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Pushing Optimal ABox Repair from $\mathcal{EL}$ Towards More Expressive Horn-DLs (Extended Version)

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Abstract

Ontologies based on Description Logic (DL) represent general background knowledge in a terminology (TBox) and the actual data in an ABox. DL systems can then be used to compute consequences (such as answers to certain queries) from an ontology consisting of a TBox and an ABox. Since both human-made and machine-learned data sets may contain errors, which manifest themselves as unintuitive or obviously incorrect consequences, repairing DL-based ontologies in the sense of removing such unwanted consequences is an important topic in DL research. Most of the repair approaches described in the literature produce repairs that are not optimal, in the sense that they do not guarantee that only a minimal set of consequences is removed. In a series of papers, we have developed an approach for computing optimal repairs, starting with the restricted setting of an $\mathcal{EL}$ instance store, extending this to the more general setting of a quantified ABox (where some individuals may be anonymous), and then adding a static $\mathcal{EL}$ TBox.

Here, we extend the expressivity of the underlying DL considerably, by adding nominals, inverse roles, regular role inclusions and the bottom concept to $\mathcal{EL}$, which yields a fragment of the well-known DL Horn-$\mathcal{SROIQ}$. The ideas underlying our repair approach still apply to this DL, though several non-trivial extensions are needed to deal with the new constructors and axioms. The developed repair approach can also be used to treat unwanted consequences expressed by certain conjunctive queries or regular path queries, and to handle Horn-$\mathcal{ALCOI}$ TBoxes with regular role inclusions.

1 Introduction

Description Logics (DLs) (Baader et al. 2017) are a prominent family of logic-based knowledge representation formalisms, which offer a good compromise between expressiveness and the complexity of reasoning and are the formal basis for the Web ontology language OWL.[1] The palette of well-investigated DLs with optimized reasoning support goes from the inexpressive and tractable DLs of the $\mathcal{EL}$ and DL-Lite families (Baader, Brandt, and Lutz 2005; Calvanese et al. 2007), on which the OWL 2 profiles OWL 2 EL and OWL 2 QL are based, all the way up to the N2ExpTime-complete DL $\mathcal{SROIQ}$ (Horrocks, Kutz, and Sattler 2006; Kazakov 2008), which is the DL underlying OWL 2. The consequence-based reasoning approach developed for the $\mathcal{EL}$ family (Baader, Brandt, and Lutz 2005) can be extended to Horn fragments of more expressive DLs, which yields practical "pay as you go" reasoning procedures for these fragments, though they are no longer tractable (Kazakov 2009; Ortiz, Rudolph, and Šimkus 2010).

Like all large human-made digital artefacts, the ontologies employed in applications often contain errors, and this problem is only exacerbated if parts of the ontology (e.g., the data) are automatically generated using inexact methods based on information retrieval or machine learning. Errors are usually detected when reasoning finds an inconsistency or generates consequences that are unintuitive or obviously wrong in the application domain. For the developers of a DL-based ontology it is often quite hard to see how the ontology needs to be modified such that the unwanted consequences no longer follow from the repaired ontology, but as few as possible other consequences are lost.

Classical DL repair approaches based on axiom pinpointing compute maximal subsets of the ontology that do not have the unwanted consequences (Parsia, Sirin, and Kalyanpur 2005; Schlobach et al. 2007; Baader and Suntisrivaraporn 2008). Such repairs depend on the syntactic form of the ontology: if a certain fact is expressed by a single strong axiom rather than an equivalent set of weaker ones, then too many consequences may be lost when removing this strong axiom. To overcome this problem, more fine-grained approaches for repairing DL-based ontologies have been developed (Horridge, Parsia, and Sattler 2008; Lam et al. 2008; Du, Qi, and Fu 2014; Troquard et al. 2018; Baader et al. 2018). These approaches are, however, still not optimal since they apply some restrictions on how the ontology can be changed, based on its syntactic form. In particular, they usually do not add new objects to the ABox.

To see why new objects may be needed to achieve optimality, assume that the ABox contains the information that Kim, who is rich and famous, is Ann's child, expressed by the assertions *Famous*(KIM), *Rich*(KIM), and $child(ANN, KIM)$, and that we want to remove the consequence $\exists\, child.(Rich \sqcap Famous)(ANN)$. If we decide to keep the assertion that Kim is Ann's child, then we need to remove either *Rich*(KIM) or *Famous*(KIM). However, if we decide that this Kim is not Ann's child after all, simply removing the role assertion $child(ANN, KIM)$ would

[1] https://www.w3.org/TR/owl2-overview/

also remove implied consequences for Ann. This can be avoided by adding the assertions $child(ANN, x)$, *Rich*(x), $child(ANN, y)$, and *Famous*(y), where x and y are anonymous individuals, which are formally represented in a quantified ABox (qABox) by existentially quantified variables. This example illustrates the main idea underlying the optimal repair approach introduced in (Baader et al. 2020): the use of quantified ABoxes and the construction of appropriate anonymous copies of individuals. The main technical problem to solve in (Baader et al. 2020) was to find out which copies with what properties are needed to achieve optimality. This work dealt with an input ontology consisting only of a qABox, and assumed that the unwanted consequences are instance relationships $C(a)$ for $\mathcal{EL}$ concepts C.

In (Baader et al. 2021a), we extended this approach to a setting where, in addition to the qABox, the ontology contains an $\mathcal{EL}$ TBox, which is assumed to be correct, and thus cannot be changed during repair. To add consequences implied by the TBox, we saturate the qABox by using the concept inclusions as rewrite rules before repairing the qABox. If, in our example, the TBox contained the concept inclusion *Celebrity* $\sqsubseteq$ $Rich \sqcap Famous$ and the ABox contained *Celebrity*(KIM) rather than *Rich*(KIM) and *Famous*(KIM), then saturation would add the latter two assertions. If then *Celebrity*(KIM) is removed in the repair, these two consequences can still be preserved. However, when repairing the saturated qABox, care must be taken that the TBox cannot re-introduce assertions that have been removed by the repair. For example, in the case where the unwanted consequence is *Rich*(KIM), it is not enough to remove this assertion from the saturated qABox: one also needs to remove *Celebrity*(KIM) since together with the TBox it implies *Rich*(KIM). The problem with saturation is that, in the presence of cyclic concept inclusions, such as $Rich \sqsubseteq \exists child.Rich$, it may not terminate. This is not just a problem of our repair *approach*, but may prevent the existence of optimal repairs (see Example 9 below). In (Baader et al. 2021a), two approaches are considered to overcome this problem. On the one hand, one can restrict the attention to TBoxes that are cycle-restricted as introduced in (Baader, Borgwardt, and Morawska 2012). On the other hand, if one is only interested in answers to instance queries, one can apply a weaker saturation operation, called IQ-saturation, which always terminates for $\mathcal{EL}$.

In this paper, we extend the expressivity of the DL used to formulate the TBox and the unwanted consequences considerably, by adding nominals, inverse roles, role inclusion axioms, and the bottom concept to $\mathcal{EL}$. To obtain a decidable DL and guarantee the existence of optimal repairs, we restrict the set of role inclusion axioms to being regular, as in the DL $\mathcal{SROIQ}$. In addition, we first consider the case without the bottom concept, and only later deal with the additional problems caused by the fact that bottom may cause the ontology to become inconsistent. Computability of the set of optimal repairs and the fact that this set *covers* all repairs (in the sense that every repair is entailed by an optimal one) follows from a "small repair" property, which can be shown using an adaptation of the well-known filtration technique (Baader et al. 2017). However, even disregarding the impracticality of an algorithm that computes the optimal repairs by looking at all qABoxes up to a certain size bound, this does not lead to a viable methods for choosing an appropriate optimal repair since it would require the knowledge engineer to choose among exponentially many repairs of exponential size. In contrast, the *canonical repairs* (which cover all optimal repairs) constructed by our extension of the repair approach in (Baader et al. 2021a) are characterized by so-called *repair seeds*, which are of polynomial size. The knowledge engineer can choose among these by answering a polynomial number of instance queries (i.e., queries about which instance relationships hold in the application domain).

The added expressivity generates new challenges, which require non-trivial adaptations of our approach for constructing canonical repairs. Since nominals in the TBox can imply equality between individuals, we extend qABoxes by equality assertions, to be able to represent such consequences in the saturated qABox, and we also must repair unwanted equalities. We deal with role inclusion axioms and inverse roles by using finite automata, which can represent the infinitely many implied role inclusions in a finite way. Technically, this is where we make use of the restriction to regular sets of role inclusion axioms. To handle inconsistency caused by bottom, we consider not only "local" unwanted consequences of the form $C(a)$, but also "global" ones of the form $\exists\{x\}.\{C(x)\}$. If, in our example, the TBox additionally says that rich and poor are disjoint, using the concept inclusion *Poor* $\sqcap$ *Rich* $\sqsubseteq \bot$, and the qABox states that Ann has an (anonymous) child that is a poor celebrity, then the entailed inconsistency can be repaired by preventing the consequence $\exists\{x\}.\{(Poor \sqcap Rich)(x)\}$.

The added expressivity also allows us to specify interesting kinds of unwanted consequences other than instance relationships. On the one hand, we can deal with regular reachability queries, which are similar to regular path queries (Calvanese, Eiter, and Ortiz 2009). On the other hand, we can also treat certain kinds of conjunctive queries. The problem of repairing w.r.t. conjunctive queries to qABoxes has already been considered, in the guise of achieving compliance for relational datasets with labelled nulls, in (Grau and Kostylev 2019). However, this work does not allow for background TBoxes, and the notion of optimality used there is different from ours since it restricts the possible changes to the qABox to a sequence of certain anonymization operations. Finally, our repair approach can also deal with Horn-$\mathcal{ALCOI}$-TBoxes together with sets of regular role inclusion axioms.

This extended version of the conference submission contains all technical details. We use roman numbers for the additional lemmas and propositions.

2 Preliminaries

First, we introduce the DL $\mathcal{ELROI}$ employed to formulate terminological background knowledge, and the quantified ABoxes with equalities used to represent the data. Then we describe how such an ABox can be saturated w.r.t. the terminological knowledge, and finally define regular sets of role inclusions and show how to represent them using finite automata.

2.1 The Description Logic $\mathcal{ELROI}$

The DL $\mathcal{ELROI}$ extends $\mathcal{EL}$ with (complex) role inclusions ($\mathcal{R}$), nominals ($\mathcal{O}$), and inverse roles ($\mathcal{I}$). Let Σ be a *signature*, i.e., a disjoint union of finite, non-empty sets Σ_I, Σ_C, and Σ_R of *individual names*, *concept names*, and *role names*, respectively. A *role* is either a role name or an *inverse role* r^- for some role name $r \in \Sigma_\mathsf{R}$. For a role R we write R^- to denote r^- if $R = r$ is a role name and r if $R = r^-$ is an inverse role. Concept descriptions C of $\mathcal{ELROI}$ are constructed using the grammar rule

$$C ::= \top \mid A \mid \{a\} \mid C \sqcap C \mid \exists R.C,$$

where A ranges over concept names, a over individual names, and R over roles. An *atom* is a concept name A, a *nominal* $\{a\}$, or an *existential restriction* $\exists R.C$. Each concept description C is a conjunction of atoms, with $\top$ corresponding to the empty conjunction. We denote the set of these atoms as $\mathsf{Conj}(C)$.

A *concept inclusion (CI)* is of the form $C \sqsubseteq D$ for concept descriptions C, D, and a *role inclusion (RI)* is of the form $\varepsilon \sqsubseteq S$ or $R_1 \circ \cdots \circ R_n \sqsubseteq S$ for roles $R_1, \ldots, R_n, S$ and $n \geq 1$. In the following, when we write $R_1 \circ \cdots \circ R_n \sqsubseteq S$, we assume that $n \geq 0$, where $R_1 \circ \cdots \circ R_n$ for $n = 0$ stands for ε. A *TBox* is a finite set of CIs, an *RBox* is a finite set of RIs, and a pair $(\mathcal{T}, \mathcal{R})$ consisting of a TBox $\mathcal{T}$ and an RBox $\mathcal{R}$ is called a *terminology*. A *concept assertion* $C(a)$ is a shorthand for the CI $\{a\} \sqsubseteq C$, and a *role assertion* $r(a, b)$ abbreviates $\{a\} \sqsubseteq \exists r.\{b\}$. Furthermore, $r^-(a, b)$ means $r(b, a)$.

The semantics of $\mathcal{ELROI}$ is defined based on interpretations, where an *interpretation* $\mathcal{I}$ of the signature Σ consists of a non-empty set $\mathsf{Dom}(\mathcal{I})$, the *domain*, and an *interpretation function* $\cdot^\mathcal{I}$ that maps each individual name a to an element $a^\mathcal{I}$ of $\mathsf{Dom}(\mathcal{I})$, each concept name A to a subset $A^\mathcal{I}$ of $\mathsf{Dom}(\mathcal{I})$, and each role name r to a binary relation $r^\mathcal{I}$ over $\mathsf{Dom}(\mathcal{I})$. We do *not* adopt the *unique name assumption*, i.e., $a^\mathcal{I} = b^\mathcal{I}$ is allowed for distinct individual names a, b. The interpretation of an inverse role is $(r^-)^\mathcal{I} := \{\, (\gamma, \delta) \mid (\delta, \gamma) \in r^\mathcal{I} \,\}$, and the interpretation $C^\mathcal{I}$ of a concept description C is recursively defined as $\top^\mathcal{I} := \mathsf{Dom}(\mathcal{I})$, $\{a\}^\mathcal{I} := \{a^\mathcal{I}\}$, $(C \sqcap D)^\mathcal{I} := C^\mathcal{I} \cap D^\mathcal{I}$, and $(\exists R.C)^\mathcal{I} := \{\, \delta \mid (\delta, \gamma) \in R^\mathcal{I} \text{ for some } \gamma \in C^\mathcal{I} \,\}$.

The CI $C \sqsubseteq D$ *holds* in $\mathcal{I}$ (denoted $\mathcal{I} \models C \sqsubseteq D$) if $C^\mathcal{I} \subseteq D^\mathcal{I}$, and the RI $R_1 \circ \cdots \circ R_n \sqsubseteq S$ *holds* in $\mathcal{I}$ (denoted $\mathcal{I} \models R_1 \circ \cdots \circ R_n \sqsubseteq S$) if $(R_1 \circ \cdots \circ R_n)^\mathcal{I} \subseteq S^\mathcal{I}$, where $\varepsilon^\mathcal{I} := \{\, (\delta, \delta) \mid \delta \in \mathsf{Dom}(\mathcal{I}) \,\}$ and $(R_1 \circ \cdots \circ R_n)^\mathcal{I} := \{\, (\delta_0, \delta_n) \mid (\delta_0, \delta_1) \in R_1^\mathcal{I}, \ldots, (\delta_{n-1}, \delta_n) \in R_n^\mathcal{I} \text{ for some } \delta_1, \ldots, \delta_{n-1} \in \mathsf{Dom}(\mathcal{I}) \,\}$. The interpretation $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$ (RBox $\mathcal{R}$) if every CI in $\mathcal{T}$ (RI in $\mathcal{R}$) holds in $\mathcal{I}$. This is written as $\mathcal{I} \models \mathcal{T}$ ($\mathcal{I} \models \mathcal{R}$). We say that the terminology $(\mathcal{T}, \mathcal{R})$ *entails* a CI or RI α (written $(\mathcal{T}, \mathcal{R}) \models \alpha$) if α holds in every model of $\mathcal{T}$ and $\mathcal{R}$. In case $(\mathcal{T}, \mathcal{R}) \models C \sqsubseteq D$ we say that C *is subsumed by* D w.r.t. $(\mathcal{T}, \mathcal{R})$, and may write $C \sqsubseteq^{\mathcal{T},\mathcal{R}} D$ to express this.

Note that other interesting axioms concerning roles can be expressed using RIs and inverse roles. Reflexivity, transitivity, and symmetry of r can respectively be enforced by the RIs $\varepsilon \sqsubseteq r$, $r \circ r \sqsubseteq r$, and $r \sqsubseteq r^-$, and *domain restrictions* $\mathsf{Dom}(r) \sqsubseteq C$ are expressible as $\exists r.\top \sqsubseteq C$ while *range restrictions* $\mathsf{Ran}(r) \sqsubseteq C$ can be expressed by CIs $\exists r^-.\top \sqsubseteq C$.

This last observation shows that subsumption in $\mathcal{ELROI}$ is actually undecidable since it was shown in (Baader, Lutz, and Brandt 2008) that subsumption in $\mathcal{EL}$ w.r.t. RIs and range restrictions is undecidable. We will avoid this problem by imposing a restriction on RBoxes (see Section 2.4).

2.2 Quantified ABoxes with Equalities

Quantified ABoxes were first introduced in (Baader et al. 2020), but they were also considered, as relational datasets with labelled nulls, in (Grau and Kostylev 2019), and their existentially quantified variables correspond to the "anonymous individuals" in the OWL 2 standard. Also, as explained in (Baader et al. 2020), quantified ABoxes are basically the same as Boolean conjunctive queries. Here, we extend this notion by allowing for equality assertions, but for simplicity still use the name "quantified ABoxes" for the extended formalism. Equality assertions are used to represent implied equality between individuals; e.g., the CI $\{a\} \sqsubseteq \{b\}$ implies that a and b must always be interpreted by the same element of the domain.

Let Σ be a signature. A *quantified ABox (qABox)* $\exists X.\mathcal{A}$ over Σ consists of a finite set X of *variables*, which is disjoint with Σ, and a *matrix* $\mathcal{A}$, which is a finite set of *concept assertions* $A(u)$, *role assertions* $r(u, v)$, and *equality assertions* $a \equiv b$, where $A \in \Sigma_\mathsf{C}$, $r \in \Sigma_\mathsf{R}$, $u, v \in \Sigma_\mathsf{I} \cup X$, and $a, b \in \Sigma_\mathsf{I}$. An *object name* of $\exists X.\mathcal{A}$ is either an element of Σ_I or a variable in X. We denote the set of these objects as $\mathsf{Obj}(\exists X.\mathcal{A})$. If X is empty, then we sometimes drop the quantifier $\exists \emptyset$. We do not allow equality assertions involving variables since otherwise each qABox can be normalized into a qABox without them.

The interpretation $\mathcal{I}$ is a *model* of $\exists X.\mathcal{A}$ (written $\mathcal{I} \models \exists X.\mathcal{A}$) if there is a *variable assignment* $\mathcal{Z} \colon X \to \mathsf{Dom}(\mathcal{I})$ such that the augmented interpretation $\mathcal{I}[\mathcal{Z}]$ that additionally maps each variable x to $\mathcal{Z}(x)$ is a model of the matrix $\mathcal{A}$, i.e., $u^{\mathcal{I}[\mathcal{Z}]} \in A^\mathcal{I}$ for each $A(u) \in \mathcal{A}$, $(u^{\mathcal{I}[\mathcal{Z}]}, v^{\mathcal{I}[\mathcal{Z}]}) \in r^\mathcal{I}$ for each $r(u, v) \in \mathcal{A}$, and $a^\mathcal{I} = b^\mathcal{I}$ for each $a \equiv b \in \mathcal{A}$. Given a terminology $(\mathcal{T}, \mathcal{R})$ and qABoxes $\exists X.\mathcal{A}$ and $\exists Y.\mathcal{B}$, we say that $\exists X.\mathcal{A}$ *entails* $\exists Y.\mathcal{B}$ w.r.t. $(\mathcal{T}, \mathcal{R})$ (written $\exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists Y.\mathcal{B}$) if every model of $\exists X.\mathcal{A}$ and $(\mathcal{T}, \mathcal{R})$ is also a model of $\exists Y.\mathcal{B}$. If both the TBox $\mathcal{T}$ and the RBox $\mathcal{R}$ are empty, then we omit the suffix "w.r.t. $(\mathcal{T}, \mathcal{R})$" and write $\models$ instead of $\models^{\mathcal{T},\mathcal{R}}$. Similar simplifications are made if one of them is empty.

For qABoxes without equality assertions, it was shown in (Baader et al. 2020) that entailment can be characterized using homomorphisms. In our extended setting, we need to adapt the definition of a homomorphism between qABoxes. To this purpose, we consider the equivalence relation $\approx_{\exists X.\mathcal{A}}$ on $\mathsf{Obj}(\exists X.\mathcal{A})$ induced by the equality assertions in $\exists X.\mathcal{A}$, which is defined as the reflexive, symmetric, transitive closure of the relation $\{\, (a, b) \mid a \equiv b \in \mathcal{A} \,\}$. We sometimes write $\approx$ for $\approx_{\exists X.\mathcal{A}}$ if the qABox is clear from the context, and denote the equivalence classes by $[u]_{\exists X.\mathcal{A}}$. Since there are no equality assertions involving variables, each equivalence class of a variable is a singleton set.

Definition 1. A *homomorphism* h from a qABox $\exists X.\mathcal{A}$ to a qABox $\exists Y.\mathcal{B}$ is a mapping $h\colon \mathsf{Obj}(\exists X.\mathcal{A}) \to \mathsf{Obj}(\exists Y.\mathcal{B})$ that satisfies the following conditions:

(Hom1) $a \approx_{\exists X.\mathcal{A}} b$ implies $h(a) \approx_{\exists Y.\mathcal{B}} h(b)$ for all individual names a, b.

(Hom2) $h(a) = a$ for each individual name a.

(Hom3) For each concept assertion $A(t) \in \mathcal{A}$, there is an object name v such that $v \approx_{\exists Y.\mathcal{B}} h(t)$ and $A(v) \in \mathcal{B}$.

(Hom4) For each role assertion $r(t,u) \in \mathcal{A}$, there are object names v, w such that $v \approx_{\exists Y.\mathcal{B}} h(t)$, $w \approx_{\exists Y.\mathcal{B}} h(u)$, and $r(v,w) \in \mathcal{B}$.

Based on this notion of homomorphism, entailment between qABoxes with equality assertions can now be characterized as follows.

Proposition 2. *The qABox $\exists X.\mathcal{A}$ is entailed by the qABox $\exists Y.\mathcal{B}$ iff there exists a homomorphism from $\exists X.\mathcal{A}$ to $\exists Y.\mathcal{B}$.*

Proof. We first show the if direction. Therefore fix a homomorphism h from $\exists X.\mathcal{A}$ to $\exists Y.\mathcal{B}$ and further let $\mathcal{I}$ be a model of $\exists Y.\mathcal{B}$, i.e., there is a variable assignment $\mathcal{Z}\colon Y \to \mathsf{Dom}(\mathcal{I})$ such that the augmented interpretation $\mathcal{I}[\mathcal{Z}]$ is a model of the matrix $\mathcal{B}$. We are going to prove that $\mathcal{I}$ is a model of $\exists X.\mathcal{A}$ as well.

We define the variable assignment $\mathcal{W}\colon X \to \mathsf{Dom}(\mathcal{I})$ where $\mathcal{W}(x) := h(x)^{\mathcal{I}[\mathcal{Z}]}$ for each $x \in X$, and first prove that $t^{\mathcal{I}[\mathcal{W}]} = h(t)^{\mathcal{I}[\mathcal{Z}]}$ holds for each object name t of $\exists X.\mathcal{A}$.

- If t is a variable, then $t^{\mathcal{I}[\mathcal{W}]} = \mathcal{W}(t) = h(t)^{\mathcal{I}[\mathcal{Z}]}$.
- If t is an individual, then $h(t) = t$ and thus $t^{\mathcal{I}[\mathcal{W}]} = t^{\mathcal{I}} = h(t)^{\mathcal{I}} = h(t)^{\mathcal{I}[\mathcal{Z}]}$.

Next, we show that the augmented interpretation $\mathcal{I}[\mathcal{W}]$ is a model of the matrix $\mathcal{A}$.

1. Consider a concept assertion $A(t)$ in $\mathcal{A}$; we must show that $t^{\mathcal{I}[\mathcal{W}]} \in A^{\mathcal{I}}$. Since h is a homomorphism, Condition (Hom3) in Definition 1 yields an object name v such that $v \approx_{\exists Y.\mathcal{B}} h(t)$ and $A(v) \in \mathcal{B}$. Since $\mathcal{I}[\mathcal{Z}]$ is a model of $\mathcal{B}$, it follows that $v^{\mathcal{I}[\mathcal{Z}]} \in A^{\mathcal{I}}$ as well as $h(t)^{\mathcal{I}[\mathcal{Z}]} = v^{\mathcal{I}[\mathcal{Z}]}$. We conclude that $t^{\mathcal{I}[\mathcal{W}]} = v^{\mathcal{I}[\mathcal{Z}]}$ and thus $t^{\mathcal{I}[\mathcal{W}]} \in A^{\mathcal{I}}$.
2. Let $r(t,u)$ be a role assertion in $\mathcal{A}$; we must show that $(t^{\mathcal{I}[\mathcal{W}]}, u^{\mathcal{I}[\mathcal{W}]}) \in r^{\mathcal{I}}$. As h is a homomorphism, we infer with Condition (Hom4) in Definition 1 that there is an object name v and there is an object name w such that $v \approx_{\exists Y.\mathcal{B}} h(t)$ and $w \approx_{\exists Y.\mathcal{B}} h(u)$ and $r(v,w) \in \mathcal{B}$. Since $\mathcal{I}[\mathcal{Z}]$ is a model of $\mathcal{B}$, we obtain that $(v^{\mathcal{I}[\mathcal{Z}]}, w^{\mathcal{I}[\mathcal{Z}]}) \in r^{\mathcal{I}}$, and further that $h(t)^{\mathcal{I}[\mathcal{Z}]} = v^{\mathcal{I}[\mathcal{Z}]}$ and $h(u)^{\mathcal{I}[\mathcal{Z}]} = w^{\mathcal{I}[\mathcal{Z}]}$. It follows that $t^{\mathcal{I}[\mathcal{W}]} = v^{\mathcal{I}[\mathcal{Z}]}$ and $u^{\mathcal{I}[\mathcal{W}]} = w^{\mathcal{I}[\mathcal{Z}]}$, and so we conclude that $(t^{\mathcal{I}[\mathcal{W}]}, u^{\mathcal{I}[\mathcal{W}]}) \in r^{\mathcal{I}}$.
3. Assume that $a \equiv b$ is an equality assertion in $\mathcal{A}$; we must show that $a^{\mathcal{I}} = b^{\mathcal{I}}$. Specifically, it holds that $a \approx_{\exists X.\mathcal{A}} b$. With h being a homomorphism, Condition (Hom1) yields that $h(a) \approx_{\exists Y.\mathcal{B}} h(b)$. Since $\mathcal{I}$ is a model of $\exists Y.\mathcal{B}$, it follows that $h(a)^{\mathcal{I}} = h(b)^{\mathcal{I}}$, and thus that $a^{\mathcal{I}} = b^{\mathcal{I}}$ by Condition (Hom2).

We continue with proving the only-if direction. For this purpose, assume that $\exists Y.\mathcal{B}$ entails $\exists X.\mathcal{A}$. We define the *canonical model* $\mathcal{I}_{\exists Y.\mathcal{B}}$ as follows.

$$\mathsf{Dom}(\mathcal{I}_{\exists Y.\mathcal{B}}) := \mathsf{Obj}(\exists Y.\mathcal{B})/{\approx}$$

$$\cdot^{\mathcal{I}_{\exists Y.\mathcal{B}}}\colon \begin{cases} a \mapsto [a]_{\approx} \\ A \mapsto \{\, [u]_{\approx} \mid A(u) \in \mathcal{B} \,\} \\ r \mapsto \{\, ([u]_{\approx}, [v]_{\approx}) \mid r(u,v) \in \mathcal{B} \,\} \end{cases}$$

The *canonical variable assignment* is $\mathcal{Z}_{\exists Y.\mathcal{B}}\colon y \mapsto [y]_{\approx}$.

We are going to show that $\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{Z}_{\exists Y.\mathcal{B}}]$ is a model of $\mathcal{B}$, which immediately implies that that $\mathcal{I}_{\exists Y.\mathcal{B}}$ is a model of $\exists Y.\mathcal{B}$. Beforehand, note that $t^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{Z}_{\exists Y.\mathcal{B}}]} = [t]_{\approx}$ for each object name t of $\exists Y.\mathcal{B}$.

- Consider a concept assertion $A(u)$ in $\mathcal{B}$. Then $[u]_{\approx} \in A^{\mathcal{I}_{\exists Y.\mathcal{B}}}$ is satisfied where $[u]_{\approx}$ equals $u^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{Z}_{\exists Y.\mathcal{B}}]}$. Thus, $\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{Z}_{\exists Y.\mathcal{B}}] \models A(u)$.
- If $\mathcal{B}$ contains a role assertion $r(u,v)$, then $([u]_{\approx}, [v]_{\approx})$ is in the extension $r^{\mathcal{I}_{\exists Y.\mathcal{B}}}$. Furthermore, we have $u^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{Z}_{\exists Y.\mathcal{B}}]} = [u]_{\approx}$ and $v^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{Z}_{\exists Y.\mathcal{B}}]} = [v]_{\approx}$. It follows that $\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{Z}_{\exists Y.\mathcal{B}}] \models r(u,v)$.
- Now let $a \equiv b$ be an equality assertion in $\mathcal{B}$. We infer that $a \approx b$ must be satisfied, i.e., $[a]_{\approx} = [b]_{\approx}$ holds. Since $a^{\mathcal{I}_{\exists Y.\mathcal{B}}} = [a]_{\approx}$ and $b^{\mathcal{I}_{\exists Y.\mathcal{B}}} = [b]_{\approx}$ are satisfied, we conclude that $\mathcal{I}_{\exists Y.\mathcal{B}} \models a \equiv b$.

We infer that $\mathcal{I}_{\exists Y.\mathcal{B}}$ is a model of $\exists X.\mathcal{A}$ as well, i.e., there exists a variable assignment $\mathcal{W}\colon X \to \mathsf{Dom}(\mathcal{I}_{\exists Y.\mathcal{B}})$ such that $\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}] \models \mathcal{A}$. We define a mapping $h\colon \mathsf{Obj}(\exists X.\mathcal{A}) \to \mathsf{Obj}(\exists Y.\mathcal{B})$ by $h(a) := a$ for each individual name a and by choosing $h(x) \in \mathcal{W}(x)$ for each variable $x \in X$. Each latter choice is possible since the value $\mathcal{W}(x)$ is an equivalence class and is thus non-empty, i.e., such a mapping h indeed exists. We first prove that $h(t) \in t^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]}$ holds for each object name t of $\exists X.\mathcal{A}$.

- If t is a variable, then $h(t) \in \mathcal{W}(t) = t^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]}$.
- If t is an individual, then $h(t) = t \in [t]_{\approx} = t^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]}$.

Next, we are going to show that h is a homomorphism.

(Hom1) Consider an individual a and an individual b such that $a \approx_{\exists X.\mathcal{A}} b$; we must show that $h(a) \approx_{\exists Y.\mathcal{B}} h(b)$. Since $\mathcal{I}_{\exists Y.\mathcal{B}}$ is a model of $\exists X.\mathcal{A}$, it holds that $a^{\mathcal{I}_{\exists Y.\mathcal{B}}} = b^{\mathcal{I}_{\exists Y.\mathcal{B}}}$. Due to $h(a) \in a^{\mathcal{I}_{\exists Y.\mathcal{B}}}$ and $h(b) \in b^{\mathcal{I}_{\exists Y.\mathcal{B}}}$, we infer that $h(a) \approx_{\exists Y.\mathcal{B}} h(b)$.

(Hom2) For each individual name a, the very definition ensures that $h(a)$ equals a.

(Hom3) Let $A(t)$ be a concept assertion in $\mathcal{A}$; we must show that there is an object v such that $v \approx_{\exists Y.\mathcal{B}} h(t)$ and $A(v) \in \mathcal{B}$. Since $\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]$ is a model of $\mathcal{A}$, it holds that $t^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]} \in A^{\mathcal{I}_{\exists Y.\mathcal{B}}}$. Due to the very definition of $\mathcal{I}_{\exists Y.\mathcal{B}}$, we infer that there is an object v in the equivalence class $t^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]}$ such that $A(v) \in \mathcal{B}$. As $h(t)$ is an element of the equivalence class $t^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]}$ as well, it follows that $v \approx_{\exists Y.\mathcal{B}} h(t)$ as needed.

