - \varphi(x)\bar{\mu}(x), \quad x \in (0,T]. \tag{3.4.9}
$$

For a smooth function φ vanishing outside $\mathbb{R}_+$, the Bernstein Riemann-Liouville fractional derivative satisfies

Lemma 3.4.7. *If $\phi \in C^1[0,T]$, then*

$$
{}^{\mathrm{Ce}}_{0}\mathrm{D}^{f}_{x}\, \phi(x) = \int_0^x (\phi(x) - \phi(x-s))\, \mu(ds) \tag{3.4.10}
$$

Proof. According to Definition 2.4.3,

$$ {}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\,\phi(x) = \int_0^\infty \left(\phi^0(x) - \phi^0(x-s)\right)\mu(ds) $$

From here, we can get the censored fractional derivative is given by

$$ \begin{aligned} {}^{\mathrm{Ce}}_{0}\mathrm{D}^{f}_{x}\,\phi(x) &= {}^{\mathrm{R}}_{0}\mathrm{D}^{f}_{x}\,\phi(x) - \phi(x)\bar{\mu}(x) \\ &= \int_0^\infty \left(\phi^0(x) - \phi^0(x-s)\right)\mu(ds) - \phi(x)\bar{\mu}(x) \\ &= \int_0^x (\phi(x) - \phi(x-s))\,\mu(ds) + \int_x^\infty \phi(x)\mu(ds) - \phi(x)\bar{\mu}(x) \\ &= \int_0^x (\phi(x) - \phi(x-s))\,\mu(ds) \end{aligned} $$

□

Definition 3.4.8. For $0 < r < x$, we define the following kernel

$$ k_j(x,r) = \begin{cases} \bar{\mu}(r)k(x-r), & j = 1; \\ \displaystyle\int_r^x k_1(x,s)k_{j-1}(s,r)\,ds, & j \geq 2. \end{cases} \tag{3.4.11} $$

Furthermore $\int_0^x k_j(x,r)\,dr = 1$ by induction.

Definition 3.4.9. For $\phi \in C[0,T]$, we define

$$ \mathcal{K}\phi(x) = \begin{cases} \displaystyle\int_0^x k_1(x,r)\phi(r)\,dr, & x > 0; \\ \varphi(0), & x = 0. \end{cases} \tag{3.4.12} $$

Furthermore $\mathcal{K}\phi(x) = {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}[\bar{\mu}\phi]$ and $\mathcal{K}$ is a linear operator preserving positivity positivity(i.e. $\mathcal{K}\phi \geq 0$ if $\phi \geq 0$). In addition $\mathcal{K}^j\phi(x) = \int_0^x k_j(x,r)\phi(r)\,dr$ by induction.

Lemma 3.4.10. *Let Assumption 1 hold. Let $k, \bar{\mu}$ be a Sonine pair and $\mathscr{L}(\bar{\mu},\lambda) = f(\lambda)/\lambda$, $\mathscr{L}(k,\lambda) = 1/f(\lambda)$. The limit $q = \lim_{x\to 0} \bar{\mu}(x)\int_0^x k(s)\,ds$ is equal to 1 if, and only if, the supremum $\sup_{x\in[0,T]} \bar{\mu}(x)\int_0^x k(s)\,ds$ is equal to 1 for some $T > 0$.*

Proof. According to Assumption 1, we have that $\bar{\mu}, k \in \mathcal{CM}$ and $\bar{\mu}, k$ are decreasing. Thus,

$$ \begin{aligned} \bar{\mu}(x)\int_0^x k(s)\,ds &= \bar{\mu}(x)\int_0^x k(x-s)\,ds \\ &\leq \int_0^x \bar{\mu}(s)k(x-s)\,ds \\ &= 1 \end{aligned} $$

Then we have $q \leq 1$. Assume that for every $T > 0$, there is some some $x_0 \in [0, T]$ such that $\bar{\mu}(x_0) \int_0^{x_0} k(s)\, ds = 1$. Since $\int_0^{x_0} \bar{\mu}(s)k(x_0 - s)\, ds = 1$, we have

$$\begin{aligned}
0 &= \bar{\mu}(x_0) \int_0^{x_0} k(s)\, ds - \int_0^{x_0} \bar{\mu}(s)k(x_0 - s)\, ds \\
&= \int_0^{x_0} \bar{\mu}(x_0)k(x_0 - s)\, ds - \int_0^{x_0} \bar{\mu}(s)k(x_0 - s)\, ds \\
&= \int_0^{x_0} (\bar{\mu}(x_0) - \bar{\mu}(s))k(x_0 - s)\, ds.
\end{aligned}$$

This means that $\bar{\mu}$ is constant in $[0, x_0]$ since $\bar{\mu}$ is decreasing. Thus for all $x \in [0, x_0]$ we see from the Sonine equation that $\int_0^x k(s)\, ds$ is a constant on $[0, x_0]$. Therefore, $k = 0$ on $[0, x_0]$. This is a contradiction to $\bar{\mu}, k$ being a Sonine pair. □

Remark 3.4.11. If we assume there exists a $\alpha > 0, x^\alpha \bar{\mu}(x)$ is decreasing and assume $p = \sup_{[0,T]} \int_0^x (\frac{s}{x})^\alpha \bar{\mu}(s)k(x - s)\, ds < 1$, then we have $q = \sup_{x \in [0,T]} \bar{\mu}(x) \int_0^x k(s)\, ds \leq p < 1$. We can use this condition to prove the inverse operator exists. For a special example i.e. stable Sonine pair, see [16].

Example 3.4.12. Let us consider the regular variation case. Assuming $\bar{\mu}$, k are regularly varying with index $-\alpha$, $\alpha - 1$ respectively. i.e. $\bar{\mu} \in \mathcal{R}_{-\alpha}$, and $k \in \mathcal{R}_{\alpha-1}$, see the definition in Section 1.3. Let $\bar{\mu}$, k be a Sonine pair as in Theorem 3.1.13, then we can calculate Lemma 3.4.10 directly and get q equals $(\alpha\Gamma(\alpha)\Gamma(1 - \alpha))^{-1}$.

Proof. Using the proof of Lemma 3.4.10, we can get that q attain its supremum at 0. Let us consider the boundary point at 0,

$$\begin{aligned}
\lim_{x \to 0} \bar{\mu}(x) \int_0^x k(s)\, ds &= \lim_{x \to 0} \frac{l_1(x)x^{-\alpha}}{\Gamma(1 - \alpha)} \int_0^x \frac{l_2(s)s^{\alpha-1}}{\Gamma(\alpha)}\, ds \\
&= \lim_{x \to 0} \frac{l_1(x)l_2(x)x^{-\alpha}}{\Gamma(1 - \alpha)} \int_0^x \frac{l_2(s)s^{\alpha-1}}{l_2(x)\Gamma(\alpha)}\, ds \\
&= \lim_{x \to 0} \frac{l_1(x)l_2(x)x^{-\alpha}}{\Gamma(1 - \alpha)} \int_0^1 \frac{l_2(xs)(xs)^{\alpha-1}}{l_2(x)\Gamma(\alpha)} x\, ds \\
&= \frac{1}{\alpha\Gamma(\alpha)\Gamma(1 - \alpha)} \\
&< 1,
\end{aligned}$$

where the last two equalities using the results of Theorem 3.1.13. In the diagram 3.4 we see that the limit is continuous in the interval for $\alpha \in (0, 1)$, but we show that we can extend the continuity to the boundary case, i.e. the condition is not satisfied for slowly varying functions. □

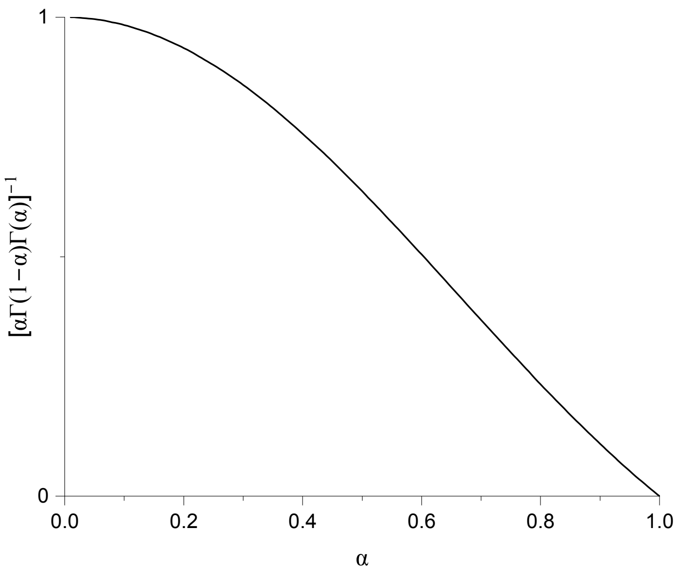

Figure 3.4

Example 3.4.13. There exist some Sonine pair of complete monotone functions which is non-regularly varying. We follow [38, Example 13.2.8], but give a different proof that the kernel is non-regularly varying. Suppose that $\alpha \in (0, 2)$. Let F be a function on $[0, \infty)$ defined by

$$F(x) = \begin{cases} 0, & 0 \le x < 1; \\ 2^n, & 2^{2(n-1)/\alpha} \le x < 2^{2n/\alpha}, \quad n \in \mathbb{N}. \end{cases} \tag{3.4.13}$$

The function F is not regularly varying and $x^{\alpha/2} \le F(x) \le 2x^{\alpha/2}$ for all $x \ge 1$, i.e for all $t > 0$,

$$\frac{t^{\alpha/2}}{2} \le \liminf_{x\to\infty} \frac{F(tx)}{F(x)} \le \limsup_{x\to\infty} \frac{F(tx)}{F(x)} \le 2t^{\alpha/2}.$$

Proof. Here we donote $a \asymp b \Longleftrightarrow$: there exists $c > 0$ such that $ca \le b \le c^{-1}a$.

We show first that $x^{\alpha/2} \le F(x) < 2x^{\alpha/2}$ for all $x \ge 1$. Let $x \in [2^{2(n-1)/\alpha}, 2^{2n/\alpha})$. We see that

$$F(x) = 2^n = (2^{2n/\alpha})^{\alpha/2} \ge x^{\alpha/2}$$

and by using the same equality, we obtain that $F(x) = 2(2^{2(n-1)/\alpha})^{\alpha/2} \le 2x^{\alpha/2}$.
Secondly, we construct a measure from the function F and show that the corresponding Stieltjes function is not regularly varying by a contradiction. The measure σ induced by the above nondecreasing regularly varying function F at ∞ is given by

$$\sigma = \sum_{n=1}^{\infty} 2^n \delta_{2^{2n/\alpha}}.$$

Since $\int_0^\infty (1+t)^{-1}\sigma(dt) < \infty$, σ is a Stieltjes measure. We denote by g the corresponding Stieltjes function and f be the complete Bernstein function given by

$$\frac{f(\lambda)}{\lambda} = g(\lambda) = \int_0^\infty \frac{1}{\lambda+t}\,\sigma(dt) = \int_0^\infty \int_0^\infty e^{-(\lambda+t)u}\,\sigma(dt)\,du = \sum_{n=1}^\infty \frac{2^n}{\lambda+2^{2n/\alpha}}.$$

We know by Bingham, Goldie and Teugels [9, Theorem 1.7.4] that the distribution function F is regularly varying at ∞ if, and only if, g is regularly varying at ∞. We show that g is not regularly varying.

We first have to prove upper and lower bounds for $\sum_1^\infty \frac{2^n}{\lambda+2^{2n/\alpha}}$, i.e. $g(\lambda) \asymp \lambda^{\alpha/2-1}$, $\lambda \to \infty$. In fact,

$$\begin{aligned}\sum_{n=1}^\infty \frac{2^n}{\lambda+2^{2n/\alpha}} &= \sum_{n=1}^\infty \int_{2^n}^{2\cdot 2^n} \frac{1}{\lambda+2^{2n/\alpha}}\,dy\\ &\le \sum_{n=1}^\infty \int_{2^n}^{2^{n+1}} \frac{1}{\lambda+(y/2)^{2/\alpha}}\,dy\\ &\le \int_2^\infty \frac{1}{\lambda+(y/2)^{2/\alpha}}\,dy\\ &\le \int_0^\infty \frac{1}{\lambda+(y/2)^{2/\alpha}}\,dy\end{aligned}$$

Now change variables $y = 2\lambda^{\alpha/2}z$ to get $\int_0^\infty \frac{1}{\lambda+\lambda z^{2/\alpha}} 2\lambda^{\alpha/2}dz \le C\lambda^{-1+\alpha/2}$. The lower bound is proved in a similar way. Assume that $\lambda \ge 1$, then

$$\begin{aligned}\sum_{n=1}^\infty \frac{2^n}{\lambda+2^{2n/\alpha}} &= \sum_{n=1}^\infty \int_{2^n}^{2\cdot 2^n} \frac{1}{\lambda+2^{2n/\alpha}}\,dy\\ &\ge \sum_{n=1}^\infty \int_{2^n}^{2^{n+1}} \frac{1}{\lambda+y^{2/\alpha}}\,dy\\ &= \int_2^\infty \frac{1}{\lambda+y^{2/\alpha}}\,dy.\end{aligned}$$

Changing variables $y = \lambda^{\alpha/2}z$ gives the following estimate:

$$\int_{2\lambda^{-\alpha/2}}^\infty \frac{1}{\lambda+\lambda z^{2/\alpha}}\lambda^{\alpha/2}\,dz \ge \int_2^\infty \frac{1}{1+z^{2/\alpha}}\lambda^{\alpha/2-1}\,dz \ge C\lambda^{\alpha/2-1}.$$

Assume that g is regularly varying, then according to the above bounds of g, the only possible index for g is $\alpha/2-1$. Take a special $\lambda = 2^{\frac{1}{\alpha}}$. From the definition of regular variation at ∞, we get $g(2^{1/\alpha}x)/g(x) \to (2^{\frac{1}{\alpha}})^{\frac{\alpha}{2}-1} = \sqrt{2}2^{-\frac{1}{\alpha}}$ as $x \to \infty$. By the definition of g, we have

$$\frac{g(2^{1/\alpha}x)}{g(x)} = \sum_{n=1}^\infty \frac{2^n}{2^{1/\alpha}x+2^{2n/\alpha}} \Big/ \sum_{n=1}^\infty \frac{2^n}{x+2^{2n/\alpha}}$$

$$= 2^{-\frac{1}{\alpha}} \sum_{n=1}^{\infty} \frac{2^n}{x + 2^{(2n-1)/\alpha}} \Big/ \sum_{n=1}^{\infty} \frac{2^n}{x + 2^{2n/\alpha}}.$$

Comparing the above limit and the quotient we obtain

$$\sum_{n=1}^{\infty} \frac{2^n}{x + 2^{(2n-1)/\alpha}} \Big/ \sum_{n=1}^{\infty} \frac{2^n}{x + 2^{2n/\alpha}} \to \sqrt{2},\ x \to \infty. \tag{3.4.14}$$

Note that the above limits holds for all α, i.e. it does not depend as α. But we will show that for $\alpha = 1$ we obtain

$$\sum_{n=1}^{\infty} \frac{2^n}{x + 2^{(2n-1)}} \Big/ \sum_{n=1}^{\infty} \frac{2^n}{x + 2^{2n}} - 1 = \sum_{n=1}^{\infty} \frac{2^{3n-1}}{(x + 2^{(2n-1)})(x + 2^{2n})} \Big/ \sum_{n=1}^{\infty} \frac{2^n}{x + 2^{2n}} \to 0. \tag{3.4.15}$$

Indeed, by using that $g(\lambda) \asymp \lambda^{\frac{\alpha}{2}-1}$, $\lambda \to \infty$, for all $\alpha \in (0, 2)$, we obtain

$$\sum_{n=1}^{\infty} \frac{2^n}{x + 2^{2n}} \asymp x^{1/2-1},$$

$$\sum_{n=1}^{\infty} \frac{2^{3n-1}}{(x + 2^{(2n-1)})(x + 2^{2n})} \leq \sum_{n=1}^{\infty} \frac{2^{3n-1}}{x^2 + 2^{4n-1}} = \sum_{n=1}^{\infty} \frac{2^{3n}}{2x^2 + 2^{4n}} \asymp (x^2)^{3/8-1} = x^{-5/4}.$$

We conclude that (3.4.15) holds true, which is a contradiction to (3.4.14). Hence, g cannot be regularly varying at ∞ i.e. F is not regularly varying at ∞. Using Theorem 3.1.15, and the connection $f(\lambda)/\lambda = \mathscr{L}(\bar{\mu}, \lambda)$, we conclude that $\bar{\mu}, k$ is not regularly varying at 0 Sonine pair. □

In fact according to the Definition 1.3.12, the above function F is an O-regularly varying function. For further example of O-regularly varying functions with different index we refer to Section 6, in Kim et.al. [39].