(Hom4) Consider a role assertion $r(t, u)$ in $\mathcal{A}$; we must show that there is an object v and there is an object w such that $v \approx_{\exists Y.\mathcal{B}} h(t)$ and $w \approx_{\exists Y.\mathcal{B}} h(u)$ and $r(v, w) \in \mathcal{B}$. With $\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]$ being a model of $\mathcal{A}$, it follows that $(t^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]}, u^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]}) \in r^{\mathcal{I}_{\exists Y.\mathcal{B}}}$. According to the very definition of $\mathcal{I}_{\exists Y.\mathcal{B}}$, there must exist an object v in the equivalence class $t^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]}$ as well as an object w in the equivalence class $u^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]}$ such that $r(v, w) \in \mathcal{B}$. Since $h(t) \in t^{\mathcal{I}_{\exists Y.\mathcal{B}}[\mathcal{W}]}$, we infer that $v \approx_{\exists Y.\mathcal{B}} h(t)$, and it similarly follows that $w \approx_{\exists Y.\mathcal{B}} h(u)$. □

As in the case of qABoxes without equality assertions, this provides us with an NP decision procedure for entailment. NP-hardness already holds without equality assertions (Baader et al. 2020).

We often need to consider the matrix $\mathcal{A}$ of a quantified ABox $\exists X.\mathcal{A}$ alone, without the quantifier prefix. We can view $\mathcal{A}$ to be an "ordinary" ABox without quantifiers (or equivalently as a qABox with empty quantifier prefix) by extending the signature to $\Sigma \cup X$, where variables are treated as individuals. This allows us to evaluate entailment expressions like $\mathcal{A} \models C(x)$, where C is a concept description and $x \in X$, using interpretations and models for the extended signature. The following result provides us with a recursive characterization of such an entailment.

Corollary I. *Let $\exists X.\mathcal{A}$ be a quantified ABox, u an object name of $\exists X.\mathcal{A}$, and C an $\mathcal{ELROI}$ concept description. Then the matrix $\mathcal{A}$ entails the concept assertion $C(u)$ iff the following conditions are fulfilled:*

1. *For each concept name $A \in \mathsf{Conj}(C)$, there is an object name u' such that $u \approx_{\exists X.\mathcal{A}} u'$ and $A(u') \in \mathcal{A}$.*
2. *For each nominal $\{a\} \in \mathsf{Conj}(C)$, it holds that $u \approx_{\exists X.\mathcal{A}} a$.*
3. *For each existential restriction $\exists R.D \in \mathsf{Conj}(C)$, there are object names u', v such that $u \approx_{\exists X.\mathcal{A}} u'$, $R(u', v) \in \mathcal{A}$, and $\mathcal{A} \models D(v)$.*

Furthermore, $\mathcal{A}$ entails the role assertion $r(u, v)$ iff there are object names u', v' such that $u \approx_{\exists X.\mathcal{A}} u'$, $v \approx_{\exists X.\mathcal{A}} v'$, and $r(u', v') \in \mathcal{A}$. Finally, $\mathcal{A}$ entails the equality assertion $u \equiv v$ iff $u \approx_{\exists X.\mathcal{A}} v$.

Taking into account that each $\mathcal{ELROI}$ concept assertion $C(a)$ can be translated into an equivalent qABox, by exhaustively applying the first three rules in Figure 1 to $\{C(a)\}$, this corollary is an easy consequence of Proposition 2. Similarly, $\exists\{x\}.\{C(x)\}$ (where C is an $\mathcal{ELROI}$ concept description) denotes the qABox obtained from it by exhaustive application of these three rules.

By induction on C we further obtain the following lemma.

Conjunction Rule. If $\mathcal{B}$ contains the assertion $(C_1 \sqcap \cdots \sqcap C_n)(t)$ for $n \neq 1$, then remove it from $\mathcal{B}$ and add the assertions $C_1(t), \ldots, C_n(t)$ to $\mathcal{B}$.

Existential Restriction Rule. If $\mathcal{B}$ contains the assertion $\exists R.C(t)$, then remove it from $\mathcal{B}$, add a fresh variable y to Y, and add the assertions $R(t, y)$ and $C(y)$ to $\mathcal{B}$.

Nominal Rule. If $\mathcal{B}$ contains the assertion $\{a\}(t)$, then remove it from $\mathcal{B}$ and, if t is an individual name, then add the equality $t \equiv a$ to $\mathcal{B}$; otherwise replace every occurrence of t in $\mathcal{B}$ by a and remove t from Y.

Concept Inclusion Rule. If $\mathcal{T}$ contains the CI $C \sqsubseteq D$ and $\mathcal{B}$ entails the concept assertion $C(t)$, but not $D(t)$, then add the concept assertion $D(t)$ to $\mathcal{B}$.

Role Inclusion Rule. If $\mathcal{R}$ contains the RI $R_1 \circ \cdots \circ R_n \sqsubseteq S$ and $\mathcal{B}$ entails the role assertions $R_1(t_0, t_1), \ldots, R_n(t_{n-1}, t_n)$, but not $S(t_0, t_n)$, then add the role assertion $S(t_0, t_n)$ to $\mathcal{B}$.

Figure 1: The saturation rules are exhaustively applied to a qABox $\exists Y.\mathcal{B}$ w.r.t. a terminology $(\mathcal{T}, \mathcal{R})$, starting with $\exists Y.\mathcal{B} := \exists X.\mathcal{A}$ for an input qABox $\exists X.\mathcal{A}$.

Lemma II. *Consider qABoxes $\exists X.\mathcal{A}$, $\exists Y.\mathcal{B}$, an object name u of $\exists X.\mathcal{A}$, and an $\mathcal{ELROI}$ concept C. If $\mathcal{A} \models C(u)$ and h is a homomorphism from $\exists X.\mathcal{A}$ to $\exists Y.\mathcal{B}$, then $\mathcal{B} \models C(h(u))$.*

2.3 Saturation

The purpose of saturation is to extend a given qABox $\exists X.\mathcal{A}$ with enough consequences derived using the terminology $(\mathcal{T}, \mathcal{R})$ such that entailment from $\exists X.\mathcal{A}$ w.r.t. $(\mathcal{T}, \mathcal{R})$ is the same as entailment from its saturation $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A}) = \exists Y.\mathcal{B}$ w.r.t. the empty terminology. The rules in Figure 1 extend the CQ-saturation rules in (Baader et al. 2021a) such that nominals, inverse roles, and RIs are taken into account. Note that, during saturation, the matrix $\mathcal{B}$ may contain complex concept assertions, but after termination all concept assertions are again restricted to concept names. The semantics of qABoxes with complex concept assertions is defined in the obvious way.

In general, application of the saturation rules need not terminate, already in the $\mathcal{EL}$ setting considered in (Baader et al. 2021a). But there the restriction to cycle-restricted TBoxes guarantees termination (in exponential time), where an $\mathcal{EL}$ TBox $\mathcal{T}$ is cycle-restricted if there is no concept C and roles $r_1, \ldots, r_n$ ($n \geq 1$) such that $C \sqsubseteq_{\mathcal{T}} \exists r_1. \cdots \exists r_n.C$. For $\mathcal{ELROI}$ terminologies, the RBox may cause non-termination even if the TBox is cycle-restricted.

Example 3. Consider the $\mathcal{ELROI}$ TBox $\mathcal{T} := \{A \sqsubseteq \exists r.\top, \exists s.\top \sqsubseteq \exists s.A\}$, the RBox $\mathcal{R} := \{r \sqsubseteq s\}$, and the qABox $\exists\emptyset.\mathcal{A}$ with $\mathcal{A} := \{A(a)\}$. The TBox $\mathcal{T}$ is cycle-restricted and saturation of $\exists\emptyset.\mathcal{A}$ with $(\mathcal{T}, \emptyset)$ terminates after a has received an r-successor x_1. However, w.r.t. $(\mathcal{T}, \mathcal{R})$, the role inclusion rule makes x_1 also an s-successor of a. The concept inclusion rule then adds an s-successor y_1 of a and the assertion $A(y_1)$. But now y_1 receives an r-successor x_2, which becomes an s-successor of y_1, etc.

Since our repair approach works on saturated qABoxes, it can only be applied in the presence of terminologies $(\mathcal{T}, \mathcal{R})$ that are terminating in the following sense.

Definition 4. The terminology $(\mathcal{T}, \mathcal{R})$ is *terminating* if, for each qABox $\exists X.\mathcal{A}$, there is a finite sequence of applications of the saturation rules in Figure 1 to $\exists X.\mathcal{A}$ resulting in a qABox to which no more rule applies. We then denote this qABox as $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$ and call it the *saturation* of $\exists X.\mathcal{A}$ w.r.t. $(\mathcal{T}, \mathcal{R})$.

We refrain here from giving our own decidable sufficient condition for termination of a terminology $(\mathcal{T}, \mathcal{R})$. Instead, we point out that one can translate the concept inclusions in $\mathcal{T}$ and the role inclusions in $\mathcal{R}$ into a set of existential rules, and that saturation then corresponds to applying the so-called chase. One can thus try to use one of the numerous acyclicity conditions guaranteeing chase termination proposed in the database and rules communities (see, e.g., (Grau et al. 2013)) to show termination of $(\mathcal{T}, \mathcal{R})$. The saturation obtained in case of termination has the following important property.

Theorem 5. *Let $(\mathcal{T}, \mathcal{R})$ be a terminating terminology and $\exists X.\mathcal{A}$ a quantified ABox. Then, for every qABox $\exists Z.\mathcal{C}$, the following statements are equivalent:*

1. $\exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists Z.\mathcal{C}$.
2. $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A}) \models \exists Z.\mathcal{C}$.
3. *There is a homomorphism from $\exists Z.\mathcal{C}$ to $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$.*

Proof. The proof is similar to the one of Theorem 2 in (Baader et al. 2021a; Baader et al. 2021b), but uses Proposition 2 instead of Proposition 2 in (Baader et al. 2020). Additionally, we provide a direct proof below.

The equivalence of Statements 2 and 3 follows from Proposition 2. We now show that Statement 2 implies Statement 1. So consider a model $\mathcal{I}$ of $\exists X.\mathcal{A}$ and $(\mathcal{T}, \mathcal{R})$. By an induction along the sequence of rule applications that produces $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$ from $\exists X.\mathcal{A}$, it can be proven that $\mathcal{I}$ is a model of each intermediate qABox and so of the final saturation too. Using the assumption $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A}) \models \exists Z.\mathcal{C}$, it follows that $\mathcal{I}$ is a model of $\exists Z.\mathcal{C}$.

It remains to show that Statement 1 implies Statement 2. For this purpose, assume that $\exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists Z.\mathcal{C}$. Recall that the canonical model $\mathcal{I}_{\exists Y.\mathcal{B}}$ of a qABox $\exists Y.\mathcal{B}$ is defined within the proof of Proposition 2, and it is shown that $\exists Y.\mathcal{B}$ entails $\exists X.\mathcal{A}$ iff $\mathcal{I}_{\exists Y.\mathcal{B}}$ is a model of $\exists X.\mathcal{A}$. The latter is useful for this proof.

Specifically, we will show that the canonical model of the saturation $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$ is a model of $\exists X.\mathcal{A}$ and $(\mathcal{T}, \mathcal{R})$. Due to the assumption, it is also a model of $\exists Z.\mathcal{C}$, and we conclude that $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$ entails $\exists Z.\mathcal{C}$.

Denote by $\mathcal{I}$ the canonical model of $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$, which is a model of $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$. Theorem 5 yields that $\exists X.\mathcal{A}$ is entailed by the saturation, and so $\mathcal{I}$ is also a model of $\exists X.\mathcal{A}$.

In order to prove that $\mathcal{I}$ is a model of the terminology $(\mathcal{T}, \mathcal{R})$, we additionally consider the canonical variable assignment $\mathcal{Z}$ of $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$, cf. the proof of Proposition 2. Then the augmented interpretation $\mathcal{I}[\mathcal{Z}]$ is the canonical model of the matrix of $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$, and $u^{\mathcal{I}[\mathcal{Z}]} = [u]_\approx$ holds for each object u of the saturation.

Let $C \sqsubseteq D$ be a concept inclusion in $\mathcal{T}$ and assume $[u]_\approx \in C^\mathcal{I}$. It follows that $u^{\mathcal{I}[\mathcal{Z}]} \in C^\mathcal{I}$, i.e., $\mathcal{I}[\mathcal{Z}]$ is a model of $C(u)$ (seen as a qABox). We infer that the matrix of $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$ entails $C(u)$. Since the concept inclusion rule is not applicable, the matrix of $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$ entails $D(u)$ as well. So $\mathcal{I}[\mathcal{Z}]$ is a model of $D(u)$, which yields that $[u]_\approx \in D^\mathcal{I}$.

Let $R_1 \circ \cdots \circ R_n \sqsubseteq S$ be a role inclusion in $\mathcal{R}$, and assume $([u_0]_\approx, [u_1]_\approx) \in R_1^\mathcal{I}, \ldots, ([u_{n-1}]_\approx, [u_n]_\approx) \in R_n^\mathcal{I}$. So $\mathcal{I}[\mathcal{Z}]$ is a model of the qABox $\{R_1(u_0, u_1), \ldots, R_n(u_{n-1}, u_n)\}$, and we infer that the latter qABox is entailed by the matrix of $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$. Since the role inclusion rule is not applicable, the matrix of $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$ must also entail $S(u_0, u_n)$. It follows that $\mathcal{I}[\mathcal{Z}]$ is a model of $S(u_0, u_n)$, i.e., $([u_0]_\approx, [u_n]_\approx) \in S^\mathcal{I}$. □

In (Baader et al. 2021a), a different kind of saturation, called IQ-saturation, was introduced, which always terminates (in polynomial time). Using IQ-saturation in the repair process was shown to be sufficient if one is only interested in instance queries. However, due to the presence of inverse roles in $\mathcal{ELROI}$, it is easy to see that finite IQ-saturations cannot always work.

Example III. Consider the qABox $\{A(a)\}$ and the TBox $\{A \sqsubseteq \exists r.A\}$. In $\mathcal{EL}$, the IQ-saturation is $\exists\{x\}.\{A(a), r(a, x), A(x), r(x, x)\}$. In $\mathcal{ELROI}$, however, it cannot be the IQ-saturation since it entails the $\mathcal{ELROI}$ concept assertion $\exists r.\exists r^-.A(a)$ that is not entailed by the given qABox and TBox.

Moreover, assume that $\exists X.\mathcal{A}$ was a finite IQ-saturation in $\mathcal{ELROI}$. It would need to entail the concept assertion $\exists r^n.A(a)$ for each $n \geq 0$, but could not contain an infinite r-chain starting from a. So there would be an r-cycle reachable from a on an r-path, but thus $\exists X.\mathcal{A}$ would entail the concept assertion $\exists r^m.\exists r^-.A(a)$ for some $m \geq 0$, which yields a contradiction since this assertion is not entailed by the above qABox and TBox.

2.4 Regular RBoxes

As pointed out at the end of Section 2.1, subsumption is undecidable in $\mathcal{ELROI}$ if arbitrary RBoxes are allowed. In (Baader, Lutz, and Brandt 2008), tractability of $\mathcal{EL}^{++}$ is ensured by restricting the interaction between range restrictions and RIs. Since, in our setting, range restrictions are expressed using inverse roles and CIs, it is not clear how to adapt this solution. Instead, we use the regularity restriction imposed in (Horrocks, Kutz, and Sattler 2006; Kazakov 2008) to make $\mathcal{SROIQ}$ decidable, which is required by our repair approach anyway.

Definition 6. An RBox $\mathcal{R}$ is *regular* if, for each role R, the language $L_\mathcal{R}(R) \coloneqq \{\, S_1 \cdots S_n \mid S_1 \circ \cdots \circ S_n \sqsubseteq^\mathcal{R} R \,\}$ is regular. The sublogic of $\mathcal{ELROI}$ that only supports regular RBoxes is denoted by $\mathcal{ELR}_{\mathsf{reg}}\mathcal{OI}$.

Since $\mathcal{ELR}_{\mathsf{reg}}\mathcal{OI}$ is a fragment of Horn-$\mathcal{SROIQ}$, it inherits the complexity upper-bound of 2ExpTime (Ortiz, Rudolph, and Šimkus 2010). The exact complexity of subsumption in $\mathcal{ELR}_{\mathsf{reg}}\mathcal{OI}$ is open, with the best lower-bound of ExpTime inherited from $\mathcal{ELI}$ (Baader, Lutz, and Brandt 2008). Additionally, the former implies that $\mathcal{ELR}_{\mathsf{reg}}\mathcal{OI}$ is itself a Horn-DL and thus has the *universal model property*: for each quantified ABox $\exists X.\mathcal{A}$ and for each (not necessarily terminating) terminology $(\mathcal{T}, \mathcal{R})$, there is an interpretation (the universal model) that is a model of exactly those quantified ABoxes that are entailed by $\exists X.\mathcal{A}$ and $(\mathcal{T}, \mathcal{R})$.

To the best of our knowledge, it is not known whether RBox regularity is decidable. Decidability of the closely related regularity problem for pure context-free grammars has been open for a long time (Maurer, Salomaa, and Wood 1980). The below lemma specifically shows that each language $L_{\mathcal{R}}(R)$ can be described by such a pure grammar. However, there exist syntactic restrictions that guarantee regularity (Horrocks, Kutz, and Sattler 2006; Kazakov 2010), and if these restrictions apply then one can effectively construct (exponentially large) finite automata accepting the regular languages $L_{\mathcal{R}}(R)$.

Lemma IV. *Let $\mathcal{R}$ be an RBox and consider the pure grammar $\mathfrak{G}_{\mathcal{R}}$ over the alphabet $\Sigma_{\mathsf{R}}^{\pm}$ with production rules*

$$\left\{ \begin{array}{r} R \to S_1 \cdots S_n, \\ R^- \to S_n^- \cdots S_1^- \end{array} \;\middle|\; S_1 \circ \cdots \circ S_n \sqsubseteq R \in \mathcal{R} \right\}.$$

It holds that $S_1 \circ \cdots \circ S_n \sqsubseteq^{\mathcal{R}} R$ iff $R \xrightarrow{*}_{\mathfrak{G}_{\mathcal{R}}} S_1 \cdots S_n$.

Proof. We start with the if direction. So assume that $R \xrightarrow{*} S_1 \cdots S_n$, i.e., there is a finite sequence of applications of the production rules that generates $S_1 \cdots S_n$ from R. This means that, for $W_0 \coloneqq R$ and $W_m \coloneqq S_1 \cdots S_n$, there are words $W_1, \cdots, W_{m-1} \in (\Sigma_{\mathsf{R}}^{\pm})^*$ such that, for each index $i \in \{1, \ldots, n\}$, there is a production rule $R^i \to S_1^i \cdots S_{n_i}^i$ and there are words $W_{i-1}^{\leftarrow}, W_{i-1}^{\rightarrow} \in (\Sigma_{\mathsf{R}}^{\pm})^*$ such that $W_{i-1} = W_{i-1}^{\leftarrow} R^i W_{i-1}^{\rightarrow}$ and $W_i = W_{i-1}^{\leftarrow} S_1^i \cdots S_{n_i}^i W_{i-1}^{\rightarrow}$.

Consider some index $i \in \{1, \ldots, n\}$. The production rule $R^i \to S_1^i \cdots S_{n_i}^i$ is used to derive W_i from W_{i-1}. Thus, the RBox $\mathcal{R}$ contains the role inclusion $S_1^i \circ \cdots \circ S_{n_i}^i \sqsubseteq R^i$ or it contains the role inclusion $(S_{n_i}^i)^- \circ \cdots \circ (S_1^i)^- \sqsubseteq (R^i)^-$. In both cases it follows that $\mathcal{R}$ entails the former RI and thus also entails the RI $\bigcirc W_i \sqsubseteq \bigcirc W_{i-1}$,[2] where we write $\bigcirc R_1 \cdots R_k$ for $R_1 \circ \cdots \circ R_k$ for roles $R_1, \ldots, R_k \in \Sigma_{\mathsf{R}}^{\pm}$.

By induction over $i \in \{1, \ldots, n\}$ we conclude that $\mathcal{R}$ entails the RI $\bigcirc W_n \sqsubseteq \bigcirc W_0$, which equals $S_1 \circ \cdots \circ S_n \sqsubseteq R$.

We proceed with proving the only-if direction. For this purpose, assume that $S_1 \circ \cdots \circ S_n \sqsubseteq R$ is entailed by $\mathcal{R}$. We construct a sequence of interpretations as follows.

1. Initialize $\mathcal{I}_0$ with $\mathsf{Dom}(\mathcal{I}_0) \coloneqq \{0, 1, \ldots, n\}$ and let all extensions be empty except that $S_i^{\mathcal{I}_0} \coloneqq \{(i-1, i)\}$ for each index $i \in \{1, \ldots, n\}$.

$$\begin{tikzcd} 0 \arrow[r, "S_1"] & 1 \arrow[r, "S_2"] & 2 \arrow[r, "S_3"] & \cdots \arrow[r, "S_n"] & n \end{tikzcd}$$

2. Saturate the sequence by means of the role inclusions in $\mathcal{R}$, that is, exhaustively apply the following rule:

   If $U_1 \circ \cdots \circ U_m \sqsubseteq T$ is a role inclusion in $\mathcal{R}$, and $d_0, \ldots, d_m \in \mathsf{Dom}(\mathcal{I}_k)$ such that $(d_{i-1}, d_i) \in U_i^{\mathcal{I}_k}$ for each index $i \in \{1, \ldots, m\}$, but $(d_0, d_m) \notin T^{\mathcal{I}_k}$,
   then define the next interpretation $\mathcal{I}_{k+1}$ as $\mathcal{I}_k$, but add (d_0, d_m) to $T^{\mathcal{I}_{k+1}}$.

$$\begin{tikzcd} d_0 \arrow[r, "U_1"] \arrow[rrrr, dashed, bend right, "T"'] & d_1 \arrow[r, "U_2"] & d_2 \arrow[r, "U_3"] & \cdots \arrow[r, "U_m"] & d_m \end{tikzcd}$$

Since Instruction 2 does not introduce fresh domain elements, the saturation must be finished after finitely many steps, say ℓ steps, i.e., the above rule is not applicable anymore to $\mathcal{I}_\ell$. Furthermore, we thus simply write $\mathsf{Dom}(\mathcal{I})$ for the domain of each interpretation in the sequence. Due to Instruction 1 we have that $(0, n) \in (S_1 \circ \cdots \circ S_n)^{\mathcal{I}_0}$, and due to Instruction 2 the last interpretation $\mathcal{I}_\ell$ is a model of the RBox $\mathcal{R}$. By construction of the sequence $\mathcal{I}_0, \ldots, \mathcal{I}_\ell$ we further have $S^{\mathcal{I}_k} \subseteq S^{\mathcal{I}_{k+1}}$ for each role $S \in \Sigma_{\mathsf{R}}^{\pm}$ and for each index $k \in \{0, \ldots, \ell - 1\}$. It follows that $(0, n) \in (S_1 \circ \cdots \circ S_n)^{\mathcal{I}_\ell}$ and $(0, n) \in R^{\mathcal{I}_\ell}$.

For each index $k \in \{0, \ldots, \ell\}$, we define the language

$$\mathsf{Words}(\mathcal{I}_k) \coloneqq \left\{ R_1 \cdots R_h \;\middle|\; \begin{array}{l} R_1, \ldots, R_h \in \Sigma_{\mathsf{R}}^{\pm} \text{ and} \\ (0, n) \in (R_1 \circ \cdots \circ R_h)^{\mathcal{I}_k} \end{array} \right\}.$$

Note that $\mathsf{Words}(\mathcal{I}_0) = \{S_1 \cdots S_n\}$ and $R \in \mathsf{Words}(\mathcal{I}_\ell)$.

Claim. Assume that $U_1 \circ \cdots \circ U_m \sqsubseteq T$ is the role inclusion used to construct $\mathcal{I}_{k+1}$ from $\mathcal{I}_k$. For each $W \in \mathsf{Words}(\mathcal{I}_{k+1})$, there is some $V \in \mathsf{Words}(\mathcal{I}_k)$ such that V can be obtained from W by a finite number of applications of the production rules $T \to U_1 \cdots U_m$ and $T^- \to U_m^- \cdots U_1^-$.

Proof of the claim. Consider a word $W = R_1 \cdots R_h \in \mathsf{Words}(\mathcal{I}_{k+1})$, i.e., there are domain elements $e_0, \ldots, e_h \in \mathsf{Dom}(\mathcal{I})$ such that $e_0 = 0$, $e_h = n$, and $(e_{j-1}, e_j) \in R_j^{\mathcal{I}_{k+1}}$ for each index $j \in \{1, \ldots, h\}$.

For the saturation step which produces $\mathcal{I}_{k+1}$ from $\mathcal{I}_k$ by means of the role inclusion $U_1 \circ \cdots \circ U_m \sqsubseteq T$, there must exist domain elements $d_0, \ldots, d_m \in \mathsf{Dom}(\mathcal{I})$ such that $(d_{j-1}, d_j) \in U_j^{\mathcal{I}_k}$ for each index $j \in \{1, \ldots, m\}$, and the extensions of all roles do not differ between $\mathcal{I}_k$ and $\mathcal{I}_{k+1}$, with the exception $T^{\mathcal{I}_{k+1}} = T^{\mathcal{I}_k} \uplus \{(d_0, d_m)\}$. It follows that $(d_0, d_m) \in (U_1 \circ \cdots \circ U_m)^{\mathcal{I}_k}$ and further that $(d_m, d_0) \in (U_m^- \circ \cdots \circ U_1^-)^{\mathcal{I}_k}$.

Now define the word $V \coloneqq V_1 \cdots V_h$ where

$$V_j \coloneqq \begin{cases} U_1 \cdots U_m & \text{if } R_j = T \text{ and } (e_{j-1}, e_j) = (d_0, d_m) \\ U_m^- \cdots U_1^- & \text{if } R_j = T^- \text{ and } (e_{j-1}, e_j) = (d_m, d_0) \\ R_j & \text{otherwise} \end{cases}$$

for each index $j \in \{1, \ldots, h\}$. By construction, V can be obtained from W by a finite number of applications of the production rules $T \to U_1 \cdots U_m$ and $T^- \to U_m^- \cdots U_1^-$

[2] Formally, we say that $\mathcal{R}$ entails $R_1 \circ \cdots \circ R_m \sqsubseteq S_1 \circ \cdots \circ S_n$ if $(R_1 \circ \cdots \circ R_m)^{\mathcal{I}} \subseteq (S_1 \circ \cdots \circ S_n)^{\mathcal{I}}$ for each model $\mathcal{I}$ of $\mathcal{R}$.

(specifically, the number of rule applications is bounded by h). Next, we show that $(e_{j-1}, e_j) \in (\bigcirc V_j)^{\mathcal{I}_k}$ for each index $j \in \{1, \ldots, h\}$.

(a) Let $R_j = T$ and $(e_{j-1}, e_j) = (d_0, d_m)$, i.e., $V_j = U_1 \cdots U_m$. Since $(d_0, d_m) \in (U_1 \circ \cdots \circ U_m)^{\mathcal{I}_k}$, we infer that $(e_{j-1}, e_j) \in (\bigcirc V_j)^{\mathcal{I}_k}$.

(b) Assume $R_j = T^-$ and $(e_{j-1}, e_j) = (d_m, d_0)$, i.e., $V_j = U_m^- \cdots U_1^-$. From $(d_m, d_0) \in (U_m^- \circ \cdots \circ U_1^-)^{\mathcal{I}_k}$ it follows that $(e_{j-1}, e_j) \in (\bigcirc V_j)^{\mathcal{I}_k}$.

(c) In the remaining case, we have $V_j = R_j$, and $R_j \neq T$ or $(e_{j-1}, e_j) \neq (d_0, d_m)$, and $R_j \neq T^-$ or $(e_{j-1}, e_j) \neq (d_m, d_0)$.
   (i) If $R_j \neq T$ and $R_j \neq T^-$, then $R_j^{\mathcal{I}_k} = R_j^{\mathcal{I}_{k+1}}$ and thus $(e_{j-1}, e_j) \in R_j^{\mathcal{I}_{k+1}}$ implies $(e_{j-1}, e_j) \in (\bigcirc V_j)^{\mathcal{I}_k}$.
   (ii) Now let $R_j \neq T$, $R_j = T^-$, and $(e_{j-1}, e_j) \neq (d_m, d_0)$. From $R_j = T^-$ and $T^{\mathcal{I}_{k+1}} = T^{\mathcal{I}_k} \uplus \{(d_0, d_m)\}$ it follows that $R_j^{\mathcal{I}_{k+1}} = R_j^{\mathcal{I}_k} \uplus \{(d_m, d_0)\}$. Since $(e_{j-1}, e_j) \in R_j^{\mathcal{I}_{k+1}}$ and $(e_{j-1}, e_j) \neq (d_m, d_0)$ we infer that $(e_{j-1}, e_j) \in R_j^{\mathcal{I}_k}$, and thus $(e_{j-1}, e_j) \in (\bigcirc V_j)^{\mathcal{I}_k}$.
   (iii) The case where $R_j = T$, $R_j \neq T^-$, and $(e_{j-1}, e_j) \neq (d_0, d_m)$ is similar to the last case.
   (iv) The remaining case where $R_j = T$ and $R_j = T^-$ is impossible.

We conclude by induction that $(0, n) = (e_0, e_h) \in (\bigcirc V)^{\mathcal{I}_k}$, i.e., $V \in \mathsf{Words}(\mathcal{I}_k)$. □

Since $R \in \mathsf{Words}(\mathcal{I}_\ell)$ and $\mathsf{Words}(\mathcal{I}_0) = \{S_1 \cdots S_n\}$, it follows from the above claim by induction that $S_1 \cdots S_n$ can be obtained from R by a finite number of applications of the production rules. □

Let $\mathcal{R}$ be a regular RBox, and for each role R, let $\mathfrak{A}_R = (Q_R, \Sigma_{\mathsf{R}}^{\pm}, i_R, \Delta_R, F_R)$ be a finite automaton (with set of states Q_R, the alphabet $\Sigma_{\mathsf{R}}^{\pm}$ of all roles, initial state i_R, transition relation Δ_R, and set of final states F_R) accepting $L_{\mathcal{R}}(R)$, i.e., such that $L(\mathfrak{A}_R) = L_{\mathcal{R}}(R)$. We assume without loss of generality (but in the worst-case paid for by another exponential blowup) that each automaton $\mathfrak{A}_R$ is deterministic, i.e., for each state q and role S, there is at most one state p such that $(q, S, p) \in \Delta_R$. In addition, we assume that $\mathfrak{A}_R$ does not contain states that are unreachable from the initial state (*only reachable states*) or from which no final state can be reached (*no dead states*), and further that the sets Q_R for different R are pairwise disjoint and are all disjoint with the signature Σ. Specifically, determinacy of the automata is needed for technical reasons when we are later concerned with constructing repairs, cf. Lemma XXVI. For each state $q \in Q_R$, the automaton $\mathfrak{A}_{\mathcal{R}}(q) := (Q_R, \Sigma_{\mathsf{R}}^{\pm}, q, \Delta_R, F_R)$ is obtained from $\mathfrak{A}_R$ by replacing the initial state i_R with q. We will use existential restrictions of the form $\exists q.C$ for $q \in Q_R$ as abbreviations for the (possibly infinite) disjunction $\bigsqcup\{\, \exists S_1.\cdots\exists S_n.C \mid S_1 \cdots S_n \in L(\mathfrak{A}_{\mathcal{R}}(q)) \,\}$, i.e., in each interpretation $\mathcal{I}$, $(\exists q.C)^{\mathcal{I}}$ is defined to be

$$\bigcup\{\, (\exists S_1.\cdots\exists S_n.C)^{\mathcal{I}} \mid S_1 \cdots S_n \in L(\mathfrak{A}_{\mathcal{R}}(q)) \,\}.$$

If q is a final state, then $(\exists q.C)^{\mathcal{I}}$ equals

$$C^{\mathcal{I}} \cup \bigcup\{\, (\exists R.\exists p.C)^{\mathcal{I}} \mid (q, R, p) \text{ is a transition} \,\},$$

and otherwise $\bigcup\{\, (\exists R.\exists p.C)^{\mathcal{I}} \mid (q, R, p) \text{ is a transition} \,\}$. Entailment for such existential restrictions can be characterized as follows.

Lemma 7. *Given a qABox $\exists X.\mathcal{A}$, an object t of it, a terminology $(\mathcal{T}, \mathcal{R})$ with regular $\mathcal{R}$, $\mathcal{ELROI}$ concept descriptions C, D, and a state q. Then the following holds:*

1. *$\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists q.C(t)$ iff there is a word $S_1 \cdots S_n \in L(\mathfrak{A}_{\mathcal{R}}(q))$ such that $\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists S_1.\cdots\exists S_n.C(t)$,*
2. *$D \sqsubseteq^{\mathcal{T},\mathcal{R}} \exists q.C$ iff there is a word $S_1 \cdots S_n \in L(\mathfrak{A}_{\mathcal{R}}(q))$ such that $D \sqsubseteq^{\mathcal{T},\mathcal{R}} \exists S_1.\cdots\exists S_n.C$.*

Proof. The first claim follows easily by using the universal model of $\exists X.\mathcal{A}$ and $(\mathcal{T}, \mathcal{R})$ (Ortiz, Rudolph, and Šimkus 2011). The second reasoning problem can be transformed into the first: it is easy to verify that $D \sqsubseteq^{\mathcal{T},\mathcal{R}} \exists q.C$ iff $\{D(a)\} \models^{\mathcal{T},\mathcal{R}} \exists q.C(a)$, where a is an individual not occurring in C, D, or $\mathcal{T}$. □

If the terminology is terminating, we can decide whether the conditions for entailment stated in Lemma 7 hold. Basically, to check whether $\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists q.C(t)$ holds, we simply need to find an accepting run of the automaton $\mathfrak{A}_{\mathcal{R}}(q)$ such that the accepted word corresponds to a path in the saturation $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$ that starts with t and ends with an instance of C. This boils down to a reachability test in the product of the automaton with the saturation.

Proposition V. *Let $(\mathcal{T}, \mathcal{R})$ be terminating and consider a state $q \in Q_R$. It holds that $\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists q.C(t)$ iff $L(\mathfrak{B}) \neq \emptyset$ where the finite automaton $\mathfrak{B} := (Q, \Sigma_{\mathsf{R}}^{\pm}, i, \Delta, F)$ has the following components:*

$Q := \mathsf{Obj}(\exists Y.\mathcal{B}) \times Q_R$
$i := (t, i_R)$
$\Delta := \{\, ((u, q), R, (v, p)) \mid R(u, v) \in \mathcal{B} \textit{ and } (q, R, p) \in \Delta_R \,\}$
$F := \{\, (u, f) \mid \mathcal{B} \models C(u) \textit{ and } f \in F_R \,\}$

where $\exists Y.\mathcal{B} := \mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$.

Next, we show that the existential restrictions $\exists R.C$ and $\exists i_R.C$ have the same behavior if the whole terminology is taken into account.

Lemma VI. *$\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists R.C(t)$ iff $\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists i_R.C(t)$.*

Proof. The only-if direction follows from $\exists R.C \sqsubseteq^{\emptyset} \exists i_R.C$. Regarding the if direction, let $\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists i_R.C(t)$. By Lemma 7 there is a word $S_1 \cdots S_n \in L(\mathfrak{A}_{\mathcal{R}}(i_R))$ such that $\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists S_1.\cdots\exists S_n.C(t)$. Since $\mathcal{A}_{\mathcal{R}}(i_R) = \mathfrak{A}_R$, it follows that $S_1 \circ \cdots \circ S_n \sqsubseteq^{\mathcal{R}} R$, and so we conclude that $\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists R.C(t)$. □

What's more, it is possible to prove that $\mathcal{A} \models^{\mathcal{R}} \exists R.C(t)$ iff $\mathcal{A} \models \exists i_R.C(t)$, but this is not needed for our purposes.

Given a state $p \in Q_R$ and a state $q \in Q_R$, we write $p \leq q$ if $L(\mathfrak{A}_{\mathcal{R}}(p)) \subseteq L(\mathfrak{A}_{\mathcal{R}}(q))$. The next lemma is easy to prove.

Lemma VII. *The following statements hold:*

1. *If $\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists p.C(t)$ and $p \leq q$, then $\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists q.C(t)$.*
2. *If $D \sqsubseteq^{\mathcal{T},\mathcal{R}} \exists p.C$ and $p \leq q$, then $D \sqsubseteq^{\mathcal{T},\mathcal{R}} \exists q.C$.*

3 Optimal and Canonical Repairs

In this section, we first extend the notion of an (optimal) repair, as introduced in (Baader et al. 2021a), to the more expressive DL $\mathcal{ELROI}$ and a setting where the repair request, which describes which consequences are to be removed, also contains global unwanted consequences. For regular sets of role inclusions, we show that every repair is entailed by a repair containing a bounded number of individuals. From this, we derive that the set of optimal repairs can effectively be computed and covers all repairs. Then, we extend the construction of canonical repairs of (Baader et al. 2021a) from $\mathcal{EL}$ to $\mathcal{ELR}_{\mathsf{reg}}\mathcal{OI}$. The set of canonical repairs can effectively be computed, covers all repairs and thus contains all optimal repairs, and a repair seed determining such a canonical repair can be chosen by answering a polynomial number of instance queries. Throughout the section, we assume (unless specified otherwise) that $\exists X.\mathcal{A}$ is a quantified ABox, $\mathcal{T}$ an $\mathcal{ELROI}$ TBox, $\mathcal{R}$ a regular RBox, all defined over the same signature Σ, and that $(\mathcal{T},\mathcal{R})$ is terminating.

Definition 8. A *repair request* $\mathcal{P}$ is a union of a finite set $\mathcal{P}_{\mathsf{loc}}$ of $\mathcal{ELROI}$ concept assertions, the *local* request, and of a finite set $\mathcal{P}_{\mathsf{glo}}$ of $\mathcal{ELROI}$ concept descriptions, the *global* request. A *repair* of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ is a quantified ABox $\exists Y.\mathcal{B}$ that fulfills the following properties:

(Rep1) $\exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists Y.\mathcal{B}$,

(Rep2) $\exists Y.\mathcal{B} \not\models^{\mathcal{T},\mathcal{R}} C(a)$ for each $C(a) \in \mathcal{P}_{\mathsf{loc}}$,

(Rep3) $\exists Y.\mathcal{B} \not\models^{\mathcal{T},\mathcal{R}} \exists\{x\}.\{D(x)\}$ for each $D \in \mathcal{P}_{\mathsf{glo}}$.

This repair is *optimal* if there is no repair $\exists Z.\mathcal{C}$ such that $\exists Z.\mathcal{C} \models^{\mathcal{T},\mathcal{R}} \exists Y.\mathcal{B}$, but $\exists Y.\mathcal{B} \not\models^{\mathcal{T},\mathcal{R}} \exists Z.\mathcal{C}$. We say that a set of repairs $\mathfrak{S}$ *covers* all repairs if every repair is entailed w.r.t. $(\mathcal{T},\mathcal{R})$ by a repair in $\mathfrak{S}$.

Obviously, $\exists X.\mathcal{A}$ has a repair for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ iff the terminology alone does not imply any of the unwanted consequences in $\mathcal{P}$, since then the empty ABox is a repair. The restriction to terminating terminologies and regular RBoxes is needed to ensure that any repair problem has a finite set of optimal repairs covering all repairs. The proof of Proposition 2 in (Baader et al. 2018) contains an example with non-terminating terminology where there is no optimal repair, though there is a repair. However, in this proof it is only shown that there cannot be an optimal repair that is an ABox. While this proof can be adapted to deal also with qABoxes, we present here a modified example with exactly one optimal repair, which however does not cover all repairs.

Example 9. Assume that $\mathcal{T} := \{A \sqsubseteq \exists r.A,\ \exists r.A \sqsubseteq A\}$, $\mathcal{R} := \emptyset$, $\mathcal{A} := \{A(a), B(a)\}$, and $\mathcal{P} := \{(A \sqcap B)(a)\}$. Then $\exists\{x\}.\{A(a), A(x), B(x)\}$ is an optimal repair of $\exists\emptyset.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$. However, there are also repairs in which the concept assertion $B(a)$ is retained, and $A(a)$ is removed. To see that there cannot be an optimal repair containing $B(a)$, first note that $\mathcal{A}$ together with $\mathcal{T}$ does not imply the existence of any role cycle, and thus no repair can contain such a cycle. Consequently, for an optimal repair $\exists Y.\mathcal{B}$ containing $B(a)$, there is an upper bound n on the length of role chains starting from a. Adding $r(a, y_1), r(y_2, y_3), \ldots, r(y_n, y_{n+1})$ for fresh existentially quantified variables $y_1, \ldots, y_{n+1}$ to $\exists Y.\mathcal{B}$ then yields a new repair that strictly implies $\exists Y.\mathcal{B}$, which contradicts the assumed optimality of this repair.

The following example shows that non-regularity of the RBox may prevent all repairs from being covered by a finite set of repairs.

Example 10. The RBox $\mathcal{R} := \{r^- \circ s \circ r \sqsubseteq s\}$ is not regular since $L_{\mathcal{R}}(s) = \{(r^-)^i s r^i \mid i \geq 0\}$ is a context-free language over the alphabet $\{r^-, s, r\}$ known to be non-regular. Together with the TBox $\mathcal{T} := \{\exists s.A \sqsubseteq A,\ \exists s.B \sqsubseteq B\}$, this RBox yields a terminating terminology. Consider the ABox $\mathcal{A} := \{r(a,a), s(a,a), A(a), B(a)\}$ and the repair request $\mathcal{P} = \mathcal{P}_{\mathsf{glo}} := \{A \sqcap B\}$. It is not hard to see that, for each $n \geq 1$, the qABox $\exists X_n.\mathcal{A}_n$ is a repair of $\exists\emptyset.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$, where $X_n := \{x_1, \ldots, x_n\}$ and

$$\begin{aligned}\mathcal{A}_n := \{\, & r(a, x_1), r(x_1, x_2), \ldots, r(x_{n-1}, x_n),\\ & s(a,a), s(x_1,x_1), \ldots, s(x_n,x_n),\\ & A(a), A(x_1), A(x_2), \ldots, A(x_{n-1}), B(x_n)\,\}.\end{aligned}$$

Assume that $\mathfrak{S}$ is a finite set of repairs of $\exists\emptyset.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ that covers all repairs, and let n be larger than the maximal number of objects occurring in the elements of $\mathfrak{S}$. Without loss of generality we assume that the elements of $\mathfrak{S}$ are saturated w.r.t. $(\mathcal{T},\mathcal{R})$. Then there must exist a repair $\exists Y.\mathcal{B}$ in $\mathfrak{S}$ such that there is a homomorphism h from $\exists X_n.\mathcal{A}_n$ to $\exists Y.\mathcal{B}$. Since $\mathcal{B}$ contains less than n objects, there must be i, j with $1 \leq i < j \leq n$ such that $h(x_i) = h(x_j)$. Consequently, $h(x_n)$ is reachable from $h(a)$ with the role r both in n steps and in $m < n$ steps, where $m = n - (j - i)$. Since $h(x_m)$ is also reachable in m steps from $h(a)$ and $s(h(a), h(a))$ must be in $\mathcal{B}$, the fact that $\exists Y.\mathcal{B}$ is saturated implies that $s(h(x_n), h(x_m))$ must belong to $\mathcal{B}$. Since $A(x_m) \in \mathcal{A}_n$ yields $A(h(x_m)) \in \mathcal{B}$, this implies that $A(h(x_n)) \in \mathcal{B}$. However, since $B(x_n) \in \mathcal{A}_n$ also yields $B(h(x_n) \in \mathcal{B}$, this contradicts our assumption that $\exists Y.\mathcal{B}$ is a repair for $\mathcal{P}$.

3.1 The Small Repair Property

If we restrict the attention to terminating terminologies with regular RBoxes $\mathcal{R}$, then we can show that the repairs of a certain bounded size cover all repairs. For an $\mathcal{ELROI}$ TBox $\mathcal{T}$ and a repair request $\mathcal{P}$, let $\mathsf{Sub}(\mathcal{T},\mathcal{P})$ denote the set of concept descriptions occurring in $\mathcal{T}$ and $\mathcal{P}$ and $\mathsf{Atoms}(\mathcal{T},\mathcal{P})$ the set of atoms in this set. To take the RBox into account, we introduce the set of *$\mathcal{R}$-extended atoms* $\mathsf{Atoms}_{\mathcal{R}}(\mathcal{T},\mathcal{P})$, which is obtained from $\mathsf{Atoms}(\mathcal{T},\mathcal{P})$ by replacing each $\exists R.C \in \mathsf{Atoms}(\mathcal{T},\mathcal{P})$ with the existential restrictions $\exists q.C$, where q ranges over Q_R (i.e., the set of states of the automaton for $L_{\mathcal{R}}(R)$).

Proposition 11. *Let $(\mathcal{T},\mathcal{R})$ be a terminating $\mathcal{ELROI}$ terminology with regular RBox, $\mathcal{P}$ an $\mathcal{ELROI}$ repair request, $\exists X.\mathcal{A}$ a (w.l.o.g) saturated qABox with m objects, and $n := |\mathsf{Atoms}(\mathcal{T},\mathcal{P}) \cup \mathsf{Atoms}_{\mathcal{R}}(\mathcal{T},\mathcal{P})|$. Then every repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ is entailed w.r.t. $(\mathcal{T},\mathcal{R})$ by a repair that contains at most $m \cdot 2^n$ objects.*

This proposition can be shown by adapting the well-known filtration technique, e.g., used in (Baader et al. 2017) to prove the finite model property for $\mathcal{ALC}$. Let $\exists Y.\mathcal{B}$ be a repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T}, \mathcal{R})$, and assume without loss of generality that it is saturated. Since $\exists X.\mathcal{A}$ entails every repair and is also assumed to be saturated, there is a homomorphism h from $\exists Y.\mathcal{B}$ to $\exists X.\mathcal{A}$. For each object u of $\exists Y.\mathcal{B}$, we define its *type* by[3]

$$\mathsf{Type}(u) := \left\{ C \;\middle|\; \begin{array}{l} C \in \mathsf{Atoms}(\mathcal{P}, \mathcal{T}) \cup \mathsf{Atoms}_{\mathcal{R}}(\mathcal{P}, \mathcal{T}) \\ \text{and } \mathcal{B} \models C(u) \end{array} \right\}$$

and define the equivalence relation $\sim$ on these objects as

$$u \sim v \text{ iff } \mathsf{Type}(u) = \mathsf{Type}(v) \text{ and } h(u) = h(v).$$

Obviously, $\sim$ has at most $m \cdot 2^n$ equivalence classes $[u]_\sim$. The *filtration* $\exists Z.\mathcal{C}$ has these equivalence classes as objects, with the class $[a]_\sim$ standing for the individual a.[4] The classes inherit their concept and role assertions from the ones of their elements in $\mathcal{B}$. Specifically, $\exists Z.\mathcal{C}$ has the following components:

1. the variable set Z consists of all equivalence classes $[y]_\sim$ for variables $y \in Y$ such that $[y]_\sim \cap \Sigma_\mathsf{I} = \emptyset$,
2. we identify each individual name a with the equivalence class $[a]_\sim$,
3. the matrix contains the following assertions:
   - $A([u]_\sim) \in \mathcal{C}$ if $A(u') \in \mathcal{B}$ for some $u' \sim u$,
   - $r([u]_\sim, [v]_\sim) \in \mathcal{C}$ if $r(u', v') \in \mathcal{B}$ for some $u' \sim u$ and some $v' \sim v$,
   - $[u]_\sim \equiv [v]_\sim \in \mathcal{C}$ if $a \equiv b \in \mathcal{B}$ where $a \in [u]_\sim \cap \Sigma_\mathsf{I}$ and $b \in [v]_\sim \cap \Sigma_\mathsf{I}$.

Recall from Section 2.2 that $\approx_{\exists Y.\mathcal{B}}$ is the reflexive, symmetric, transitive closure of the equality assertions in $\mathcal{B}$, and analogously for $\exists Z.\mathcal{C}$. It follows that $[u]_\sim \approx_{\exists Z.\mathcal{C}} [v]_\sim$ iff $u \sim v$ or $a \approx_{\exists Y.\mathcal{B}} b$ where $a \in [u]_\sim \cap \Sigma_\mathsf{I}$ and $b \in [v]_\sim \cap \Sigma_\mathsf{I}$.

In Lemma VIII we will show, for all $C \in \mathsf{Atoms}(\mathcal{T}, \mathcal{P}) \cup \mathsf{Atoms}_{\mathcal{R}}(\mathcal{T}, \mathcal{P})$ and for all $u \in \mathsf{Obj}(\exists Y.\mathcal{B})$, that

$$\mathcal{C} \models C([u]_\sim) \text{ iff } \mathcal{B} \models C(u).$$

Since $\exists Y.\mathcal{B}$ is a saturated repair, this implies that its filtration $\exists Z.\mathcal{C}$ is saturated w.r.t. $\mathcal{T}$ and does not entail (w.r.t. $\mathcal{T}$) any of the unwanted consequences specified by $\mathcal{P}$. The filtration $\exists Z.\mathcal{C}$ need not be saturated w.r.t. $\mathcal{R}$, but we will show in Lemma IX that its saturation w.r.t. $\mathcal{R}$ does not entail additional instance relationships for atoms in $\mathsf{Atoms}(\mathcal{T}, \mathcal{P}) \cup \mathsf{Atoms}_{\mathcal{R}}(\mathcal{T}, \mathcal{P})$. This implies that, also w.r.t. $(\mathcal{T}, \mathcal{R})$, the filtration does not entail any of the unwanted consequences in $\mathcal{P}$. Finally, it is easy to check that $u \mapsto [u]_\sim$ is a homomorphism from the repair $\exists Y.\mathcal{B}$ to the filtration $\exists Z.\mathcal{C}$, and that $[u]_\sim \mapsto h(u)$ is a homomorphism from $\exists Z.\mathcal{C}$ to the input qABox $\exists X.\mathcal{A}$ (independence of representatives follows from the very definition of $\sim$). Thus, the filtration $\exists Z.\mathcal{C}$ is a repair with at most $m \cdot 2^n$ objects that entails $\exists Y.\mathcal{B}$.

Lemma VIII. $\mathsf{Type}_{\exists Y.\mathcal{B}}(u) = \mathsf{Type}_{\exists Z.\mathcal{C}}([u]_\sim)$.

[3] In order to distinguish types w.r.t. different qABoxes, we sometimes add the respective qABox as subscript and write $\mathsf{Type}_{\exists Y.\mathcal{B}}(u)$ instead.

[4] Since h satisfies Condition (Hom2), $h(a) \neq h(b)$ holds for each two individuals a and b, and thus every equivalence class w.r.t. $\sim$ contains at most one individual.

Proof. The proof is by induction on C.

- Assume that $\mathsf{Type}_{\exists Z.\mathcal{C}}([u]_\sim)$ contains the concept name A, i.e., the matrix $\mathcal{C}$ entails $A([u]_\sim)$. With Corollary I it follows that $\mathcal{C}$ must contain $A([\bar{u}]_\sim)$ for some $[\bar{u}]_\sim \approx_{\exists Z.\mathcal{C}} [u]_\sim$, and thus the above definition of the filtration yields that there is some $u' \sim \bar{u}$ with $A(u') \in \mathcal{B}$. The latter implies that $\mathsf{Type}_{\exists Y.\mathcal{B}}(u')$ contains A. Since $\bar{u}$ and u' are equivalent, they specifically have the same type and so A is also contained in $\mathsf{Type}_{\exists Y.\mathcal{B}}(\bar{u})$.
  - If $\bar{u} \sim u$, then it immediately follows that $A \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$.
  - Otherwise, there are individuals $\bar{a} \in [\bar{u}]_\sim \cap \Sigma_\mathsf{I}$ and $a \in [u]_\sim \cap \Sigma_\mathsf{I}$ such that $\bar{a} \approx_{\exists Y.\mathcal{B}} a$. We infer from $\bar{a} \sim \bar{u}$, $\bar{a} \approx_{\exists Y.\mathcal{B}} a$, and $a \sim u$ that $\mathsf{Type}_{\exists Y.\mathcal{B}}(\bar{u}) = \mathsf{Type}_{\exists Y.\mathcal{B}}(\bar{a}) = \mathsf{Type}_{\exists Y.\mathcal{B}}(a) = \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$, and thus $A \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$.

  Conversely, assume that $A \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$, i.e., the matrix $\mathcal{B}$ entails $A(u)$, from which we infer by means of Corollary I that $A(\bar{u})$ must be contained in $\mathcal{B}$ for some $\bar{u} \approx_{\exists Y.\mathcal{B}} u$. Since $\bar{u} \sim \bar{u}$ holds, the above definition of the filtration immediately implies that $A([\bar{u}]_\sim)$ is in the matrix $\mathcal{C}$. Furthermore, $\bar{u} \approx_{\exists Y.\mathcal{B}} u$ implies $[\bar{u}]_\sim \approx_{\exists Z.\mathcal{C}} [u]_\sim$, and thus $A \in \mathsf{Type}_{\exists Z.\mathcal{C}}([u]_\sim)$.
- Next, we are concerned with the case where C is a nominal $\{a\}$. First let $\{a\} \in \mathsf{Type}_{\exists Z.\mathcal{C}}([u]_\sim)$. It follows that $[a]_\sim \approx_{\exists Z.\mathcal{C}} [u]_\sim$, and so there is an individual name $b \in [u]_\sim \cap \Sigma_\mathsf{I}$ where $a \approx_{\exists Y.\mathcal{B}} b$. We infer that $\{a\}$ is in $\mathsf{Type}_{\exists Y.\mathcal{B}}(b)$, and also in $\mathsf{Type}_{\exists Y.\mathcal{B}}(u)$ since $b \sim u$.
  Regarding the opposite direction, let $\{a\} \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$. It follows that $a \approx_{\exists Y.\mathcal{B}} u$ and u must be an individual name. The definition of the filtration yields $[a]_\sim \approx_{\exists Z.\mathcal{C}} [u]_\sim$, and so we obtain that $\{a\} \in \mathsf{Type}_{\exists Z.\mathcal{C}}([u]_\sim)$.
- Assume that C is an existential restriction $\exists R.D$. If $\exists R.D \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$, then by Corollary I there is $\bar{u} \approx_{\exists Y.\mathcal{B}} u$ such that $R(\bar{u}, v) \in \mathcal{B}$ and $\mathsf{Conj}(D) \subseteq \mathsf{Type}_{\exists Y.\mathcal{B}}(v)$. The induction hypothesis yields $\mathsf{Conj}(D) \subseteq \mathsf{Type}_{\exists Y.\mathcal{B}}([v]_\sim)$. According to the definition of the filtration we further have $R([\bar{u}]_\sim, [v]_\sim) \in \mathcal{C}$, and so $\exists R.D \in \mathsf{Type}_{\exists Z.\mathcal{C}}([\bar{u}]_\sim)$. From $\bar{u} \approx_{\exists Y.\mathcal{B}} u$ we infer that $[\bar{u}]_\sim \approx_{\exists Z.\mathcal{C}} [u]_\sim$ and thus $\exists R.D \in \mathsf{Type}_{\exists Z.\mathcal{C}}([u]_\sim)$.
  Conversely, assume $\exists R.D \in \mathsf{Type}_{\exists Z.\mathcal{C}}([u]_\sim)$. So there is $[\bar{u}]_\sim \approx_{\exists Z.\mathcal{C}} [u]_\sim$ where $R([\bar{u}]_\sim, [v]_\sim) \in \mathcal{C}$ and $\mathsf{Conj}(D) \subseteq \mathsf{Type}_{\exists Z.\mathcal{C}}([v]_\sim)$, cf. Corollary I. By induction hypothesis we obtain $\mathsf{Conj}(D) \subseteq \mathsf{Type}_{\exists Y.\mathcal{B}}(v)$. Furthermore, we have $R(u', v') \in \mathcal{B}$ for some $u' \sim \bar{u}$ and some $v' \sim v$, and thus $\exists R.D \in \mathsf{Type}_{\exists Y.\mathcal{B}}(\bar{u})$.
  - If $\bar{u} \sim u$, then it immediately follows that $\exists R.D \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$.
  - Otherwise there is an individual name $\bar{a} \sim \bar{u}$ and an individual name $a \sim u$ where $\bar{a} \approx_{\exists Y.\mathcal{B}} a$. Then $\mathsf{Type}_{\exists Y.\mathcal{B}}(\bar{u}) = \mathsf{Type}_{\exists Y.\mathcal{B}}(\bar{a}) = \mathsf{Type}_{\exists Y.\mathcal{B}}(a) = \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$, which implies $\exists R.D \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$.
- Last, let $\exists q.C \in \mathsf{Type}_{\exists Z.\mathcal{C}}([u]_\sim)$, i.e., $\mathcal{C} \models \exists q.C([u]_\sim)$. So there is a word $S_1 \cdots S_n \in L(\mathfrak{A}_{\mathcal{R}}(q))$ such that $\mathcal{C} \models \exists S_1. \cdots \exists S_n.C([u]_\sim)$. According to Corollary I there are

role assertions $S_1([v_0]_\sim, [w_1]_\sim)$, $S_2([v_1]_\sim, [w_2]_\sim)$, $\ldots$, $S_n([v_{n-1}]_\sim, [w_n]_\sim)$ in $\mathcal{C}$ such that

– $[u]_\sim \approx_{\exists Z.\mathcal{C}} [v_0]_\sim$,

– $[w_i]_\sim \approx_{\exists Z.\mathcal{C}} [v_i]_\sim$ for each index $i \in \{1, \ldots, n-1\}$,

– and $\mathcal{C} \models C([w_n]_\sim)$.

$S_1 \cdots S_n \in L(\mathfrak{A}_\mathcal{R}(q))$ implies that there are transitions $(q_0, S_1, q_1), \ldots, (q_{n-1}, S_n, q_n)$ where $q_0 = q$ and q_n is final. Furthermore, $\mathsf{Conj}(C) \subseteq \mathsf{Type}_{\exists Z.\mathcal{C}}([w_n]_\sim)$, and the induction hypothesis yields $\mathsf{Conj}(C) \subseteq \mathsf{Type}_{\exists Y.\mathcal{B}}(w_n)$, and thus $\exists q_n.C \in \mathsf{Type}_{\exists Y.\mathcal{B}}(w_n)$.

We show by induction along the above transitions that $\exists q_0.C \in \mathsf{Type}_{\exists Y.\mathcal{B}}(v_0)$. Consider an index i, starting with the largest one, n, and then in decreasing order. Since $S_i([v_{i-1}]_\sim, [w_i]_\sim)$ is in the filtration matrix $\mathcal{C}$, there is a role assertion $S_i(\bar{v}_{i-1}, \bar{w}_i)$ in $\mathcal{B}$ where $\bar{v}_{i-1} \sim v_{i-1}$ and $\bar{w}_i \sim w_i$. We thus infer from $\exists q_i.C \in \mathsf{Type}_{\exists Y.\mathcal{B}}(w_i)$ that first $\exists q_i.C \in \mathsf{Type}_{\exists Y.\mathcal{B}}(\bar{w}_i)$, then $\exists q_{i-1}.C \in \mathsf{Type}_{\exists Y.\mathcal{B}}(\bar{v}_{i-1})$, and thus $\exists q_{i-1}.C \in \mathsf{Type}_{\exists Y.\mathcal{B}}(v_{i-1})$. We continue with a case distinction why $[w_{i-1}]_\sim \approx_{\exists Z.\mathcal{C}} [v_{i-1}]_\sim$ holds.

– If $w_{i-1} \sim v_{i-1}$, then it directly follows that $\exists q_{i-1}.C \in \mathsf{Type}_{\exists Y.\mathcal{B}}(w_{i-1})$.

– Otherwise, there are individual names $a_{i-1} \sim v_{i-1}$ and $b_{i-1} \sim w_{i-1}$ where $a_{i-1} \approx_{\exists Y.\mathcal{B}} b_{i-1}$. Then the types $\mathsf{Type}_{\exists Y.\mathcal{B}}(v_{i-1})$, $\mathsf{Type}_{\exists Y.\mathcal{B}}(a_{i-1})$, $\mathsf{Type}_{\exists Y.\mathcal{B}}(b_{i-1})$, and $\mathsf{Type}_{\exists Y.\mathcal{B}}(w_{i-1})$ are equal, and we obtain $\exists q_{i-1}.C \in \mathsf{Type}_{\exists Y.\mathcal{B}}(w_{i-1})$ as well.

We have shown that $\exists q.C \in \mathsf{Type}_{\exists Y.\mathcal{B}}(v_0)$. With a similar case distinction on $[u]_\sim \approx_{\exists Z.\mathcal{C}} [v_0]_\sim$ as above, we infer $\exists q.C \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$.

It remains to show the converse direction. Consider $\exists q.C \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$. So there is a word $S_1 \cdots S_n \in L(\mathfrak{A}_\mathcal{R}(q))$ and role assertions $S_1(v_0, w_1)$, $S_2(v_1, w_2)$, $\ldots$, $S_n(v_{n-1}, w_n)$ in $\mathcal{B}$ such that

– $u \approx_{\exists Y.\mathcal{B}} v_0$,

– $w_i \approx_{\exists Y.\mathcal{B}} v_i$ for each index $i \in \{1, \ldots, n-1\}$,

– and $\mathsf{Conj}(C) \subseteq \mathsf{Type}_{\exists Y.\mathcal{B}}(w_n)$.

Furthermore, $S_1 \cdots S_n \in L(\mathfrak{A}_\mathcal{R}(q))$ implies that there are transitions (q_0, S_1, q_1), $\ldots$, (q_{n-1}, S_n, q_n) where $q_0 = q$ and q_n is final.

According to the definition of the filtration, its matrix $\mathcal{C}$ contains the role assertions $S_1([v_0]_\sim, [w_1]_\sim)$, $S_2([v_1]_\sim, [w_2]_\sim), \ldots, S_n([v_{n-1}]_\sim, [w_n]_\sim)$, and it further holds that $[u]_\sim \approx_{\exists Z.\mathcal{C}} [v_0]_\sim$ and $[w_i]_\sim \approx_{\exists Z.\mathcal{C}} [v_i]_\sim$ for each index $i \in \{1, \ldots, n-1\}$. The induction hypothesis yields $\mathsf{Conj}(C) \subseteq \mathsf{Type}_{\exists Z.\mathcal{C}}([w_n]_\sim)$. By induction, it follows that $\exists q.C \in \mathsf{Type}_{\exists Z.\mathcal{C}}([u]_\sim)$. □

Let $\exists Z.\mathcal{C}'$ be the $\mathcal{R}$-saturation of the filtration $\exists Z.\mathcal{C}$.

Lemma IX. $\mathsf{Type}_{\exists Y.\mathcal{B}}(u) = \mathsf{Type}_{\exists Z.\mathcal{C}'}([u]_\sim)$

Proof. We show the claim by an induction along the sequence of applications of the RI Rule. Therefore let $\mathcal{C} =: \mathcal{C}_0 \to \mathcal{C}_1 \to \cdots \to \mathcal{C}_n \coloneqq \mathcal{C}'$ be the sequence of matrices such that $\mathcal{C}_{i+1}$ is obtained from $\mathcal{C}_i$ by one application of the RI Rule. Note that applying the RI Rule does not introduce new objects or new equality assertions, i.e., $\exists Z.\mathcal{C}$, $\exists Z.\mathcal{C}'$, and all qABoxes $\exists Z.\mathcal{C}_i$ in the sequence have the same equivalence relation on objects, which we simply denote as $\approx_{\exists Z.\mathcal{C}}$.

The induction base follows from Lemma VIII.

Assume that $\mathcal{C}_{i+1}$ is produced from $\mathcal{C}_i$ by applying the RI Rule for $R_1 \circ \cdots \circ R_n \sqsubseteq S$ at $([v]_\sim, [w]_\sim)$, i.e., there are role assertions $R_1([x_0]_\sim, [y_1]_\sim)$, $\ldots$, $R_n([x_{n-1}]_\sim, [y_n]_\sim)$ in $\mathcal{C}_i$ where

- $[x_0]_\sim \approx_{\exists Y.\mathcal{C}} [v]_\sim$,
- $[x_j]_\sim \approx_{\exists Y.\mathcal{C}} [y_j]_\sim$ for each index $j \in \{1, \ldots, n-1\}$,
- and $[y_n]_\sim \approx_{\exists Y.\mathcal{C}} [w]_\sim$,

and the new assertion $S([v]_\sim, [w]_\sim)$ is added to the matrix, yielding $\mathcal{C}_{i+1}$.

It is easy to see that $\mathsf{Type}_{\exists Z.\mathcal{C}_i}([u]_\sim)$ is always a subset of $\mathsf{Type}_{\exists Z.\mathcal{C}_{i+1}}([u]_\sim)$, and so the induction hypothesis yields that $\mathsf{Type}_{\exists Y.\mathcal{B}}(u) \subseteq \mathsf{Type}_{\exists Z.\mathcal{C}_{i+1}}([u]_\sim)$.

In the opposite direction, we show by induction on C that $C \in \mathsf{Type}_{\exists Z.\mathcal{C}_{i+1}}([u]_\sim)$ implies $C \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$.