Lemma 3.4.14. *Let Assumption 1 and* $q = \lim_{x \to 0} \bar{\mu}(x) \int_0^x k(s)\,ds < 1$ *hold true. Let* $k, \bar{\mu}$ *be a Sonine pair and* $\mathscr{L}(\bar{\mu}, \lambda) = f(\lambda)/\lambda$, $\mathscr{L}(k, \lambda) = 1/f(\lambda)$. *Assume that* $\varphi \in C[0, T]$ *and* $|\varphi(x)| \leq M \int_0^x k(s)\,ds$, *where* M *is a positive constant, then for all* $x \in [0, T]$, $\mathcal{K}\varphi(x) \leq Mq \int_0^x k(s)\,ds$. *Furthermore,* $|\mathcal{K}^j \varphi(x)| \leq Mq^j \int_0^x k(s)\,ds$, $x \in [0, T]$ *and* $\mathcal{K}\varphi(x) \in C[0, T]$.

Proof. According to Assumption 1 and from Lemma 3.4.1, we have that $\bar{\mu}, k \in \mathcal{CM}$ and $\bar{\mu}, k$ are decreasing. For the first claim, we have

$$|\mathcal{K}\varphi(x)| = \left| \int_0^x \bar{\mu}(r) k(x - r) \varphi(r)\,dr \right|$$
$$\leq \sup_{r \leq x} \varphi(r) \bar{\mu}(r) \int_0^x k(s)\,ds$$

$$\leq M \int_0^x k(s)\,ds \underbrace{\sup_{r\leq x} \bar{\mu}(r) \int_0^r k(s)\,ds}_{\overset{\text{Lemma 3.4.10}}{=} q}$$

$$= Mq \int_0^x k(s)\,ds,$$

where we use the condition $|\varphi(x)| \leq M \int_0^x k(s)\,ds$ and $q = \lim_{x\to 0} \bar{\mu}(x) \int_0^x k(s)\,ds$ for the estimate, see Lemma 3.4.10. For the second claim, we use the first claim and iterate it,

$$\begin{aligned}
|\mathcal{K}^j \varphi(x)| &\overset{(3.4.12)}{=} |\mathcal{K}^{j-1}\mathcal{K} \int_0^x \bar{\mu}(r)k(x-r)\varphi(r)\,dr| \\
&\leq Mq\mathcal{K}^{j-1} \int_0^x k(r)\,dr \\
&= Mq\mathcal{K}^{j-2}\mathcal{K} \int_0^x k(r)\,dr \\
&= Mq\mathcal{K}^{j-2} \int_0^x k(x-s)\bar{\mu}(s) \int_0^s k(r)\,dr\,ds \\
&\leq Mq^2\mathcal{K}^{j-2} \int_0^x k(x-s)\,ds \\
&= Mq^2\mathcal{K}^{j-2} \int_0^x k(s)\,ds \\
&= \ldots \\
&\leq Mq^j \int_0^x k(s)\,ds.
\end{aligned}$$

This finishes the second claim.

Now we prove $\mathcal{K}\varphi$ is continuous in $[0, T]$. The first case is if $x = 0$, we show $\lim_{x\to 0} \mathcal{K}\varphi(x) = \varphi(0)$.

$$\begin{aligned}
|\mathcal{K}\varphi(x) - \varphi(0)| &\overset{(3.4.12)}{=} \left| \int_0^x \bar{\mu}(r)k(x-r)(\varphi(r) - \varphi(0))\,dr \right| \\
&\leq \int_0^x \bar{\mu}(r)k(x-r)\,|\varphi(r) - \varphi(0)|\,dr \\
&< \varepsilon.
\end{aligned}$$

The last inequality is because the continuity of φ at 0. For the second case, assume $x > 0$ and $x > y$; we prove $\mathcal{K}\varphi$ is continuous in $(0, T]$. For all $\delta > 0$, we consider $C[\delta, T]$

$$\begin{aligned}
|\mathcal{K}\varphi(x) - \mathcal{K}\varphi(y)| &\overset{(3.4.12)}{=} \left| \int_0^x \bar{\mu}(r)k(x-r)\varphi(r)\,dr - \int_0^y \bar{\mu}(r)k(y-r)\varphi(r)\,dr \right| \\
&= \left| \int_0^x \bar{\mu}(x-r)k(r)\varphi(x-r)\,dr - \int_0^y \bar{\mu}(y-r)k(r)\varphi(y-r)\,dr \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\int_0^y \bar{\mu}(x-r)k(r)\varphi(x-r)\,dr - \int_0^y \bar{\mu}(y-r)k(r)\varphi(y-r)\,dr\right| \\
&\quad + \left|\int_y^x \bar{\mu}(x-r)k(r)\varphi(x-r)\,dr\right| \\
&= \text{I+II}.
\end{aligned}
$$

For Part II, due to $\varphi \in C[0,T]$, k being finite on $[\delta, T]$ and $\bar{\mu}$ being locally integrable, Part II goes to 0, as x goes to y. For Part I, we have the following estimation,

$$
\begin{aligned}
&\left|\int_0^y \bar{\mu}(x-r)k(r)\varphi(x-r)\,dr - \int_0^y \bar{\mu}(y-r)k(r)\varphi(y-r)\,dr\right| \\
&= \left|\int_0^y \bar{\mu}(x-r)k(r)\varphi(x-r)\,dr - \int_0^y \bar{\mu}(y-r)k(r)\varphi(x-r)\,dr\right. \\
&\quad \left. + \int_0^y \bar{\mu}(y-r)k(r)\varphi(x-r)\,dr - \int_0^y \bar{\mu}(y-r)k(r)\varphi(y-r)\,dr\right| \\
&\leq \left|\int_0^y \bar{\mu}(x-r)k(r)\varphi(x-r)\,dr - \int_0^y \bar{\mu}(y-r)k(r)\varphi(x-r)\,dr\right| \\
&\quad + \left|\int_0^y \bar{\mu}(y-r)k(r)\varphi(x-r)\,dr - \int_0^y \bar{\mu}(y-r)k(r)\varphi(y-r)\,dr\right| \\
&\leq \int_0^y |\bar{\mu}(x-r) - \bar{\mu}(y-r)|\, k(r)\varphi(x-r)\,dr + \int_0^y |\varphi(x-r) - \varphi(y-r)|\, \bar{\mu}(y-r)k(r)\,dr \\
&= \text{III+IV}
\end{aligned}
$$

Due to $\varphi \in C[0,T]$, by Lebesgue's dominated convergence theorem, we obtain that Part IV goes to 0, as x goes to y. For Part III, for all $\varepsilon > 0$,

$$
\begin{aligned}
&\int_0^y |\bar{\mu}(x-r) - \bar{\mu}(y-r)|\, k(r)\varphi(x-r)\,dr \\
&= \int_0^{y-\varepsilon} |\bar{\mu}(x-r) - \bar{\mu}(y-r)|\, k(r)\varphi(x-r)\,dr \\
&\quad + \int_{y-\varepsilon}^y |\bar{\mu}(x-r) - \bar{\mu}(y-r)|\, k(r)\varphi(x-r)\,dr \\
&= \int_0^{y-\varepsilon} \left|1 - \frac{\bar{\mu}(x-r)}{\bar{\mu}(y-r)}\right| \bar{\mu}(y-r)k(r)\varphi(x-r)\,dr \\
&\quad + \int_{y-\varepsilon}^y |\bar{\mu}(x-r) - \bar{\mu}(y-r)|\, k(r)\varphi(x-r)\,dr \\
&= \text{V+VI}
\end{aligned}
$$

Due to $\varphi \in C[0,T]$ and $\bar{\mu} \in \mathcal{CM}$, using Lebesgue's dominated convergence theorem, we can get that Part V goes to 0, as x goes to y. Finally, for Part VI

$$
\int_{y-\varepsilon}^y |\bar{\mu}(x-r) - \bar{\mu}(y-r)|\, k(r)\varphi(x-r)\,dr
$$

$$
\begin{aligned}
&\leq 2\int_{y-\varepsilon}^{y} \bar{\mu}(y-r)k(r)\varphi(x-r)\,dr \\
&\leq 2\|\varphi\|_\infty \int_{y-\varepsilon}^{y} \bar{\mu}(y-r)k(r)\,dr \\
&\to 0, \quad \varepsilon \to 0.
\end{aligned}
$$

For the last convergence, due to $\varphi \in C[0,T]$, k has no singularity (at $y \neq 0$) and $\bar{\mu}$ is locally integrable, as ε goes to 0. Thus, Part VI goes to 0. For the case $x < y$, we can use the same strategy. □

Lemma 3.4.15. *Let Assumption 1 and $q = \lim_{x\to 0} \bar{\mu}(x)\int_0^x k(s)\,ds < 1$ hold true. Let $k, \bar{\mu}$ be a Sonine pair and $\mathscr{L}(\bar{\mu},\lambda) = f(\lambda)/\lambda$, $\mathscr{L}(k,\lambda) = 1/f(\lambda)$. Assume that for all $\varphi \in C[0,T]$ and for all $x \in [0,T]$, $|\varphi(x)| \leq M\int_0^x k(s)\,ds$. Then for all $\phi \in C[0,T]$, $\sum_{j=1}^\infty \mathcal{K}^j\varphi \in C[0,T]$ and for all $x \in [0,T]$,*

$$
|\sum_{j=1}^{\infty} \mathcal{K}^j\varphi(x)| \leq \sum_{j=1}^{\infty} |\mathcal{K}^j\varphi(x)| \leq M\sum_{j=1}^{\infty} q^j \int_0^x k(s)\,ds.
$$

Proof. By the proof of Lemma 3.4.14, $\mathcal{K}\varphi \in C[0,T]$ and $|\mathcal{K}^j\varphi(x)| \leq Mq^j\int_0^x k(s)\,ds$. Summing over j, we get

$$
\left|\sum_{j=1}^{\infty} \mathcal{K}^j\varphi(x)\right| \leq \sum_{j=1}^{\infty} \left|\mathcal{K}^j\varphi(x)\right| \leq M\sum_{j=1}^{\infty} q^j \int_0^x k(s)\,ds.
$$

The series on the right hand side is uniformly convergent for $x \in [0,T]$. Hence we get $\sum_{j=1}^\infty \mathcal{K}^j\varphi(x)$ uniformly convergent to a limit in $C[0,T]$. □

Lemma 3.4.16. *Let Assumption 1 and $q = \lim_{x\to 0} \bar{\mu}(x)\int_0^x k(s)\,ds < 1$ hold true. Let $k, \bar{\mu}$ be a Sonine pair and $\mathscr{L}(\bar{\mu},\lambda) = f(\lambda)/\lambda$, $\mathscr{L}(k,\lambda) = 1/f(\lambda)$. Then*

$$
|{}_0^{\mathrm{R}}\mathrm{I}_x^f\,\varphi| \leq \|\varphi\|_\infty \int_0^x k(s)\,ds \quad \textit{for all} \quad \varphi \in C[0,T].
$$

Proof. By the definition of the integral, we can easily get the following results,

$$
\begin{aligned}
|{}_0^{\mathrm{R}}\mathrm{I}_x^f\,\varphi| &= \int_0^x \varphi(s)k(x-s)\,ds \\
&\leq \int_0^x |\varphi(s)|k(x-s)\,ds \\
&\leq \|\varphi\|_\infty \int_0^x k(s)\,ds
\end{aligned}
$$
□

Next we begin with the censored initial value problem (3.4.16) with $g \in C[0,T]$ and $\phi(0) \in \mathbb{R}$.

$$
{}_0^{\mathrm{R}}\mathrm{D}_x^f\,\phi = \begin{cases} g(x) + \phi(x)\bar{\mu}(x), & x > 0; \\ \phi(0), & x = 0. \end{cases} \tag{3.4.16}
$$

Theorem 3.4.17. *Let Assumption 1 and* $q = \lim_{x\to 0} \bar{\mu}(x) \int_0^x k(s)\,ds < 1$ *hold true. Let* $k, \bar{\mu}$ *be a Sonine pair and* $\mathscr{L}(\bar{\mu}, \lambda) = f(\lambda)/\lambda$, $\mathscr{L}(k, \lambda) = 1/f(\lambda)$. $\phi_0 \in \mathbb{R}$, $g \in C[0, T]$, *then there exists a unique function* $\phi \in C_{\bar{\mu}}[0, T]$ *satisfying* (3.4.16)*, and it has the series representation*

$$\phi(x) - \phi_0 = {}^{\mathrm{Ce}}_{\ 0}\mathrm{I}^f_x\, g = \sum_{j=0}^{\infty} \mathcal{K}^j\, {}^{\mathrm{R}}_{0}\mathrm{I}^f_x\, g. \tag{3.4.17}$$

Remark 3.4.18. We see from the proof of Theorem 3.4.17 that if $g \in C[0, T]$, then we have ${}^{\mathrm{Ce}}_{\ 0}\mathrm{I}^f_{\cdot}\, g \in C_{\bar{\mu}}[0, T]$.

Proof. Step 1. we show the existence of ϕ using Picard iteration. Set $\bar{\phi}(x) = \phi(x) - \phi(0)$, we have the following equality.

$${}^{\mathrm{R}}_{0}\mathrm{D}^f_x\, \bar{\phi} = \begin{cases} g(x) + \bar{\phi}(x)\bar{\mu}(x), & x > 0; \\ 0, & x = 0. \end{cases} \tag{3.4.18}$$

We define the following sequence

$$\begin{cases} \bar{\phi}_{n+1}(x) = {}^{\mathrm{R}}_{0}\mathrm{I}^f_x \left[g + \bar{\phi}_n \bar{\mu}\right] \\ \bar{\phi}_0(x) = 0 \end{cases} \tag{3.4.19}$$

$$\begin{aligned}
\bar{\phi}_1(x) &= {}^{\mathrm{R}}_{0}\mathrm{I}^f_x[g] \\
&= \mathcal{K}^0\left[{}^{\mathrm{R}}_{0}\mathrm{I}^f_x[g]\right] \\
\bar{\phi}_2(x) &= {}^{\mathrm{R}}_{0}\mathrm{I}^f_x \left[g + \bar{\mu}\, {}^{\mathrm{R}}_{0}\mathrm{I}^f_x[g]\right] \\
&= {}^{\mathrm{R}}_{0}\mathrm{I}^f_x[g] + {}^{\mathrm{R}}_{0}\mathrm{I}^f_x \left[\bar{\mu}\, {}^{\mathrm{R}}_{0}\mathrm{I}^f_x[g]\right] \\
&= \mathcal{K}^0\left[{}^{\mathrm{R}}_{0}\mathrm{I}^f_x[g]\right] + \mathcal{K}^1\left[{}^{\mathrm{R}}_{0}\mathrm{I}^f_x[g]\right] \\
&, \ldots, \\
\bar{\phi}_{n+1}(x) &= \sum_{j=0}^{n} \mathcal{K}^j\left[{}^{\mathrm{R}}_{0}\mathrm{I}^f_x[g]\right]
\end{aligned}$$

For every x the limit $\lim_{n\to\infty} \bar{\phi}_{n+1}(x) = \bar{\phi}(x) = \sum_{j=0}^{\infty} \mathcal{K}^j\left[{}^{\mathrm{R}}_{0}\mathrm{I}^f_x[g]\right]$ exists due to Lemma 3.4.16 and Lemma 3.4.15. We get

$$\bar{\phi}(x) = \sum_{j=0}^{\infty} \mathcal{K}^j\left[{}^{\mathrm{R}}_{0}\mathrm{I}^f_x[g]\right] \quad \text{i.e.} \quad {}^{\mathrm{Ce}}_{\ 0}\mathrm{I}^f_x\, g = \sum_{j=0}^{\infty} \mathcal{K}^j\, {}^{\mathrm{R}}_{0}\mathrm{I}^f_x[g].$$

Step 2. Before we continue our proof, let us show the following auxiliary statement and estimate:

Lemma 3.4.19. *Under the conditions of Theorem 3.4.17,* ${}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x}\, g$ *can be equivalently represented as*

$$ {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x}\, g = {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \left[\bar{\mu} \sum_{j=0}^{\infty} \mathcal{K}^{j} \left[\bar{\mu}^{-1} g \right] \right] \tag{3.4.20} $$

$$ = \sum_{j=0}^{\infty} \mathcal{K}^{j+1} \left[\bar{\mu}(x)^{-1} g(x) \right] \tag{3.4.21} $$