The only interesting cases are where C is an existential restriction involving the role S or an automaton concept such that the new role assertion $S([v]_\sim, [w]_\sim)$ is used to entail that C has $[u]_\sim$ as an instance w.r.t. $\mathcal{C}_{i+1}$.

- Assume $\mathcal{C}_{i+1} \models \exists S.D([u]_\sim)$. According to Corollary I there is a role assertion $S([v']_\sim [w']_\sim)$ in $\mathcal{C}_{i+1}$ such that $[u]_\sim \approx_{\exists Z.\mathcal{C}} [v']_\sim$ and $\mathsf{Conj}(D) \subseteq \mathsf{Type}_{\exists Z.\mathcal{C}_{i+1}}([w']_\sim)$. The inner induction hypothesis yields that $\mathsf{Conj}(D) \subseteq \mathsf{Type}_{\exists Y.\mathcal{B}}(w')$. We have already seen above that $\mathsf{Type}_{\exists Y.\mathcal{B}}(w') \subseteq \mathsf{Type}_{\exists Z.\mathcal{C}_i}([w']_\sim)$, and thus $\mathsf{Conj}(D) \subseteq \mathsf{Type}_{\exists Z.\mathcal{C}_i}([w']_\sim)$.
  If this role assertion is not the new one, it also contained in $\mathcal{C}_i$. It then immediately follows that $\exists S.D \in \mathsf{Type}_{\exists Z.\mathcal{C}_i}([v']_\sim)$. Since $[u]_\sim \approx_{\exists Z.\mathcal{C}} [v']_\sim$, the latter type is equal to $\mathsf{Type}_{\exists Z.\mathcal{C}_i}([u]_\sim)$. Thus the outer induction hypothesis yields $\exists S.D \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$.
  Now assume that $S([v']_\sim [w']_\sim)$ equals the new role assertion, i.e., $v \sim v'$ and $w \sim w'$. Since $R_1 \circ \cdots \circ R_n \sqsubseteq S$ is an RI in the RBox, the automaton for S contains transitions $(q_0, R_1, q_1), (q_1, R_2, q_2), \ldots, (q_{n-1}, R_n, q_n)$ where $q_0 = i_S$ is the initial state and q_n is a final state. From $\mathsf{Conj}(D) \subseteq \mathsf{Type}_{\exists Z.\mathcal{C}_i}([w]_\sim)$ and $w \sim x_n$ we infer that $\exists q_n.D \in \mathsf{Type}_{\exists Z.\mathcal{C}_i}([x_n]_\sim)$.
  We now consider each index j in decreasing order, starting with n. Since $\mathcal{C}_i$ contains the role assertion $R_j([x_{j-1}]_\sim, [x_j]_\sim)$, it holds that $\exists q_j.D \in \mathsf{Type}_{\exists Z.\mathcal{C}_i}([x_j]_\sim)$ implies $\exists q_{j-1}.D \in \mathsf{Type}_{\exists Z.\mathcal{C}_i}([x_{j-1}]_\sim)$.
  By induction we obtain $\exists q_0.D \in \mathsf{Type}_{\exists Z.\mathcal{C}_i}([x_0]_\sim)$, and thus $q_0 = i_S$ and $x_0 \sim v$ implies $\exists i_S.D \in \mathsf{Type}_{\exists Z.\mathcal{C}_i}([v]_\sim)$. With $[u]_\sim \approx_{\exists Z.\mathcal{C}} [v]_\sim$ we conclude that the latter type equals $\mathsf{Type}_{\exists Z.\mathcal{C}_i}([u]_\sim)$.
  The outer induction hypothesis yields $\exists i_S.D \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$. Since $\exists Y.\mathcal{B}$ is saturated w.r.t. $(\mathcal{T}, \mathcal{R})$, we obtain by Lemma VI that $\exists S.D \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$.
- The case where $C = \exists S^-.D$ is similar.

- Now let $C = \exists q.D \in \mathsf{Type}_{\exists Z.\mathcal{C}_{i+1}}([u]_\sim)$, i.e., there is a word $S_1 \cdots S_m \in L(\mathfrak{A}_\mathcal{R}(q))$ such that $\mathcal{C}_{i+1} \models \exists S_1. \cdots \exists S_m.D([u]_\sim)$. By Corollary I there are role assertions $S_1([v_0]_\sim,[w_1]_\sim)$, $S_2([v_1]_\sim,[w_2]_\sim)$, $\dots$, $S_m([v_{m-1}]_\sim,[w_m]_\sim)$ in $\mathcal{C}_{i+1}$ where
  - $[u]_\sim \approx_{\exists Z.\mathcal{C}} [v_0]_\sim$,
  - $[w_j]_\sim \approx_{\exists Z.\mathcal{C}} [v_j]_\sim$ for each index $j \in \{1,\dots,m-1\}$,
  - and $\mathsf{Conj}(D) \subseteq \mathsf{Type}_{\exists Z.\mathcal{C}_{i+1}}([w_m]_\sim)$.

  The inner induction hypothesis yields $\mathsf{Conj}(D) \subseteq \mathsf{Type}_{\exists Y.\mathcal{B}}(w_m)$ and with $\mathsf{Type}_{\exists Y.\mathcal{B}}(w_m) \subseteq \mathsf{Type}_{\exists Z.\mathcal{C}_i}([w_m]_\sim)$ we infer that $\mathsf{Conj}(D) \subseteq \mathsf{Type}_{\exists Z.\mathcal{C}_i}([w_m]_\sim)$.
  In the sequence of role assertions $S_1([v_0]_\sim,[w_1]_\sim)$, $S_2([v_1]_\sim,[w_2]_\sim)$, $\dots$, $S_m([v_{m-1}]_\sim,[w_m]_\sim)$, which are all in the matrix $\mathcal{C}_{i+1}$, we replace
  - each occurrence of the new role assertion $S([v]_\sim,[w]_\sim)$ by the subsequence $R_1([x_0]_\sim,[y_1]_\sim)$, $\dots$, $R_n([x_{n-1}]_\sim,[y_n]_\sim)$, and
  - each occurrence of the inverse new role assertion $S^-([w]_\sim,[v]_\sim)$ by the subsequence $R_n^-([y_n]_\sim,[x_{n-1}]_\sim),\dots,R_1^-([y_1]_\sim,[x_0]_\sim)$.

  We so obtain a sequence of role assertions in the matrix $\mathcal{C}_i$, say $S'_1([v'_0]_\sim,[w'_1]_\sim)$, $S'_2([v'_1]_\sim,[w'_2]_\sim)$, $\dots$, $S'_\ell([v'_{\ell-1}]_\sim,[w'_\ell]_\sim)$, where
  - $[v_0]_\sim \approx_{\exists Z.\mathcal{C}} [v'_0]_\sim$,
  - $[w'_j]_\sim \approx_{\exists Z.\mathcal{C}} [v'_j]_\sim$ for each index $j \in \{1,\dots,\ell-1\}$,
  - and $[w'_\ell]_\sim \approx_{\exists Z.\mathcal{C}} [w_m]_\sim$.

  Since $S_1 \cdots S_m \in L(\mathfrak{A}_\mathcal{R}(q))$, the RBox $\mathcal{R}$ contains the role inclusion $R_1 \circ \cdots \circ R_n \sqsubseteq S$, and the automaton $\mathfrak{A}_\mathcal{R}$ is deterministic, it then further holds that $S'_1 \cdots S'_\ell \in L(\mathfrak{A}_\mathcal{R}(q))$, and so there are transitions (q_0,S'_1,q_1), $\dots$, $(q_{\ell-1},S'_\ell,q_\ell)$ where $q_0 = q$ and q_ℓ is final.
  From $\mathsf{Conj}(D) \subseteq \mathsf{Type}_{\exists Z.\mathcal{C}_i}([w_m]_\sim)$ and $[w'_\ell]_\sim \approx_{\exists Z.\mathcal{C}} [w_m]_\sim$ it follows that $\exists q_\ell.D \in \mathsf{Type}_{\exists Z.\mathcal{C}_i}([w'_\ell]_\sim)$. By induction, it follows that $\exists q_0.D \in \mathsf{Type}_{\exists Z.\mathcal{C}_i}([v'_0]_\sim)$. Due to $q_0 = q$ and $[u]_\sim \approx_{\exists Z.\mathcal{C}} [v'_0]_\sim$ we get $\exists q.D \in \mathsf{Type}_{\exists Z.\mathcal{C}_i}([u]_\sim)$. The outer induction yields $\exists q.D \in \mathsf{Type}_{\exists Y.\mathcal{B}}(u)$. □

Since, for a fixed signature and up to renaming of variables, there are only finitely many qABoxes containing at most $m \cdot 2^n$ objects, we can effectively construct the set of optimal repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ by enumerating these qABoxes, then removing the ones that are not repairs, and finally removing from the remaining set the elements that are strictly entailed by an other element.

Theorem 12. *Let $\exists X.\mathcal{A}$ be a qABox, $(\mathcal{T},\mathcal{R})$ a terminating $\mathcal{ELROI}$ terminology with regular RBox whose associated automata can effectively be computed, and $\mathcal{P}$ an $\mathcal{ELROI}$ repair request. Then the set of all optimal repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ can, up to equivalence, effectively be computed, and every repair is entailed by an optimal repair.*

The following example shows that the "automata atoms" in $\mathsf{Atoms}_\mathcal{R}(\mathcal{T},\mathcal{P})$ are needed for the filtration.

Example 13. Assume that $\mathcal{T} \coloneqq \emptyset$, $\mathcal{R} \coloneqq \{r \circ r \sqsubseteq s\}$, $\mathcal{A} \coloneqq \{r(a,b), r(b,c), s(a,c)\}$, and $\mathcal{P} \coloneqq \{\exists s.\top(a)\}$. The qABox $\exists\{x\}.\{r(a,b), r(x,c)\}$ is a (saturated) repair of $\exists\emptyset.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$. If we used only $\mathsf{Atoms}(\mathcal{T},\mathcal{P}) = \{\exists s.\top\}$ for the filtration, then the objects b and x would be identified since they behave the same w.r.t. this concept in the repair. Thus, $[a]_\sim$ would have $[x]_\sim = [b]_\sim$ as r-successor in the filtration, which in turn would have $[c]_\sim$ as r-successor. This shows that the filtration would have $\exists s.\top(a)$ as a consequence, and thus would not be a repair. The regular language $L_\mathcal{R}(s) = \{rr, s\}$ is accepted by a deterministic automaton with three states, q_0, q_1, q_2, where q_0 is initial and q_2 is final, r-transitions from q_0 to q_1 and from q_1 to q_2, and an s-transition from q_0 to q_2. Since the object x belongs to $\exists q_1.\top$, but b does not, they are not identified in the filtration that takes the concepts in $\mathsf{Atoms}_\mathcal{R}(\mathcal{T},\mathcal{P}) = \{\exists q_i.\top \mid 0 \le i \le 2\}$ into account.

3.2 Repair Types and Repair Seeds

Instead of blindly searching for optimal repairs among the very large set of "small" repairs, we now show how the considerably smaller set of canonical repairs, which contains all optimal repairs, can be constructed from repair seeds. Such a repair seed is of polynomial size, and it basically specifies which atoms in $\mathsf{Atoms}(\mathcal{T},\mathcal{P}) \cup \mathsf{Atoms}_\mathcal{R}(\mathcal{T},\mathcal{P})$ need to be removed for each individual.

From now on, we assume that $\exists Y.\mathcal{B}$ is the saturation of $\exists X.\mathcal{A}$ w.r.t. $(\mathcal{T},\mathcal{R})$. Our canonical repairs will actually be computed from $\exists Y.\mathcal{B}$. This guarantees that no consequences are lost which would, in the original qABox $\exists X.\mathcal{A}$, only follow from removed assertions but which cannot participate in violating the repair request. As mentioned in the introduction, to achieve optimality, it is not sufficient to remove assertions from this qABox. We must also generate anonymous copies of its objects. Basically, these copies are induced by pairs $(u,\mathcal{K})$ where u is an object in $\mathcal{B}$ and $\mathcal{K} \subseteq \mathsf{Atoms}_\mathcal{R}(\mathcal{T},\mathcal{P})$ is a set of atoms C such that u is an instance of C in $\exists Y.\mathcal{B}$. Putting an atom into $\mathcal{K}$ means that the copy of u induced by $(u,\mathcal{K})$ should not be an instance of C.

Recall that $\mathsf{Conj}(C)$ is the set of all top-level conjuncts of a concept description C. The set $\mathsf{Conj}_\mathcal{R}(C)$ is obtained from $\mathsf{Conj}(C)$ by replacing each existential restriction $\exists R.D \in \mathsf{Conj}(C)$ with $\exists i_R.D$.[5] We will use the following mapping $(\cdot)_\mathcal{R}$ from atoms to $\mathcal{R}$-extended atoms: $(A)_\mathcal{R} \coloneqq A$ for concept names, $(\{a\})_\mathcal{R} \coloneqq \{a\}$ for nominals, and $(\exists R.D)_\mathcal{R} \coloneqq \exists i_R.D$ for existential restrictions. We further set $(\exists q.D)_\mathcal{R} \coloneqq \exists q.D$ if q is a state. With that, we have $\mathsf{Conj}_\mathcal{R}(C) = \{\, (D)_\mathcal{R} \mid D \in \mathsf{Conj}(C) \,\}$.

The sets $\mathcal{K}$ used to construct copies of u must be repair types for u.

Definition 14. Let u be an object name of $\exists Y.\mathcal{B}$. A *repair type* for u is a set $\mathcal{K} \subseteq \mathsf{Atoms}_\mathcal{R}(\mathcal{T},\mathcal{P})$ satisfying:

(RT1) If $C \in \mathcal{K}$, then $\mathcal{B} \models C(u)$.

(RT2) If $D \in \mathsf{Sub}(\mathcal{T},\mathcal{P}) \cup \mathsf{Atoms}_\mathcal{R}(\mathcal{T},\mathcal{P})$ with $\mathcal{B} \models D(u)$ and $C \in \mathcal{K}$ with $D \sqsubseteq^{\mathcal{T},\mathcal{R}} C$, then $\mathsf{Conj}_\mathcal{R}(D) \cap \mathcal{K} \neq \emptyset$.

(RT3) If $E \in \mathcal{P}_{\mathsf{glo}}$ and $\mathcal{B} \models E(u)$, then $\mathsf{Conj}_\mathcal{R}(E) \cap \mathcal{K} \neq \emptyset$.

[5] Recall that i_R is the initial state of the automaton $\mathfrak{A}_R$.

The first condition says that only concept assertions that really hold for u need to be removed. The second condition ensures that concept assertions that are removed for u cannot be reintroduced by the terminology. The third condition has the effect that no copy can be an instance of a concept description that occurs in the global request. Each two individual names a and b with $a \approx_{\exists Y.\mathcal{B}} b$ have the same repair types.

Note that Condition (RT2) could also be formulated without taking the $\mathcal{R}$-extended atoms in $\mathsf{Atoms}_{\mathcal{R}}(\mathcal{T}, \mathcal{P})$ into account. The canonical repairs will then still have enough structure such that Theorem 16 can be proven. Specifically, the $\mathcal{R}$-extended atoms in $\mathsf{Atoms}_{\mathcal{R}}(\mathcal{T}, \mathcal{P})$ would then also not need to be considered in Definition XIII and Lemmas XXIII and XXIV, and the second condition in Definition XVI would not be necessary.

In the canonical repairs, one of the copies of each individual will stand for this individual, whereas the other copies are variables. In addition, some individuals that are equal w.r.t. $\exists Y.\mathcal{B}$ may no longer be equal in the repair. The repair seed makes these decisions explicit.

Definition 15. A *repair seed* $\mathcal{S}$ consists of an equivalence relation $\approx_{\mathcal{S}}$ on $\mathsf{Obj}(\exists Y.\mathcal{B})$ that is a refinement of $\approx_{\exists Y.\mathcal{B}}$ (i.e., $\approx_{\mathcal{S}} \subseteq \approx_{\exists Y.\mathcal{B}}$) and of a function that maps each equivalence class $[a]_{\mathcal{S}}$ of an individual a w.r.t. $\approx_{\mathcal{S}}$ to a repair type $\mathcal{S}_{[a]_{\mathcal{S}}}$ for a, such that the following conditions are fulfilled:

(RS1) If $C(a) \in \mathcal{P}_{\mathsf{loc}}$ and $\mathcal{B} \models C(a)$, then $\mathsf{Conj}_{\mathcal{R}}(C) \cap \mathcal{S}_{[a]_{\mathcal{S}}} \neq \emptyset$.

(RS2) If a, b are individuals and $\{a\} \in \mathsf{Atoms}(\mathcal{T}, \mathcal{P})$, then $\{a\} \in \mathcal{S}_{[b]_{\mathcal{S}}}$ iff $a \approx_{\exists Y.\mathcal{B}} b$ and $a \not\approx_{\mathcal{S}} b$.

The first condition guarantees that the repair induced by the seed satisfies the local request. The second condition ensures that the decision made by the seed that two individuals should no longer be equal is respected in the repair.

Example X. We choose the TBox $\{A \sqsubseteq \{b\}\}$, the empty RBox, the (saturated) qABox $\{a \equiv b, A(a)\}$, and the repair request $\{\{a\}(b)\}$. The individuals a and b are equivalent. So we have two choices for a refinement: either $a \approx_{\mathcal{S}} b$ or $a \not\approx_{\mathcal{S}} b$. In the former case, there is only one equivalence class $\{a, b\}$. Condition (RS1) would require that $\mathcal{S}_{\{a,b\}}$ contains $\{a\}$, which violates Condition (RS2).

Now consider the latter case, where $\{a\}$ and $\{b\}$ are the equivalence classes of a repair seed. Condition (RS1) enforces $\{a\} \in \mathcal{S}_{\{b\}}$, and thus Condition (RS2) requires $\{b\} \in \mathcal{S}_{\{a\}}$. With Condition (RT2) we further get $A \in \mathcal{S}_{\{a\}}$. The latter ensures that a is specifically repaired for A and thus inference with the TBox and RBox cannot restore the equality $a \equiv b$.

Computing a refinement of the equivalence relation $\approx_{\exists Y.\mathcal{B}}$ according to the unwanted equalities expressed in the repair request has a strong connection to a well-known problem in graph theory. First of all, we can construct an undirected graph (V, E) in which the vertices are the objects of $\exists Y.\mathcal{B}$ and where two objects are joined by an edge if they are equivalent w.r.t. $\approx_{\exists Y.\mathcal{B}}$. Since $\approx_{\exists Y.\mathcal{B}}$ is transitive, the graph (V, E) must be a disjoint union of cliques. Specifically, each clique represents an equivalence class. Now for each unwanted equality expressed by the repair request, we remove the corresponding edge in the graph (V, E), which can be seen as a request to split an equivalence class into smaller classes. In order to construct a refinement of $\approx_{\exists Y.\mathcal{B}}$ that adheres to the unwanted equalities, we compute a *clique cover* of the modified graph (V, E), which is a partition of the vertex set V into cliques of (V, E). The cliques are then the refined equivalence classes.

Such a clique cover is called *minimum clique cover* if there is no clique cover that partitions V into fewer cliques. The *clique cover number* of (V, E) is the number of cliques in a minimum clique cover. The problem consisting of all pairs $((V, E), n)$ where (V, E) has a clique cover number not exceeding n is NP-complete (Karp 1972). It follows that computing a minimum clique cover is NP-hard. Clearly, the coarsest refinements correspond to the minimum clique covers. While a coarsest refinement might be desirable to retain as many equalities as possible, there is always a trade-off to retaining other consequences.

Given a repair seed $\mathcal{S}$, the copies of objects u have $\approx_{\mathcal{S}}$-equivalence classes $[u]_{\mathcal{S}}$ as first component.[6] We write such a copy, consisting of an equivalence class $[u]_{\mathcal{S}}$ and a repair type $\mathcal{K}$ for u as $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$, and call it the *$\mathcal{K}$-copy* of u. The role assertions between these copies are determined by the next definition.

Definition XI. Given copies $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ and $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$, we write $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle \xrightarrow{r} \langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$, and occasionally also $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle \xrightarrow{r^-} \langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$, if the following conditions hold:

(RA1) $\mathcal{B} \models r(u, v)$.

(RA2) For each $\exists q.C \in \mathcal{K}$, if there is a transition (q, r, p) in some automaton $\mathfrak{A}_R$ such that $\mathcal{B} \models \exists p.C(v)$, then there is a state $p' \in Q_R$ such that $p \leq p'$ and $\exists p'.C \in \mathcal{L}$.[7]

(RA3) For each $\exists q.C \in \mathcal{K}$, if there is a transition (q, r, f) in some automaton $\mathfrak{A}_R$ for a final state f and $\mathcal{B} \models C(v)$, then $\mathsf{Conj}_{\mathcal{R}}(C) \cap \mathcal{L} \neq \emptyset$.

(RA4) For each $\exists q.C \in \mathcal{L}$, if there is a transition (q, r^-, p) in some automaton $\mathfrak{A}_R$ such that $\mathcal{B} \models \exists p.C(u)$, then there is a state $p' \in Q_R$ such that $p \leq p'$ and $\exists p'.C \in \mathcal{K}$.

(RA5) For each $\exists q.C \in \mathcal{L}$, if there is a transition (q, r^-, f) in some automaton $\mathfrak{A}_R$ for a final state f and $\mathcal{B} \models C(u)$, then $\mathsf{Conj}_{\mathcal{R}}(C) \cap \mathcal{K} \neq \emptyset$.

The first condition says that $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$ can only be an r-successor of $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ if v is an r-successor of u in the saturation. The other four conditions require that fillers of existential restrictions in the two repair types are propagated between each other, ensuring that in the repair the copies will not be instances of these existential restrictions.

To see why this works, reconsider Proposition V and Lemma VI. Specifically, since $\exists Y.\mathcal{B}$ is saturated w.r.t. $(\mathcal{T}, \mathcal{R})$, it holds that $\mathcal{B} \models \exists R.C(t)$ iff $\mathcal{B} \models \exists i_R.C(t)$, where i_R is the initial state of the automaton $\mathfrak{A}_R$. This means that, if we want to ensure that an object t is no instance of $\exists R.C$, it suffices to modify it such that it is no instance of $\exists i_R.C$ anymore. For this reason, we add in

[6] For variables x, their equivalence class is the singleton set $\{x\}$.

[7] Recall that, for states p and q in Q_R, we write $p \leq q$ if $L(\mathfrak{A}_{\mathcal{R}}(p)) \subseteq L(\mathfrak{A}_{\mathcal{R}}(q))$.

Conditions (RT3) and (RS1) the atom $\exists i_R.C$ to the repair type (instead of $\exists R.C$). Furthermore, by synchronously traversing the transitions and the role assertions, we have $\mathcal{B} \models \exists q.C(u)$ iff one of the following statements is fulfilled:

- There is a transition (q, R, p) and there is an an object v such that $\mathcal{B} \models R(u, v)$ and $\mathcal{B} \models \exists p.C(v)$.
- There is a transition (q, R, f) where f is a final state and there is an object v such that $\mathcal{B} \models R(u, v)$ and $\mathcal{B} \models C(v)$.

So to make an object u no instance of $\exists q.C$, we must modify each R-successor of u such that it is no instance of $\exists p.C$ for each transition (q, R, p), see Conditions (RA2) and (RA4). At the same time, each R-successor of u must not be an instance of C for each transition (q, R, f) where f is a final state, see Conditions (RA3) and (RA5).

In contrast to the case for $\mathcal{EL}$, not every repair seed induces a repair, as illustrated by the next example.

Example XII. Consider the qABox $\{r(a, b), B(b)\}$ and the TBox $\{\{a\} \sqsubseteq \exists r.B,\ B \sqsubseteq \{b\}\}$. The qABox is already saturated. For the repair request $\{B(b)\}$, there is the repair seed with $\mathcal{S}_{[a]_\mathcal{S}} = \emptyset$ and $\mathcal{S}_{[b]_\mathcal{S}} = \{B\}$ (which is the only one), but no repair exists since the TBox already entails $B(b)$.

To characterize the repair seeds that do induce repairs, we need to introduce the following notions.

Definition XIII. Let $\mathcal{K}$ be a repair type for u. The *residual* of $\mathcal{K}$ w.r.t. u is defined as

$$\mathcal{K}^+(u) := \left\{\, D \;\middle|\; \begin{array}{l} D \in \mathsf{Sub}(\mathcal{T}, \mathcal{P}) \cup \mathsf{Atoms}_\mathcal{R}(\mathcal{T}, \mathcal{P}), \\ \mathcal{B} \models D(u), \text{ and} \\ D \not\sqsubseteq^{\mathcal{T},\mathcal{R}} C \text{ for all } C \in \mathcal{K} \end{array} \right\}.$$

Intuitively, $\mathcal{K}^+(u)$ contains the subconcepts D of which the copy $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$ should still be an instance after the repair.

Lemma XIV. *Let $\mathcal{K}$ be a repair type for u. For each $C \in \mathsf{Sub}(\mathcal{T}, \mathcal{P}) \cup \mathsf{Atoms}_\mathcal{R}(\mathcal{T}, \mathcal{P})$ where $\mathcal{B} \models C(u)$, we have either $C \in \mathcal{K}^+(u)$ or $\mathsf{Conj}_\mathcal{R}(C) \cap \mathcal{K} \neq \emptyset$.*

Proof. Consider a concept C in $\mathsf{Sub}(\mathcal{T}, \mathcal{P}) \cup \mathsf{Atoms}_\mathcal{R}(\mathcal{T}, \mathcal{P})$. If C is not in the residual $\mathcal{K}^+(u)$, then there is some atom D in the repair type $\mathcal{K}$ such that $C \sqsubseteq^{\mathcal{T},\mathcal{R}} D$. Condition (RT2) in Definition 14 enforces that $\mathsf{Conj}_\mathcal{R}(C) \cap \mathcal{K} \neq \emptyset$.

It remains to show that the two conditions are mutually exclusive. Assume that both would hold. From $\mathsf{Conj}_\mathcal{R}(C) \cap \mathcal{K} \neq \emptyset$ it would follow that there is a top-level conjunct $E \in \mathsf{Conj}(C)$ such that $(E)_\mathcal{R} \in \mathcal{K}$. Since $C \sqsubseteq^\emptyset E \sqsubseteq^\emptyset (E)_\mathcal{R}$, it would follow that $C \sqsubseteq^{\mathcal{T},\mathcal{R}} D$ for some $D \in \mathcal{K}$, i.e., $C \notin \mathcal{K}^+(u)$, a contradiction. □

Lemma XV. *Each residual is closed under subsumers w.r.t. $\mathcal{T}$, i.e., for each $C \in \mathcal{K}^+(u)$ and for each $D \in \mathsf{Sub}(\mathcal{T}, \mathcal{P})$ where $C \sqsubseteq^\mathcal{T} D$, it holds that $D \in \mathcal{K}^+(u)$.*

Proof. Consider two concepts $C \in \mathcal{K}^+(u)$ and $D \in \mathsf{Sub}(\mathcal{T}, \mathcal{P})$ such that $C \sqsubseteq^\mathcal{T} D$. From $C \in \mathcal{K}^+(u)$ it follows by Definition 14 that $\mathcal{B} \models C(u)$. Since $\mathcal{B}$ is $\mathcal{T}$-saturated, $C \sqsubseteq^\mathcal{T} D$ implies $\mathcal{B} \models D(u)$. Now consider some atom $E \in \mathcal{K}$; we must show that $D \not\sqsubseteq^{\mathcal{T},\mathcal{R}} E$. Assuming the contrary would immediately yield the contradiction that $C \sqsubseteq^{\mathcal{T},\mathcal{R}} E$ — this cannot hold as $C \in \mathcal{K}^+(u)$. □

Definition XVI. Let $\mathcal{S}$ be a repair seed and consider a subset Γ of

$$\begin{aligned} &\{\, \langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle \mid a \in \Sigma_\mathsf{I} \,\} \\ &\cup \left\{ \langle\!\langle [a]_\mathcal{S}, \mathcal{K} \rangle\!\rangle \;\middle|\; \begin{array}{r} a \in \Sigma_\mathsf{I},\ \mathcal{K} \text{ is a repair type for } a, \text{ and} \\ \{b\} \in \mathcal{K} \text{ for each } \{b\} \in \mathsf{Atoms}_\mathcal{R}(\mathcal{T}, \mathcal{P}) \\ \text{where } \exists Y.\mathcal{B} \models a \equiv b \end{array} \right\} \\ &\cup \{\, \langle\!\langle [x]_\mathcal{S}, \mathcal{K} \rangle\!\rangle \mid x \in Y \text{ and } \mathcal{K} \text{ is a repair type for } x \,\}. \end{aligned}$$

We say that Γ is *saturated* if the following conditions are fulfilled for each $\langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle \in \Gamma$:

(S1) for each $\exists R.C \in \mathcal{L}^+(v)$, there is some $\langle\!\langle [w]_\mathcal{S}, \mathcal{M} \rangle\!\rangle \in \Gamma$ such that $\langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle \xrightarrow{R} \langle\!\langle [w]_\mathcal{S}, \mathcal{M} \rangle\!\rangle$ and $C \in \mathcal{M}^+(w)$,

(S2) for each $\exists q.C \in \mathcal{L}^+(v)$, there is some $R_1 \cdots R_n \in L(\mathfrak{A}_\mathcal{R}(q))$ and $\langle\!\langle [w_1]_\mathcal{S}, \mathcal{M}_1 \rangle\!\rangle, \dots, \langle\!\langle [w_n]_\mathcal{S}, \mathcal{M}_n \rangle\!\rangle \in \Gamma$ such that $\langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle \xrightarrow{R_1} \langle\!\langle [w_1]_\mathcal{S}, \mathcal{M}_1 \rangle\!\rangle \xrightarrow{R_2} \cdots \xrightarrow{R_n} \langle\!\langle [w_n]_\mathcal{S}, \mathcal{M}_n \rangle\!\rangle$ and $C \in \mathcal{M}_n^+(w_n)$.

The whole set in the above definition contains all copies that, at least in principle, would make sense in a repair. First of all, these are the copies $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$ that will stand for the individual names. However, we must not consider copies $\langle\!\langle [a]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$ where the repair type $\mathcal{K}$ does not ensure that inference with the TBox and RBox cannot restore equality with a; the remaining ones are in the second part. Since equalities can never involve variables, it is unproblematic to consider copies of variables (those in the third part).

To see that, in the second part, we cannot replace the condition $\exists Y.\mathcal{B} \models a \equiv b$ with $a \approx_\mathcal{S} b$, recall that the residual $\mathcal{K}^+(a)$ should contain all subconcepts that are still satisfied by the copy $\langle\!\langle [a]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$ in the repair. If we would now have the situation where the saturation $\exists Y.\mathcal{B}$ entails $a \equiv b$ but the repair seed $\mathcal{S}$ is chosen such that $a \not\approx_\mathcal{S} b$, and further the nominal $\{b\}$ is in $\mathsf{Atoms}_\mathcal{R}(\mathcal{T}, \mathcal{P})$, then with the replaced condition we would allow a copy $\langle\!\langle [a]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$ where the repair type $\mathcal{K}$ does not contain $\{b\}$ — but then the residual $\mathcal{K}^+(a)$ could contain $\{b\}$, which means that the copy $\langle\!\langle [a]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$ would be identified with b, which is represented by $\langle\!\langle [b]_\mathcal{S}, \mathcal{S}_{[b]_\mathcal{S}} \rangle\!\rangle$. This could lead to undesired effects. The proof of Lemma XXIV gives a more sophisticated answer.

The notion of saturatedness is closely connected to the notion of a residual. Recall that we want each copy $\langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle$ to be an instance of all concepts in the residual $\mathcal{L}^+(v)$. In order to ensure this, we require above that each existential restriction $\exists R.C$ in $\mathcal{L}^+(v)$ has a *witness*, which is an R-successor $\langle\!\langle [w]_\mathcal{S}, \mathcal{M} \rangle\!\rangle$ where $C \in \mathcal{M}^+(w)$, and similarly for each automaton concept $\exists q.C$ in $\mathcal{L}^+(v)$.

It is easy to see that each union of saturated sets is saturated. We infer that there is a largest saturated set.

Definition XVII. The largest saturated set is denoted by $\Omega(\mathcal{S})$, and we call its elements the *admissible copies*.

The set $\Omega(\mathcal{S})$ can be computed by starting with the whole set in the above definition and then subsequently deleting copies that violate the saturatedness condition. Repair seeds should only assign repair types $\mathcal{S}_{[a]_\mathcal{S}}$ to individuals a such that the resulting copies $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$ are admissible.