Proof. Using the definition of the above Step 1, and the definition of the operator $\mathcal{K}$ in Definition 3.4.9, we get

$$ \begin{aligned} {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x}\, g &= \sum_{j=0}^{\infty} \mathcal{K}^{j} \left[{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}[g] \right] \\ &= \sum_{j=0}^{\infty} \int_{0}^{x} k_{j}(x,r)\, {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{r}[g(r)]\, dr \\ &= \sum_{j=0}^{\infty} \int_{0}^{x} k_{j}(x,r) \mathcal{K}\left[\bar{\mu}(r)^{-1} g(r) \right] dr \\ &= \sum_{j=0}^{\infty} \int_{0}^{x} k_{j+1}(x,r) \bar{\mu}(r)^{-1} g(r)\, dr \\ &= \sum_{j=0}^{\infty} \mathcal{K}^{j+1} \left[\bar{\mu}(x)^{-1} g(x) \right] \\ &= \sum_{j=0}^{\infty} \mathcal{K} \left[\bar{\mu}(x)^{-1} \bar{\mu}(x) \mathcal{K}^{j} \left[\bar{\mu}(x)^{-1} g(x) \right] \right] \\ &= \sum_{j=0}^{\infty} {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \left[\bar{\mu} \mathcal{K}^{j} \left[\bar{\mu}^{-1} g \right] \right] \\ &= {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \left[\sum_{j=0}^{\infty} \bar{\mu} \mathcal{K}^{j} \left[\bar{\mu}^{-1} g \right] \right]. \end{aligned} $$

□

We will now continue with the proof of Theorem 3.4.17. Denote $\hat{g} = \bar{\mu} \sum_{j=0}^{\infty} \mathcal{K}^{j} \left[\bar{\mu}^{-1} g \right]$, we have the following estimation:

$$ \mathcal{K}^{j} \left[\bar{\mu}^{-1} g \right](x) \leq q^{j-1} \|g\|_{\infty} \int_{0}^{x} k(r)\, dr. \tag{3.4.22} $$

We prove the (3.4.22) by induction. Indeed, when $j = 1$, using the definition of the operator $\mathcal{K}$ we have

$$ \begin{aligned} \mathcal{K} \bar{\mu}^{-1}(x) g(x) &= \int_{0}^{x} \bar{\mu}(r)(\bar{\mu}^{-1}(r) g(r) k(x-r)\, dr \\ &\leq \|g\|_{\infty} \int_{0}^{x} k(r)\, dr. \end{aligned} $$

We assume that (3.4.22) holds for some $j \in \mathbb{N}$. For $j \rightsquigarrow j+1$, we obtain

$$\begin{aligned}\mathcal{K}^{j+1}\bar{\mu}^{-1}(x)g(x) &= \mathcal{K}^j\mathcal{K}\bar{\mu}^{-1}(x)g(x)\\ &\leq \|g\|_\infty \mathcal{K}^j \int_0^x k(r)\,dr\\ &\leq \|g\|_\infty q^j \int_0^x k(r)\,dr\\ &\leq \|g\|_\infty q^j \int_0^T k(r)\,dr,\end{aligned}$$

where the second inequality follows by Lemma 3.4.15. This finishes the proof of induction.

Step 3. We show ${}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{\cdot}\, g \in C_{\bar{\mu}}[0,T]$. By Lemma 3.4.15, Lemma 3.4.16 and (3.4.17), we have ${}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{\cdot}\, g \in C[0,T]$ since g is continuous. Next we need to show ${}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{\cdot}\, g \in C_{\bar{\mu}}(0,T]$. For this, we write ${}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{\cdot}\, g = {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot}\, \hat{g}$, with $\hat{g} = \bar{\mu}\sum_{j=0}^{\infty}\mathcal{K}^j\left[\bar{\mu}^{-1}\varphi\right]$ as in Step 2 and show that $\hat{g} \in C(0,T] \cap L^1(0,T]$ see Theorem 3.4.4. In fact, integrability of $\hat{g}$ follows by

$$\begin{aligned}&\int_0^T \left|\sum_{j=0}^{\infty} \bar{\mu}(x)\mathcal{K}^j\left[\bar{\mu}(x)^{-1}g(x)\right]\right| dx\\ &\leq \int_0^T \bar{\mu}(x)\sum_{j=0}^{\infty}\left|\mathcal{K}^j\left[\bar{\mu}(x)^{-1}g(x)\right]\right| dx\\ &\leq \frac{\|g\|_\infty}{1-q}\int_0^T\int_0^x k(s)\,ds\,\bar{\mu}(x)\,dx\\ &\leq \frac{\|g\|_\infty}{1-q}\int_0^T \bar{\mu}(x)\,dx\int_0^T k(s)\,ds\\ &< \infty,\end{aligned}$$

where the first inequality follows by (3.4.22). To see the continuity of $\hat{g}$, it suffices to show that $\sum_{j=0}^{\infty}\mathcal{K}^j\left[\bar{\mu}^{-1}g\right]$ is continuous, since $\bar{\mu}$ is continuous on $C(0,T]$ by Assumption 1. Indeed, to show that $\sum_{j=0}^{\infty}\mathcal{K}^j\left[\bar{\mu}^{-1}g\right]$ is uniformly convergent, we can use the following estimate:

$$\begin{aligned}&\sum_{j=0}^{\infty}\mathcal{K}^j\left[\bar{\mu}^{-1}g\right]\\ &= \bar{\mu}^{-1}g + \sum_{j=1}^{\infty}\mathcal{K}^j\left[\bar{\mu}^{-1}g\right].\end{aligned}$$

Clearly $\bar{\mu}^{-1}g \in C(0,T]$. Moreover

$$\sum_{j=1}^{\infty}\mathcal{K}^j\left[\bar{\mu}^{-1}g\right] \overset{(3.4.22)}{\leq} \|g\|_\infty + \|g\|_\infty\int_0^x k(r)\,dr + ... + q^{j-1}\|g\|_\infty\int_0^x k(r)\,dr + ...,$$

$$\leq \frac{\|g\|_\infty}{1-q}\int_0^T k(r)\,dr.$$

Thus, $\sum_{j=1}^{\infty}\mathcal{K}^j\left[\bar{\mu}^{-1}g\right]$ and $\sum_{j=0}^{\infty}\mathcal{K}^j\left[\bar{\mu}^{-1}g\right]$ is continuous and in $C(0,T]$. Combining the above results, we can get ${}^{\mathrm{Ce}}_{0}\mathrm{I}^f_{\cdot}\,g \in C[0,T]$ and ${}^{\mathrm{Ce}}_{0}\mathrm{I}^f_{\cdot}\,g \in C_{\bar{\mu}}(0,T]$, then we obtain ${}^{\mathrm{Ce}}_{0}\mathrm{I}^f_{\cdot}\,g \in C_{\bar{\mu}}[0,T]$.

Step 4. We show that the solution is unique in $C_{\bar{\mu}}[0,T]$. Let $\phi_1,\phi_2 \in C_{\bar{\mu}}[0,T]$ be two solutions to the problem (3.4.16). By the linearity of the operator ${}^{\mathrm{Ce}}_{0}\mathrm{D}^f_{\cdot}$, $\psi = \phi_1 - \phi_2 \in C_{\bar{\mu}}[0,T]$ satisfies the following equation

$$ {}^{\mathrm{R}}_{0}\mathrm{D}^f_x\,\psi = \begin{cases} \psi(x)\bar{\mu}(x), & x > 0; \\ 0, & x = 0. \end{cases} \tag{3.4.23}$$

Using Theorem 3.4.4, Definition 3.4.9 and Lemma 3.4.5 , apply the inverse operator of ${}^{\mathrm{R}}_{0}\mathrm{D}^f_x$ on both sides, we get $\psi(x) = {}^{\mathrm{R}}_{0}\mathrm{I}^f_x\left[\psi\bar{\mu}\right] = \mathcal{K}\psi(x)$. If $\psi \equiv 0$ on $[0,T]$, we get uniqueness. Assume, to be contrary, that $\psi \neq 0$. Because of $\psi(x) = \mathcal{K}\psi(x)$ we have $\int_0^x(\psi(x)-\psi(r))\bar{\mu}(r)k(x-r)\,dr = 0$, for all $x \in [0,T]$. This is impossible: Take $\xi \in \arg\max_{r\in[0,T]}|\psi(r)|$, we have $\int_0^{\xi}(\psi(\xi)-\psi(r))\bar{\mu}(r)k(\xi-r)\,dr = 0$. This is impossible, and we have reached a contradiction. □

3.5 Linear censored initial value problem

In this section we assume the condition Assumption 1 hold true and the condition from Lemma 3.4.10, $q = \lim_{x\to 0}\bar{\mu}(x)\int_0^x k(s)\,ds < 1$ plays an important role again in the context of this section. Let $k, \bar{\mu}$ be a Sonine pair and $\mathscr{L}(\bar{\mu},\lambda) = f(\lambda)/\lambda, \mathscr{L}(k,\lambda) = 1/f(\lambda)$.

Lemma 3.5.1. *If φ is increasing and positive, then ${}^{\mathrm{R}}_{0}\mathrm{I}^f_x\,\varphi$ is increasing.*

Proof. Let $y \geq x$. Using definition of ${}^{\mathrm{R}}_{0}\mathrm{I}^f_{\cdot}$, we obtain

$$\begin{aligned} {}^{\mathrm{R}}_{0}\mathrm{I}^f_x\,\varphi &= \int_0^x \varphi(s)k(x-s)\,ds = \int_0^x \varphi(x-s)k(s)\,ds \\ &\leq \int_0^x \varphi(y-s)k(s)\,ds \leq \int_0^y \varphi(y-s)k(s)\,ds \\ &= {}^{\mathrm{R}}_{0}\mathrm{I}^f_y\,\varphi. \end{aligned}$$

□

Theorem 3.5.2. *Let Assumption 1 and $q = \lim_{x\to 0}\bar{\mu}(x)\int_0^x k(s)\,ds < 1$ hold true. Let $k, \bar{\mu}$ be a Sonine pair and $\mathscr{L}(\bar{\mu},\lambda) = f(\lambda)/\lambda$, $\mathscr{L}(k,\lambda) = 1/f(\lambda)$. Moreover we assume that f is unbounded. For any $\phi_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$, the following initial value problem*

$$\begin{cases} {}^{\mathrm{Ce}}_{0}\mathrm{D}^f_x\,\phi(x) = \lambda\phi(x), & x \in (0,T] \\ \phi(x) = \phi_0, & x = 0 \end{cases} \tag{3.5.1}$$

has a unique solution ϕ in $C_{\bar{\mu}}[0,T]$ given by $\phi(x) = \phi_0\sum_{j=0}^{\infty}(\lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^f_x)^j\mathbb{1}$.

Proof. Without loss of generality we assume $\lambda > 0$, if $\lambda < 0$ we take the modulus to get the results. The proof relies on an iteration scheme, i.e. we define the following sequence

$$\begin{cases} \phi_n(x) = \lambda \, {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} \, \phi_{n-1} + \phi_0 \\ \phi_0(x) = \phi_0 \end{cases} \tag{3.5.2}$$

$$\begin{aligned} \phi_1(x) &= \lambda \, {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} \, \phi_0 + \phi_0, \\ \phi_2(x) &= \lambda \, {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} \, \phi_1 + \phi_0 \\ &= \lambda \, {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} [\lambda \, {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} \, \phi_0 + \phi_0] + \phi_0 \\ &= \lambda^2 ({}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x})^2 \phi_0 + {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} \, \phi_0 + \phi_0, \\ &, \ldots, \\ \phi_{n+1}(x) &= \sum_{j=0}^{n} (\lambda \, {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x})^j \phi_0. \end{aligned}$$

We have to show that $\sum_{j=0}^{\infty} (\lambda \, {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x})^j \mathbb{1}$ converges.
We use induction to show

$$(\lambda \, {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x})^j \mathbb{1} \leq \left(\frac{\lambda}{1-q} \right)^j \left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right)^{j-1} \int_0^{\bullet} k(y) \, dy \tag{3.5.3}$$

Recall the definition ${}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} \, \mathbb{1} = \sum_{i=0}^{\infty} \mathcal{K}^i \, {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \, \mathbb{1}$ in Theorem 3.4.17 (3.4.17). For $j = 1$, we get from Lemma 3.4.14 and from the the following inequality,

$$\mathcal{K}^i \, {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \, \mathbb{1} \leq q^i \int_0^x k(y) \, dy$$

and from the above definition of ${}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} \, \mathbb{1}$ that

$${}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} \, \mathbb{1} = \sum_{i=0}^{\infty} \mathcal{K}^i \, {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \, \mathbb{1} \leq \sum_{i=0}^{\infty} q^i \int_0^x k(y) \, dy = \frac{1}{1-q} \int_0^x k(y) \, dy.$$

Using (3.5.3) as induction assumption for some $j \in \mathbb{N}$, we get for $j \rightsquigarrow j+1$, we obtain

$$\begin{aligned} \left({}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} \right)^{j+1} \mathbb{1} &= {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} \left({}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} \right)^{j} \mathbb{1} \\ &\leq \frac{1}{(1-q)^j} \, {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x} \left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right)^{j-1} \int_0^{\bullet} k(y) \, dy \\ &\overset{(3.4.17)}{=} \frac{1}{(1-q)^j} \sum_{i=0}^{\infty} \mathcal{K}^i \, {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \left[\left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot} \right)^{j-1} \int_0^{\bullet} k(y) \, dy \right] \\ &\leq \frac{1}{(1-q)^{j+1}} \left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right)^{j} \int_0^{\bullet} k(y) \, dy. \end{aligned}$$

The last inequality follows from the calculation below:

$$\mathcal{K}^i \, {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \left[\left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot} \right)^{j-1} \int_0^{\bullet} k(y) \, dy \right]$$

$$
\begin{aligned}
&= \mathcal{K}^i \left[\left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right)^j \int_0^{\bullet} k(y)\, dy \right] \\
&= \mathcal{K}^{i-1} \mathcal{K} \left[\left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right)^j \int_0^{\bullet} k(y)\, dy \right] \\
&= \mathcal{K}^{i-1} \left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right) \bar{\mu}(\cdot) \left[\left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot} \right)^j \int_0^{\bullet} k(y)\, dy \right] \\
&\overset{(*)}{\leq} \mathcal{K}^{i-1} \left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right) \left[\left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{\cdot} \right)^j \bar{\mu}(\cdot) \int_0^{\bullet} k(y)\, dy \right] \\
&\leq q \mathcal{K}^{i-1} \left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right)^{j+1} \mathbb{1} \\
&= q \mathcal{K}^{i-1} \left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right)^j \int_0^{\bullet} k(y).
\end{aligned}
$$

The step marked by $(*)$ we use the monotonicity of $\bar{\mu}$. Repeated use of the above calculation, yields

$$
\mathcal{K}^i \, {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \left[\left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right)^{j-1} \int_0^{\bullet} k(y)\, dy \right] \leq q^i \left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right)^j \int_0^{x} k(y)\, dy.
$$

This finishes the proof of (3.5.3).