Definition XVIII. Let $\mathcal{S}$ be a repair seed. We say that $\mathcal{S}$ is *admissible* if it additionally satisfies the following condition:

(RS3) $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle \in \Omega(\mathcal{S})$ for each $a \in \Sigma_\mathsf{I}$.

This restriction ensures that instance relationships for existential restrictions in the residual remain satisfied in the repair. Of course, a repair seed $\mathcal{S}$ is admissible iff there exists a saturated set containing $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$ for each individual a.

Example XIX. Fix the TBox $\{B \sqsubseteq \{b\},\ \exists r.B \sqsubseteq A\}$, the empty RBox, the (saturated) qABox $\{A(a),\ r(a,b),\ B(b)\}$, and the repair request $\{B(b)\}$. There are no equalities in the qABox and thus each equivalence class is a singleton. It follows that the equivalence relation $\approx_\mathcal{S}$ in a repair seed $\mathcal{S}$ is the reflexive relation on $\{a, b\}$. A particular repair seed is now obtained with the mappings $\mathcal{S}_{[a]_\mathcal{S}} := \emptyset$ and $\mathcal{S}_{[b]_\mathcal{S}} := \{B\}$. We are going to show that $\mathcal{S}$ is not admissible.

Consider the copy $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$; the residual $\mathcal{S}^+_{[a]_\mathcal{S}}(a)$ contains the atom $\exists r.B$. For each saturated subset Γ containing $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$, there must thus be a copy $\langle\!\langle [b]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$ such that $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle \xrightarrow{r} \langle\!\langle [b]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$ and the residual $\mathcal{K}^+(b)$ contains B. (As b is the only r-successor of a, the copy $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$ cannot have other r-successors). Since the repair type $\mathcal{S}_{[b]_\mathcal{S}}$ contains B, its residual w.r.t. b cannot contain B. Now consider another repair type $\mathcal{K}$ for b that satisfies the condition in Definition XVI, i.e., it must contain the nominal $\{b\}$. With Condition (RT2) it follows that $\mathcal{K}$ must also contain B, and thus the residual $\mathcal{K}^+(b)$ cannot contain B either. So there does not exist a saturated subset Γ that contains $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$.

In contrast, the repair seed $\mathcal{S}'$ where $\mathcal{S}'_{[a]_{\mathcal{S}'}} := \{\exists r.B\}$ and $\mathcal{S}'_{[b]_{\mathcal{S}'}} := \{B\}$ is admissible.

Furthermore, if we exchange the CI $B \sqsubseteq \{b\}$ with $\{b\} \sqsubseteq B$, then there is no repair seed at all. In particular, Condition (RS1) would enforce $B \in \mathcal{S}''_{[b]_{\mathcal{S}''}}$ and so Condition (RT2) would further require $\{b\} \in \mathcal{S}''_{[b]_{\mathcal{S}''}}$, which contradicts Condition (RS2).

Compared to the $\mathcal{EL}$ case in (Baader et al. 2021a), we needed to impose further substantial conditions on the seeds that induce the repairs. However, this is only a conservative extension: if the quantified ABox $\exists X.\mathcal{A}$ does not contain equalities, the TBox $\mathcal{T}$ as well as the repair request $\mathcal{P}$ are formulated in $\mathcal{EL}$, and the RBox $\mathcal{R}$ is empty, then each repair seed is admissible. The proof of the if direction of Lemma XII in (Baader et al. 2021b) specifically shows that, for each repair seed $\mathcal{S}$, the set $\Omega(\mathcal{S})$ consists of all $\langle\!\langle [t]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$ where t is an object of $\exists Y.\mathcal{B}$ and $\mathcal{K}$ is a repair type for t.

3.3 Canonical Repairs

We are now ready to define the repairs induced by admissible repair seeds. As in (Baader et al. 2021a), we call the repairs obtained this way "canonical." Recall that $\exists Y.\mathcal{B}$ denotes the saturation of $\exists X.\mathcal{A}$ w.r.t. the terminating terminology $(\mathcal{T}, \mathcal{R})$.

Definition XX. Let $\mathcal{S}$ be an admissible repair seed. The *canonical repair* of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T}, \mathcal{R})$ *induced by* $\mathcal{S}$ is defined as the qABox $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A}, \mathcal{S}) := \exists Z.\mathcal{C}$ that is constructed as follows:

(CR1) Add each admissible copy $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle \in \Omega(\mathcal{S})$ to Z, except if u is an individual and $\mathcal{K} = \mathcal{S}_{[u]_\mathcal{S}}$.

(CR2) Add the concept assertion $A(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle)$ to $\mathcal{C}$ for each $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle \in \Omega(\mathcal{S})$ where $\mathcal{B} \models A(u)$ and $A \notin \mathcal{K}$.

(CR3) Add the role ass. $r(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle, \langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle)$ to $\mathcal{C}$ for all $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle, \langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle \in \Omega(\mathcal{S})$ s.t. $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle \xrightarrow{r} \langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle$.

(CR4) For each equivalence class $[a]_\mathcal{S}$ where $a \in \Sigma_\mathsf{I}$,
- choose a representative $a' \in [a]_\mathcal{S}$,
- replace each occurrence of $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$ in $\mathcal{C}$ with a',
- and add the equality assertion $a' \equiv b$ to $\mathcal{C}$ for each individual $b \in [a]_\mathcal{S} \setminus \{a'\}$.

Afterwards, we treat $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$ and a' as synonyms.

In the remainder of this section we are going to prove several technical lemmas that hold for canonical repairs. Therefore assume that $\mathcal{S}$ is an admissible repair seed and that $\exists Z.\mathcal{C}$ is the canonical repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T}, \mathcal{R})$ induced by $\mathcal{S}$. Further recall that $\exists Y.\mathcal{B}$ is the saturation of $\exists X.\mathcal{A}$ w.r.t. $(\mathcal{T}, \mathcal{R})$.

We begin with proving some rather simple consequences of the interplay between Instructions (CR2), (CR3), and (CR4).

Lemma XXI. *The following statements hold.*

1. *For each individual a and for each individual b, the following are equivalent:*
   (a) $\mathcal{C} \models a \equiv b$
   (b) $\mathcal{C} \models \langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle \equiv b$
   (c) $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle \equiv b \in \mathcal{C}$ *or* $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle = b$.
   (d) $a \approx_\mathcal{S} b$
   (e) $a \approx_{\exists Z.\mathcal{C}} b$
2. *For each concept name A and for each individual a, the following are equivalent:*
   (a) $\mathcal{C} \models A(a)$
   (b) $\mathcal{C} \models A(\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle)$
   (c) $A(\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle) \in \mathcal{C}$
3. *For each role name r, for each individual a, and for each individual b, the following are equivalent:*
   (a) $\mathcal{C} \models r(a, b)$
   (b) $\mathcal{C} \models r(\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle, \langle\!\langle [b]_\mathcal{S}, \mathcal{S}_{[b]_\mathcal{S}} \rangle\!\rangle)$
   (c) $r(\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle, \langle\!\langle [b]_\mathcal{S}, \mathcal{S}_{[b]_\mathcal{S}} \rangle\!\rangle) \in \mathcal{C}$

Proof. 1. Statements 1a and 1b are equivalent since either $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$ equals a (namely if a was chosen as the representative of $[a]_\mathcal{S}$ in Instruction (CR4)) or the matrix $\mathcal{C}$ contains the equality assertion $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle \equiv a$ by Instruction (CR4). Statements 1a and 1e are equivalent by Corollary I. Furthermore, Statements 1b to 1d are equivalent by Instruction (CR4).

2. Recall that either the chosen representative $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$ equals a or $\mathcal{C}$ contains $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle \equiv a$ by Instruction (CR4). It follows that Statements 2a and 2b are equivalent. It is further trivial that Statement 2c implies Statement 2b.
   Now assume that the matrix $\mathcal{C}$ entails the concept assertion $A(\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle)$. According to Corollary I there is an object u' of $\exists Z.\mathcal{C}$ such that $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle \approx_{\exists Z.\mathcal{C}} u'$ and $A(u') \in \mathcal{C}$. The latter concept assertion $A(u')$ must

have been added to $\mathcal{C}$ by Instruction (CR2), i.e., u' must be of the form $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$.
Since no variable can occur in an equality assertion, $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ must be a synonym of an individual. Due to Instruction (CR4), the equivalence class of the representative $\langle\!\langle [a]_{\mathcal{S}}, \mathcal{S}_{[a]_{\mathcal{S}}} \rangle\!\rangle$ does not contain other synonyms than itself. We conclude that $\langle\!\langle [a]_{\mathcal{S}}, \mathcal{S}_{[a]_{\mathcal{S}}} \rangle\!\rangle$ equals $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$, which yields that $\mathcal{C}$ contains $A(\langle\!\langle [a]_{\mathcal{S}}, \mathcal{S}_{[a]_{\mathcal{S}}} \rangle\!\rangle)$.

3. The last claim can be proven in a similar manner as the second claim. $\square$

Next, we show that each canonical repair is entailed by the input qABox $\exists X.\mathcal{A}$ w.r.t. $(\mathcal{T}, \mathcal{R})$. According to Theorem 5 it therefore suffices to prove that a homomorphism from the canonical repair $\exists Z.\mathcal{C}$ to the saturation $\exists Y.\mathcal{B}$ exists.

Lemma XXII. *Each mapping h where $h(a) := a$ for each individual name $a \in \Sigma_\mathsf{I}$ and $h(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle) \in [u]_{\exists Y.\mathcal{B}}$ for each variable $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle \in Z$ is a homomorphism from $\exists Z.\mathcal{C}$ to $\exists Y.\mathcal{B}$, and there is at least one such mapping.*

Proof. Consider a mapping h as above. Since $\approx_{\mathcal{S}}$ is a refinement of $\approx_{\exists Y.\mathcal{B}}$, each equivalence class $[u]_{\mathcal{S}}$ is a subset of $[u]_{\exists Y.\mathcal{B}}$ and thus the choice of each value $h(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$ is independent of the representative of $[u]_{\mathcal{S}}$, i.e., h is well-defined. There is at least one such mapping as equivalence classes are never empty.

(Hom1) Consider an individual a and an individual b where $a \approx_{\exists Z.\mathcal{C}} b$. According to Lemma XXI the equivalence relations $\approx_{\exists Z.\mathcal{C}}$ and $\approx_{\mathcal{S}}$ are equal on individuals, and so we obtain $a \approx_{\mathcal{S}} b$. According to Definition 15, $\approx_{\mathcal{S}}$ is a refinement of $\approx_{\exists Y.\mathcal{B}}$ and thus $a \approx_{\exists Y.\mathcal{B}} b$. Since $h(a) = a$ and $h(b) = b$, we conclude that $h(a) \approx_{\exists Y.\mathcal{B}} h(b)$.

(Hom2) By the very definition of h, it holds that $h(a) = a$ for each individual a.

(Hom3) Consider a concept assertion $A(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$ in $\mathcal{C}$, which can only have been created by Instruction (CR2). It follows that $\mathcal{B}$ entails the concept assertion $A(u)$ and so Corollary I yields an object $u' \approx_{\exists Y.\mathcal{B}} u$ such that $A(u') \in \mathcal{B}$. With $h(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle) \in [u]_{\exists Y.\mathcal{B}}$ we infer that $u' \approx_{\exists Y.\mathcal{B}} h(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$.

(Hom4) Consider a role assertion $r(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle, \langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle)$ in $\mathcal{C}$, which can only have been introduced by Instruction (CR3). Due to Definition XI it follows that $\mathcal{B}$ entails the role assertion $r(u, v)$ and thus Corollary I yields an object $u' \approx_{\exists Y.\mathcal{B}} u$ as well as an object $v' \approx_{\exists Y.\mathcal{B}} v$ such that $r(u', v') \in \mathcal{B}$. Since $h(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle) \in [u]_{\exists Y.\mathcal{B}}$ and $h(\langle\!\langle [v]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle) \in [v]_{\exists Y.\mathcal{B}}$, we conclude that $u' \approx_{\exists Y.\mathcal{B}} h(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$ and $v' \approx_{\exists Y.\mathcal{B}} h(\langle\!\langle [v]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$. $\square$

The following lemma is key to proving correctness of the repairs. Specifically, it shows that each copy $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ is no instance of each atom in the repair type $\mathcal{K}$.

Lemma XXIII. *For each copy $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle \in \Omega(\mathcal{S})$ and for each atom $C \in \mathsf{Atoms}(\mathcal{T}, \mathcal{P}) \cup \mathsf{Atoms}_{\mathcal{R}}(\mathcal{T}, \mathcal{P})$, it holds that $(C)_{\mathcal{R}} \in \mathcal{K}$ implies $\mathcal{C} \not\models C(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$.*

Proof. We show the claim by induction over C.

1. Consider a concept name A that is contained in $\mathcal{K}$. By Instruction (CR2) we obtain $A(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle) \notin \mathcal{C}$. According to Lemma XXI it follows that $\mathcal{C} \not\models A(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$.
2. Consider a nominal $\{a\}$ that is contained in $\mathcal{K}$. We start with the case where $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ is no synonym of an individual name, i.e., either u is a variable of $\exists Y.\mathcal{B}$ or the repair type $\mathcal{K}$ is not equal to $\mathcal{S}_{[u]_{\mathcal{S}}}$. Since in Instruction (CR4) of Definition XX we only add to $\mathcal{C}$ equalities involving individual names, $\mathcal{C}$ cannot entail $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle \equiv a$. It follows that $\mathcal{C} \not\models \{a\}(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$.
In the remaining case, assume that $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ is a synonym of the individual name b and thus u must be an individual such that $u \approx_{\mathcal{S}} b$ and the repair type $\mathcal{K}$ must be equal to $\mathcal{S}_{[u]_{\mathcal{S}}}$, cf. Instruction (CR4). It follows that $\{a\} \in \mathcal{S}_{[b]_{\mathcal{S}}}$ and so Condition (RS2) in Definition 15 yields $a \not\approx_{\mathcal{S}} b$. We infer with Lemma XXI that the equality $a \equiv b$ is not entailed by $\mathcal{C}$, and so we conclude that $\mathcal{C} \not\models \{a\}(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$.
3. Consider an existential restriction $\exists R.C$ such that $(\exists R.C)_{\mathcal{R}} = \exists i_R.C$ is contained in $\mathcal{K}$. We need to prove that $\mathcal{C}$ does not entail $\exists R.C(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$. According to Corollary I, it suffices to show that, for each role assertion in $\mathcal{C}$ with the role R where the object in first position is equivalent to $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$, the object in second position is no instance of the filler C. Due to Instruction (CR3), the objects in both positions can only be in $\Omega(\mathcal{S})$. Furthermore, within its equivalence class $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ is the only element in $\Omega(\mathcal{S})$. We thus need to consider only $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ in first position.
So, let $R(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle, \langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle)$ be a role assertion in $\mathcal{C}$, i.e., $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle \xrightarrow{R} \langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$ holds by Instruction (CR3). We will show that $\mathcal{C}$ does not entail $C(\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle)$.
Since $R \sqsubseteq^{\mathcal{R}} R$, there is a transition (i_R, R, f) towards a final state f. By Condition (RA3) or (RA5) in Definition XI (depending on whether R is an inverse role or not) it follows that $\mathsf{Conj}_{\mathcal{R}}(C) \cap \mathcal{L} \neq \emptyset$, i.e., there is a top-level conjunct D of C where $(D)_{\mathcal{R}} \in \mathcal{L}$. As the role depth of D is smaller than the role depth of C, we can apply the induction hypothesis and so obtain that $\mathcal{C}$ does not entail $D(\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle)$. It follows that $\mathcal{C}$ cannot entail $C(\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle)$ either.
4. Last, consider an automaton concept $\exists q.C$ in $\mathcal{K}$. We show that $\mathcal{C} \not\models \exists R_1. \cdots \exists R_n.C(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$ for each word $R_1 \cdots R_n \in L(\mathfrak{A}_{\mathcal{R}}(q))$. Consider such a word; there are transitions (q_0, R_1, q_1), (q_1, R_2, q_2), $\ldots$, (q_{n-1}, R_n, q_n) where $q_0 = q$ and q_n is final. If there is no copy $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$ where $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle \xrightarrow{R_1} \cdots \xrightarrow{R_n} \langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$, then the entailment can obviously not hold, cf. Corollary I. Otherwise, it is a finger exercise to show by induction and using Definition XI that either $\mathcal{B} \not\models C(v)$ or $\mathsf{Conj}_{\mathcal{R}}(C) \cap \mathcal{L} \neq \emptyset$ must hold. By Lemma XXII the former implies $\mathcal{C} \not\models C(\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle)$, and by induction hypothesis the latter implies $\mathcal{C} \not\models C(\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle)$ as well. $\square$

Next, we prove that the residual of each copy consists exactly of those subconcepts the copy is an instance of w.r.t. the canonical repair. It will be used to show that each canonical repair is already saturated w.r.t. the concept inclusions in $\mathcal{T}$.

Lemma XXIV. *For each copy $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle \in \Omega(\mathcal{S})$ and for each concept description $D \in \mathsf{Sub}(\mathcal{T}, \mathcal{P}) \cup \mathsf{Atoms}_{\mathcal{R}}(\mathcal{T}, \mathcal{P})$, it holds that $D \in \mathcal{K}^{+}(u)$ iff $\mathcal{C} \models D(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$.*

Proof. We first show the only-if direction. Therefore consider an admissible copy $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ and a concept D contained in the residual $\mathcal{K}^{+}(u)$. Recall from Definition XIII that the latter implies $\mathcal{B} \models D(u)$ and $D \not\sqsubseteq^{\mathcal{T},\mathcal{R}} C$ for each $C \in \mathcal{K}$. We prove $\mathcal{C} \models D(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$ by induction over D.

- The case $D = \top$ is trivial.
- Let $D = A$ be a concept name. It follows that $\mathcal{B}$ entails the assertion $A(u)$, and further that $\mathcal{K}$ does not contain A (otherwise we get the immediate contradiction $D \sqsubseteq^{\mathcal{T},\mathcal{R}} A \in \mathcal{K}$). According to Instruction (CR2), the matrix $\mathcal{C}$ contains the concept assertion $A(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$ and so it must also be entailed.
- Assume that $D = \{a\}$ is a nominal. We conclude that $\mathcal{B}$ entails the equality $u \equiv a$, and that $\mathcal{K}$ does not contain $\{a\}$. Since no variables occur in equalities, u must be an individual. Since $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ is an admissible copy, it follows from Definition XVI that either $\mathcal{K} = \mathcal{S}_{[u]_{\mathcal{S}}}$ or $\{b\} \in \mathcal{K}$ for each nominal $\{b\} \in \mathsf{Atoms}(\mathcal{T}, \mathcal{P})$ where $\exists Y.\mathcal{B} \models u \equiv b$. So $\{a\} \notin \mathcal{K}$ implies $\mathcal{K} = \mathcal{S}_{[u]_{\mathcal{S}}}$.
  It further follows that $\{a\} \notin \mathcal{S}_{[u]_{\mathcal{S}}}$, i.e., $a \approx_{\mathcal{S}} u$ by Condition (RS2) of Definition 15. An application of Lemma XXI shows that $\mathcal{C} \models a \equiv u$. Furthermore, $\mathcal{C}$ contains the equality assertion $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle \equiv u$ by Instruction (CR4) or $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ equals u (i.e., u is the chosen representative of $[u]_{\mathcal{S}}$). We conclude that $\mathcal{C}$ entails $a \equiv \langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$, i.e., $\mathcal{C} \models \{a\}(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$.
- If D is a conjunction, then the claim easily follows by an application of the induction hypothesis for each top-level conjunct of D.
- Consider an existential restriction $D = \exists R.E$ in the residual $\mathcal{K}^{+}(u)$. According to Definition XVI there is some admissible copy $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle \in \Omega(\mathcal{S})$ such that $E \in \mathcal{L}^{+}(v)$ and $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle \xrightarrow{R} \langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$. From the former we infer by an application of the induction hypothesis that $\mathcal{C}$ entails $E(\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle)$, and the latter implies that $\mathcal{C}$ contains the role assertion $R(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle, \langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle)$, cf. Instruction (CR3) in Definition XX. We conclude that $\mathcal{C}$ entails $\exists R.E(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$.
- Last, assume that $D = \exists q.E$ is an automaton concept in the residual $\mathcal{K}^{+}(u)$. Definition XVI implies that there is a word $R_1 \cdots R_n \in L(\mathfrak{A}_{\mathcal{R}}(q))$ and there is an admissible copy $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle \in \Omega(\mathcal{S})$ where $E \in \mathcal{L}^{+}(v)$ and $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle \xrightarrow{R_1} \cdots \xrightarrow{R_n} \langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$. By induction hypothesis the former implies $\mathcal{C} \models E(\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle)$, and the latter means that the matrix $\mathcal{C}$ contains an $R_1 \cdots R_n$-chain of role assertions from $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ to $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$. From this we infer with Corollary I that $\mathcal{C} \models \exists R_1. \cdots \exists R_n.E(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$, and finally Lemma 7 yields that $\mathcal{C} \models \exists q.E(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$.

It remains to prove the if direction. First of all, $\mathcal{C} \models D(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$ implies $\mathcal{B} \models D(u)$ by Lemmas II and XXII, and additionally by Lemma 7 if D is an automaton concept. Now assume that D were not in the residual $\mathcal{K}^{+}(u)$, i.e., according to Definition XIII there would be some atom $C \in \mathcal{K}$ such that $D \sqsubseteq^{\mathcal{T},\mathcal{R}} C$. By Condition (RT2) in Definition 14 it would follow that $\mathsf{Conj}_{\mathcal{R}}(D) \cap \mathcal{K} \neq \emptyset$. Then Lemma XXIII would yield the contradiction that $\mathcal{C} \not\models D(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$. ↯ ☐

After the two structural lemmas our next steps will be to prove that, firstly, each canonical repair is saturated w.r.t. $\mathcal{T}$ and, secondly, each canonical repair is saturated w.r.t. $\mathcal{R}$.

Lemma XXV. *The concept inclusion rule from Figure 1 is not applicable to $\exists Z.\mathcal{C}$.*

Proof. Consider a concept inclusion $C \sqsubseteq D$ in $\mathcal{T}$ as well as an object name $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ of $\exists Z.\mathcal{C}$. Assume that $\mathcal{C}$ entails $C(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$. Then Lemma XXIV yields that the residual $\mathcal{K}^{+}(u)$ contains C. Since $\mathcal{T}$ contains $C \sqsubseteq D$, Lemma XV implies that the residual $\mathcal{K}^{+}(u)$ contains D as well. An application of Lemma XXIV yields that $\mathcal{C} \models D(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$, i.e., the concept inclusion rule is not applicable to $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ for $C \sqsubseteq D$.

The concept inclusion rule is also not applicable to an individual name a since otherwise it would be applicable to the admissible copy $\langle\!\langle [a]_{\mathcal{S}}, \mathcal{S}_{[a]_{\mathcal{S}}} \rangle\!\rangle$. ☐

Lemma XXVI. *The role inclusion rule from Figure 1 is not applicable to $\exists Z.\mathcal{C}$.*

Proof. Consider a role inclusion $S_1 \circ \cdots \circ S_n \sqsubseteq R$ in the RBox $\mathcal{R}$, and further let $\langle\!\langle [u_0]_{\mathcal{S}}, \mathcal{K}_0 \rangle\!\rangle \xrightarrow{S_1} \langle\!\langle [u_1]_{\mathcal{S}}, \mathcal{K}_1 \rangle\!\rangle$, $\ldots$, $\langle\!\langle [u_{n-1}]_{\mathcal{S}}, \mathcal{K}_{n-1} \rangle\!\rangle \xrightarrow{S_n} \langle\!\langle [u_n]_{\mathcal{S}}, \mathcal{K}_n \rangle\!\rangle$. We show that $\langle\!\langle [u_0]_{\mathcal{S}}, \mathcal{K}_0 \rangle\!\rangle \xrightarrow{R} \langle\!\langle [u_n]_{\mathcal{S}}, \mathcal{K}_n \rangle\!\rangle$. We consider the case where R is a role name, otherwise the proof of Condition (RA2) becomes the proof of Condition (RA4) and vice versa, and accordingly for Conditions (RA3) and (RA5).

(RA1) We infer from the assumption that the saturation $\exists Y.\mathcal{B}$ must entail the role assertion $S_i(u_{i-1}, u_i)$ for each index $i \in \{1, \ldots, n\}$, cf. Instruction (CR3) and Definition XI. It follows that $\exists Y.\mathcal{B}$ entails the role assertion $R(u_0, u_n)$ since it is $\mathcal{R}$-saturated.

(RA2) Consider an existential restriction $\exists q_0.C \in \mathcal{K}_0$ as well as a transition (q_0, R, p) where $\mathcal{B} \models \exists p.C(u_n)$. We must prove that there is a state p' such that $p \leq p'$ and $\exists p'.C \in \mathcal{K}_n$.

1. Specifically, let T be a role such that $p \in Q_T$. Since the finite automaton $\mathfrak{A}_T$ does not contain unreachable states, there must be some word x such that $\mathfrak{A}_T$ reaches q_0 from its initial state i_T when reading x. Furthermore, as $\mathfrak{A}_T$ does not contain dead states, there must be some word y such that $\mathfrak{A}_T$ reaches one of its final states from p when reading y. We conclude that $\mathfrak{A}_T$ accepts the word xRy, and thus $\mathcal{R}$ entails $xRy \sqsubseteq T$. Since $\mathcal{R}$ contains the role inclusion $S_1 \circ \cdots \circ S_n \sqsubseteq R$, it follows that $\mathcal{R}$ entails $xS_1 \cdots S_n y \sqsubseteq T$.
   Due to determinacy of $\mathfrak{A}_T$, the accepting run for $xS_1 \cdots S_n y$ must reach q_0 from i_T after reading the prefix x, and we have $S_1 \cdots S_n y \in L(\mathfrak{A}_{\mathcal{R}}(q_0))$. So there must be (unique) transitions (q_0, S_1, p_1^0), (p_1^0, S_2, p_2^0), $\ldots$, (p_{n-1}^0, S_n, p_n^0), where $y \in L(\mathfrak{A}_{\mathcal{R}}(p_n^0))$. Due to determinacy this holds for

all y, and so we have $L(\mathfrak{A}_\mathcal{R}(p)) \subseteq L(\mathfrak{A}_\mathcal{R}(p_n^0))$, i.e., $p \leq p_n^0$.

With Lemma VII we infer from $\mathcal{B} \models \exists p.C(u_n)$ that $\mathcal{B} \models \exists p_n^0.C(u_n)$. By induction it follows that $\mathcal{B} \models \exists p_1^0.C(u_1)$, namely due to the transitions $(p_1^0, S_2, p_2^0), \ldots, (p_{n-1}^0, S_n, p_n^0)$ and the role assertions $S_2(u_1, u_2), \ldots, S_n(u_{n-1}, u_n)$.

From $\langle\!\langle [u_0]_\mathcal{S}, \mathcal{K}_0 \rangle\!\rangle \xrightarrow{S_1} \langle\!\langle [u_1]_\mathcal{S}, \mathcal{K}_1 \rangle\!\rangle$, $\exists q_0.C \in \mathcal{K}_0$, the transition (q_0, S_1, p_1^0), and $\mathcal{B} \models \exists p_1^0.C(u_1)$, we infer with Condition (RA2) or (RA4) in Definition XI (depending on whether S_1 is a role name or an inverse role) that there is some state q_1 such that $p_1^0 \leq q_1$ and $\exists q_1.C \in \mathcal{K}_1$.

2. We continue with an induction over $i \in \{1, \ldots, n\}$, for which the induction base is above. The whole induction is visualized in Figure 2.

   Since $p_i^{i-1} \leq q_i$, we infer that the role language $\{S_{i+1} \cdots S_n\} \circ L(\mathfrak{A}_\mathcal{R}(p_n^{i-1}))$ is a subset of $L(\mathfrak{A}_\mathcal{R}(q_i))$. Due to determinacy of $\mathfrak{A}_\mathcal{R}$, there are transitions $(q_i, S_{i+1}, p_{i+1}^i)$, $(p_{i+1}^i, S_{i+2}, p_{i+2}^i)$, $\ldots$, (p_{n-1}^i, S_n, p_n^i) such that $L(\mathfrak{A}_\mathcal{R}(p_n^{i-1})) \subseteq L(\mathfrak{A}_\mathcal{R}(p_n^i))$, i.e., where $p_n^{i-1} \leq p_n^i$.

   With Lemma VII we infer from $\mathcal{B} \models \exists p_n^{i-1}.C(u_n)$ that $\mathcal{B} \models \exists p_n^i.C(u_n)$. By induction it follows that $\mathcal{B} \models \exists p_{i+1}^i.C(u_{i+1})$, using the transitions $(p_{i+1}^i, S_{i+2}, p_{i+2}^i), \ldots, (p_{n-1}^i, S_n, p_n^i)$ and the role assertions $S_{i+2}(u_{i+1}, u_{i+2}), \ldots, S_n(u_{n-1}, u_n)$.

   From $\langle\!\langle [u_i]_\mathcal{S}, \mathcal{K}_i \rangle\!\rangle \xrightarrow{S_{i+1}} \langle\!\langle [u_{i+1}]_\mathcal{S}, \mathcal{K}_{i+1} \rangle\!\rangle$, $\exists q_i.C \in \mathcal{K}_i$, the transition $(q_i, S_{i+1}, p_{i+1}^i)$, and $\mathcal{B} \models \exists p_{i+1}^i.C(u_{i+1})$, we infer from Condition (RA2) or (RA4) in Definition XI (depending on whether S_{i+1} is a role name or an inverse role) that there is some state q_{i+1} such that $p_{i+1}^i \leq q_{i+1}$ and $\exists q_{i+1}.C \in \mathcal{K}_{i+1}$.

3. Specifically, $\mathcal{K}_n$ contains $\exists q_n.C$. Furthermore, it holds that $p \leq p_n^0 \leq p_n^1 \leq \cdots \leq p_n^{n-1} \leq q_n$ and so we are done.

(RA3) Consider an existential restriction $\exists q_0.C \in \mathcal{K}_0$ as well as a transition (q_0, R, f) where f is a final state and $\mathcal{B} \models C(u_n)$. We need to show that $\mathsf{Conj}_\mathcal{R}(C) \cap \mathcal{K}_n \neq \emptyset$. From the assumption we get that $R \in L(\mathfrak{A}_\mathcal{R}(q_0))$.

Similarly as above we conclude that $S_1 \cdots S_n \in L(\mathfrak{A}_\mathcal{R}(q_0))$, and we can find a sequence of states $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$ such that (q_{i-1}, S_i, p_i) is a transition, and $S_{i+1} \cdots S_n \in L(\mathfrak{A}_\mathcal{R}(p_i))$, and $p_i \leq q_i$, and $\exists q_i.C \in \mathcal{K}_i$ for each index i. In particular, we have $\exists q_{n-1}.C \in \mathcal{K}_{n-1}$, and (q_{n-1}, S_n, p_n) is a transition, and $\varepsilon \in L(\mathfrak{A}_\mathcal{R}(p_n))$. The latter implies that p_n must be a final state. By means of Condition (RA3) or (RA5) in Definition XI (depending on whether R is a role name or an inverse role) it follows that $\mathsf{Conj}_\mathcal{R}(C) \cap \mathcal{K}_n \neq \emptyset$ as needed.

(RA4), (RA5) The remaining two conditions can be proved similarly. $\square$

We close this section with our first proposition, namely that each canonical repair is in fact a repair.

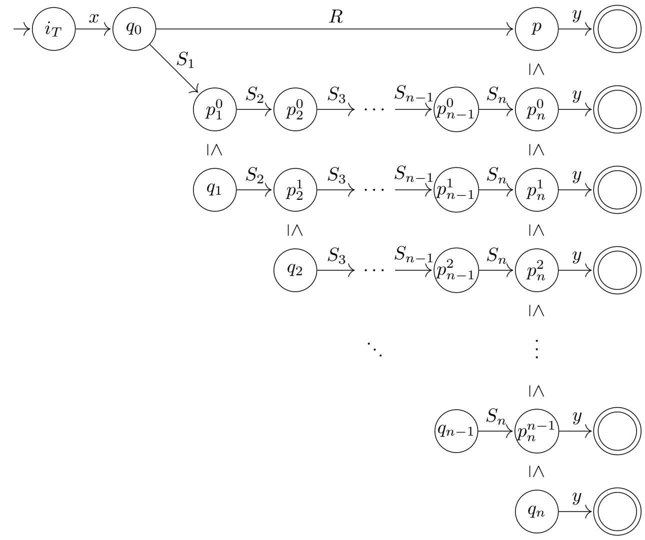

Figure 2: The induction that yields a state q_n such that $p \leq q_n$ and $\exists q_n.C \in \mathcal{K}_n$

Proposition XXVII. *For each admissible repair seed $\mathcal{S}$, the canonical repair induced by $\mathcal{S}$ is a repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T}, \mathcal{R})$.*

Proof. Let $\exists Z.\mathcal{C}$ be the canonical repair induced by an admissible repair seed $\mathcal{S}$. We are going to verify that it satisfies the three conditions in Definition 8.