Now we show how the assertion of the Theorem follows from (3.5.3). Taking the Laplace transform on the right hand side of (3.5.3), we get of the theorem:

$$
\begin{aligned}
&\mathscr{L}_{x\to s} \left(\left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right)^{j-1} \int_0^{\bullet} k(y)\, dy, s \right) \\
&= \left(\mathscr{L}_{x\to s} [k(x), s] \right)^{j-1} \mathscr{L}_{x\to s} \left(\int_0^{\bullet} k(y)\, dy, s \right) \\
&= \frac{1}{(f(s))^{j-1}} \frac{1}{s f(s)} \\
&= \frac{1}{s\,(f(s))^j}
\end{aligned}
$$

From here we get

$$
\begin{aligned}
\mathscr{L}_{x\to s} \left[F_n(x), s \right] &:= \mathscr{L}_{x\to s} \Bigg[\underbrace{\sum_{j=0}^{n} \left(\frac{\lambda}{1-q} \right)^j \left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right)^{j-1} \int_0^{\bullet} k(y)\, dy}_{F_n(x)}, s \Bigg] \\
&= \sum_{j=0}^{n} \left(\frac{\lambda}{1-q} \right)^j \mathscr{L}_{x\to s} \left[\left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x} \right)^{j-1} \int_0^{\bullet} k(y)\, dy, s \right] \\
&= \sum_{j=0}^{n} \left(\frac{\lambda}{1-q} \right)^j \frac{1}{s\,(f(s))^j}
\end{aligned}
$$

If $s \geq s_0$ is large enough to guarantee that $\lambda/[f(s)(1-q)] < 1$, we see that

$$
\mathscr{L}_{x\to s}[F_n(x), s] \to \mathscr{L}_{x\to s}[F(x), s], \quad \forall s \geq s_0.
$$

For all n, m big enough and $s \geq s_0$, we obtain

$$\mathscr{L}_{x\to s_0}\left[|F_n(x) - F_m(x)|, s_0\right] = \int_0^\infty e^{-s_0 x}|F_n(x) - F_m(x)|\,dx$$
$$= \|F_n - F_m\|_{L^1(e^{-s_0 x}\,dx)}$$

Using the diagonal method, we get a subsequence F_{n_l} which converges pointwise almost everywhere to a function F. Without loss of generality we can assume that $F_{n_l}(T)$ converges to $F(T)$, otherwise we take another value for T. Assuming $m > n$ and by Lemma 3.5.1, we have

$$\begin{aligned}
\sup_{x\in[0,T]}\left|F_n(x) - F_m(x)\right| &= \sup_{x\in[0,T]}\left|\sum_{n+1}^{m}\left(\frac{\lambda}{1-q}\right)^j \left({}_0^{\mathrm{R}}\mathrm{I}_x^f\right)^{j-1}\int_0^{\bullet} k(s)\,ds\right| \\
&\leq \left|\sum_{n+1}^{m}\left(\frac{\lambda}{1-q}\right)^j \left({}_0^{\mathrm{R}}\mathrm{I}_T^f\right)^{j-1}\int_0^{\bullet} k(s)\,ds\right| \\
&= \left|F_n(T) - F_m(T)\right| \\
&\leq \left|F_{n_l}(T) - F_{m_l}(T)\right| \\
&\to 0, n, m \to \infty.
\end{aligned}$$

where $n_l \leq n + 1 < m \leq m_l$ are such that n_l, m_l goes to ∞. Now we define

$$G_n(x) = \sum_{j=0}^{n}(\lambda\, {}_0^{\mathrm{Ce}}\mathrm{I}_x^f)^j \mathbb{1}.$$

Using (3.5.3), we get $G_n(x) \leq F_n(x)$ and furthermore, assuming $m > n$ we obtain uniformly for all $x \in [0, T]$,

$$\begin{aligned}
\left|G_n(x) - G_m(x)\right| &= \left|\sum_{n+1}^{m}(\lambda\, {}_0^{\mathrm{Ce}}\mathrm{I}_x^f)^j \mathbb{1}\right| \\
&\leq \left|\sum_{n+1}^{m}\left(\frac{\lambda}{1-q}\right)^j \left({}_0^{\mathrm{R}}\mathrm{I}_T^f\right)^{j-1}\int_0^{\bullet} k(s)\,ds\right| \\
&= \left|F_n(T) - F_m(T)\right| \\
&\to 0.
\end{aligned}$$

From this we get that G_n converges to G (locally) uniformly. Thus,

$$G(x) = \phi_0 \sum_{j=0}^{\infty}(\lambda\, {}_0^{\mathrm{Ce}}\mathrm{I}_x^f)^j \mathbb{1} \quad \text{and} \quad G \in C[0, T].$$

Note that from Remark 3.4.18

$$G(x) = \phi_0 \sum_{j=1}^{\infty}(\lambda\, {}_0^{\mathrm{Ce}}\mathrm{I}_x^f)^j \mathbb{1} + \phi_0 = \phi_0 \lambda\, {}_0^{\mathrm{Ce}}\mathrm{I}_x^f \left[\sum_{j=0}^{\infty}(\lambda\, {}_0^{\mathrm{Ce}}\mathrm{I}_x^f)^j \mathbb{1}\right] + \phi_0,$$

by Theorem 3.4.17, $G \in C_{\bar{\mu}}[0,T]$ and it solves (3.5.1).

Next we show the uniqueness of (3.5.1). Assume ϕ_1, ϕ_2 are two solutions of (3.5.1). By the linearity of the fractional derivative we have that $\phi = \phi_1 - \phi_2$ is the solution of the following equation

$$\begin{cases} {}^{\mathrm{Ce}}_{0}\mathrm{D}^{f}_{x}\,\phi = 0, & x \in (0,T] \\ \phi(x) = 0, & x = 0 \end{cases}$$

Using the strategy of Theorem 3.4.17 Step 4, we get the uniqueness. □

Remark 3.5.3. We give an alternative proof for the last part of the proof of Theorem 3.5.2 which is more direct and interesting in its own right. We are going to establish (3.5.3). Since k is a completely monotone Sonine kernel, cf. Assumption 1, we know that k is decreasing. Fix ϵ, set $k = k\mathbb{1}_{(0,\epsilon]} + k\mathbb{1}_{(\epsilon,\infty)}$ and note that $\int_0^\epsilon k(s)\,ds$ goes to 0 as ϵ converges to 0 and $k(\epsilon)$ converges to ∞, as ϵ goes to 0. [1] For the right side of (3.5.3), we obtain

$$\begin{aligned}
&\left\| \left({}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\right)^{j-1} \int_0^{\bullet} k(s)\,ds \right\|_{L^\infty(0,T)} \\
&= \| \underbrace{k * k ... * k}_{j} * \mathbb{1} \|_{L^\infty(0,T)} \\
&= \| \underbrace{(k\mathbb{1}_{(0,\epsilon]} + k\mathbb{1}_{(\epsilon,\infty)}) * (k\mathbb{1}_{(0,\epsilon]} + k\mathbb{1}_{(\epsilon,\infty)}}_{j} * \mathbb{1} \|_{L^\infty(0,T)} \\
&= \left\| \sum_{i=0}^{j} \binom{j}{i} \left(k\mathbb{1}_{(0,\epsilon]}\right)^{*i} \left(k\mathbb{1}_{(\epsilon,\infty)}\right)^{*(j-i)} * \mathbb{1} \right\|_{L^\infty(0,T)} \\
&= \sum_{i=0}^{j} \binom{j}{i} \left\| \left(k\mathbb{1}_{(0,\epsilon]}\right)^{*i} \left(k\mathbb{1}_{(\epsilon,\infty)}\right)^{*(j-i)} * \mathbb{1} \right\|_{L^\infty(0,T)} \\
&\overset{(5)}{\leq} \sum_{i=0}^{j} \binom{j}{i} \left\| \left(k\mathbb{1}_{(0,\epsilon]}\right) \|_{L^1}^{i} \| \left(k\mathbb{1}_{(\epsilon,\infty)}\right)^{*(j-i)} * \mathbb{1} \right\|_{L^\infty(0,T)} \\
&\overset{(6)}{\leq} \sum_{i=0}^{j} \binom{j}{i} \left\| \left(k\mathbb{1}_{(0,\epsilon]}\right) \right\|_{L^1(0,T)}^{i} k(\epsilon)^{j-i} \| \underbrace{\mathbb{1} * ... * \mathbb{1}}_{j-i+1} \|_{L^\infty(0,T)} \\
&= \sum_{i=0}^{j} \binom{j}{i} \left\| \left(k\mathbb{1}_{(0,\epsilon]}\right) \right\|_{L^1(0,T)}^{i} k(\epsilon)^{j-i} \frac{T^{j-i+1}}{(j-i+1)!} \\
&\leq \sum_{i=0}^{j} \binom{j}{i} \left(\int_0^\epsilon k(s)\,ds \right)^{i} k(\epsilon)^{j-i} \frac{T^{j-i+1}}{(j-i+1)!}
\end{aligned}$$

[1] By definition, $1/f(\lambda) = \mathscr{L}(k,\lambda)$. Thus, $\int_0^1 k(s)\,ds < \infty$ and we get $\lim_{\epsilon\to 0} \int_0^\epsilon k(s)\,ds = 0$ by montone convergence. On the other hand, by a change of variable $\lambda/f(\lambda) = \int_0^\infty e^{-tk(t/\lambda)}\,dt$ i.e. $\lim_{\lambda\to\infty} k(t/\lambda) = k(0+) = \infty$ if $\lim_{\lambda\to\infty} \lambda/f(\lambda) = \infty$, see Assumption 1.

$$
\begin{aligned}
&= \underbrace{\sum_{i=0}^{j\theta} \binom{j}{i} \left(\int_0^{\varepsilon} k(s)\, ds \right)^i k(\varepsilon)^{j-i} \frac{T^{j-i+1}}{(j-i+1)!}}_{\mathrm{I}} \\
&\quad + \underbrace{\sum_{i=j\theta}^{j} \binom{j}{i} \left(\int_0^{\varepsilon} k(s)\, ds \right)^i k(\varepsilon)^{j-i} \frac{T^{j-i+1}}{(j-i+1)!}}_{\mathrm{II}},
\end{aligned}
$$

where the (5) and (6) steps were obtained using the Young's inequality. We are now going to choose $\theta \in (0,1)$ in a suitable way. For Part I, since $i \le j\theta$ and $j - i \ge j - j\theta$, we get by the binomial theorem and the monotonicity of the Γ function,

$$
\begin{aligned}
&\sum_{i=0}^{j\theta} \binom{j}{i} \left(\int_0^{\varepsilon} k(s)\, ds \right)^i k(\varepsilon)^{j-i} \frac{T^{j-i+1}}{(j-i+1)!} \\
&\le \frac{T}{\Gamma[j(1-\theta)]} \sum_{i=0}^{j} \binom{j}{i} \left(\int_0^{\varepsilon} k(s)\, ds \right)^i [Tk(\varepsilon)]^{j-i} \\
&= \frac{T}{\Gamma[j(1-\theta)]} \left(\int_0^{\varepsilon} k(s)\, ds + Tk(\varepsilon) \right)^j .
\end{aligned}
$$

For Part II, using $x^n/n! \le e^x$, we obtain

$$
\begin{aligned}
&\sum_{i=j\theta}^{j} \binom{j}{i} \left(\int_0^{\varepsilon} k(s)\, ds \right)^i k(\varepsilon)^{j-i} \frac{T^{j-i+1}}{(j-i+1)!} \\
&\le \sum_{j\theta}^{j} \binom{j}{i} \left(\int_0^{\varepsilon} k(s)\, ds \right)^i k(\varepsilon)^{j-i} \frac{T^{j-i+1}}{(j-i)!} \\
&\le Te^{Tk(\varepsilon)} \sum_{i=j\theta}^{j} \binom{j}{i} \left(\int_0^{\varepsilon} k(s)\, ds \right)^i \\
&\le Te^{Tk(\varepsilon)} 2^j \left(\int_0^{\varepsilon} k(s)\, ds \right)^{j\theta} .
\end{aligned}
$$

Combing the estimates for Part I and Part II, we have

$$
\left\| \left({}_0^{\mathrm{R}}\mathrm{I}_x^f \right)^{j-1} \int_0^{\cdot} k(s)\, ds \right\|_{\infty} \le Te^{Tk(\varepsilon)} 2^j \left(\int_0^{\varepsilon} k(s)\, ds \right)^{j\theta} + \frac{T \left(\int_0^{\varepsilon} k(s)\, ds + Tk(\varepsilon) \right)^j}{\Gamma[j(1-\theta)]}. \tag{3.5.4}
$$

Now we choose ε such that $2|\lambda/(1-q)| \left(\int_0^{\varepsilon} k(s)\, ds \right)^{\theta} < 1$ and some j^* such that for all

$$
j > j^*, \quad \frac{T \left(\lambda/(1-q) \int_0^{\varepsilon} k(s)\, ds + T\lambda/(1-q) k(\varepsilon) \right)^j}{\Gamma[j(1-\theta)]} \le \frac{M^j}{(j!)^{1-\theta}},
$$

for some constant $M > 0$. Then summing over $j = 0, ..., \infty$ in (3.5.4) gives us two convergent series on the right hand side, so $\sum_{j=0}^{\infty} (\lambda\, {}_0^{\mathrm{Ce}}\mathrm{I}_x^f)^j \mathbb{1}$ is (locally) uniformly convergent. Using the same argument as in Theorem 3.4.17 we can get that $\sum_{j=0}^{\infty} (\lambda\, {}_0^{\mathrm{Ce}}\mathrm{I}_x^f)^j \mathbb{1}$ is the unique solution in $C_{\bar{\mu}}[0,T]$ and it solves (3.4.17).

For inhomogeneous initial value problem.

Theorem 3.5.4. *Let Assumption 1 and* $q = \lim_{x\to 0} \bar{\mu}(x) \int_0^x k(s)\,ds < 1$ *hold true. Let* $k, \bar{\mu}$ *be a Sonine pair and* $\mathscr{L}(\bar{\mu}, \lambda) = f(\lambda)/\lambda, \mathscr{L}(k, \lambda) = 1/f(\lambda)$*. For any* $\phi_0 \in \mathbb{R}$*,* $\lambda \in \mathbb{R}$*,* $g \in C[0, T]$*, the linear initial value problem*

$$\begin{cases} {}^{\mathrm{Ce}}_{\;0}\mathrm{D}^{f}_{x}\, \phi(x) = \lambda\phi(x) + g(x), x \in (0, T] \\ \qquad \phi(x) = \phi_0, \qquad\qquad x = 0, \end{cases} \tag{3.5.5}$$

has a unique solution in $C_{\bar{\mu}}[0, T]$ *given by*

$$\phi(x) = \phi_0 \sum_{j=0}^{\infty} (\lambda\, {}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x})^{j} \mathbb{1} + \sum_{j=0}^{\infty} \lambda^{j} ({}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x})^{j+1} g.$$

Proof. Using the proof of Theorem 3.5.2, we can show

$$\sum_{j=0}^{\infty} \lambda^{j} ({}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x})^{j+1} g \quad \text{and} \quad \sum_{j=0}^{\infty} \lambda^{j} ({}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x})^{j+1} \mathbb{1}$$

are (locally) uniformly convergence in $C[0, T]$. Applying ${}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x}$ on both side of (3.5.5) gives after inserting the concrete form of ϕ,

$$\begin{aligned} \lambda\, {}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x}\, \phi + {}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x}\, g &= \lambda\phi_0\, {}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x} \sum_{j=0}^{\infty} (\lambda\, {}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x})^{j} 1 + \lambda\, {}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x} \sum_{j=0}^{\infty} \lambda^{j} ({}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x})^{j+1} g + {}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x}\, g \\ &\overset{(*)}{=} \phi_0 \sum_{j=0}^{\infty} (\lambda\, {}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x})^{j+1} 1 + \sum_{j=0}^{\infty} \lambda^{j+1} ({}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x})^{j+2} g + {}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x}\, g \\ &= \phi_0 + \sum_{j=1}^{\infty} (\lambda\, {}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x})^{j+1} 1 + \sum_{j=1}^{\infty} \lambda^{j+1} ({}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x})^{j+2} g \\ &= \phi(x) - \phi_0 \end{aligned}$$

In the step marked by (*) we use the continuity of ${}^{\mathrm{Ce}}_{\;0}\mathrm{I}^{f}_{x}$ which is clear from its construction using the Banach fixed–point theorem. Using Theorem 3.4.17, ϕ solves (3.5.5) and this solution is unique. □

Chapter 4

Censored process

In this chapter, we consider a Bernstein function f with a Sonine pair $(\bar{\mu}, k)$

$$f(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\mu(dx).$$

We assume that f satisfies all assumptions from the previous Chapter 3. By $S = S^1 = (S_t^1)_{t \geq 0}$ we denote the subordinator associated with f. In section 4.1, we give a idea to construct a Markov process and explore its properties. Section 4.2 is devoted to study the probabilistic representation about censored Bernstein fractional derivative.

4.1 Construction

The starting point of the censored process is always assumed to be some fixed $x > 0$. We will define the censored decreasing subordinator S^c by piecing together. This construction guarantees that we will get a right continuous strong Markov process by using Theorem 1.1 in [54]. The construction is : run $x - S_t^1$ until τ_1, the time when S^1 first exits 0, where S^1 is an increasing subordinator given by f and starting at 0. If $x - S_{\tau_1}^1$ is less or equal than 0, then kill the process, otherwise piece together an independent copy S^2 of S^1 started at $x - S_{\tau_1}^1$ and repeat the same procedure for at most countably many times. Thus, we prove the above construction S_t^c is a Feller process and which has generator given by (3.4.10), i.e ${}_0^{\mathrm{Ce}}\mathrm{D}_\cdot^f$.

Set $\tau_j = \sum_{i=0}^j \sigma_i$, we define the censored decreasing process S^c as

$$S_t^c = \begin{cases} x - S_t^1, & 0 \leq t < \sigma_1, \quad j = 1, \\ S_{\tau_{j-1}-}^c - S_{t-\tau_{j-1}}^j, & \tau_{j-1} \leq t < \tau_j, \quad j \geq 2, \\ \partial, & t \geq \tau_j, \end{cases}$$

where ∂ is the grave yard (cemetery point) and

$$\sigma_j = \begin{cases} 0 & \text{if } j = 0\,, \\ \inf\{t > 0 : S_{t+\sigma_{j-1}}^c \leq 0\} & \text{if } j \in \mathbb{N}, \end{cases}$$

where $\{S^j\}_{j\in\mathbb{N}}$ is an i.i.d. collection of an increasing subordinators given by f. Assume that the generalized inverse of $y - S^j_\bullet$ is $E_j(y) = \inf\{s : y - S^j_s < 0\}$, for all $j \in \mathbb{N}$, $y > 0$. $U(dx) = k(x)\,dx$ is the potential measure of S^1, then we have the following relation $\mathbb{E}[E_1(y)] = U(y)$ see Theorem 1.5.12.

Theorem 4.1.1. *The above construction gives a strong Markov process $S^c_\bullet$.*

Proof. Set $A_j = \{\tau_{j-1} \le \eta < \tau_j\}$ and $B_i = \{\tau_{i-1} \le \eta + s < \tau_i\}$. Clearly, $\Omega = \cup_{j=1}^\infty \cup_{i=1}^\infty A_j \cap B_i$ and the union is a union of mutually disjoint sets. Therefore it is enough to consider (4.1.1) on $A_j \cap B_i$.

$$\mathbb{E}^x[\phi(S^c_{\eta+s})|\mathcal{F}_\eta] = \mathbb{E}^{S^c_\eta}[\phi(S^c_s)]. \tag{4.1.1}$$

Case 1 When $i = j$ we have

$$\begin{aligned}
&\mathbb{E}^x\left[\phi(S^c_{\eta+s})\mathbb{1}_{A_j}\mathbb{1}_{B_i}|\mathcal{F}_\eta\right]\\
&= \mathbb{1}_{A_j}\mathbb{E}^x\left[\phi\left(S^c_{\tau_{j-1}-} - S^j_{\eta-\tau_{j-1}} + (S^j_{\eta-\tau_{j-1}} - S^j_{\eta+s-\tau_{j-1}})\right)\mathbb{1}_{B_i}|\mathcal{F}_\eta\right]\\
&\overset{(*)}{=} \mathbb{E}^x\left[\phi\big(S^c_\eta + \underbrace{(S^j_{\eta-\tau_{j-1}} - S^j_{\eta+s-\tau_{j-1}})}_{\sim S^c_s = -S^1_s}\big)\mathbb{1}_{B_i}\mathbb{1}_{A_j}|\mathcal{F}_\eta\right]\\
&\overset{(*)}{=} \mathbb{E}^{S^c_\eta}\left[S^c_s\mathbb{1}_{B_i}\mathbb{1}_{A_j}\right].
\end{aligned}$$

In the steps marked with (*) we use the following facts

(a) $\mathbb{E}^x[g(X,Y)|\mathcal{F}] = \mathbb{E}^x[g(z,Y)]\,|_{z=X}$ if g is bounded and measurable and X is $\mathcal{F}-$measurable and Y is independent of $\mathcal{F}$.

(b) $S^c_{\tau_{j-1}-} - S^j_{\eta-\tau_j} = S^c_\eta$ on A_j.

(c) $S^j_{\eta-\tau_{j-1}} - S^j_{\eta+s-\tau_{j-1}} \sim -S^j_s \sim -S^1_s \sim S^c_s$ on $A_j \cap B_i$.

Case 2 When $j \le i$ we have

$$\begin{aligned}
&\mathbb{E}^x[\phi(S^c_{\eta+s})\mathbb{1}_{A_j}\mathbb{1}_{B_i}|\mathcal{F}_\eta]\\
&= \mathbb{1}_{A_j}\mathbb{E}^x[\phi(S^c_{\eta+s})\mathbb{1}_{B_i}|\mathcal{F}_\eta]\\
&\overset{\text{tower}}{=} \mathbb{1}_{A_j}\mathbb{E}^x\left\{\mathbb{E}^x\left[\phi(S^c_{\eta+s})\mathbb{1}_{B_i}|\mathcal{F}_{\tau_{i-1}}\right]|\mathcal{F}_\eta\right\}\\
&= \mathbb{1}_{A_j}\mathbb{E}^x\left\{\mathbb{E}^{S^c_{\tau_{i-1}}}\left[\phi(-S^c_{\eta+s-\tau_{i-1}})\mathbb{1}_{B_i}\right]|\mathcal{F}_\eta\right\},
\end{aligned}$$

where in the last equality, we use similar steps to Case 1.
We may iterate this argument, conditioning (tower property) on $\mathcal{F}_{\tau_{i-2}}$. Since $\sigma_l = \tau_l - \tau_{l-1}$, we have

$$\mathbb{E}^x\left[\phi(S^c_{\eta+s})\mathbb{1}_{A_j}\mathbb{1}_{B_i}|\mathcal{F}_\eta\right]$$

$$= \quad \mathbb{1}_{A_j}\mathbb{E}^x\left[\mathbb{E}^{S^c_{\tau_{k-2}}}\left[\mathbb{1}_{B_i}\mathbb{E}^{S^{i-1}_{\sigma_{k-1}-}}\left[\phi(-S^k_\rho)\right]_{\rho=\eta+s-\tau_{i-1}}\right]|\mathcal{F}_\eta\right]$$

$$= \quad ....$$

$$= \quad \mathbb{1}_{A_j}\mathbb{E}^x\left[\mathbb{1}_{B_i}\mathbb{E}^{S^c_{\tau_j}}\left[\phi(S^{j+1}_{\sigma_{j+1}-}+...+S^{i-1}_{\sigma_{i-1}-}-S^k_\rho)\right]_{\rho=\eta+s-\tau_{i-1}}|\mathcal{F}_\eta\right]$$

$$\overset{\text{as in Case 1}}{=} \mathbb{E}^{S^c_\eta}\left[\mathbb{1}_{A_j}\mathbb{1}_{B_i}\phi(S^c_s)\right].$$

Since on $A_j \cap B_i$, S^c_s and $S^j_{\tau_j-\eta} + S^{j+1}_{\sigma_{j+1}-} + ... + S^i_{\eta+s-\tau_{i-1}}$ have the same law. □

Remark 4.1.2. Using [71] Theorem 14.8, we can show the censored process is strong Markov process. Take transfer kernel $K(S^c_{\tau_j}, dy) = \delta_{S^c_{\tau_j-}}(dy)$, where δ is the $\delta-$ distribution.

Recall $U(dy) = k(y)\,dy$ is the potential measure, $1/f(\lambda) = \mathscr{L}(k,\lambda) = \mathscr{L}(U,\lambda)$ see Remark 1.5.11 and Theorem 1.5.12 and $k_1(x,r) = \bar{\mu}(r)k(x-r)$ see Definition 3.4.9.

Lemma 4.1.3. *For any $x > 0$ and $j \in \mathbb{N}$, assuming $S^c_0 = x$, we have*

(i) $\mathbb{E}^x[\tau_j] < \infty$, $\mathbb{P}[S^c_{\tau_j} \in (0,x)] = 1$ *and* $S^c_{\tau_j}$ *has the density* $k_j(x,\cdot)$, *as defined in* (3.4.11)*;*

(ii) $\mathbb{E}^x[\sigma_{j+1}] = \mathbb{E}[E_{j+1}(S^c_{\tau_j})] = \int_0^x U(y)k_j(x,y)\,dy$*;*

(iii) $\mathbb{P}^x[\lim_{j\to\infty}\tau_j < \infty] = 1$ *and* $\mathbb{P}^x[S^c_{\lim_{j\to\infty}\tau_j-} = 0] = 1.$

Proof. (i) We use induction. If $j = 1$ we have

$$\mathbb{E}[\tau_1(y)] = \mathbb{E}[E_1(y)] = U(y) < \infty,$$

where we use $E_1(y) = \inf\{s : y - S^1_s < 0\}$ and $\mathbb{E}[E_1(y)] = U(y)$ see Theorem 1.5.12. From [6] Chapter 3, Proposition 2 we know that

$$\mathbb{P}(x - S_{\sigma_1-} \in dy, x - S_{\tau_1} \in dz) = U(dy)\mu(dz-y),$$

where $\tau(x) = \inf\{t : S_t \geq x\}$. As $\tau_1 = \tau(x)$ under $\mathbb{P}^x$, we get for $0 \leq a \leq x$,

$$\begin{aligned}
&\mathbb{P}^x(S^c_{\tau_1-} \in (a,x]) \\
&= \mathbb{P}(x - S^c_{\tau(x)-} \in (a,x]) \\
&= \mathbb{P}(x - S^c_{\tau(x)-} \in (a,x], S_{\tau(x)} \geq x) \\
&= \mathbb{P}(S^c_{\tau(x)-} \in ([0,,x-a), S_{\tau(x)} \geq x) \\
&= \int_0^{x-a} \bar{\mu}(x-y)k(y)\,dy.
\end{aligned}$$

This shows that $k_1(x,r) = \bar{\mu}(x-y)k(y)$ is the density of $S^c_{\tau_1-}$ under $\mathbb{P}^x$.

Assume $j \geq 1$, $\tau_j < \infty$, $S^c_{\tau_j} > 0$ and $S^c_{\tau_j}$ independent of S^{j+1}. According to the strong Markov property, we have

$$\begin{aligned}\sigma_{j+1} &= \inf\{r > 0 : S^c_{\tau_j} < S^{j+1}_r\} \\ &= E_{j+1}(S^c_{\tau_j})\end{aligned}$$

Using $S^c_{\tau_j} < x$, we can get the following estimate

$$\mathbb{E}[\tau_{j+1}] = \mathbb{E}[E_{j+1}(S^c_{\tau_j})] + \mathbb{E}[\tau_j] \leq \mathbb{E}[E_{j+1}(x)] + \mathbb{E}[\tau_j] < \infty.$$

Next we show $S^c_{\tau_{j+1}} \in (0, x)$. Using the above equality,

$$S^c_{\tau_{j+1}} = S^c_{\tau_j} - S^{j+1}_{\sigma_{j+1}-} = S^c_{\tau_j} - S^{j+1}_{E_{j+1}(S^c_{\tau_j})-} \in (0, S^c_{\tau_j}) \subset (0, x).$$

Finally, we show $S^c_{\tau_{j+1}}$ has density $k_{j+1}(x, \cdot)$. For any bounded measurable ϕ, we have

$$\begin{aligned}\mathbb{E}^x\left[\phi(S^c_{\tau_{j+1}})\right] &= \mathbb{E}^x\left[\phi(S^c_{\tau_j} - S^{j+1}_{E_{j+1}(S^c_{\tau_j})-})\right] \\ &= \mathbb{E}\left[\int_0^x \phi(y - S^{j+1}_{E_{j+1}(S^c_{\tau_j})-})\right]\mathbb{P}(S^c_{\tau_j} \in dy) \\ &= \mathbb{E}\left[\int_0^x \phi(y - S^{j+1}_{E_{j+1}(S^c_{\tau_j})-})\right] k_j(x, y)\, dy \\ &= \int_0^x \mathbb{E}^y\left[\phi(-S^{j+1}_{E_{j+1}(S^c_{\tau_j})-})\right] k_j(x, y)\, dy \\ &= \int_0^x \int_0^y \phi(z) k_1(y, z)\, dz\, k_j(x, y)\, dy \\ &\overset{\text{Fubini}}{=} \int_0^x \phi(z) \int_z^x k_j(x, y) k_1(y, z)\, dy\, dz\end{aligned}$$

The fourth equality holds because $\{-S^j\}_{j\in\mathbb{N}}$ is an i.i.d. collection of subordinators and $-S^{j+1}_{E_{j+1}(S^c_{\tau_j})} \overset{\mathbb{P}^y}{\sim} -S^1_{E_1}(y)$ has density $k_1(y, \cdot)$ and $S^c_{\sum_{i=1}^j \sigma_i}$ is independent of S^{j+1}. Using the Definition 3.4.8, we know that $S^c_{\sum_{i=1}^{j+1} \sigma_i}$ has density $k_{j+1}(x, \cdot)$.

(ii) Using the results of (i), we have $\mathbb{E}[\sigma_{j+1}] = \mathbb{E}[E_{j+1}(S^c_{\tau_j})]$ and $S^c_{\tau_j}$ is independent of S^{j+1}. By the definition of E_j we have,

$$\begin{aligned}\mathbb{E}^x\left[\sigma_{j+1}\right] &= \mathbb{E}^x\left[E_{j+1}(S^c_{\tau_j})\right] \\ &= \mathbb{E}\left[\int_0^x E_{j+1}(y) k_j(x, y)\, dy\right] \\ &= \int_0^x \mathbb{E}\left[E_{j+1}(y)\right] k_j(x, y)\, dy \\ &= \int_0^x U(y) k_j(x, y)\, dy.\end{aligned}$$

(iii) From the Markov inequality

$$n\mathbb{P}(\tau_\infty > n) \le \mathbb{E}^x[\tau_\infty], \quad n \to \infty.$$

We see that it is enough to show that $\mathbb{E}^x(\tau_\infty) < \infty$ to get $\mathbb{P}(\tau_\infty < \infty)$. Now by Beppo Levi

$$\begin{aligned}
\mathbb{E}^x(\tau_\infty) &= \lim_{j\to\infty} \mathbb{E}\left[\sum_{i=1}^{j} \sigma_i\right] = \lim_{j\to\infty} \sum_{i=1}^{j} \mathbb{E}[\sigma_i] \\
&= \lim_{j\to\infty} \sum_{i=1}^{j} \int_0^x U(y) k_{i-1}(x,y)\,dy \\
&= \sum_{i=1}^{\infty} \int_0^x U(y) k_{i-1}(x,y)\,dy \\
&= \sum_{i=1}^{\infty} \mathcal{K}^{i-1} U(x) \\
&< \infty,
\end{aligned}$$

where the last inequality, we use the definition of $U(y) = \int_0^y k(x)\,dx$ and it satisfies the condition of Lemma 3.4.15.

Next we show $\mathbb{P}^x[S^c_{\lim_{j\to\infty}\tau_j-} > 0] = 0$.

$$\mathbb{P}^x\left[S^c_{\lim_{j\to\infty}\tau_j-} > 0\right] \le \sum_{n=1}^{\infty} \mathbb{P}\left[S^c_{\lim_{j\to\infty}\tau_j-} > \frac{1}{n}\right],$$

and for each $n \in \mathbb{N}$,, using the following results $\left\{S^c_{\lim_{j\to\infty}\tau_{j+1}} > \frac{1}{n}\right\} \subset \left\{S^c_{\tau_j} > \frac{1}{n}\right\}$,

$$\mathbb{P}^x\left[S^c_{\lim_{j\to\infty}\tau_j-} > \frac{1}{n}\right] \le \mathbb{P}^x\left[\cap_{j=1}^{\infty}\{S^c_{\tau_j} > \frac{1}{n}\}\right] = \lim_{j\to\infty} \mathbb{P}^x\left[S^c_{\tau_j} > \frac{1}{n}\right].$$

Let F be a strictly increasing function, e.g. $F(y) = \int_0^y k(s)\,ds$, $F(0) = 0$, then by Chebyshev's inequality,

$$\begin{aligned}
\mathbb{P}^x\left[S^c_{\tau_j} > \frac{1}{n}\right] &= \mathbb{P}^x\left[F(S^c_{\tau_j}) > F(\frac{1}{n})\right] \\
&\le \frac{1}{F(\frac{1}{n})}\mathbb{E}\left[F(S^c_{\tau_j})\right] \\
&= \frac{1}{F(\frac{1}{n})} \int_0^x k_j(x,y) \int_0^y k(s)\,ds\,dy \\
&\to 0, \quad \text{as} \quad j \to \infty.
\end{aligned}$$

Indeed, since $\sum_{j=0}^{\infty} \int_0^x k_j(x,y) \int_0^y k(s)\,ds\,dy$ is finite, by the results of Lemma 3.4.15, we obtain $\int_0^x k_j(x,y) \int_0^y k(s)\,ds\,dy$ goes to 0, as j goes to ∞. □

4.2 Probabilistic representation

Let $(S_t)_{t\geq 0}$ be the subordinator corresponding to the Bernstein function f. As before, $U(dx)$ is the potential measure i.e. $1/f(\lambda) = \mathscr{L}(k,\lambda) = \mathscr{L}(U,\lambda)$ and $\tau_1 = \tau(x)$ is the first time S_t cross the level x. Again we assume that Assumption 1 from Section 3.4 holds.