(Rep1) We infer $\exists Y.\mathcal{B} \models \exists Z.\mathcal{C}$ from Lemma XXII and Proposition 2. Thus, Theorem 5 yields that $\exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists Z.\mathcal{C}$.

(Rep2) Consider an unwanted concept assertion $C(a) \in \mathcal{P}_{\mathsf{loc}}$. We need to show that $\exists Z.\mathcal{C} \not\models^{\mathcal{T},\mathcal{R}} C(a)$. According to Lemmas XXV and XXVI neither the concept inclusion rule nor the role inclusion rule from Figure 1 is applicable to the canonical repair $\exists Z.\mathcal{C}$, i.e., it equals its saturation w.r.t. $\mathcal{T}$ and $\mathcal{R}$. By Theorem 5 it thus suffices to prove that $\exists Z.\mathcal{C} \not\models C(a)$.

If $\mathcal{B} \not\models C(a)$, then $\mathcal{C} \not\models C(a)$ by Lemma XXII. Otherwise, assume $\mathcal{B} \models C(a)$. By Condition (RS1) in Definition 15 we have $\mathsf{Conj}_\mathcal{R}(C) \cap \mathcal{S}_{[a]_\mathcal{S}} \neq \emptyset$. Thus Lemma XXIII yields that $\mathcal{C} \not\models C(\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle)$. If in Instruction (CR4) we chose a as the representative of $[a]_\mathcal{S}$, then a and $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$ are synonyms and it thus follows that $\mathcal{C} \not\models C(a)$. Otherwise, $\mathcal{C}$ contains the equality assertion $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle \equiv a$ and we conclude that $\mathcal{C} \not\models C(a)$.

(Rep3) Let $C \in \mathcal{P}_{\mathsf{glo}}$; we must show that $\exists Z.\mathcal{C} \not\models^{\mathcal{T},\mathcal{R}} \exists\{x\}.\{C(x)\}$. Similarly as above, it suffices to prove that $\exists Z.\mathcal{C} \not\models \exists\{x\}.\{C(x)\}$. According to Proposition 2, this is satisfied if no object of $\exists Z.\mathcal{C}$ is an instance of C.

Consider an admissible copy $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$, i.e., $\mathcal{K}$ is a repair type for u. If $\mathcal{B} \not\models C(u)$, then Lemma XXII implies that $\mathcal{C} \not\models C(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle)$. Otherwise, due to Condition (RT3) it holds that $\mathsf{Conj}_\mathcal{R}(C) \cap \mathcal{K} \neq \emptyset$, and so Lemma XXIII yields that $\mathcal{C} \not\models C(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle)$.

Finally, also no individual b can be an instance of C as otherwise the admissible copy $\langle\!\langle [b]_{\mathcal{S}}, \mathcal{S}_{[b]_{\mathcal{S}}} \rangle\!\rangle$ would be an instance of C, which contradicts the above. $\square$

3.4 Proof of Completeness

Our second proposition formulates completeness of the canonical repairs in the sense that each repair is entailed by a canonical one. A brief summary of the proof is as follows. Assume that $\exists W.\mathcal{D}$ is a repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T}, \mathcal{R})$, which is w.l.o.g. saturated. So there is a homomorphism h from $\exists W.\mathcal{D}$ to the saturation of $\exists X.\mathcal{A}$. For each object name t of $\exists W.\mathcal{D}$, we define a set $\mathcal{F}(t)$ that consists of all atoms of which t is no instance w.r.t. (the matrix of) the repair $\exists W.\mathcal{D}$ but of which $h(t)$ is an instance w.r.t. (the matrix of) the saturation of $\exists X.\mathcal{A}$, and then we show that $\mathcal{F}(t)$ is a repair type for $h(t)$. An admissible repair seed $\mathcal{S}$ is then obtained by defining its equivalence relation by $a \approx_{\mathcal{S}} b$ iff $a \approx_{\exists W.\mathcal{D}} b$, and by defining $\mathcal{S}_{[a]_{\mathcal{S}}}$ as $\mathcal{F}(a)$ for each individual name a. Finally, we prove that the mapping $t \mapsto \langle\!\langle [h(t)]_{\mathcal{S}}, \mathcal{F}(t) \rangle\!\rangle$ is a homomorphism from the repair $\exists W.\mathcal{D}$ to the canonical repair induced by $\mathcal{S}$.

In principle, this means that we gather together objects of the repair $\exists W.\mathcal{D}$ that are mapped by h to the same object and that do *not* satisfy the same atoms. However, we obtain the canonical repair not by filtration of $\exists W.\mathcal{D}$, but it is instead directly constructed from the saturation of $\exists X.\mathcal{A}$ and the admissible repair seed $\mathcal{S}$ as per Definitions XI and XX.

Within the below completeness proof we need the next lemma, which shows that each atom E and its corresponding $\mathcal{R}$-extended atom $(E)_{\mathcal{R}}$ have the same instances in the saturation $\exists Y.\mathcal{B}$ of $\exists X.\mathcal{A}$.

Lemma XXVIII. *Let $\exists Y.\mathcal{B}$ be saturated w.r.t. $(\mathcal{T}, \mathcal{R})$. For each atom $E \in \mathsf{Atoms}(\mathcal{T}, \mathcal{P}) \cup \mathsf{Atoms}_{\mathcal{R}}(\mathcal{T}, \mathcal{P})$ and for each object t of $\exists Y.\mathcal{B}$, it holds that $\mathcal{B} \models E(t)$ iff $\mathcal{B} \models (E)_{\mathcal{R}}(t)$.*

Proof. The claim is trivial if E is a concept name, a nominal, or an automaton concept. For an existential restriction it follows from Lemma VI since $\exists Y.\mathcal{B}$ is saturated w.r.t. $(\mathcal{T}, \mathcal{R})$. $\square$

Proposition XXIX. *Each repair is entailed by a canonical repair. Specifically, for each repair $\exists W.\mathcal{D}$ of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T}, \mathcal{R})$, there is an admissible repair seed $\mathcal{S}$ such that the induced canonical repair $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A}, \mathcal{S})$ entails $\exists W.\mathcal{D}$.*

Proof. Consider a repair $\exists W'.\mathcal{D}'$ of $\exists X.\mathcal{A}$. Further let $\exists W.\mathcal{D}$ be the saturation of $\exists W'.\mathcal{D}'$. We are going to show that there exists an admissible repair seed $\mathcal{S}$ such that $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A}, \mathcal{S})$ entails $\exists W.\mathcal{D}$. Since $\exists W.\mathcal{D}$ entails $\exists W'.\mathcal{D}'$, it then follows that $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A}, \mathcal{S})$ also entails $\exists W'.\mathcal{D}'$ as claimed.

Since $\exists W'.\mathcal{D}'$ is a repair, it holds that $\exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists W'.\mathcal{D}'$. Furthermore, we infer with Theorem 5 that $\exists W'.\mathcal{D}' \models^{\mathcal{T},\mathcal{R}} \exists W.\mathcal{D}$. It follows that $\exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists W.\mathcal{D}$. Recall that $\exists Y.\mathcal{B}$ is the saturation of $\exists X.\mathcal{A}$ and thus Theorem 5 yields that $\exists Y.\mathcal{B} \models \exists W.\mathcal{D}$. With an application of Proposition 2 we infer that there exists a homomorphism h from $\exists W.\mathcal{D}$ to $\exists Y.\mathcal{B}$.

We subdivide the remainder of the proof into three steps, indicated with bold roman numbers.

I. For each object name t of $\exists W.\mathcal{D}$, we define

$$\mathcal{F}(t) \coloneqq \left\{ C \;\middle|\; \begin{array}{l} C \in \mathsf{Atoms}_{\mathcal{R}}(\mathcal{T}, \mathcal{P}),\ \mathcal{D} \not\models C(t), \\ \text{and } \mathcal{B} \models C(h(t)) \end{array} \right\}.$$

We verify that each $\mathcal{F}(t)$ is a repair type for $h(t)$, by checking the three conditions in Definition 14.

(RT1) We already have by definition that $\mathcal{B} \models C(h(t))$ for each atom $C \in \mathcal{F}(t)$.

(RT2) Let $C \in \mathcal{F}(t)$ and $D \in \mathsf{Sub}(\mathcal{T}, \mathcal{P}) \cup \mathsf{Atoms}_{\mathcal{R}}(\mathcal{T}, \mathcal{P})$ where $D \sqsubseteq^{\mathcal{T},\mathcal{R}} C$ and $\mathcal{B} \models D(h(t))$. From $C \in \mathcal{F}(t)$ we infer that $\mathcal{D} \not\models C(t)$. Since $\exists W.\mathcal{D}$ is saturated w.r.t. $(\mathcal{T}, \mathcal{R})$, it follows that $\mathcal{D} \not\models^{\mathcal{T},\mathcal{R}} C(t)$. Thus $D \sqsubseteq^{\mathcal{T},\mathcal{R}} C$ implies that $\mathcal{D} \not\models^{\mathcal{T},\mathcal{R}} D(t)$.
It follows that $\mathcal{D} \not\models D(t)$ and so there is a top-level conjunct $E \in \mathsf{Conj}(D)$ such that $\mathcal{D} \not\models E(t)$. According to Lemma XXVIII we obtain that $\mathcal{D} \not\models (E)_{\mathcal{R}}(t)$.
Of course, $\mathcal{B}$ must entail $E(h(t))$. Thus Lemma XXVIII yields that $\mathcal{B} \models (E)_{\mathcal{R}}(h(t))$. Summing up, we can conclude that $\mathcal{F}(t)$ contains $(E)_{\mathcal{R}}$.

(RT3) Consider a concept $C \in \mathcal{P}_{\mathsf{glo}}$ where $\mathcal{B} \models C(h(t))$. Since $\exists W.\mathcal{D}$ satisfies Condition (Rep3) in Definition 8, it follows that $\mathcal{D} \not\models C(t)$. So there is a top-level conjunct $D \in \mathsf{Conj}(C)$ such that $\mathcal{D} \not\models D(t)$. Lemma XXVIII yields that $\mathcal{D} \not\models (D)_{\mathcal{R}}(t)$, and Lemma XXVIII implies that $\mathcal{B} \models (D)_{\mathcal{R}}(h(t))$. We conclude that $\mathcal{F}(t)$ contains $(D)_{\mathcal{R}}$.

II. Next, we show that $\mathcal{S}$ is an admissible repair seed, where $\approx_{\mathcal{S}}$ is defined as the binary relation $\{ (a, b) \mid a, b \in \Sigma_{\mathsf{I}} \text{ and } a \approx_{\exists W.\mathcal{D}} b \} \cup \{ (x, x) \mid x \in Y \}$ on $\mathsf{Obj}(\exists Y.\mathcal{B})$, and where $\mathcal{S}_{[a]_{\mathcal{S}}} \coloneqq \mathcal{F}(a)$ for each individual a. We do so by verifying the conditions in Definitions 15 and XVIII.

Since $\approx_{\exists W.\mathcal{D}}$ is an equivalence relation, it follows that also $\approx_{\mathcal{S}}$ is one. We must show that $\approx_{\mathcal{S}}$ is a refinement of $\approx_{\exists Y.\mathcal{B}}$. Therefore let $a \approx_{\mathcal{S}} b$, i.e., $a \approx_{\exists W.\mathcal{D}} b$ by definition of $\approx_{\mathcal{S}}$ and so $\exists W.\mathcal{D} \models a \equiv b$ by Corollary I. Since $\exists Y.\mathcal{B} \models \exists W.\mathcal{D}$, it follows that $\exists Y.\mathcal{B} \models a \equiv b$ and thus $a \approx_{\exists Y.\mathcal{B}} b$ by Corollary I.

We further prove that the mapping $\mathcal{S}$ is well-defined. Assume $a \approx_{\mathcal{S}} b$. Then both $\exists W.\mathcal{D}$ and $\exists Y.\mathcal{B}$ entail the equality $a \equiv b$. Condition (Hom2) in Definition 1 further yields $h(a) = a$ and $h(b) = b$. It follows that $\mathcal{F}(a) = \mathcal{F}(b)$.

(RS1) Let $C(a) \in \mathcal{P}_{\mathsf{loc}}$ where $\mathcal{B} \models C(a)$. Since $\exists W.\mathcal{D}$ is a repair, it holds that $\mathcal{D} \not\models^{\mathcal{T},\mathcal{R}} C(a)$. We infer that $\mathcal{D} \not\models C(a)$. So there must be a top-level conjunct $E \in \mathsf{Conj}(C)$ such that $\mathcal{D} \not\models E(a)$. An application of Lemma XXVIII yields that $\mathcal{D} \not\models (E)_{\mathcal{R}}(a)$.
From $\mathcal{B} \models C(a)$ we infer by means of Lemma XXVIII that $\mathcal{B} \models (E)_{\mathcal{R}}(a)$. We conclude that $\mathcal{F}(a)$ contains $(E)_{\mathcal{R}}$, and so $\mathsf{Conj}_{\mathcal{R}}(C) \cap \mathcal{S}_{[a]_{\mathcal{S}}}$ is not empty.

(RS2) Recall that we must show the following: for each individual a such that $\{a\} \in \mathsf{Atoms}(\mathcal{T}, \mathcal{P})$ and for each individual b, it holds that $\{a\} \in \mathcal{S}_{[b]_\mathcal{S}}$ iff $a \approx_{\exists Y.\mathcal{B}} b$ but $a \not\approx_\mathcal{S} b$.

$$\begin{array}{ll} & \{a\} \in \mathcal{S}_{[b]_\mathcal{S}} \\ \text{iff} & \{a\} \in \mathcal{F}(b) \\ \text{iff} & \mathcal{D} \not\models \{a\}(b) \text{ and } \mathcal{B} \models \{a\}(h(b)) \\ \text{iff} & \mathcal{D} \not\models \{a\}(b) \text{ and } \mathcal{B} \models \{a\}(b) \\ \text{iff} & \mathcal{D} \not\models a \equiv b \text{ and } \mathcal{B} \models a \equiv b \\ \text{iff} & a \not\approx_{\exists W.\mathcal{D}} b \text{ and } a \approx_{\exists Y.\mathcal{B}} b \\ \text{iff} & a \not\approx_\mathcal{S} b \text{ and } a \approx_{\exists Y.\mathcal{B}} b \end{array}$$

The statements are equivalent since $\mathcal{S}_{[b]_\mathcal{S}} = \mathcal{F}(b)$, by definition of $\mathcal{F}$, by Condition (Hom2) (i.e., $h(b) = b$), since the concept assertion $\{a\}(b)$ is equivalent to the equality assertion $a \equiv b$, by Corollary I, and finally by definition of $\approx_\mathcal{S}$.

(RS3) Consider the set

$$\Gamma \coloneqq \{\, \langle\!\langle [h(t)]_\mathcal{S}, \mathcal{F}(t) \rangle\!\rangle \mid t \text{ is an object name of } \exists W.\mathcal{D} \,\}.$$

First of all, we prove that Γ is a subset of the whole set defined in Definition XVI. This is obvious for pairs $\langle\!\langle [h(a)]_\mathcal{S}, \mathcal{F}(a) \rangle\!\rangle$ for individuals a as well as for pairs $\langle\!\langle [h(t)]_\mathcal{S}, \mathcal{F}(t) \rangle\!\rangle$ where $h(t)$ is a variable, since no special restrictions are then imposed in Definition XVI. Now, let $\langle\!\langle [h(t)]_\mathcal{S}, \mathcal{F}(t) \rangle\!\rangle \in \Gamma$ where $h(t) = a$ is an individual, and $\mathcal{F}(t) \neq \mathcal{S}_{[a]_\mathcal{S}}$. We must show that $\mathcal{F}(t)$ contains each nominal $\{b\}$ in $\mathsf{Atoms}_\mathcal{R}(\mathcal{T}, \mathcal{P})$ where $\exists Y.\mathcal{B} \models a \equiv b$. Since we have defined the repair seed $\mathcal{S}$ by $\mathcal{S}_{[a]_\mathcal{S}} \coloneqq \mathcal{F}(a)$, it follows that $t \neq a$. By Condition (Hom2), t must be a variable. Since qABoxes do not contain equalities involving variables, we infer that $\mathcal{D} \not\models \{b\}(t)$. Furthermore, $\exists Y.\mathcal{B} \models a \equiv b$ implies $\mathcal{B} \models \{b\}(h(t))$, and so $\mathcal{F}(t)$ contains $\{b\}$.

As next step, we are going to prove that Γ is saturated, cf. Definition XVI. For this purpose, we first prove the following three statements:

(α) If $t \approx_{\exists W.\mathcal{D}} t'$, then $\langle\!\langle [h(t)]_\mathcal{S}, \mathcal{F}(t) \rangle\!\rangle = \langle\!\langle [h(t')]_\mathcal{S}, \mathcal{F}(t') \rangle\!\rangle$.

If t is a variable, then $t \approx_{\exists W.\mathcal{D}} t'$ implies $t = t'$ as no variables can occur in equalities. Then $h(t) = h(t')$ and $\mathcal{F}(t) = \mathcal{F}(t')$.

Otherwise, t as well as t' must be an individual, which implies $h(t) = t$ and $h(t') = t'$ by Condition (Hom2) and we further infer from $t \approx_{\exists W.\mathcal{D}} t'$ that $t \approx_\mathcal{S} t'$, i.e., $h(t) \approx_\mathcal{S} h(t')$. Furthermore, $t \approx_{\exists W.\mathcal{D}} t'$ implies $h(t) \approx_{\exists Y.\mathcal{B}} h(t')$ by Condition (Hom1), and thus $\mathcal{F}(t) = \mathcal{F}(t')$.

(β) If $\mathcal{D}$ contains the role assertion $R(t, u)$, then $\langle\!\langle [h(t)]_\mathcal{S}, \mathcal{F}(t) \rangle\!\rangle \xrightarrow{R} \langle\!\langle [h(u)]_\mathcal{S}, \mathcal{F}(u) \rangle\!\rangle$.

We only treat the case where R is a role name. If R is an inverse role instead, then we can simply swap the proof of Condition (RA2) and the proof of Condition (RA4), and likewise for Conditions (RA3) and (RA5).

(RA1) By Condition (Hom4), the assumption $R(t, u) \in \mathcal{D}$ implies that there are object names v, w such that $v \approx_{\exists Y.\mathcal{B}} h(t)$, $w \approx_{\exists Y.\mathcal{B}} h(u)$, and $R(v, w) \in \mathcal{B}$. Corollary I yields that $\mathcal{B}$ entails $R(h(t), h(u))$.

(RA2) Now let $\exists q.D \in \mathcal{F}(t)$ and consider a transition (q, R, p) where $\mathcal{B} \models \exists p.D(h(u))$. We need to show that there is a state p' such that $p \leq p'$ and $\exists p'.D \in \mathcal{F}(u)$.

From $\exists q.D \in \mathcal{F}(t)$ it follows that $\mathcal{D} \not\models \exists q.D(t)$. Recall that $\mathcal{D}$ contains the role assertion $R(t, u)$. Due to the transition (q, R, p), we infer that $\mathcal{D} \not\models \exists p.D(u)$. We conclude that $\exists p.D$ is in the repair type $\mathcal{F}(u)$.

(RA3) Now let $\exists q.D \in \mathcal{F}(t)$ and consider a transition (q, R, f) where f is a final state and $\mathcal{B} \models D(h(u))$. We need to show that $\mathsf{Conj}_\mathcal{R}(D) \cap \mathcal{F}(u) \neq \emptyset$.

From $\exists q.D \in \mathcal{F}(t)$ it follows that $\mathcal{D} \not\models \exists q.D(t)$. Recall that $\mathcal{D}$ contains the role assertion $R(t, u)$. Due to the transition (q, R, f) where f is final, we infer that $\mathcal{D} \not\models D(u)$. It follows that there must exist a top-level conjunct $E \in \mathsf{Conj}(D)$ such that $\mathcal{D} \not\models E(u)$. By Lemma XXVIII we infer that $\mathcal{D} \not\models (E)_\mathcal{R}(u)$, where $(E)_\mathcal{R} \in \mathsf{Conj}_\mathcal{R}(D)$. Since $\mathcal{B} \models D(h(u))$, it follows that by Lemma XXVIII that $\mathcal{B} \models (E)_\mathcal{R}(h(u))$. Summing up, we conclude that $(E)_\mathcal{R}$ is in the repair type $\mathcal{F}(u)$.

(RA4), (RA5) The other two conditions that treat existential restrictions $\exists q.D \in \mathcal{F}(u)$ and transitions (q, R^-, p) can be proved similarly.

(γ) If $\mathcal{D} \models C(u)$ where $C \in \mathsf{Sub}(\mathcal{T}, \mathcal{P})$, then $C \in \mathcal{F}(u)^+(h(u))$.

Using the homomorphism h, we conclude that $\mathcal{B} \models C(h(u))$ by Lemma II. Now consider an atom $D \in \mathcal{F}(u)$; we must show that $C \not\sqsubseteq^{\mathcal{T},\mathcal{R}} D$. Assuming the contrary would yield that $\mathcal{D} \models^{\mathcal{T},\mathcal{R}} D(u)$, and with $\exists W.\mathcal{D}$ being $(\mathcal{T}, \mathcal{R})$-saturated we could infer that $\mathcal{D} \models D(u)$ — a contradiction since $D \in \mathcal{F}(u)$ implies the contrary.

We now verify that the conditions in Definition XVI are fulfilled.

(S1) Consider a copy $\langle\!\langle [h(t)]_\mathcal{S}, \mathcal{F}(t) \rangle\!\rangle$ in Γ and let $\exists R.C$ be in the residual $\mathcal{F}(t)^+(h(t))$ — we need to show that there is a copy $\langle\!\langle [h(u)]_\mathcal{S}, \mathcal{F}(u) \rangle\!\rangle$ in Γ such that $\langle\!\langle [h(t)]_\mathcal{S}, \mathcal{F}(t) \rangle\!\rangle \xrightarrow{R} \langle\!\langle [h(u)]_\mathcal{S}, \mathcal{F}(u) \rangle\!\rangle$ and the residual $\mathcal{F}(u)^+(h(u))$ contains C.

We first show that $\mathcal{D} \models \exists R.C(t)$. Assume the contrary. Then Lemma XXVIII would imply $\mathcal{D} \not\models \exists i_R.C(t)$. Furthermore, we could infer from $\exists R.C \in \mathcal{F}(t)^+(h(t))$ that $\mathcal{B} \models \exists R.C(h(t))$ and $\exists R.C \not\sqsubseteq^{\mathcal{T},\mathcal{R}} D$ for each $D \in \mathcal{F}(t)$, and the former would imply $\mathcal{B} \models \exists i_R.C(h(t))$ by Lemma XXVIII. It would follow that the repair type $\mathcal{F}(t)$ contains the atom $\exists i_R.C$, which would produce a contradiction since $\exists R.C \sqsubseteq^{\mathcal{T},\mathcal{R}} \exists i_R.C$.

According to Corollary I, we can now infer that there is an object t' and there is an object u where $t \approx_{\exists W.\mathcal{D}} t'$ and $R(t', u) \in \mathcal{D}$ and $\mathcal{D} \models C(u)$. Statement α yields $\langle\!\langle [h(t)]_\mathcal{S}, \mathcal{F}(t) \rangle\!\rangle = \langle\!\langle [h(t')]_\mathcal{S}, \mathcal{F}(t') \rangle\!\rangle$, Statement β shows that $\langle\!\langle [h(t')]_\mathcal{S}, \mathcal{F}(t') \rangle\!\rangle \xrightarrow{R} \langle\!\langle [h(u)]_\mathcal{S}, \mathcal{F}(u) \rangle\!\rangle$, and Statement γ yields that the residual $\mathcal{F}(u)^+(h(u))$ contains C.

(S2) Now consider a copy $\langle\!\langle [h(t)]_{\mathcal{S}}, \mathcal{F}(t) \rangle\!\rangle$ in Γ and an automaton concept $\exists q.C$ in the residual $\mathcal{F}(t)^{+}(h(t))$. We must show that there is a copy $\langle\!\langle [h(u)]_{\mathcal{S}}, \mathcal{F}(u) \rangle\!\rangle$ in Γ and a word $R_1 \cdots R_n \in L(\mathfrak{A}_{\mathcal{R}}(q))$ such that $\langle\!\langle [h(t)]_{\mathcal{S}}, \mathcal{F}(t) \rangle\!\rangle \xrightarrow{R_1} \cdots \xrightarrow{R_n} \langle\!\langle [h(u)]_{\mathcal{S}}, \mathcal{F}(u) \rangle\!\rangle$ and $C \in \mathcal{F}(u)^{+}(h(u))$.

Since $\exists q.C$ is in the residual $\mathcal{F}(t)^{+}(h(t))$, it follows that $\mathcal{B} \models \exists q.C(h(t))$ and also that $\exists q.C$ is not in the repair type $\mathcal{F}(t)$. We infer that $\mathcal{D} \models \exists q.C(t)$.

According to Lemma 7 there is a role word $R_1 \cdots R_n \in L(\mathfrak{A}_{\mathcal{R}}(q))$ with $\mathcal{D} \models \exists R_1. \cdots \exists R_n.C(t)$. Corollary I further yields role assertions $R_1(t_0, u_1)$, $R_2(t_1, u_2)$, $\ldots$, $R_n(t_{n-1}, u_n)$ in $\mathcal{D}$ such that

- $t \approx_{\exists W.\mathcal{D}} t_0$,
- $t_i \approx_{\exists W.\mathcal{D}} u_i$ for each index $i \in \{1, \ldots, n-1\}$,
- and $\mathcal{D} \models C(u_n)$.

With Statements α, β, and γ we obtain that

- $\langle\!\langle [h(t)]_{\mathcal{S}}, \mathcal{F}(t) \rangle\!\rangle = \langle\!\langle [h(t_0)]_{\mathcal{S}}, \mathcal{F}(t_0) \rangle\!\rangle$,
- $\langle\!\langle [h(t_i)]_{\mathcal{S}}, \mathcal{F}(t_i) \rangle\!\rangle = \langle\!\langle [h(u_i)]_{\mathcal{S}}, \mathcal{F}(u_i) \rangle\!\rangle$ for each $i \in \{1, \ldots, n-1\}$,
- $\langle\!\langle [h(t_{i-1})]_{\mathcal{S}}, \mathcal{F}(t_{i-1}) \rangle\!\rangle \xrightarrow{R_i} \langle\!\langle [h(u_i)]_{\mathcal{S}}, \mathcal{F}(u_i) \rangle\!\rangle$ for each $i \in \{1, \ldots, n\}$,
- and $C \in \mathcal{F}(u_n)^{+}(h(u_n))$.

With defining $u := u_n$, the claim follows.

Since Γ is saturated, it is a subset of $\Omega(\mathcal{S})$. Furthermore, Γ contains $\langle\!\langle [a]_{\mathcal{S}}, \mathcal{S}_{[a]_{\mathcal{S}}} \rangle\!\rangle = \langle\!\langle [h(a)]_{\mathcal{S}}, \mathcal{F}(a) \rangle\!\rangle$ for each individual a, and so $\mathcal{S}$ is admissible.

III. It remains to show that there is a homomorphism from $\exists W.\mathcal{D}$ to the canonical repair induced by $\mathcal{S}$, which we denote by $\exists Z.\mathcal{C}$. We are going to verify that the mapping k where $k(a) := a$ for each individual a and $k(x) := \langle\!\langle [h(x)]_{\mathcal{S}}, \mathcal{F}(x) \rangle\!\rangle$ for each variable x is a homomorphism.

(Hom1) Consider an individual a and an individual b where $a \approx_{\exists W.\mathcal{D}} b$. We need to prove that $k(a) \approx_{\exists Z.\mathcal{C}} k(b)$. The precondition $a \approx_{\exists W.\mathcal{D}} b$ implies $a \approx_{\mathcal{S}} b$ and thus $a \approx_{\exists Z.\mathcal{C}} b$ by Lemma XXI. Due to $k(a) = a$ and $k(b) = b$, we conclude that $k(a) \approx_{\exists Z.\mathcal{C}} k(b)$.

(Hom2) We already have by definition that $k(a) = a$ for each individual a.

(Hom3) Consider a concept assertion $A(t)$ in $\mathcal{D}$. By Condition (Hom3) there is an object v such that $v \approx_{\exists Y.\mathcal{B}} h(t)$ and $\mathcal{B}$ contains $A(v)$, i.e., $\mathcal{B}$ entails $A(h(t))$. Furthermore, $\mathcal{D}$ entails $A(t)$, and so the repair type $\mathcal{F}(t)$ cannot contain A by the very definition of $\mathcal{F}$. We infer that the matrix $\mathcal{C}$ contains the concept assertion $A(\langle\!\langle [h(t)]_{\mathcal{S}}, \mathcal{F}(t) \rangle\!\rangle)$, cf. Instruction (CR2).

(Hom4) Last, assume that $r(t, u)$ is a role assertion in $\mathcal{D}$. Statement β shows that $\langle\!\langle [h(t)]_{\mathcal{S}}, \mathcal{F}(t) \rangle\!\rangle \xrightarrow{r} \langle\!\langle [h(u)]_{\mathcal{S}}, \mathcal{F}(u) \rangle\!\rangle$. Due to Instruction (CR3) in Definition XX it follows that the matrix $\mathcal{C}$ contains the role assertion $r(\langle\!\langle [h(t)]_{\mathcal{S}}, \mathcal{F}(t) \rangle\!\rangle, \langle\!\langle [h(u)]_{\mathcal{S}}, \mathcal{F}(u) \rangle\!\rangle)$. ☐

The next theorem summarizes our results in Lemmas XXV and XXVI and Propositions XXVII and XXIX.

Theorem 16. *For every admissible repair seed $\mathcal{S}$, the induced canonical repair $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A}, \mathcal{S})$ can effectively be computed, is saturated w.r.t. $(\mathcal{T}, \mathcal{R})$, and is a repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T}, \mathcal{R})$. Conversely, every repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T}, \mathcal{R})$ is entailed by such a canonical repair.*

As an easy consequence of this theorem we obtain that the set of canonical repairs induced by the admissible repair seeds contains (up to equivalence) every optimal repair. Since all admissible repair seeds can effectively be generated, Theorem 12 can also be obtained as a corollary to this theorem. Even in the case without a terminology, not all canonical repairs need to be optimal (Baader et al. 2020), but we expect even the non-optimal ones to be quite good w.r.t. preserving consequences. One advantage of canonical repairs is that each one can be characterized by a polynomial-size repair seed, which can be generated by the knowledge engineers by making a polynomial number of decisions based on their domain knowledge. Another advantage is that the optimized approach for generating a canonical repair from a repair seed introduced in (Baader et al. 2021a) for $\mathcal{EL}$ can be extended to $\mathcal{ELROI}$. More details on these advantages can be found in the following two sections.

Finally note that, if the input qABox does not contain individual names and the terminology alone does not imply any of the unwanted consequences in the repair request, then there is exactly one repair seed, namely where the equivalence relation $\approx_{\mathcal{S}}$ equals $\approx_{\exists Y.\mathcal{B}}$ and where each repair type $\mathcal{S}_{[a]_{\mathcal{S}}}$ is empty, and thus there is a unique optimal repair.

Alternative proof of Theorem 12. Proposition XXIX shows that each repair is entailed by a canonical repair, and Proposition XXVII shows that each canonical repair is in fact a repair as per Definition 8. Since only finitely many repair seeds exist, there are only finitely many canonical repairs. Now let $\mathfrak{R}$ be the set of all canonical repairs that are not strictly entailed by another canonical repair. It then follows that each repair is entailed by a can. repair in $\mathfrak{R}$. To see this, assume the contrary, i.e., there was a repair $\exists Y.\mathcal{B}$ that is not entailed by a canonical repair in $\mathfrak{R}$. Then Proposition XXIX would yield a canonical repair $\exists Z.\mathcal{C}$ that entails $\exists Y.\mathcal{B}$, which means that $\exists Z.\mathcal{C}$ is not in $\mathfrak{R}$. But then, due to the very definition of $\mathfrak{R}$, there would be a can. repair $\exists Z'.\mathcal{C}'$ in $\mathfrak{R}$ that entails $\exists Z.\mathcal{C}$, and thus also entails $\exists Y.\mathcal{B}$ — a contradiction.