Lemma 4.2.1. *Assume S_t^1 has transition probability $\mathbb{P}^x(-S_t^1 \in dy) = p_t(x-dy)$, we have the following equality:*

$$ {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, g(x) = \mathbb{E}^x\left[\int_0^{\tau_1} g(-S_s^1)\, ds\right] \tag{4.2.1}$$

Proof. The results come from a direct calculation, we have

$$\begin{aligned}
&\mathbb{E}^x\left[\int_0^{\tau_1} g(-S_s^1) ds\right] \\
&= \mathbb{E}^x\left[\int_0^{\infty} g(-S_s^1) 1_{\{s\leq\tau_1\}}\, ds\right] \\
&= \mathbb{E}^x\left[\int_0^{\infty} g(-S_s^1) 1_{\{x-S_s^1\leq 0\}}\, ds\right] \\
&= \int_0^{\infty}\int_0^{x} g(y)\mathbb{P}^x(-S_s^1 \in dy)\, ds \\
&= \int_0^{x}\int_0^{\infty} g(y)\, p_s(x-dy)\, ds \\
&= \int_0^{x} g(y) \int_0^{\infty} p_s(x-dy)\, ds \\
&\overset{(1.5.13)}{=} \int_0^{x} g(y)\, U(x-dy).
\end{aligned}$$

Since the Sonine pair $(\bar{\mu}, k)$ consists of functions because of Assumption 1, we have $U(dx) = k(x)\, dx$, so

$$\mathbb{E}^x\left[\int_0^{\tau_1} g(-S_s^1) ds\right] = \int_0^{x} g(y) k(x-y)\, dy.$$

Using the definition of Riemann–Liouville integral we get

$$ {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, g(x) = \mathbb{E}^x\left[\int_0^{\tau_1} g(-S_s^1)\, ds\right]. \qquad \square$$

Remark 4.2.2. Here we use (1.5.13). It also true $U(dx) = \int_0^\infty p_s(dx)\, ds$ in the vague topology. i.e. $\langle U, \phi\rangle = \langle \int_0^\infty p_s(\cdot)\, ds, \phi\rangle$ where $\langle m, \phi\rangle = \int \phi\, dm$, for all $\phi \in C_c^\infty$.

Theorem 4.2.3. *If g satisfies the conditions of Theorem 3.4.17, we have the following representation*

$$ {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x}\, g(x) = \sum_{j=0}^{\infty} \mathbb{E}^x\left[{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{S^c_{\tau_j}}\, g(S^c_{\tau_j})\right] = \mathbb{E}^x\left[\int_0^{\tau_\infty} g(S_s^c)\, ds\right], \tag{4.2.2}$$

where $\tau_1, \tau_2 \ldots.$ are the stopping times from the construction of Section 4.1.

Proof. First, we assume $g \geq 0$. We have

$$\begin{aligned}
\mathbb{E}^x\left[\int_0^{\tau_\infty} g(S^c_s)\,ds\right] &= \mathbb{E}^x \sum_{j=0}^{\infty}\left[\int_{\tau_j}^{\tau_{j+1}} g(S^c_s)\,ds\right]\\
&= \sum_{j=0}^{\infty}\mathbb{E}^x\left[\int_{\tau_j}^{\tau_{j+1}} g(S^c_s)\,ds\right]\\
&\overset{s-\tau_j=u}{=} \sum_{j=0}^{\infty}\mathbb{E}^x\left[\int_0^{\tau_{j+1}-\tau_j} g(S^c_{u+\tau_j})\,du\right]\\
&= \sum_{j=0}^{\infty}\mathbb{E}^x\left[\int_0^{E_{j+1}(S^c_{\tau_j})} g(S^c_{u+\tau_j})\,du\right]\\
&= \sum_{j=0}^{\infty}\mathbb{E}^x\left[\int_0^{E_{j+1}(S^c_{\tau_j})} g(S^{j+1}_u + S^c_{\tau_j})\,du\right]\\
&= \sum_{j=0}^{\infty}\mathbb{E}^x\left[\mathbb{E}\left[\int_0^{E_{j+1}(S^c_{\tau_j})} g(S^{j+1}_u + S^c_{\tau_j})\,du \,\middle|\, S^c_{\tau_j}\right]\right]\\
&= \sum_{j=0}^{\infty}\mathbb{E}^x\left[\mathbb{E}^{S^c_{\tau_j}}\left[\int_0^{E_{j+1}(S^c_{\tau_j})} g(S^{j+1}_u)\,du\right]\right]\\
&\overset{\text{Lemma 4.2.1}}{=} \sum_{j=0}^{\infty}\mathbb{E}^x\left[{}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{S^c_{\tau_j}}\, g(S^c_{\tau_j})\right]\\
&= \sum_{j=0}^{\infty}\mathcal{K}^j\, {}^{\mathrm{R}}_{0}\mathrm{I}^{f}_{x}\, g(x)\\
&= {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x}\, g(x).
\end{aligned}$$

□

Recall that $C_\infty(0,T] = \overline{C_c(0,T]}^{\|\cdot\|_\infty} = \{u \in C(0,T] : u(0+) = 0\}$.

Theorem 4.2.4. *For any $T > 0$, the process S^c_t gives a Feller semigroup on $C_\infty(0,T]$, whose generator is $(-{}^{\mathrm{Ce}}_{0}\mathrm{D}^{f}_{\bullet}, {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{\bullet}\, C_\infty(0,T])$.*

Proof. Recall Definition 1.4.3 and Theorem 4.1.1. We know that $S^c_\cdot$ is a Markov process, so $P^c_t\phi(x) = \mathbb{E}^x[\phi(S^c_t)]$ is a positivity preserving contraction semigroup on the Borel–measurable functions $\mathcal{B}[0,T]$. If we can show that P^c_t is a Feller operator i.e. $P^c_t : C_\infty(0,T] \to C_\infty(0,T]$ and $t \mapsto P^c_t$ is strongly continuous, then we can show P^c_t is a Feller semigroup.

First we show P^c_t is strongly continuous on $C_\infty(0,T]$. Assume $\phi \in C_\infty(0,T]$, and define $\phi(x) = \phi(0+)$, $x \in (-\infty, 0]$,

$$\begin{aligned}
|P^c_t\phi(x) - \phi(x)| &= \left|\mathbb{E}^x[\phi(S^c_t)] - \phi(x)\right|\\
&= \left|\mathbb{E}^x[\phi(S^c_t)] - \mathbb{E}^x[\phi(x)]\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left|\mathbb{E}^x\left[\phi(S_t^c)(1_{\{t<\tau_1\}}+1_{\{t\geq\tau_1\}})\right] - \mathbb{E}^x\left[\phi(x)(1_{\{t<\tau_1\}}+1_{\{t\geq\tau_1\}})\right]\right| \\
&\leq \left|\mathbb{E}^x\left[(\phi(S_t^c)-\phi(x))1_{\{t<\tau_1\}}\right]\right| + \left|\mathbb{E}^x\left[(\phi(S_t^c)-\phi(x))1_{\{t\geq\tau_1\}}\right]\right| \\
&= \left|\mathbb{E}^0\left[(\phi(S_t^1+x)-\phi(x))1_{\{t<\tau_1\}}\right]\right| + \left|\mathbb{E}^x\left[(\phi(S_t^c)-\phi(x))1_{\{t\geq\tau_1\}}\right]\right|.
\end{aligned}
$$

According to the construction of Section 4.1, we can get that the first summand vanishes uniformly in x when $t \to 0$ because S_t^1 is a subordinator. For the second one we have

$$
\begin{aligned}
\left|\mathbb{E}^x\left[(\phi(S_t^c)-\phi(x))1_{\{t\geq\tau_1\}}\right]\right| &\leq 2\|\phi\|_{C[0,x]}\mathbb{P}^0[t\geq\tau_1] \\
&\leq 2\|\phi\|_{C[0,x]}\mathbb{P}^0[t\geq E_1(x)] \\
&\leq 2\|\phi\|_{C[0,x]}\mathbb{P}^0[x-S_t^1\leq 0] \\
&\leq 2\|\phi\|_{C[0,x]}\mathbb{P}^0[x\leq S_t^1].
\end{aligned}
$$

Because of $\phi(0+) = 0$, we have for any $\varepsilon > 0$ and choose δ such that $0 \leq x \leq \delta$, $\|\phi\|_{C[0,x]} \leq \varepsilon$ and we have $\mathbb{P}[x \leq S_t^1] \leq 1$. If $t \to 0, x > \delta$ we have $\mathbb{P}[x \leq S_t^1] \to 0$. Thus we have the following estimate as $t \to 0$,

$$
\|\phi\|_{C[0,x]}\mathbb{P}[x\leq S_t^1] \leq \begin{cases} \varepsilon, & 0\leq x\leq\delta, \\ \varepsilon\|\phi\|_{C[0,T]}, & \delta < x \leq T. \end{cases}
$$

This proves that P_t^c is strongly continuous on $C_\infty(0,T]$.

Next, we show $P_t^c : C_\infty(0,T] \to C_\infty(0,T]$. We begin by showing that $(P_t^c\phi)(0+) = 0$. Clearly $\phi \in C_\infty(0,T]$ satisfies $\phi(0+) = 0$, thus our previous calculations show

$$
\left|P_t^c\phi(x)-\phi(x)\right| \leq \left|\mathbb{E}^0\left[(\phi(S_t^1+x)-\phi(x))\mathbb{1}_{\{t\leq\tau_1\}}\right]\right| + 2\|\phi\|_{C[0,x]}\mathbb{P}^0(x\leq S_t^1).
$$

Since $S_t^1 \leq x$ if $t \leq \tau_1 = \tau(x)$ we get

$$
\left|P_t^c\phi(x)-\phi(x)\right| \leq 4\|\phi\|_{C[0,x]} \to 0, \quad \text{as} \quad x \to 0.
$$

Now we show that $x \mapsto P_t^c\phi(x)$ is continuous. Pick $\phi \in C_\infty(0,T]$ and assume, without loss of generality, that $0 \leq x \leq y$. Let $h > 0$ be small and write ${}^zS_t^c$ for the process S_t^c with $S_0^c = z$. We have

$$
\begin{aligned}
&\left|P_t^c\phi(x)-P_t^c\phi(y)\right| \\
&= \left|\mathbb{E}^x\phi(S_t^c)-\mathbb{E}^y\phi(S_t^c)\right| \\
&= \left|\mathbb{E}\left[\mathbb{1}_{\{{}^xS_t^c-{}^yS_t^c\leq h\}}\left[\phi({}^xS_t^c)-\phi({}^yS_t^c)\right]\right] + \mathbb{E}\left[1_{\{{}^xS_t^c-{}^yS_t^c>h\}}\left[\phi({}^xS_t^c)-\phi({}^yS_t^c)\right]\right]\right| \\
&\leq \left|\mathbb{E}\left[\mathbb{1}_{\{{}^xS_t^c-{}^yS_t^c\leq h\}}\left[\phi({}^xS_t^c)-\phi({}^yS_t^c)\right]\right]\right| + \left|\mathbb{E}\left[1_{\{{}^xS_t^c-{}^yS_t^c>h\}}\left[\phi({}^xS_t^c)-\phi({}^yS_t^c)\right]\right]\right| \\
&\leq \left|\mathbb{E}\left[\mathbb{1}_{\{{}^xS_t^c-{}^yS_t^c\leq h\}}\left[\phi({}^xS_t^c)-\phi({}^yS_t^c)\right]\right]\right| + 2\|\phi\|_{C[0,T]}\left|\mathbb{E}\left[\mathbb{1}_{\{{}^xS_t^c-{}^yS_t^c>h\}}\right]\right| \\
&\leq \varepsilon + 2\|\phi\|_{C[0,T]}\mathbb{P}({}^xS_t^c-{}^yS_t^c>h).
\end{aligned}
$$

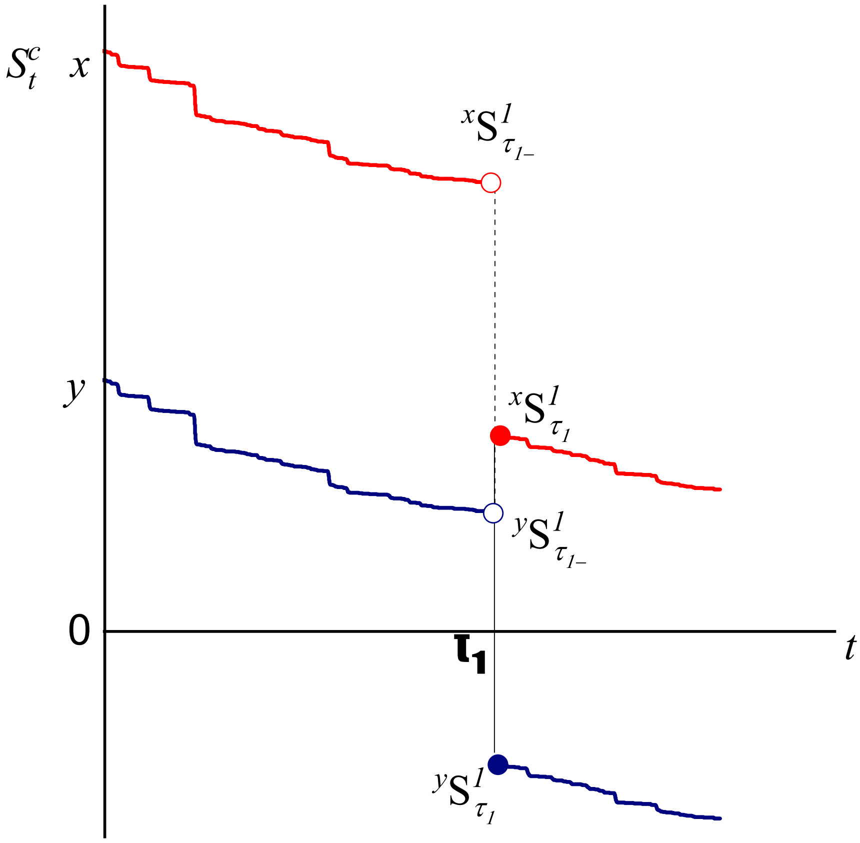

Figure 4.1

In the last step we use that $\phi \in C_\infty(0, T]$ is uniformly continuous. Note that $h = h_\varepsilon$. We have the following two cases:

1. If $0 \leq y \leq x \leq h$, it is clear that $\mathbb{P}({}^xS^c_t - {}^yS^c_t > h) \to 0$ as $x - y \to 0$.

2. If $x > h$, it is enough for us to consider the first censoring time $\tau_y = \inf\{t \geq 0 : {}^yS^c_t \leq 0\}$ for ${}^yS^c_t$ and ${}^xS^c_{\tau_y} \geq 0$ see Figure 4.1. If $t < \tau_y$ we have $\mathbb{P}({}^xS^1_t - {}^yS^1_t > h) \to 0$ as $x - y \to 0$. Using the construction of $S^c_\cdot$, we have the following, (see also Figure 4.1): write $\Delta S_t = S_t - S_{t-}$ for the jump at time t, then,

$$\begin{aligned}
&\mathbb{P}\left[0 \in (-\Delta S^1_{\tau_y} + ({}^yS^1_{\tau_y -}, {}^xS^1_{\tau_y})\right] \\
&= \mathbb{P}\left[\Delta S^1_{\tau_y} \in ({}^yS^1_{\tau_y -}, {}^xS^1_{\tau_y -})\right] \\
&= \mathbb{P}\left[\Delta S^1_{\tau_y} \in ({}^yS^1_{\tau_y -}, {}^yS^1_{\tau_y -} + x - y)\right] \\
&= \mathbb{P}\left[{}^yS^1_{\tau_y -} + \Delta S^1_{\tau_y} \in (2\, {}^yS^1_{\tau_y -}, 2\, {}^yS^1_{\tau_y -} + x - y)\right] \\
&= \mathbb{P}\left[{}^yS^1_{\tau_y} \in (2\, {}^yS^1_{\tau_y -}, 2\, {}^yS^1_{\tau_y -} + x - y)\right] \\
&= \int_0^{x+y} \int_0^y 1_{\{2u \leq v \leq 2u + x - y\}} \mathbb{P}\left[{}^yS^1_{\tau_y} \in dv, {}^yS^1_{\tau_y -} \in du\right]
\end{aligned}$$

$$= \int_0^{x+y} \int_0^{y} 1_{\{2u \le v \le 2u+x-y\}} U(du)\, \bar{\mu}(dv-u)$$
$$\to 0 \quad \text{as} \quad x-y \to 0,$$

where the second to last equality, uses Proposition 2 [6, P76]. By Assumption 1 our jump measure is (absolutely) continuous, so the last limits holds.