Furthermore, $\mathfrak{R}$ is the set of all optimal repairs, up to equivalence. We verify this by showing that the contrary implies a contradiction. Therefore consider an optimal repair $\exists Y.\mathcal{B}$ that is not equivalent to a can. repair in $\mathfrak{R}$. According to the above, there is a canonical repair $\exists Z.\mathcal{C}$ in $\mathfrak{R}$ where $\exists Z.\mathcal{C} \models \exists Y.\mathcal{B}$. Now optimality would imply that $\exists Z.\mathcal{C}$ and $\exists Y.\mathcal{B}$ are equivalent — a contradiction.

In order to compute the set $\mathfrak{R}$, we first need to compute all admissible seed functions, then the induced canonical repairs, and finally filter out the non-optimal ones.

1. Since the saturation is finite, the set $\mathsf{Atoms}_{\mathcal{R}}(\mathcal{T}, \mathcal{P})$ is finite, and the instance problem and the subsumption prob-

lem are decidable, we can enumerate all repair types in finite time. From the repair types we can then construct all possible repair seeds. Checking the so obtained finitely many repair seeds for admissibility terminates as well.

2. Given an admissible repair seed, the induced can. repair can be constructed in finite time.
3. Since the entailment problem for qABoxes is decidable, see Proposition 2, we can effectively filter out the non-optimal can. repairs. □

3.5 Interactive Selection of a Canonical Repair in Polynomial Time

A qABox and a repair request $\mathcal{P}$ as input usually do not determine a unique optimal repair. The source of non-determinism are conjunctions that are either directly in $\mathcal{P}$ or imply a concept in $\mathcal{P}$. In the worst case, there may even be exponentially many optimal repairs. As a result, it is impractical to compute all these repairs first, and then expect the knowledge engineer or domain expert (called *user* in the following) to choose a suitable one. We have seen that each canonical repair computed by our approach is induced by a polynomial-size repair seed. To construct an appropriate repair with reasonable effort, the corresponding repair seed should be identified by interacting with the user. This means that the users should utilize their domain knowledge to determine which repair is constructed, rather than to select one from all possible optimal repairs. However, Definition 15 is probably too technical to specify such a seed directly based on it, and even if the user would be able to do so, there would be the remaining problem that not every repair seed is admissible, i.e., the user-defined seed might not induce a repair. As an alternative, we introduce the notion of a *repair template*, which basically describes which consequences should *not* be removed by the repair.

Repair Templates Such a template is an ordinary ABox, which may contain only assertions of particular forms, as defined below. By exploiting a correspondence between seeds and templates, we show that every (optimal) canonical repair is induced by a polynomial-size repair template.

Definition XXX. Let $\exists X.\mathcal{A}$ be a qABox, $(\mathcal{T},\mathcal{R})$ a terminology, and $\mathcal{P}$ a repair request. A *repair template* of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ is an (ordinary) ABox $\mathcal{B}$ that, firstly, is a repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ as per Definition 8 and, secondly, only contains assertions of the following forms:

- equality assertions $a \equiv b$ where $a, b \in \Sigma_{\mathsf{I}}$,
- concept assertions $E(a)$ where $E \in \mathsf{Atoms}_{\mathcal{R}}(\mathcal{T},\mathcal{P})$ and $a \in \Sigma_{\mathsf{I}}$, but E is not an automaton concept $\exists q.F$,
- concept assertions $\exists R_1.\cdots\exists R_n.F(a)$ where $\exists q.F \in \mathsf{Atoms}_{\mathcal{R}}(\mathcal{T},\mathcal{P})$ and $R_1\cdots R_n \in L(\mathfrak{A}_{\mathcal{R}}(q))$ and $a \in \Sigma_{\mathsf{I}}$.

The Completeness Proof (see Proposition XXIX) shows in a constructive way that every repair, and thus also each repair template, induces an admissible repair seed. Specifically, given a repair template $\mathcal{B}$,[8] the induced repair seed $\mathcal{S}_{\mathcal{B}}$ is defined as follows:

- $a \approx_{\mathcal{S}_{\mathcal{B}}} b$ iff the saturation of $\mathcal{B}$ w.r.t. $(\mathcal{T},\mathcal{R})$ entails $a \equiv b$,
- for each individual name a, the repair type $(\mathcal{S}_{\mathcal{B}})_{[a]_{(\mathcal{S}_{\mathcal{B}})}}$ consists of all atoms $C \in \mathsf{Atoms}_{\mathcal{R}}(\mathcal{T},\mathcal{P})$ such that the saturation of $\exists X.\mathcal{A}$ entails $C(a)$ but the saturation of $\mathcal{B}$ does not entail $C(a)$.

We say that a canonical repair is *induced* by $\mathcal{B}$ if it is induced by $\mathcal{S}_{\mathcal{B}}$.

Lemma XXXI. *Each canonical repair, and specifically every optimal one, is induced by a polynomial-size repair template.*

Proof. Consider the canonical repair induced by the admissible repair seed $\mathcal{S}$. We define the ABox $\mathcal{B}_{\mathcal{S}}$ as follows.

- Add the equality assertion $a \equiv b$ to $\mathcal{B}_{\mathcal{S}}$ if the canonical repair $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S})$ entails it and $a \neq b$.
- Add the concept assertion $E(a)$ to $\mathcal{B}_{\mathcal{S}}$ if $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S})$ entails it and E is in $\mathsf{Atoms}_{\mathcal{R}}(\mathcal{T},\mathcal{P})$, but is no automaton concept.
- If $\exists q.F$ is an automaton concept in $\mathsf{Atoms}_{\mathcal{R}}(\mathcal{T},\mathcal{P})$ such that $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S})$ entails $\exists q.F(a)$, then choose an arbitrary shortest role word $R_1\cdots R_n \in L(\mathfrak{A}_{\mathcal{R}}(q))$ and add the concept assertion $\exists R_1.\cdots\exists R_n.F(a)$ to $\mathcal{B}_{\mathcal{S}}$.

It is easy to verify that $\mathcal{B}_{\mathcal{S}}$ is a repair template, namely because it is entailed by the canonical repair $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S})$. Furthermore, $\mathcal{B}_{\mathcal{S}}$ has polynomial size for the following reasons:

- Each individual name in the canonical repair $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S})$ is also contained in the saturation $\mathsf{sat}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A})$, and must thus be contained in the input qABox $\exists X.\mathcal{A}$ or in the terminology $(\mathcal{T},\mathcal{R})$. It follows that $\mathcal{B}_{\mathcal{S}}$ can contain at most polynomially many equality assertions $a \equiv b$.
- Due to the last point and since the set $\mathsf{Atoms}_{\mathcal{R}}(\mathcal{T},\mathcal{P})$ is polynomial in the size of the repair request $\mathcal{P}$, the TBox $\mathcal{T}$, and the automata for $\mathcal{R}$, it follows that $\mathcal{B}_{\mathcal{S}}$ can contain at most polynomially many concept assertions.
- For each automaton concept $\exists q.F$ and each individual name a, further note that we choose at most one shortest role word $R_1\cdots R_n$ in the language $L(\mathfrak{A}_{\mathcal{R}}(q))$ and add the assertion $\exists R_1.\cdots\exists R_n.F(a)$ to $\mathcal{B}_{\mathcal{S}}$. Since the role word is a shortest one, the according accepting run in $\mathfrak{A}_{\mathcal{R}}(q)$ cannot contain each state more than once, and so the length of $R_1\cdots R_n$ is bounded by the number of states in Q_R, where $q \in Q_R$.

In order to show that the canonical repair $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S})$ is induced by $\mathcal{B}_{\mathcal{S}}$, we prove that the repair seeds $\mathcal{S}$ and $\mathcal{S}_{\mathcal{B}_{\mathcal{S}}}$ are the same.

We first show that the equivalence relations $\approx_{\mathcal{S}}$ and $\approx_{\mathcal{S}_{\mathcal{B}_{\mathcal{S}}}}$ are the same. Consider two individual names a and b.

- If $a \approx_{\mathcal{S}} b$, then $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S}) \models a \equiv b$ and thus $a \equiv b \in \mathcal{B}_{\mathcal{S}}$, which implies $a \approx_{\mathcal{S}_{\mathcal{B}_{\mathcal{S}}}} b$.

[8] More precisely, we would first need to transform a repair template $\mathcal{B}$ into an equivalent qABox $\exists Y.\mathcal{B}'$ by means of the first three saturation rules in Figure 1. According to Definition XXX, this qABox is also a repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$.

- Conversely, assume $a \approx_{\mathcal{S}_{\mathcal{B}_\mathcal{S}}} b$, i.e., $\mathcal{B}_\mathcal{S} \models^{\mathcal{T},\mathcal{R}} a \equiv b$. By definition, we have $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S}) \models \mathcal{B}_\mathcal{S}$. According to Lemmas XXV and XXVI each canonical repair is saturated w.r.t. $(\mathcal{T},\mathcal{R})$, and we infer that $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S}) \models a \equiv b$. By Corollary I and Lemma XXI we conclude that $a \approx_\mathcal{S} b$.

Next, we prove that the repair type $\mathcal{S}_{[a]_\mathcal{S}}$ is a subset of $(\mathcal{S}_{\mathcal{B}_\mathcal{S}})_{[a]_{(\mathcal{S}_{\mathcal{B}_\mathcal{S}})}}$. For this purpose, consider an atom C in $\mathcal{S}_{[a]_\mathcal{S}}$. Lemma XXIII yields that the canonical repair induced by $\mathcal{S}$ does not entail $C(a)$. It follows that the above defined repair template $\mathcal{B}_\mathcal{S}$ does not entail $C(a)$ w.r.t. $(\mathcal{T},\mathcal{R})$. (To see this, assume the contrary, i.e., $\mathcal{B}_\mathcal{S} \models^{\mathcal{T},\mathcal{R}} C(a)$. With $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S}) \models \mathcal{B}_\mathcal{S}$ it would follow that $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S}) \models^{\mathcal{T},\mathcal{R}} C(a)$. Since canonical repairs are saturated for the underlying terminology, see Lemmas XXV and XXVI, we would infer the contradiction that $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S}) \models C(a)$.) Furthermore, the saturation of $\exists X.\mathcal{A}$ entails $C(a)$ since $\mathcal{S}_{[a]_\mathcal{S}}$ satisfies Condition (RT1). According to the above definition of the repair seed $\mathcal{S}_{\mathcal{B}_\mathcal{S}}$, we immediately conclude that the repair type $(\mathcal{S}_{\mathcal{B}_\mathcal{S}})_{[a]_{(\mathcal{S}_{\mathcal{B}_\mathcal{S}})}}$ contains C.

It remains to show that also $(\mathcal{S}_{\mathcal{B}_\mathcal{S}})_{[a]_{(\mathcal{S}_{\mathcal{B}_\mathcal{S}})}}$ is a subset of $\mathcal{S}_{[a]_\mathcal{S}}$. Therefore let $C \in (\mathcal{S}_{\mathcal{B}_\mathcal{S}})_{[a]_{(\mathcal{S}_{\mathcal{B}_\mathcal{S}})}}$, i.e., $\exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} C(a)$ and $\mathcal{B}_\mathcal{S} \not\models^{\mathcal{T},\mathcal{R}} C(a)$.

- If C is no automaton concept, then the latter implies that the canonical repair $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S})$ does not entail $C(a)$ as otherwise $C(a)$ would be contained in the repair template $\mathcal{B}_\mathcal{S}$.
- Now let $C = \exists q.D$ be an automaton concept. If the canonical repair $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S})$ would entail $\exists q.D(a)$, then the repair template $\mathcal{B}_\mathcal{S}$ would contain an assertion $\exists R_1.\cdots\exists R_n.D(a)$ for some role word $R_1\cdots R_n \in L(\mathfrak{A}_\mathcal{R}(q))$, and thus $\mathcal{B}_\mathcal{S}$ would entail $\exists q.D(a)$, a contradiction.

In both cases, it follows that $\mathsf{rep}^{\mathcal{T},\mathcal{R}}(\exists X.\mathcal{A},\mathcal{S})$ does not entail $C(a)$. By Lemma XXIV the residual $\mathcal{S}^{+}_{[a]_\mathcal{S}}(a)$ does not contain C, which means that $\mathcal{S}_{[a]_\mathcal{S}}$ must contain an atom D where $C \sqsubseteq^{\mathcal{T},\mathcal{R}} D$, cf. Definition XIII. As $\mathcal{S}_{[a]_\mathcal{S}}$ fulfills Condition (RT2), we conclude that $C \in \mathcal{S}_{[a]_\mathcal{S}}$. □

User Interaction We have seen that each optimal repair can be described by a repair template of polynomial size. Next, we explain how this fact can be used to determine a unique repair by interacting with the user. Deciding whether an assertion of the form $a \equiv b$ or $E(a)$ holds in the application domain is delegated to the user. For this purpose, these assertions are enumerated in a suitable order. The user is presented with each assertion, one after another, and needs to accept, reject, or ignore it. The accepted assertions are collected in the repair template and the rejected assertions are added to the repair request. It is then guaranteed that the induced canonical repair entails all accepted assertions and does not entail any rejected assertion. We will show that the canonical repair obtained this way is unique if the user has not ignored a question.

Note that the user does not need to check each single assertion. If an assertion already follows from previously accepted assertions, it must be accepted as well. Furthermore, an assertion must be rejected if that assertion together with all previously accepted assertions would violate the repair request or would entail a previously rejected assertion. Only the remaining assertions need to be decided by the user.

To describe this approach in more detail, we fix a qABox $\exists X.\mathcal{A}$, a terminating $\mathcal{ELROI}$ terminology $(\mathcal{T},\mathcal{R})$ with regular RBox, and an $\mathcal{ELROI}$ repair request $\mathcal{P}$. We assume that a repair exists, i.e., the terminology alone does not entail any assertion in the request. Initially let $\mathcal{B}$ be the empty repair template and $\mathcal{Q} \coloneqq \mathcal{P}$ be the original repair request. By interacting with the user, we will now enlarge both sets. The invariant is that $\mathcal{B}$ is always a repair of $\exists X.\mathcal{A}$ for $\mathcal{Q}$ w.r.t. $(\mathcal{T},\mathcal{R})$.

For each assertion α that is of the form $a \equiv b$ or $E(a)$ where $a, b \in \Sigma_\mathsf{I}$ and $E \in \mathsf{Atoms}_\mathcal{R}(\mathcal{T},\mathcal{P})$ and that is entailed by $\exists X.\mathcal{A}$ w.r.t. $(\mathcal{T},\mathcal{R})$, ask the user whether α holds in the underlying domain of interest. The user can either accept, reject, or ignore α. In case of acceptance α is added to $\mathcal{B}$ — except if E is an automaton concept $\exists q.F$, then the user must additionally choose a preferably short role word $R_1\cdots R_n \in L(\mathfrak{A}_\mathcal{R}(q))$ and the assertion $\exists R_1.\cdots\exists R_n.F(a)$ is instead added to $\mathcal{B}$[9] — and in case of rejection α is added to $\mathcal{Q}$.

In order to fulfill the invariant, α must be accepted if it already follows from the current repair template $\mathcal{B}$. Furthermore, α must be rejected if $\mathcal{B}\cup\{\alpha\}$ would violate the current repair request $\mathcal{Q}$. The user simply need not be asked if such an assertion α is under consideration.

Note that the above iteration can never reach a deadlock. To see this assume that the current repair template $\mathcal{B}$ and the current repair request $\mathcal{Q}$ satisfy the invariant, i.e., $\mathcal{B}$ is a repair of $\exists X.\mathcal{A}$ for $\mathcal{Q}$ w.r.t. $(\mathcal{T},\mathcal{R})$. Now, let α be the next assertion to be decided by the user, i.e., α does not follow from $\mathcal{B}$ and $\mathcal{B} \cup \{\alpha\}$ entails no assertion in $\mathcal{Q}$. The user can freely choose to accept, ignore, or reject α. Since no assertion in $\mathcal{Q}$ follows from $\mathcal{B} \cup \{\alpha\}$, the user can safely accept α, by which it is added to $\mathcal{B}$. It is guaranteed that the enlarged repair template $\mathcal{B} \cup \{\alpha\}$ is a repair of $\exists X.\mathcal{A}$ for the repair request $\mathcal{Q}$ w.r.t. the terminology. Since $\mathcal{B}$ does not entail α, the user can also safely reject α, by which it is added to $\mathcal{Q}$. The repair template $\mathcal{B}$ is then still a repair of $\exists X.\mathcal{A}$ for the enlarged repair request $\mathcal{Q} \cup \{\alpha\}$ w.r.t. the terminology. Ignoring α does not enlarge $\mathcal{B}$ or $\mathcal{Q}$ and is thus unproblematic for the invariant.

Lemma XXXII. *If the user does not ignore any assertion, then the obtained canonical repair is unique.*

Proof. Assume that, after the user has answered all questions without ignoring any assertions, we have obtained the repair template $\mathcal{B}$ and the (enlarged) repair request $\mathcal{Q}$. Further let $\mathcal{S}$ and $\mathcal{S}'$ be admissible repair seeds such that the induced canonical repairs entail every assertion in $\mathcal{B}$ but none in $\mathcal{Q}$. We first show that the equivalence relations $\approx_\mathcal{S}$ and

[9] It would also suffice to automatically choose an arbitrary shortest role word $R_1\cdots R_n \in L(\mathfrak{A}_\mathcal{R}(q))$.

$\approx_{\mathcal{S}'}$ are equal and then show that, for each individual name a, the repair type $\mathcal{S}_{[a]_\mathcal{S}}$ coincides with the repair type $\mathcal{S}'_{[a]_{\mathcal{S}'}}$.

Consider an equality assertion $a \equiv b$. If it was accepted, it is entailed by $\mathcal{B}$ and thus also by every canonical repair for $\mathcal{Q}$ that entails $\mathcal{B}$. Otherwise, it is contained in $\mathcal{Q}$, and thus is not entailed by any canonical repair for $\mathcal{Q}$ that entails $\mathcal{B}$. Thus, the equivalence relation on objects is uniquely determined.

Consider an atom C in the repair type $\mathcal{S}_{[a]_\mathcal{S}}$. Lemma XXIII yields that the canonical repair induced by $\mathcal{S}$ does not entail $C(a)$. It follows that also the repair template $\mathcal{B}$ does not entail $C(a)$, which means that the user did not accept the assertion $C(a)$ and it was also not entailed by previously accepted assertions. Since the user did not ignore any assertion, she or he must have rejected $C(a)$ and so the (enlarged) repair request $\mathcal{Q}$ contains $C(a)$. Now recall that C is an atom, and thus Condition (RS1) implies that also the other repair type $\mathcal{S}'_{[a]_{\mathcal{S}'}}$ contains C. The converse subset inclusion follows in the same manner. □

3.6 Optimized Repairs, which have Exponential Size only in the Worst Case

Canonical repairs are always of exponential size, measured in the size of the TBox, the repair request, and the automata for the RBox. It is thus often impractical to compute them in their full form. Similar to the case of $\mathcal{EL}$ (Baader et al. 2021a), most canonical repairs are equivalent to a considerably smaller sub-qABox, and the objects of this sub-qABox can be enumerated using a rule-based approach.

Assume that the input consists of a qABox $\exists X.\mathcal{A}$, a terminology $(\mathcal{T}, \mathcal{R})$, and a repair request $\mathcal{P}$. Further let $\mathcal{S}$ be an admissible repair seed and denote by $\exists Y.\mathcal{B}$ the canonical repair induced by $\mathcal{S}$. A sub-qABox equivalent to it must at least contain all individual names, which includes all synonyms $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$ where a is an individual, and all copies $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$ where $\mathcal{K}$ is a minimal repair type for u[10] such that $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle \in \Omega(\mathcal{S})$. The former are needed as otherwise no homomorphism can exist due to Condition (Hom2), and the latter are needed since these are the copies with the fewest modifications and can thus not be homomorphically mapped to a copy with more modifications.

Starting from this set, we try to define a homomorphism from $\exists Y.\mathcal{B}$ to the sub-qABox induced by it.[11] Since all individuals are already there, all equality assertions in the canonical repair are also contained in this sub-qABox. If we now map each individual to itself and map each admissible copy $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$ in the canonical repair to the synonym $\langle\!\langle [u]_\mathcal{S}, \mathcal{S}_{[u]_\mathcal{S}} \rangle\!\rangle$ if u is an individual and $\mathcal{S}_{[u]_\mathcal{S}} \subseteq \mathcal{K}$, and otherwise to $\langle\!\langle [u]_\mathcal{S}, \mathcal{L} \rangle\!\rangle$ where $\mathcal{L}$ is a minimal repair type for u such that $\mathcal{L} \subseteq \mathcal{K}$ and $\langle\!\langle [u]_\mathcal{S}, \mathcal{L} \rangle\!\rangle \in \Omega(\mathcal{S})$, then also the concept assertions in the full canonical repair are properly treated, and the so defined mapping satisfies Conditions (Hom1), (Hom2), and (Hom3). It might, however, be that Condition (Hom4) is not fulfilled, i.e., there is a role assertion $r(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle, \langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle)$ in $\exists Y.\mathcal{B}$ the image of which is not a role assertion in the induced sub-qABox. We call such a role assertion a *defect* and show in the following how all defects can be resolved one by one, namely by extending the set of objects that induce the sub-qABox and by modifying the mapping from $\exists Y.\mathcal{B}$ to it.

Besides the set Σ_I of all individuals, which contains all synonyms $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$, let Y_0 consist of all copies $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$ where $\mathcal{K}$ is a minimal repair type for u such that $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle \in \Omega(\mathcal{S})$. We define the mapping $h_0\colon \mathsf{Obj}(\exists Y.\mathcal{B}) \to \Sigma_\mathsf{I} \cup Y_0$ as follows:

- $h_0(a) := a$ for each individual name a,
- if a is an individual and $\mathcal{S}_{[a]_\mathcal{S}} \subseteq \mathcal{K}$, then $h_0(\langle\!\langle [a]_\mathcal{S}, \mathcal{K} \rangle\!\rangle) := \langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$,[12]
- otherwise, if u is no individual or $\mathcal{S}_{[u]_\mathcal{S}} \not\subseteq \mathcal{K}$, then $h_0(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle) := \langle\!\langle [u]_\mathcal{S}, \mathcal{L} \rangle\!\rangle$ where $\mathcal{L}$ is a minimal repair type for u such that $\mathcal{L} \subseteq \mathcal{K}$ and $\langle\!\langle [u]_\mathcal{S}, \mathcal{L} \rangle\!\rangle \in \Omega(\mathcal{S})$.

Starting with h_0 and Y_0, we will construct finite sequences of mappings $h_0, h_1, h_2, \ldots, h_n$ and of sets $Y_0, Y_1, Y_2, \ldots, Y_n$ such that the following conditions are satisfied:

1. Each mapping h_i is of type $\mathsf{Obj}(\exists Y.\mathcal{B}) \to \Sigma_\mathsf{I} \cup Y_i$.
2. $h_i(a) = a$ for each individual name a,
3. For each copy $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle$, there is a repair type $\mathcal{K}_i$ for u such that $h_i(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle) = \langle\!\langle [u]_\mathcal{S}, \mathcal{K}_i \rangle\!\rangle$ and $\mathcal{K}_i \subseteq \mathcal{K}$.
4. If $h_i(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle) = \langle\!\langle [u]_\mathcal{S}, \mathcal{K}_i \rangle\!\rangle$ and $h_{i+1}(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle) = \langle\!\langle [u]_\mathcal{S}, \mathcal{K}_{i+1} \rangle\!\rangle$, then $\mathcal{K}_i \subseteq \mathcal{K}_{i+1}$.
5. $Y_i \subseteq \Omega(\mathcal{S})$ and $Y_i \subseteq Y_{i+1}$ for each index i.
6. The last mapping h_n is a homomorphism from the canonical repair $\exists Y.\mathcal{B}$ to the sub-qABox induced by $\Sigma_\mathsf{I} \cup Y_n$.

A *defect* of a mapping h_i is a role assertion $r(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle, \langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle)$ in the canonical repair $\exists Y.\mathcal{B}$ such that $r(h_i(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle), h_i(\langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle))$ is not in $\exists Y.\mathcal{B}$, i.e., $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle \xrightarrow{r} \langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle$ holds but $h_i(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle) \xrightarrow{r} h_i(\langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle)$ does not hold.

If h_i has no defects, then it is a homomorphism (see the below proof) and no next mapping needs to be constructed. Otherwise, let $r(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle, \langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle)$ be the defect. We define the subsequent mapping h_{i+1} by case distinction why $h_i(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle) \xrightarrow{r} h_i(\langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle)$ does not hold. Assume that $\langle\!\langle [u]_\mathcal{S}, \mathcal{K}_i \rangle\!\rangle := h_i(\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle)$ and $\langle\!\langle [v]_\mathcal{S}, \mathcal{L}_i \rangle\!\rangle := h_i(\langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle)$.

(RA2) Assume that there is an extended atom $\exists q.C$ in $\mathcal{K}_i$ and a transition (q, r, p) such that $\mathcal{B} \models \exists p.C(v)$, but there is no state p' with $p \leq p'$ and $\exists p'.C \in \mathcal{L}_i$.

Since $\mathcal{K}_i$ is a subset of $\mathcal{K}$ and $\langle\!\langle [u]_\mathcal{S}, \mathcal{K} \rangle\!\rangle \xrightarrow{r} \langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle$ holds, we have $\exists p'.C \in \mathcal{L}$ for some state $p' \geq p$. Now let $\mathcal{L}_{i+1}$ be a minimal repair type for v such that $\mathcal{L}_i \cup \{\exists p'.C\} \subseteq \mathcal{L}_{i+1} \subseteq \mathcal{L}$ and $\langle\!\langle [v]_\mathcal{S}, \mathcal{L}_{i+1} \rangle\!\rangle \in \Omega(\mathcal{S})$. Further define $Y_{i+1} := Y_i \cup \{\langle\!\langle [v]_\mathcal{S}, \mathcal{L}_{i+1} \rangle\!\rangle\}$ and $h_{i+1} := h_i$ except $h_{i+1}(\langle\!\langle [v]_\mathcal{S}, \mathcal{L} \rangle\!\rangle) := \langle\!\langle [v]_\mathcal{S}, \mathcal{L}_{i+1} \rangle\!\rangle$.

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[10]That is, $\mathcal{K}$ is a repair type for u and no strict subset of $\mathcal{K}$ is a repair type for u.

[11]Formally: the sub-qABox of $\exists Y.\mathcal{B}$ induced by a subset $W \subseteq \mathsf{Obj}(\exists Y.\mathcal{B})$ has the same variables and its matrix consists of all assertions in $\mathcal{B}$ that involve only objects in W.

[12]Note that, if $\langle\!\langle [a]_\mathcal{S}, \mathcal{S}_{[a]_\mathcal{S}} \rangle\!\rangle$ and a' have been selected as synonyms in the construction of the canonical repair as per Definition XX, then we do not accidentally overwrite the previous assignment $h_0(a') := a'$ here.

(RA3) Next, consider the case where $\mathcal{K}_i$ contains an extended atom $\exists q.C$ and there is a transition (q, r, f) to a final state f such that $\mathcal{B} \models C(v)$, but $\mathsf{Conj}_{\mathcal{R}}(C) \cap \mathcal{L}_i = \emptyset$. From $\mathcal{K}_i \subseteq \mathcal{K}$ and $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle \xrightarrow{r} \langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$ we infer that there is an extended atom $D \in \mathsf{Conj}_{\mathcal{R}}(C) \cap \mathcal{L}$. Let now $\mathcal{L}_{i+1}$ be a minimal repair type for v that fulfills $\mathcal{L}_i \cup \{D\} \subseteq \mathcal{L}_{i+1} \subseteq \mathcal{L}$ and $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L}_{i+1} \rangle\!\rangle \in \Omega(\mathcal{S})$. Then define $Y_{i+1} \coloneqq Y_i \cup \{\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L}_{i+1} \rangle\!\rangle\}$ and $h_{i+1} \coloneqq h_i$ except $h_{i+1}(\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle) \coloneqq \langle\!\langle [v]_{\mathcal{S}}, \mathcal{L}_{i+1} \rangle\!\rangle$.

(RA4), (RA5) are analogous to (RA2) and (RA3).

We show that the last mapping h_n, which has no defects, is a homomorphism from the canonical repair $\exists Y.\mathcal{B}$ to the sub-qABox induced by the objects in $\Sigma_{\mathsf{I}} \cup Y_n$.

(Hom2) Consider an individual a. If it does not have a synonym, then $h_i(a) = a$ holds for all i since such values $h_i(a)$ are never changed, and so $h_n(a) = a$.

Otherwise, assume that the individual a' and $\langle\!\langle [a]_{\mathcal{S}}, \mathcal{S}_{[a]_{\mathcal{S}}} \rangle\!\rangle$ are synonyms. We have defined $h_0(\langle\!\langle [a]_{\mathcal{S}}, \mathcal{S}_{[a]_{\mathcal{S}}} \rangle\!\rangle)$ as $\langle\!\langle [a]_{\mathcal{S}}, \mathcal{S}_{[a]_{\mathcal{S}}} \rangle\!\rangle$. The above Invariants 3 and 4 yield that $\mathcal{S}_{[a]_{\mathcal{S}}} \subseteq \mathcal{K}_i \subseteq \mathcal{S}_{[a]_{\mathcal{S}}}$ for $h_i(\langle\!\langle [a]_{\mathcal{S}}, \mathcal{S}_{[a]_{\mathcal{S}}} \rangle\!\rangle) =: \langle\!\langle [a]_{\mathcal{S}}, \mathcal{K}_i \rangle\!\rangle$, i.e., we have not accidentally violated the invariant that $h_i(a')$ is mapped to a'. We conclude that $h_n(a') = a'$.

(Hom1) Since the set $\Sigma_{\mathsf{I}} \cup Y_n$ contains all individual names, the matrix of the induced sub-qABox contains the same equality assertions as the canonical repair. Thus the relation $\approx_{\exists Y.\mathcal{B}}$ coincides with the relation $\approx$ for the induced sub-qABox.

(Hom3) Consider a concept assertion $A(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$ in the canonical repair, i.e., $A(u)$ is in the saturation of the input qABox and $A \notin \mathcal{K}$. Due to Invariant 3, we have $\mathcal{K}_n \subseteq \mathcal{K}$ where $h_n(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle) =: \langle\!\langle [u]_{\mathcal{S}}, \mathcal{K}_n \rangle\!\rangle$. It follows that $A \notin \mathcal{K}_n$, i.e., $A(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K}_n \rangle\!\rangle)$ is in the canonical repair as well and thus also in the induced sub-qABox.

(Hom4) Let $r(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle, \langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle)$ be a role assertion in the canonical repair $\exists Y.\mathcal{B}$. Since h_n is free of defects, $\exists Y.\mathcal{B}$ also contains the image $r(h_n(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle), h_n(\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle))$. According to the above inductive construction, the two objects $h_n(\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle)$ and $h_n(\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle)$ are in $\Sigma_{\mathsf{I}} \cup Y_n$, and thus the induced sub-qABox contains the latter role assertion.

The above approach can also be understood as a compression technique for canonical repairs. Specifically, the compressed repair is obtained as the image of the constructed homomorphism h_n. It is, however, impractical to shrink repairs in this way, simply because the mappings are defined on the set of all objects of the canonical repair, which are exponentially many. As an alternative, it suffices to construct a subset of the objects that induces a sub-qABox to which a homomorphism exists — similar as done above.

We therefore employ the following rules, that are exhaustively applied, starting with the set $\Sigma_{\mathsf{I}} \cup Y_0$ from above.

Object Rule 2. If $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ and $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$ are in $\Sigma_{\mathsf{I}} \cup Y_i$, the role assertion $r(u, v)$ is in $\mathcal{B}$, the extended atom $\exists q.C$ is in $\mathcal{K}$, there is the transition (q, r, p), and $\mathcal{B} \models C(v)$, but $\exists p'.C \notin \mathcal{L}$ for each state p' where $p \leq p'$, then do the following:[13]
Initialize Y_{i+1} as Y_i, and then add the object $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L}' \rangle\!\rangle$ for each minimal repair type $\mathcal{L}'$ for v such that $\mathcal{L} \subseteq \mathcal{L}'$ and $\exists p'.C \in \mathcal{L}'$ for some state $p' \geq p$ and $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L}' \rangle\!\rangle \in \Omega(\mathcal{S})$, but $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L}' \rangle\!\rangle \notin \Sigma_{\mathsf{I}} \cup Y_i$.