Finally we show that the generator of $P_t^c \phi(x) = \mathbb{E}^x[\phi(S_t^c)]$ is $-{}_0^{\mathrm{Ce}}\mathrm{D}_\cdot^f$. Using Theorem 4.2.3, we know the potential of P_t^c is ${}_0^{\mathrm{Ce}}\mathrm{I}_\cdot^f$ and by [17, P26] and Theorem 3.4.17, we can get the above results. □

Lemma 4.2.5. *If $\phi \in C^1(0,T] \cap L^1(0,T]$, then $-{}_0^{\mathrm{Ce}}\mathrm{D}_\cdot^f$ satisfies the positive maximum principle.*

Proof. Assume $\phi(x_0) = \sup_{(0,T]} \phi(x) \ge 0$. Using the Lemma 3.4.7, we have

$$\begin{aligned} -{}_0^{\mathrm{Ce}}\mathrm{D}_{x_0}^f\, \varphi(x_0) &= -\int_0^{x_0} \big(\varphi(x_0) - \varphi(x_0 - s)\big)\, \mu(ds) \\ &= \int_0^{x_0} (\varphi(x_0 - r) - \varphi(x_0))\, \mu(ds) \\ &\le 0 \end{aligned}$$
□

Remark 4.2.6. (1) $-{}_0^{\mathrm{Ce}}\mathrm{D}_\cdot^f$ is dissipative. Every operator satisfying the positive maximum principle is dissipative.

(2) Using standard theory e.g. [17, P.28] Theorem 1.3., the following evolution equation has the unique solution $\phi(t,x) = \mathbb{E}^x[g(S_t^c)]$, $g \in \mathcal{D}(-{}_0^{\mathrm{Ce}}\mathrm{D}_x^f)$.

$$\begin{cases} \partial_t \phi(t,x) = -{}_0^{\mathrm{Ce}}\mathrm{D}_x^f\, \phi(t,x), \\ \phi(0,x) = g(x). \end{cases} \tag{4.2.3}$$

Theorem 4.2.7. *Assuming $\lambda < 0$, $T > 0$ and $g \in C[0,T]$, we have*

$$\mathbb{E}^x\left[\int_0^{\tau_\infty} e^{\lambda t} g(S_t^c)\, dt\right] = \sum_{j=0}^{\infty} \lambda^j ({}_0^{\mathrm{Ce}}\mathrm{I}_x^f)^{j+1} g. \tag{4.2.4}$$

In particular, $\mathbb{E}^x[e^{-\lambda\tau_\infty}] = \sum_{j=0}^{\infty} \lambda^j ({}_0^{\mathrm{Ce}}\mathrm{I}_x^f)^j \mathbb{1}$.

Proof. If $g \in C_\infty(0,T]$, using Theorem 4.2.4 and Theorem 1.4.5, the equality holds true as the following equation has a unique solution:

$$\begin{cases} {}_0^{\mathrm{Ce}}\mathrm{D}_x^f\, \phi(x) &= \lambda\phi(x) + g(x), \quad x \in (0,T], \\ \phi(x) &= 0, \qquad\qquad\quad\;\; x = 0. \end{cases}$$

Now, for any $g \in C[0,T]$, take $g_n \in C_\infty(0,\infty]$ such that g_n converges to g locally uniformly in $(0,T]$. For any fixed $x \in (0,T]$ and $t > 0$, we have

$$\mathbb{E}^x[g_n(S_t^c)] \to \mathbb{E}^x[g(S_t^c)], \quad \text{as} \quad n \to \infty,$$

by using the dominated convergence theorem, since $\sup_n \|g\|_{C[0,T]} < \infty$. With the dominating function $\sup_n \|g_n\|_{C[0,T]} e^{\lambda t} < \infty$ and using Lemma 4.1.3 $\mathbb{E}(\tau_\infty) < \infty$, we apply again dominated convergence theorem. This gives

$$\mathbb{E}^x \left[\int_0^{\tau_\infty} e^{-\lambda t} g_n(S_t^c)\, dt \right] \to \mathbb{E}^x \left[\int_0^{\tau_\infty} e^{-\lambda t} g(S_t^c)\, dt \right], \quad \text{as} \quad n \to \infty.$$

Assume, for a moment that $g \geq 0$. Clearly we can choose $0 \leq g_n \leq g$ such that $g = \sup_n g_n$ (increasing limit). Since ${}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{\bullet}$ is positivity preserving and linear, we have ${}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x}\, g_n \uparrow {}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x}\, g$ as n goes to ∞. and so we have

$$\sum_{j=0}^{\infty} \lambda^j ({}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x})^{j+1} g_n \to \sum_{j=0}^{\infty} \lambda^j ({}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x})^{j+1} g, \quad \text{as} \quad n \to \infty.$$

The general case follows by considering positive and negative parts: $g = g^+ + g^-$ with $g \in C[0,T]$ and $g_n \to g^+$, $h_n \to g^-$ as n goes to ∞. Therefore we have proved (4.2.4) for all functions $g \in C[0,T]$.

To prove the special case we take $g = \lambda$, so the left side of (4.2.4) become,

$$\mathbb{E}^x \left[\int_0^{\tau_\infty} e^{\lambda t} \lambda\, dt \right] = \mathbb{E}^x e^{-\lambda \tau_\infty} - 1,$$

and the right side of (4.2.4) gives

$$\sum_{j=0}^{\infty} \lambda^{j+1} ({}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x})^{j+1} \mathbb{1} = \sum_{j=0}^{\infty} \lambda^{j} ({}^{\mathrm{Ce}}_{0}\mathrm{I}^{f}_{x})^{j} \mathbb{1} - 1.$$

This finishes the proof. □

Chapter 5

Application

In this chapter, we discuss applications of the existence and uniqueness of the linear censored equation, which we established in the previous chapter. In the first section, we investigate the function space that involves regularly varying functions and contains the Hölder space under specific conditions; see Theorem 5.1.1. We then explore the linear censored initial value problem for regularly varying kernels using induction methods. This approach is slightly different from that in Chapter 3. Throughout this chapter we assume that $f \in \mathcal{CBF}$ and recall that

$$\mathscr{L}[\bar{\mu}, \lambda] = f(\lambda)/\lambda, \quad \mathscr{L}[k, \lambda] = 1/f(\lambda)$$

Then $k, \bar{\mu}$ is a Sonine pair see Chapter 3 Section 3.1.

5.1 Censored subordinator for a regularly varying kernel

In this section, we assume that f is regularly varying at infinity. According to Theorem 3.1.15, $k, \bar{\mu}$ are also regularly varying at 0, and we may assume

$$\bar{\mu}(x) = \frac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)}, \quad k(x) = \frac{l_2(x)x^{\beta-1}}{\Gamma(\beta)}, \quad 0 < \beta < 1.$$

If we replace in the definition of the Bernstein Riemann-Liouville integral (3.4.6) and Bernstein Riemann-Liouville derivative (3.4.5) the kernels by the above regularly varying kernels, we have

$$ {}^{\mathrm{R}}_{0}\mathrm{I}^{\beta,l_2}_{x}\, \varphi = \int_0^x \varphi(s) \frac{l_2(x-s)(x-s)^{\beta-1}}{\Gamma(\beta)}\, ds \tag{5.1.1}$$

$$ {}^{\mathrm{R}}_{0}\mathrm{D}^{\beta,l_1}_{x}\, \varphi = \frac{d}{dx} \int_0^x \varphi(s)\, \frac{l_1(x-s)(x-s)^{-\beta}}{\Gamma(1-\beta)}\, ds, \tag{5.1.2}$$

Let us recall the following function spaces (see Chapter 3, Section 3.3)

$$C_{\bar{\mu}}(0,T] = \left\{\varphi \in C \cap L^1(0,T] \,:\, (\bar{\mu} * \varphi)(\cdot) \in C^1(0,T]\right\},$$

$$C_{\bar{\mu}}[0,T] = C[0,T] \cap C_{\bar{\mu}}(0,T].$$

In the present setting this simplifies to

$$C_{\beta,l_1}(0,T] = \left\{ \varphi \in C \cap L^1(0,T] \,:\, \left(\frac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)} * \varphi\right)(\cdot) \in C^1(0,T] \right\},$$

$$C_{\beta,l_1}[0,T] = C[0,T] \cap C_{\beta,l_1}(0,T].$$

Theorem 5.1.1. *Assume $\alpha > \beta$, then $C^\alpha[0,T] \subset C_{\beta,l_1}(0,T]$.*

Proof. Let $\phi \in C^\alpha[0,T]$. We have to show

$$\left(\frac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)} * \phi\right)(x) \in C_{\beta,l_1}(0,T] \quad \text{i.e.} \quad \frac{d}{dx}\left(\frac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)} * \phi\right)(x) \in C(0,T].$$

We have for x and ε

$$\begin{aligned}
&\int_0^{x+\varepsilon} l_1(x+\varepsilon-r)(x+\varepsilon-r)^{-\beta}\phi(r)\,dr - \int_0^x l_1(x-r)(x-r)^{-\beta}\phi(r)\,dr \\
&= \int_0^{x+\varepsilon} l_1(x+\varepsilon-r)(x+\varepsilon-r)^{-\beta}(\phi(r)-\phi(x))\,dr - \Bigg(\int_0^x l_1(x-r)(x-r)^{-\beta}\phi(x)\,dr \\
&\quad - \int_0^{x+\varepsilon} l_1(x+\varepsilon-r)(x+\varepsilon-r)^{-\beta}\phi(x)\,dr\Bigg) - \int_0^x l_1(x-r)(x-r)^{-\beta}(\phi(r)-\phi(x))\,dr \\
&= \mathrm{I} - \mathrm{II} - \mathrm{III}
\end{aligned}$$

It is easy to deal with Part II, and we obtain the following result:

$$\begin{aligned}
\mathrm{II} &= \int_0^x l_1(x-r)(x-r)^{-\beta}\phi(x)\,dr - \int_0^{x+\varepsilon} l_1(x+\varepsilon-r)(x+\varepsilon-r)^{-\beta}\phi(x)\,dr \\
&= \phi(x)\Bigg(\int_0^x l_1(r)(r)^{-\beta}\,dr - \int_0^{x+\varepsilon} l_1(r)r^{-\beta}\,dr\Bigg) \\
&= -\phi(x)\int_x^{x+\varepsilon} l_1(r)r^{-\beta}\,dr.
\end{aligned}$$

For Part I and Part III, we have the following:

$$\begin{aligned}
&\mathrm{I}-\mathrm{III} \\
&= \int_0^{x+\varepsilon} l_1(x+\varepsilon-r)(x+\varepsilon-r)^{-\beta}(\phi(r)-\phi(x))\,dr \\
&\quad - \int_0^x l_1(x-r)(x-r)^{-\beta}(\phi(r)-\phi(x))\,dr
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\varepsilon}^{x} l_1(\varepsilon + r)(\varepsilon + r)^{-\beta}(\phi(x-r) - \phi(x))\, dr - \int_0^x l_1(r) r^{-\beta}(\phi(x-r) - \phi(x))\, dr \\
&= \int_0^x \big(l_1(\varepsilon + r)(\varepsilon + r)^{-\beta} - l_1(r) r^{-\beta}\big)(\phi(x-r) - \phi(x))\, dr \\
&\quad + \int_{-\varepsilon}^{0} l_1(\varepsilon + r)(\varepsilon + r)^{-\beta}(\phi(x-r) - \phi(x))\, dr.
\end{aligned}
$$

From the above calculations we obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left[\left(\frac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)} * \phi \right)(x+\varepsilon) - \left(\frac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)} * \phi \right)(x) \right] \\
&= \lim_{\varepsilon\to 0} \left[\frac{1}{\varepsilon} \left(\int_{-\varepsilon}^{0} l_1(\varepsilon + r)(\varepsilon + r)^{-\beta}(\phi(x-r) - \phi(x))\, dr + \phi(x) \int_x^{x+\varepsilon} l_1(r) r^{-\beta}\, dr \right) \right. \\
&\quad \left. + \frac{1}{\varepsilon} \int_0^x \big(l_1(\varepsilon + r)(\varepsilon + r)^{-\beta} - l_1(r) r^{-\beta}\big)(\phi(x-r) - \phi(x))\, dr \right] \frac{1}{\Gamma(1-\beta)} \\
&= \frac{1}{\Gamma(1-\beta)} \left(\phi(x) l_1(x) x^{-\beta} + \int_0^x \big(l_1(r) r^{-\beta}\big)' (\phi(x-r) - \phi(x))\, dr \right) \\
&= \mathrm{IV}.
\end{aligned}
$$

In the above calculation we use the following facts: The first term vanishes, which can be seen from the following consideration:

$$
\begin{aligned}
&\left| \int_{-\varepsilon}^{0} l_1(\varepsilon + r)(\varepsilon + r)^{-\beta}(\phi(x-r) - \phi(x))\, dr \right| \\
&= \left| \int_{-\varepsilon}^{0} l_1(\varepsilon + r)(\varepsilon + r)^{-\beta} r^{\alpha} \frac{(\phi(x-r) - \phi(x))}{r^{\alpha}}\, dr \right| \\
&\le \varepsilon^{\alpha} \int_{-\varepsilon}^{0} \left| l_1(\varepsilon + r)(\varepsilon + r)^{-\beta} \frac{(\phi(x-r) - \phi(x))}{r^{\alpha}} \right| dr \\
&\le C\varepsilon^{\alpha} \int_{-\varepsilon}^{0} \left| l_1(\varepsilon + r)(\varepsilon + r)^{-\beta} \right| dr
\end{aligned}
$$

For the second term we use the boundedness and continuity of the integrand, and for the third term we use dominated convergence theorem in combination with the monotone density theorem (note that the kernel $l_1(x)x^{-\beta}$ is completely monotone and regularly varying). Thus,

$$
\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{0} l_1(\varepsilon + r)(\varepsilon + r)^{-\beta}(\phi(x-r) - \phi(x))\, dr = 0.
$$

Since $\phi \in C^{\alpha}[0,T]$ and $\alpha > \beta$, it is easy to see that the function appearing in IV is continuous, which finishes our proof. □

We will now use Definitions 2.4.5 and 3.4.6 and the potential form of the kernel $\bar{\mu}(x) = \frac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)}$ to re-define the general fractional derivative operators.

Remark 5.1.2. (1) For any $\varphi \in C_{\bar{\mu}}(0,T] = C_{\beta,l_1}(0,T]$, we have

$$\begin{aligned} {}^{\mathrm{Ce}}_{0}\mathrm{D}^{\beta,l_1}_{x}\,\varphi &= {}^{\mathrm{R}}_{0}\mathrm{D}^{\beta,l_1}_{x}\,\varphi - \varphi(x)\bar{\mu}(x) \\ &= \frac{d}{dx}\int_0^x \varphi(s)\frac{l_1(x-s)(x-s)^{-\beta}}{\Gamma(1-\beta)}\,ds - \frac{\varphi(x)l_1(x)x^{-\beta}}{\Gamma(1-\beta)} \end{aligned}$$

The corresponding kernels (see Definitions 3.4.8) have the following form

(2) For $0 < r < x$, we define the following kernel

$$k_j^{\mathrm{rv}}(x,r) = \begin{cases} \frac{l_1(r)r^{-\beta}}{\Gamma(1-\beta)}\frac{l_2(x-r)(x-r)^{\beta-1}}{\Gamma(\beta)}, & j = 1; \\ \int_r^x k_1^{\mathrm{rv}}(x,s)k_{j-1}^{\mathrm{rv}}(s,r)\,ds, & j \geq 2. \end{cases} \tag{5.1.3}$$

Clearly, $\int_0^x k_j^{\mathrm{rv}}(x,r)\,dr = 1$, which follows from induction.

Definition 5.1.3. For $\varphi \in C[0,T]$, we define

$$\mathcal{K}_{\mathrm{rv}}\varphi(x) = \begin{cases} \int_0^x k_1^{\mathrm{rv}}(x,r)\varphi(r)\,dr, & x > 0; \\ \varphi(0), & x = 0. \end{cases} \tag{5.1.4}$$

Furthermore $\mathcal{K}_{\mathrm{rv}}\varphi(x) = {}^{\mathrm{R}}_{0}\mathrm{I}^{\beta,l_2}_{x}[\bar{\mu}(x)\varphi(x)]$ and $\mathcal{K}_{\mathrm{rv}}$ is a linear operator which is positivity preserving (i.e. $\mathcal{K}_{\mathrm{rv}}\varphi \geq 0$ if $\varphi \geq 0$).

Recall the Theorem 3.4.17, when we apply the regularly varying Sonine pair as mentioned in Section 5.1

$$\bar{\mu}(x) = \frac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)}, \quad k(x) = \frac{l_2(x)x^{\beta-1}}{\Gamma(\beta)}, \quad 0 < \beta < 1.$$

We can get the inverse operator of ${}^{\mathrm{Ce}}_{0}\mathrm{D}^{\beta,l_1}_{x}$ and denote it by ${}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x}$ and its domain is $C_{\beta,l_1}[0,T]$ (see the definition in Section 5.1). The definition of ${}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x}$ given by for $g \in C_{\beta,l_1}[0,T]$

$${}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x}\,g = \sum_{j=0}^{\infty}\mathcal{K}^j_{\mathrm{rv}}\,{}^{\mathrm{R}}_{0}\mathrm{I}^{\beta,l_2}_{x}\,g \tag{5.1.5}$$

5.2 Linear censored initial value problem for regularly varying kernels

In this section, we still work under the assumptions of Section 5.1. Let $f \in \mathcal{CBF}$, and write $k, \bar{\mu}$ for the Sonine pair and we assume that f is regularly varying at infinity. According to Theorem 3.1.15, $k, \bar{\mu}$ are also regularly varying at 0, and we may assume

$$\bar{\mu}(x) = \frac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)}, \quad k(x) = \frac{l_2(x)x^{\beta-1}}{\Gamma(\beta)}, \quad 0 < \beta < 1.$$

The following lemmas are the main technical tools for the main results of this section, Theorem 5.2.3.

Lemma 5.2.1. *Assume that f, $\bar{\mu}$ and k are as above, then the limit $l_1(0+)$ exists if, and only if, the limit $l(\infty-)$ exists.*

Proof. We have,

$$\begin{aligned}\frac{f(\lambda)}{\lambda} &= \lambda^{\beta-1}l(\lambda)\\ &= \int_0^\infty e^{-\lambda x}\frac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)}\,dx\\ &= \int_0^\infty e^{-y}\frac{l_1(y/\lambda)(y/\lambda)^{-\beta}}{\lambda\Gamma(1-\beta)}\,dy.\end{aligned}$$

From this calculation we see $l(\lambda) = \int_0^\infty e^{-y}\frac{l_1(y/\lambda)y^{-\beta}}{\Gamma(1-\beta)}\,dy$. Now it is easy to see with the donaminated convergence theorem that $\lim_{\lambda\to\infty} l(\lambda)$ exists and is finite exactly when the limit $\lim_{\lambda\to\infty} l_1(y/\lambda) = l_1(0+)$ exists and is finite. □

Recall that $B(\alpha,\beta)$ is Euler's beta function.

Lemma 5.2.2. *Assume $\beta > 0$, $j \in \mathbb{N}$ and $l_2(0+) < \infty$. Then there exists $T > 0$ such that for all $x \in [0,T]$, we have*

$$({}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x})^j\mathbb{1}(x) \le \frac{1}{\beta}\frac{C_1^jC_{l_2}^j}{\Gamma(\beta)^j}x^{j\beta}\prod_{i=1}^{j}B(i\beta+1,\beta) \le C\frac{2^jC_1^jC_{l_2}^jx^{j\beta}}{\beta((j+1)!\beta^j)^\beta}. \tag{5.2.1}$$

Proof. We will prove

$$({}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x})^j\mathbb{1}(x) \le C_1^j\underbrace{({}^{\mathrm{R}}_{0}\mathrm{I}^{\beta,l_2}_{x})^{j-1}\int_0^x\frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr}_{\mathrm{I}}. \tag{5.2.2}$$

holds for all $j \in \mathbb{N}$. From the proof of Theorem 3.5.2 and Example 3.4.12, we get

$$\mathcal{K}^i_{\mathrm{rv}}\,{}^{\mathrm{R}}_{0}\mathrm{I}^{\beta,l_2}_{x}\,\mathbb{1}(x) \le \frac{1}{[\beta\Gamma(\beta)\Gamma(1-\beta)]^i}\int_0^x\frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr. \tag{5.2.3}$$

Using the results of eq (5.2.3), we obtain the estimate in eq (5.2.1) when $j = 1$.

$$\begin{aligned}{}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x}\,\mathbb{1}(x) &= \sum_{i=0}^{\infty}\mathcal{K}^i_{\mathrm{rv}}\,{}^{\mathrm{R}}_{0}\mathrm{I}^{\beta,l_2}_{x}\,\mathbb{1}(x)\\ &\le \sum_{i=0}^{\infty}\frac{1}{[\beta\Gamma(\beta)\Gamma(1-\beta)]^i}\int_0^x\frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr\\ &= C_1\int_0^x\frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr.\end{aligned}$$

Here $C_1 = \left(1 - \frac{1}{\beta\Gamma(\beta)\Gamma(1-\beta)}\right)^{-1}$. Now we proceed by induction: We assume that (5.2.3) holds for some $j \in \mathbb{N}$. For $j \rightsquigarrow j+1$, we obtain

$$\begin{aligned}
({}^{\text{Ce}}_{\ 0}\mathrm{I}^{\text{rv}}_x)^{j+1}\mathbb{1} &= {}^{\text{Ce}}_{\ 0}\mathrm{I}^{\text{rv}}_x \left({}^{\text{Ce}}_{\ 0}\mathrm{I}^{\text{rv}}_x\right)^j \mathbb{1} \\
&\leq C_1^j\, {}^{\text{Ce}}_{\ 0}\mathrm{I}^{\text{rv}}_x \left({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x\right)^j \int_0^x \frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr \\
&\leq C_1^j \sum_{i=0}^{\infty} \mathcal{K}^i_{\text{rv}}\, {}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x \left[({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x)^j \int_0^x \frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr\right] \\
&\leq C_1^{j+1} \left({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x\right)^{j+1} \int_0^x \frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr.
\end{aligned}$$

In the last inequality we use the monotonicity property of $\bar{\mu}(x) = \dfrac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)}$ at step 4 to get

$$\begin{aligned}
&\mathcal{K}^i_{\text{rv}}\, {}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x \left[({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x)^j \int_0^x \frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr\right] \\
&= \mathcal{K}^i_{\text{rv}} \left[({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x)^{j+1} \int_0^x \frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr\right] \\
&= \mathcal{K}^{i-1}_{\text{rv}}\mathcal{K}_{\text{rv}} \left[({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x)^{j+1} \int_0^x \frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr\right] \\
&= \mathcal{K}^{i-1}_{\text{rv}}\, {}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x \left\{\frac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)} \left[({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x)^{j+1} \int_0^x \frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr\right]\right\} \\
&\leq \mathcal{K}^{i-1}_{\text{rv}}\, {}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x \left\{({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x)^{j+1} \left[\frac{l_1(x)x^{-\beta}}{\Gamma(1-\beta)} \int_0^x \frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr\right]\right\} \\
&\leq \frac{1}{[\beta\Gamma(\beta)\Gamma(1-\beta)]}\mathcal{K}^{i-1}_{\text{rv}} \left[({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x)^{j+2}\mathbb{1}(x)\right] \\
&= \frac{1}{[\beta\Gamma(\beta)\Gamma(1-\beta)]}\mathcal{K}^{i-1}_{\text{rv}} \left[({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x)^{j+1}\, {}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x\, \mathbb{1}(x)\right] \\
&= \frac{1}{[\beta\Gamma(\beta)\Gamma(1-\beta)]}\mathcal{K}^{i-1}_{\text{rv}} \left[({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x)^{j+1} \int_0^x \frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr\right]
\end{aligned}$$

Repeated applications of the above calculation, give

$$\mathcal{K}^i_{\text{rv}}\, {}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x \left[({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x)^j \int_0^x \frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr\right] \leq \frac{1}{[\beta\Gamma(\beta)\Gamma(1-\beta)]^i}({}^{\text{R}}_{0}\mathrm{I}^{\beta,l_2}_x)^{j+1} \int_0^x \frac{r^{\beta-1}l_2(r)}{\Gamma(\beta)}\,dr.$$

This finishes the proof of (5.2.2).

Now let us estimate the right side of (5.2.2). By induction we show that

$$({}^{\text{Ce}}_{\ 0}\mathrm{I}^{\text{rv}}_x)^j\mathbb{1}(x) \leq \frac{1}{\beta}\frac{C_1^j C_{l_2}^j}{\Gamma(\beta)^j} x^{j\beta} \prod_{i=1}^{j} B(i\beta+1,\beta) \tag{5.2.4}$$

where the $C_{l_2} = \sup_{[0,T]} l_2$.

Now let us calculate the expression I, for $j = 1$. We have

$$\begin{aligned}
&\int_0^x \frac{(x-r)^{\beta-1} l_2(x-r)}{\Gamma(\beta)} \int_0^r \frac{s^{\beta-1} l_2(s)}{\Gamma(\beta)}\, ds\, dr \\
&\leq \frac{C_{l_2}^2}{\Gamma(\beta)^2} \int_0^x \int_0^r s^{\beta-1}\, ds\, (x-r)^{\beta-1}\, dr \\
&\leq \frac{C_{l_2}^2}{\beta\Gamma(\beta)^2} \int_0^x r^{\beta}(x-r)^{\beta-1}\, dr \\
&= \frac{C_{l_2}^2 B(\beta+1,\beta)}{\beta\Gamma(\beta)^2} x^{2\beta},
\end{aligned}$$

where we use the condition $l_2(0+) < \infty$ and Potter's bounds see Theorem 1.3.4 to get C_{l_2}. Assume that the above result (5.2.4) holds for $j \in \mathbb{N}$. For $j \rightsquigarrow j+1$, we obtain

$$\begin{aligned}
&({}^{\mathrm{R}}_{0}\mathrm{I}^{\beta,l_2}_{x})^{j+1} \int_0^x \frac{r^{\beta-1} l_2(r)}{\Gamma(\beta)}\, dr \\
&= {}^{\mathrm{R}}_{0}\mathrm{I}^{\beta,l_2}_{x} \left[({}^{\mathrm{R}}_{0}\mathrm{I}^{\beta,l_2}_{x})^{j} \int_0^x \frac{r^{\beta-1} l_2(r)}{\Gamma(\beta)}\, dr \right] \\
&\leq {}^{\mathrm{R}}_{0}\mathrm{I}^{\beta,l_2}_{x} \left[\frac{1}{\beta} C_{l_2}^j x^{j\beta} \prod_{i=1}^{j} B(i\beta+1,\beta) \right] \\
&= \frac{1}{\beta} \frac{C_{l_2}^j}{\Gamma(\beta)^j} \prod_{i=1}^{j} B(i\beta+1,\beta) \int_0^x r^{j\beta}(x-r)^{\beta-1} l_2(x-r)\, dr \\
&\leq \frac{1}{\beta} \frac{C_{l_2}^{j+1} B(j\beta+1,\beta) x^{j\beta+1}}{\Gamma(\beta)^j} \prod_{i=1}^{j} B(i\beta+1,\beta) \\
&\leq \frac{1}{\beta} \frac{C_{l_2}^{j+1}}{\Gamma(\beta)^j x^{j\beta+1}} \prod_{i=1}^{j+1} B(i\beta+1,\beta).
\end{aligned}$$

This finishes the above proof of (5.2.4). Using Stirling's formula for the gamma function, i.e.

$$\Gamma(z) = \left(\frac{2\pi}{z}\right)^{0.5} \left(\frac{z}{e}\right)^{z} \left(1 + O\left(z^{-1}\right)\right),$$

we get the following relation

$$\frac{\Gamma(i\beta+1)}{\Gamma((i-1)\beta+1)} = (i\beta)^{\beta} \left(1 + O(\frac{1}{i\beta})\right), \tag{5.2.5}$$

Using the definition of the Beta function and the above estimation (5.2.5), we have

$$\begin{aligned}
B(i\beta+1,\beta) &= \frac{\Gamma(i\beta+1)\Gamma(\beta)}{\Gamma((i+1)\beta+1)} \\
&= \Gamma(\beta) \Big/ \frac{\Gamma((i+1)\beta+1)}{\Gamma(i\beta+1)} \\
&= \Gamma(\beta) \Big/ \left(((i+1)\beta)^{\beta} \left(1 + O(\frac{1}{(i+1)\beta})\right)\right) \\
&\leq 2\Gamma(\beta) \Big/ ((i+1)\beta)^{\beta}.
\end{aligned} \tag{5.2.6}$$

Multiplying over i on both sides of (5.2.6), we have

$$\frac{1}{\Gamma(\beta)^j}\prod_{i=1}^{j} B(i\beta+1,\beta) \le \frac{1}{\Gamma(\beta)^j}\prod_{i=1}^{j}\frac{2\Gamma(\beta)}{((i+1)\beta)^{\beta}} = \frac{2^j}{((j+1)!\beta^j)^{\beta}}.$$

So Lemma 5.2.2 holds for all $j \in \mathbb{N}$. □

Theorem 5.2.3. *For any $\phi_0, \lambda \in \mathbb{R}$, the linear initial value problem*

$$\begin{cases} {}^{\mathrm{Ce}}_{0}\mathrm{D}^{\beta,l_1}_{x}\,\phi(x) & = \lambda\phi(x), \quad x \in (0,T], \\ \phi(x) & = \phi_0, \qquad x = 0. \end{cases} \tag{5.2.7}$$

has a unique solution in $C_{\beta,l_1}[0,T]$ given by $\sum_{j=0}^{\infty}(\lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x})^j\mathbb{1}$.

Proof. We define the following sequence

$$\begin{cases} \phi_n(x) = \lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^{rv}_{x}\,\phi_{n-1} + \phi_0, \\ \phi_0(x) = \phi_0. \end{cases} \tag{5.2.8}$$

$$\begin{aligned} \phi_1(x) &= \lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x}\,\phi_0 + \phi_0 \\ \phi_2(x) &= \lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x}\,\phi_1 + \phi_0 \\ &= \lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x}[\lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x}\,\phi_0 + \phi_0] + \phi_0 \\ &= \lambda^2({}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x})^2\phi_0 + {}^{\mathrm{Ce}}_{0}\mathrm{I}^{rv}_{x}\,\phi_0 + \phi_0 \\ &,\ldots, \\ \phi_{n+1}(x) &= \sum_{j=0}^{n}(\lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x})^j\phi_0. \end{aligned}$$

Let $n \to \infty$, then

$$\phi_n(x) \to \phi(x) = \sum_{j=0}^{\infty}(\lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x})^j\phi_0 = \phi_0\sum_{j=0}^{\infty}(\lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x})^j\mathbb{1}.$$

Using Lemma 5.2.2, $\sum_{j=0}^{\infty}(\lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x})^j\mathbb{1}$ converges uniformly on $[0,T]$. Using a similar argument as in the proof of Theorem 3.5.2, we can get that $\sum_{j=0}^{\infty}(\lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x})^j\mathbb{1}$ solves (5.2.7). □

Now consider the inhomogeneous initial value problem.

Theorem 5.2.4. *For any $\phi_0, \lambda \in \mathbb{R}$, and $g \in C[0,T]$. The inhomogeneous linear initial value problem*

$$\begin{cases} {}^{\mathrm{Ce}}_{0}\mathrm{D}^{\beta,l_1}_{x}\,\phi(x) & = \lambda\phi(x) + g(x), \quad x \in (0,T], \\ \phi(x) & = \phi_0, \qquad x = 0. \end{cases} \tag{5.2.9}$$

has a unique solution in $C_{\beta,l_1}[0,T]$ given by

$$\phi(x) = \phi_0\sum_{j=0}^{\infty}(\lambda\,{}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x})^j\mathbb{1} + \sum_{j=0}^{\infty}\lambda^j({}^{\mathrm{Ce}}_{0}\mathrm{I}^{\mathrm{rv}}_{x})^{j+1}g. \tag{5.2.10}$$

Proof. According to the assumptions, g is continuous and we will get all sun (5.2.10) uniformly convergent by using Lemma 5.2.2. So the function given by eq (5.2.10) is in $C[0,T]$. Using a similar argument as in the proof of Theorem 3.5.4, we can get the desired results. □

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