Object Rule 3. If $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ and $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$ are in $\Sigma_{\mathsf{I}} \cup Y_i$, the role assertion $r(u, v)$ is in $\mathcal{B}$, the extended atom $\exists q.C$ is in $\mathcal{K}$, there is the transition (q, r, f) where f is a final state, and $\mathcal{B} \models C(v)$, but $\mathsf{Conj}_{\mathcal{R}}(C) \cap \mathcal{L} = \emptyset$, then do the following:
Initialize Y_{i+1} as Y_i, and then add the object $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L}' \rangle\!\rangle$ for each minimal repair type $\mathcal{L}'$ for v such that $\mathcal{L} \subseteq \mathcal{L}'$ and $D \in \mathcal{L}'$ for some $D \in \mathsf{Conj}_{\mathcal{R}}(C)$ and $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L}' \rangle\!\rangle \in \Omega(\mathcal{S})$, but $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L}' \rangle\!\rangle \notin \Sigma_{\mathsf{I}} \cup Y_i$.

Object Rule 4. If $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ and $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$ are in $\Sigma_{\mathsf{I}} \cup Y_i$, the role assertion $r(u, v)$ is in $\mathcal{B}$, the extended atom $\exists q.C$ is in $\mathcal{L}$, there is the transition (q, r^-, p), and $\mathcal{B} \models C(u)$, but $\exists p'.C \notin \mathcal{K}$ for each state p' where $p \leq p'$, then do the following:
Initialize Y_{i+1} as Y_i, and then add the object $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K}' \rangle\!\rangle$ for each minimal repair type $\mathcal{K}'$ for u such that $\mathcal{K} \subseteq \mathcal{K}'$ and $\exists p'.C \in \mathcal{K}'$ for some state $p' \geq p$ and $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K}' \rangle\!\rangle \in \Omega(\mathcal{S})$, but $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K}' \rangle\!\rangle \notin \Sigma_{\mathsf{I}} \cup Y_i$.

Object Rule 5. If $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K} \rangle\!\rangle$ and $\langle\!\langle [v]_{\mathcal{S}}, \mathcal{L} \rangle\!\rangle$ are in $\Sigma_{\mathsf{I}} \cup Y_i$, the role assertion $r(u, v)$ is in $\mathcal{B}$, the extended atom $\exists q.C$ is in $\mathcal{L}$, there is the transition (q, r^-, f) where f is a final state, and $\mathcal{B} \models C(u)$, but $\mathsf{Conj}_{\mathcal{R}}(C) \cap \mathcal{K} = \emptyset$, then do the following:
Initialize Y_{i+1} as Y_i, and then add the object $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K}' \rangle\!\rangle$ for each minimal repair type $\mathcal{K}'$ for u such that $\mathcal{K} \subseteq \mathcal{K}'$ and $D \in \mathcal{K}'$ for some $D \in \mathsf{Conj}_{\mathcal{R}}(C)$ and $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K}' \rangle\!\rangle \in \Omega(\mathcal{S})$, but $\langle\!\langle [u]_{\mathcal{S}}, \mathcal{K}' \rangle\!\rangle \notin \Sigma_{\mathsf{I}} \cup Y_i$.

The final set $\Sigma_{\mathsf{I}} \cup Y_n$, to which none of the above rules is applicable, is large enough to induce a sub-qABox to which a homomorphism from the full canonical repair exists. Specifically, Object Rule i ensures that there are no defects that violate Condition (RAi) for each $i \in \{2, 3, 4, 5\}$.

Last, to get even smaller repairs, one could first apply the above rules exhaustively to determine an equivalent first sub-qABox, say $\exists Y'.\mathcal{B}'$, and then use the step-by-step approach to construct a homomorphism from $\exists Y'.\mathcal{B}'$ to a second sub-qABox (which is often feasible since the first sub-qABox $\exists Y'.\mathcal{B}'$ does not have exponential size anymore). The image of this homomorphism is a sub-qABox of the canonical repair that can be even smaller than $\exists Y'.\mathcal{B}'$, since it only contains those copies to which the homomorphism maps.

4 Extensions and Applications

In this section we present several extensions to the repair framework. Section 4.1 shows how inconsistencies can be repaired that come into play when the bottom concept $\bot$ is added. Section 4.2 deals with repair requests formulated as conjunctive queries. Finally, Section 4.3 briefly mentions additional extensions that cannot be presented in detail here.

[13]The rule is not applicable if Y_{i+1} would be equal to Y_i, and similarly for the other rules.

4.1 Adding the Bottom Concept and Repairing Inconsistencies

The DL $\mathcal{ELROI}(\bot)$ ($\mathcal{ELR}_{\mathsf{reg}}\mathcal{OI}(\bot)$) extends $\mathcal{ELROI}$ ($\mathcal{ELR}_{\mathsf{reg}}\mathcal{OI}$) with the bottom concept $\bot$, which is always interpreted as the empty set. If $\bot$ is available in the TBox, then qABoxes may become inconsistent w.r.t. terminologies. We call the quantified ABox $\exists X.\mathcal{A}$ *consistent* w.r.t. a terminology $(\mathcal{T},\mathcal{R})$ if there is a model of $\exists X.\mathcal{A}$ and $(\mathcal{T},\mathcal{R})$, and *inconsistent* otherwise. For instance, the qABox $\{A(a)\}$ is inconsistent w.r.t. the TBox $\{A \sqsubseteq \bot\}$

Since any concept assertion (qABox) is entailed w.r.t. $(\mathcal{T},\mathcal{R})$ by a qABox that is inconsistent w.r.t. $(\mathcal{T},\mathcal{R})$, any non-empty repair request to an inconsistent qABox requires us also to get rid of the inconsistency. In addition, the definition of what is a repair needs to be revised since (Rep1) is trivially satisfied in case $\exists X.\mathcal{A}$ is inconsistent w.r.t. $(\mathcal{T},\mathcal{R})$. Any qABox $\exists Y.\mathcal{B}$ satisfying (Rep2) and (Rep3) is thus a repair, even if $\exists Y.\mathcal{B}$ is completely unrelated to $\exists X.\mathcal{A}$. Hence, there cannot be an optimal repair since we can always extend a given repair by adding completely unrelated assertions.

Fortunately, in $\mathcal{ELROI}(\bot)$ we can divide the TBox into a positive and an "unsatisfiable" part, where the unsatisfiable part plays a rôle when an inconsistency is derived, but has no effect otherwise. To be more precise, consider an $\mathcal{ELROI}(\bot)$ TBox $\mathcal{T}$. Since each concept containing $\bot$ is equivalent to $\bot$, we can assume without loss of generality that each concept description occurring in $\mathcal{T}$ is either $\bot$ or does not contain $\bot$ as a subconcept. After removing tautological CIs $\bot \sqsubseteq C$, it follows that $\mathcal{T}$ is a disjoint union of a TBox $\mathcal{T}_+$ in which $\bot$ does not occur (the *positive part*) and of a TBox $\mathcal{T}_\bot$ containing only CIs of the form $C \sqsubseteq \bot$ where C does not contain $\bot$ (the *unsatisfiable part*). We can characterize inconsistency by means of this partitioning of $\mathcal{T}$, and show that $\mathcal{T}_\bot$ is only relevant for causing an inconsistency.

Proposition 17. *The following holds for every $\mathcal{ELROI}(\bot)$ terminology $(\mathcal{T},\mathcal{R})$:*

1. *The quantified ABox $\exists X.\mathcal{A}$ is inconsistent w.r.t. $(\mathcal{T},\mathcal{R})$ iff there is a CI $C \sqsubseteq \bot$ in $\mathcal{T}_\bot$ such that $\exists X.\mathcal{A} \models^{\mathcal{T}_+,\mathcal{R}} \exists\{x\}.\{C(x)\}$.*
2. *If $\exists X.\mathcal{A}$ is consistent w.r.t. $(\mathcal{T},\mathcal{R})$, then $\exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists Y.\mathcal{B}$ iff $\exists X.\mathcal{A} \models^{\mathcal{T}_+,\mathcal{R}} \exists Y.\mathcal{B}$.*

Proof. 1. To prove the if direction, assume that there is a CI $C \sqsubseteq \bot$ in $\mathcal{T}_\bot$ such that $\exists X.\mathcal{A} \models^{\mathcal{T}_+,\mathcal{R}} \exists\{x\}.\{C(x)\}$. We must show that $\exists X.\mathcal{A}$ is inconsistent w.r.t. $(\mathcal{T},\mathcal{R})$. Assume to the contrary that $\mathcal{I}$ is a model of $\exists X.\mathcal{A}$ and $(\mathcal{T},\mathcal{R})$. Then $\mathcal{I}$ is a model of $\exists X.\mathcal{A}$ and $(\mathcal{T}_+,\mathcal{R})$, and thus it satisfies $\exists\{x\}.\{C(x)\}$, i.e., there is an element $d \in \mathsf{Dom}(\mathcal{I})$ such that $d \in C^{\mathcal{I}}$. This shows that $\mathcal{I}$ does not satisfy the CI $C \sqsubseteq \bot$, which contradicts our assumption that $\mathcal{I}$ is a model of $\mathcal{T}$.
To show the only-if direction, assume that $\exists X.\mathcal{A}$ is inconsistent w.r.t. $(\mathcal{T},\mathcal{R})$. Since $\exists X.\mathcal{A}$ is consistent w.r.t. $(\mathcal{T}_+,\mathcal{R})$, there is a *universal* model $\mathcal{I}_u$ of $\exists X.\mathcal{A}$ and $(\mathcal{T}_+,\mathcal{R})$ (Ortiz, Rudolph, and Šimkus 2011), i.e., a model that satisfies exactly those qABoxes that are entailed by $\exists X.\mathcal{A}$ w.r.t. $(\mathcal{T}_+,\mathcal{R})$. By our inconsistency assumption, $\mathcal{I}_u$ cannot be a model of $\mathcal{T}_\bot$, and thus there is a CI $C \sqsubseteq \bot$ in $\mathcal{T}_\bot$ that is not satisfied by $\mathcal{I}_u$, i.e., $C^{\mathcal{I}_u} \neq \bot^{\mathcal{I}_u} = \emptyset$. This show that there is an element $d \in \mathsf{Dom}(\mathcal{I}_u)$ such that $d \in C^{\mathcal{I}_u}$, and thus $\mathcal{I}$ satisfies $\exists\{x\}.\{C(x)\}$. Universality of $\mathcal{I}_u$ then yields $\exists X.\mathcal{A} \models^{\mathcal{T}_+,\mathcal{R}} \exists\{x\}.\{C(x)\}$.
2. The if direction is trivial since $\mathcal{T}_+$ is a subset of $\mathcal{T}$. To prove the only-if direction, we assume that $\exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists Y.\mathcal{B}$. Let $\mathcal{I}_u$ be the universal model of $\exists X.\mathcal{A}$ and $(\mathcal{T}_+,\mathcal{R})$ (Ortiz, Rudolph, and Šimkus 2011). According to Proposition 17, it holds that $\exists X.\mathcal{A} \not\models^{\mathcal{T}_+,\mathcal{R}} \exists\{x\}.\{C(x)\}$ for each $C \sqsubseteq \bot \in \mathcal{T}_\bot$, and thus $\mathcal{I}_u$ does not satisfy any of the qABoxes $\exists\{x\}.\{C(x)\}$ for $C \sqsubseteq \bot \in \mathcal{T}_\bot$. This implies that $C^{\mathcal{I}_u} = \emptyset$ for each $C \sqsubseteq \bot \in \mathcal{T}_\bot$, and thus $\mathcal{I}_u$ is also a model of $(\mathcal{T},\mathcal{R})$. Now the assumption $\exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists Y.\mathcal{B}$ yields that $\mathcal{I}_u$ is a model of $\exists Y.\mathcal{B}$. By universality of $\mathcal{I}_u$ we can thus conclude that $\exists X.\mathcal{A} \models^{\mathcal{T}_+,\mathcal{R}} \exists Y.\mathcal{B}$. □

Motivated by the second statement of this proposition, we now use $\mathcal{T}_+$ rather than $\mathcal{T}$ in (Rep1), and of course additionally require the repair to be consistent. Also note that it does not make sense to use $\bot$ in the repair request.

Definition 18. Consider a qABox $\exists X.\mathcal{A}$, an $\mathcal{ELROI}$ repair request $\mathcal{P}$, and an $\mathcal{ELROI}(\bot)$ terminology $(\mathcal{T},\mathcal{R})$. An *inconsistency repair* of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ is a qABox $\exists Y.\mathcal{B}$ such that

(IRep1) $\exists X.\mathcal{A} \models^{\mathcal{T}_+,\mathcal{R}} \exists Y.\mathcal{B}$

(IRep2) $\exists Y.\mathcal{B}$ is consistent w.r.t. $(\mathcal{T},\mathcal{R})$,

(IRep3) $\exists Y.\mathcal{B} \not\models^{\mathcal{T},\mathcal{R}} C(a)$ for each $C(a) \in \mathcal{P}_{\mathsf{loc}}$, and

(IRep4) $\exists Y.\mathcal{B} \not\models^{\mathcal{T},\mathcal{R}} \exists\{x\}.\{D(x)\}$ for each $D \in \mathcal{P}_{\mathsf{glo}}$.

This inconsistency repair is *optimal* if it is not strictly entailed by another inconsistency repair w.r.t. $(\mathcal{T},\mathcal{R})$.

Due to the second statement in Proposition 17, the notion of an inconsistency repair coincides with that of a repair as introduced in Definition 8 if $\exists X.\mathcal{A}$ is consistent w.r.t. $(\mathcal{T},\mathcal{R})$. If $\exists X.\mathcal{A}$ is inconsistent w.r.t. $(\mathcal{T},\mathcal{R})$, then the first statement in Proposition 17 shows that (IRep2) can be enforced by extending the global request with the concepts C for which $C \sqsubseteq \bot \in \mathcal{T}_\bot$. Given a repair request $\mathcal{P}$, we denote the extended request obtained this way as $\mathcal{P}^{\mathcal{T}_\bot}$.

Theorem 19. *Consider a qABox $\exists X.\mathcal{A}$, an $\mathcal{ELROI}$ repair request $\mathcal{P}$, and an $\mathcal{ELROI}(\bot)$ terminology $(\mathcal{T},\mathcal{R})$. If $(\mathcal{T},\mathcal{R})$ is inconsistent, then there are no inconsistency repairs of $\exists X.\mathcal{A}$ w.r.t. $(\mathcal{T},\mathcal{R})$. Otherwise, the (optimal) inconsistency repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ coincide with the (optimal) repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}^{\mathcal{T}_\bot}$ w.r.t. $(\mathcal{T}_+,\mathcal{R})$.*

Proof. It is easy to see that each inconsistency repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ is a repair of $\exists X.\mathcal{A}$ for $\mathcal{P}_\bot$ w.r.t. $(\mathcal{T}_+,\mathcal{R})$, cf. Definitions 8 and 18.

Now we show the opposite direction. Let $\exists Y.\mathcal{B}$ be a repair of $\exists X.\mathcal{A}$ for $\mathcal{P}_\bot$ w.r.t. $(\mathcal{T}_+,\mathcal{R})$.

(IRep1) Since $\exists Y.\mathcal{B}$ satisfies Condition (Rep1), it also satisfies Condition (IRep1).

(IRep2) Since the global part of $\mathcal{P}_\bot$ contains C for each concept inclusion $C \sqsubseteq \bot$ in the unsatisfiable part $\mathcal{T}_\bot$, Condition (Rep3) implies that $\exists Y.\mathcal{B} \not\models^{\mathcal{T}_+,\mathcal{R}} \exists\{x\}.\{C(x)\}$ for each such C. Proposition 17 yields that $\exists Y.\mathcal{B}$ is consistent w.r.t. $(\mathcal{T},\mathcal{R})$, i.e., Condition (IRep2) is fulfilled.

(IRep3) Consider a concept assertion $D(a)$ in the local request $\mathcal{P}_{\mathsf{loc}}$. In order to verify Condition (IRep3) we must show that $\exists Y.\mathcal{B} \not\models^{\mathcal{T},\mathcal{R}} D(a)$. According to Condition (Rep2) and since $D(a)$ is also in the local part of $\mathcal{P}_\bot$, it holds that $\exists Y.\mathcal{B} \not\models^{\mathcal{T}_+,\mathcal{R}} D(a)$. With Proposition 17 we conclude that $\exists Y.\mathcal{B} \not\models^{\mathcal{T},\mathcal{R}} D(a)$.

(IRep4) It remains to show Condition (IRep4). Assume that $E \in \mathcal{P}_{\mathsf{glo}}$. Then E is also in the global part of $\mathcal{P}_\bot$, and so Condition (Rep3) yields that $\exists Y.\mathcal{B} \not\models^{\mathcal{T}_+,\mathcal{R}} \exists\{x\}.\{E(x)\}$. By means of Proposition 17 we infer that $\exists Y.\mathcal{B} \not\models^{\mathcal{T},\mathcal{R}} \exists\{x\}.\{E(x)\}$.

Finally, Proposition 17 shows that the optimal repairs coincide. □

If $\mathcal{R}$ is regular and $(\mathcal{T}_+,\mathcal{R})$ is terminating, then we can apply the approach described in the previous section to compute all optimal inconsistency repairs.

Adding the Unique Name Assumption In this paper, we do not make the *unique name assumption*, which requires $a^\mathcal{I} \neq b^\mathcal{I}$ for distinct individual names a, b. However, in the presence of nominals and bottom, we can enforce that two individuals are interpreted by different elements using the CI $\{a\} \sqcap \{b\} \sqsubseteq \bot$. Thus, our repair approach also works if the unique name assumption is made.

4.2 Repairs for Conjunctive Queries

Until now, we have only allowed the use of $\mathcal{ELROI}$ concept queries in the repair request. We now extend this to conjunctive queries (CQs). More precisely, we employ Boolean conjunctive queries (BCQs), i.e., CQs without answer variables. This is in line with the fact that we only considered concept queries where the answer variable was either instantiated with an individual or existentially quantified. In (Grau and Kostylev 2019), CQs with answer variables are employed in the policy (which corresponds to our repair request), with the meaning that such a CQ should not have any answer tuple in the repair. This can clearly be expressed using the finitely many BCQs obtained by instantiating the answer variables with all answer tuples. As already mentioned above, BCQs and qABoxes are merely syntactic variants of each other (Baader et al. 2020). For this reason, we avoid introducing BCQs formally and use qABoxes instead.

Definition 20. A *qABox repair request* $\mathcal{P}$ is a finite set of qABoxes. Given a qABox $\exists X.\mathcal{A}$, a terminology $(\mathcal{T},\mathcal{R})$, and a qABox repair request $\mathcal{P}$, a *repair* of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ is a qABox $\exists Y.\mathcal{B}$ that satisfies

(CQRep1) $\exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists Y.\mathcal{B}$, and

(CQRep2) $\exists Y.\mathcal{B} \not\models^{\mathcal{T},\mathcal{R}} \exists Z.\mathcal{C}$ for each $\exists Z.\mathcal{C} \in \mathcal{P}$.

It is *optimal* if it is not strictly entailed by another repair.

Since both concept assertions $C(a)$ and global repair requests $\exists\{x\}.\{C(x)\}$ for $\mathcal{ELROI}$ concept descriptions C can be rewritten into equivalent qABoxes, using the first three rules in Figure 1, the repair requests and repairs introduced in Definition 8 are a special case of the qABox repair requests and repairs introduced here. We will now investigated under what conditions a rewriting in the other direction is possible.

Definition 21. An $\mathcal{ELROI}$ *rewriting* of the qABox $\exists Z.\mathcal{C}$ is an $\mathcal{ELROI}$ concept description C such that $\exists Z.\mathcal{C}$ and $\exists\{x\}.\{C(x)\}$ are equivalent.

By adapting the notion of c-acyclicity introduced in (Alexe et al. 2011), we can give (effectively checkable) conditions characterizing the existence of such a rewriting. Basically, the qABox is translated into an appropriate undirected graph, and the condition for c-acyclicity says that every cycle must contain an individual.

Given a qABox $\exists X.\mathcal{A}$, we define the undirected graph $\mathcal{G}_{\exists X.\mathcal{A}} := (V, E)$ where the vertex set V consists of all equivalence classes $[t]_{\exists X.\mathcal{A}}$ for objects $t \in \mathsf{Obj}(\exists X.\mathcal{A})$, and the edge set E contains an undirected edge $\{[t],[u]\}$ between $[t]$ and $[u]$ if there are representatives $t' \in [t]$ and $u' \in [u]$ that occur together in a role assertion in $\mathcal{A}$, i.e.,

$$E := \{\, \{[t],[u]\} \mid R(t',u') \in \mathcal{A} \text{ for some } t' \in [t], u' \in [u] \,\}.$$

A *path* is of the form $[t_0] \to [t_1] \to \cdots \to [t_\ell]$ where $\{[t_{j-1}],[t_j]\}$ is an edge in E for each $j \in \{1,\ldots,\ell\}$, and we call $[t_0]$ its *source*, $[t_\ell]$ its *target*, and ℓ its *length*. We call $\exists X.\mathcal{A}$ *connected* if the graph $\mathcal{G}_{\exists X.\mathcal{A}}$ is connected, i.e., if for each two vertices $[t]$ and $[u]$, there is a path with source $[t]$ and target $[u]$. The qABox $\exists X.\mathcal{A}$ is *c-acyclic* if

- every cycle in the graph $\mathcal{G}_{\exists X.\mathcal{A}}$ contains a vertex $[a]$ for an individual a, where a *cycle* is a path with same source and target that has non-zero length (i.e., also loops $\{[t]\}$ are cycles), and
- each two variables $x, y \in X$ occur together in at most one role assertion in $\mathcal{A}$.

We further define an edge labeling L by[14]

$$L([t],[u]) := \{\, R \mid R(t',u') \in \mathcal{A} \text{ for some } t' \in [t], u' \in [u] \,\}.$$

Specifically, $L([t],[u])$ consists of all labels when the undirected edge $\{[t],[u]\}$ is treated as a directed edge from $[t]$ to $[u]$. It thus holds that $L([u],[t]) = \{\, R^- \mid R \in L([t],[u]) \,\}$.

It follows that, for each c-acyclic qABox $\exists X.\mathcal{A}$, we have $L([x],[y]) \leq 1$ for all variables $x, y \in X$, but there is no such restriction for label sets $L([t],[u])$ where t or u is an individual (or both).

Furthermore, we need the notion of a *core*, which is a qABox such that each endomorphism on it is bijective. Each qABox $\exists X.\mathcal{A}$ has a computable and (up to renaming) unique core to which it is equivalent (Hell and Nešetřil 1992). It will be denoted in the following as $\mathsf{core}(\exists X.\mathcal{A})$.

Example XXXIII. Consider the qABox $\exists\{x,y\}.\{r(a,b), s(b,b), B(b), r(a,x), B(x), s(x,y), B(y), s(y,x)\}$. It is equivalent to the sub-qABox $\{r(a,b), s(b,b), B(b)\}$ due to

[14] Recall that $r^-(t,u)$ stands for $r(u,t)$.

the homomorphism that sends a to a and sends the other objects b, x, y to b. The latter qABox is the core since it is itself not equivalent to a proper sub-qABox.

Example XXXIV. The qABox $\exists\{x, y\}.\{r(a, x), B(x), r(a, y), B(y)\}$ has two cores, namely $\exists\{x\}.\{r(a, x), B(x)\}$ and $\exists\{y\}.\{r(a, y), B(y)\}$, which are equivalent.

The next proposition characterizes when $\mathcal{ELROI}$ rewritings exist.

Proposition 22. *A qABox has an $\mathcal{ELROI}$ rewriting iff its core is connected and c-acyclic.*

Proof. Assume that the $\mathcal{ELROI}$ concept description C is a rewriting of $\exists X.\mathcal{A}$, which means that $\exists X.\mathcal{A}$ is equivalent to the qABox $\exists Y.\mathcal{B}$ obtained from $\exists\{x\}.\{C(x)\}$ by exhaustively applying the first three rules from Figure 1, where w.l.o.g. we apply the Nominal Rule with lowest priority. Obviously, the latter qABox is connected.

Further recall that new variables are only introduced by applications of the Existential Restriction Rule. Let $\exists Y'.\mathcal{B}'$ be the intermediate qABox obtained by exhaustive applications of the Conjunction Rule and the Existential Restriction Rule, but before the Nominal Rules is applied for the first time. Then $\exists Y'.\mathcal{B}'$ is connected, it is tree-shaped (with root x) and thus acyclic, and each two variables occur in at most one role assertion together. Now the final qABox $\exists Y.\mathcal{B}$ is obtained by exhaustively applying the Nominal Rule to $\exists Y'.\mathcal{B}'$, which might identify several variables by replacing them with one individual (or rather: with individuals from the same equivalence class). Each resulting cycle in $\mathcal{G}_{\exists Y.\mathcal{B}}$ must thus contain an equivalence class represented by individual, i.e., $\exists Y.\mathcal{B}$ is c-acyclic. It follows that also the core of $\exists Y.\mathcal{B}$ is connected and c-acyclic. Since $\exists X.\mathcal{A}$ and $\exists Y.\mathcal{B}$ are equivalent and cores are unique up to renaming of variables, we conclude that also the core of $\exists X.\mathcal{A}$ must be connected and c-acyclic.

Conversely, assume that a qABox has a connected c-acyclic core $\exists X.\mathcal{A}$, to which it is equivalent. As first step, we transform the undirected graph $\mathcal{G}_{\exists X.\mathcal{A}} = (V, E)$ into a tree. The $\mathcal{ELROI}$ rewriting of $\exists X.\mathcal{A}$ is afterwards obtained from this tree.

The tree will be constructed inductively, namely as the final element of a sequence of trees $T_0, T_1, \dots$ where $T_{i-1} \subset T_i$ for each index i, and every tree T_i is a prefix-closed[15] set of (directed) paths in the graph $\mathcal{G}_{\exists X.\mathcal{A}}$.

We choose an arbitrary vertex $[t_0]$ from V and then set $T_0 \coloneqq \{[t_0]\}$ and $F_0 \coloneqq \emptyset$. Afterwards, we apply the following two rules as often as possible, where the first rule has higher precedence. In a nutshell, the strategy is to always first to try to unravel at variables, and only if this is not possible to unravel at individuals.

Variable Rule. If there is a path $p \in T_i$ with target $[x]$ for some variable $x \in X$ and there is a vertex $[t]$ such that $\{[x], [t]\} \in E \setminus F_i$, then define:

$$T_{i+1} \coloneqq T_i \cup \{\, p \to [u] \mid \{[x], [u]\} \in E \setminus F_i \,\}$$

$$F_{i+1} \coloneqq F_i \cup \{\, \{[x], [u]\} \mid \{[x], [u]\} \in E \setminus F_i \,\}$$

Individual Rule. If there is a path $p \in V_i$ with target $[a]$ for some individual $a \in \Sigma_\mathsf{I}$ and there is a vertex $[t]$ such that $\{[a], [t]\} \in E \setminus F_i$, then define:

$$T_{i+1} \coloneqq T_i \cup \{p \to [t]\}$$
$$F_{i+1} \coloneqq F_i \cup \{\{[x], [t]\}\}$$

After no rule is applicable, say after n iterations, denote by T the final tree T_n and likewise let $F \coloneqq F_n$. Since the graph (V, E) is connected and neither of the two rules can be applied to T and F, it holds that $F = E$. It follows that all edges in E are represented in the tree T, i.e., for each edge $\{[t], [u]\}$ in E, there is a path p in T such that

- either $\mathsf{target}(p) = [t]$ and $p \to [u]$ is in T,
- or $\mathsf{target}(p) = [u]$ and $p \to [t]$ is in T.

Next, we show the following claim.

Claim. For each vertex $[x]$ where x is a variable in X, there is a unique path $p \in T$ with target $[x]$.

In order to prove the claim, assume that p_1 and p_2 are distinct paths in T, where w.l.o.g. p_1 has been created before p_2. Further let p be the longest common prefix of p_1 and p_2. We continue with a case distinction.

1. In the first case, assume that p_1 is a prefix of p_2. We infer a contradiction, since after applying the Variable Rule to p_1 there are no edges involving $[x]$ left over in $E \setminus F_i$, i.e., afterwards the path p_2 could not have been constructed anymore.

$$[t_0] \xrightarrow{\quad p_1 \quad} [x] \xrightarrow{\; p_2 - p_1 \;} [x]$$

   Conversely, p_2 cannot be a prefix of p_1, since p_1 has been created before p_2.

2. Now let $p_1 \neq p \neq p_2$, and further assume that the target of p is $[y]$ for a variable y in X. Then there is a cycle involving $[x]$ and $[y]$. Due to c-acyclicity, there is an individual a such that $[a]$ is the target of a path q where either $p < q < p_1$ or $p < q < p_2$.[16]

$$\begin{tikzcd} {[t_0]} \arrow[r, "p"] & {[y]} \arrow[r, bend left, "p_1 - p"] \arrow[r, bend right, "p_2 - p"'] & {[x]} \end{tikzcd}$$

   If $q < p_1$, then p_2 must have a prefix q' with target $[a']$ for some individual a', since otherwise p_2 would have been created before p_1. In both cases, p_2 has a prefix q'' of which the last element is of the form $[a'']$ for some individual a''. But this means that the Variable Rule is first applied to p_1 before the Individual Rule can be applied to q'', and thus afterwards there are no edges involving $[x]$ left over, i.e., the path p_2 cannot exist.

[15]A set of paths is *prefix-closed* if it also contains all prefixes of each path in it.

[16]We write $p < q$ if p is a strict prefix of q.

3. In the last case, we have $p_1 \neq p \neq p_2$ and there is an individual a such that $[a]$ is the target of p. Thus, before another successor can be added to p, yielding a prefix of p_2, the Variable Rule is applied to p_1, and after that there are no edges involving $[x]$ left over, which could be used to construct the path p_2, i.e., p_2 does not exist.

$$\begin{tikzcd} {[t_0]} \arrow[r, "p"] & {[a]} \arrow[r, bend left, "p_1 - p"] \arrow[r, bend right, "p_2 - p"'] & {[x]} \end{tikzcd}$$

From the final tree T, we construct the concept $C \coloneqq C_{[t_0]}$ where the concepts C_p for paths $p \in T$ are recursively defined as follows:

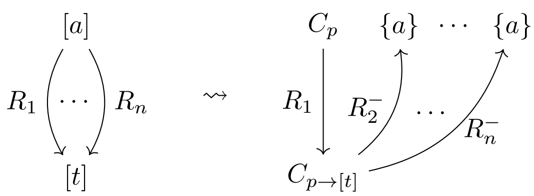
- If $\mathsf{target}(p) = [a]$ for an individual a, then let

$$\begin{aligned} C_p \coloneqq {} & \bigsqcap \{ \, \{b\} \mid b \in [a] \, \} \\ & \sqcap \bigsqcap \{ \, A \mid A(b) \in \mathcal{A} \text{ for some } b \in [a] \, \} \\ & \sqcap \bigsqcap \{ \, D_{p \to [t]} \mid p \to [t] \in T \, \} \end{aligned}$$

  where

$$D_{p\to[t]} \coloneqq \exists R_1.(C_{p\to[t]} \sqcap \exists R_2^-.\{a\} \sqcap \cdots \sqcap \exists R_n^-.\{a\})$$

  for some arbitrary enumeration $\{R_1, \ldots, R_n\}$ of the label set $L([a],[t])$. The construction is illustrated below for a successor $p \to [t]$.



  Firstly, we choose an arbitrary role R_1 from $L([a],[t])$ and add to C_p the existential restriction $\exists R_1.C_{p\to[t]}$. Secondly, we add to the filler of this existential restriction the atoms $\exists R_i^-.\{a\}$ for each remaining role R_i in $L([a],[t])$. Since C_p also contains the nominal $\{a\}$, each such atom $\exists R_i^-.\{a\}$ essentially represents that $C_{p\to[t]}$ is also an R_i-successor of C_p.

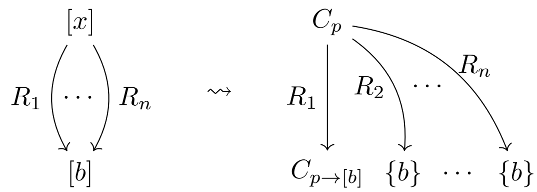
- If otherwise $\mathsf{target}(p) = [x]$ for a variable x, then define[17]

$$\begin{aligned} C_p \coloneqq {} & \bigsqcap \{ \, A \mid A(x) \in \mathcal{A} \, \} \\ & \sqcap \bigsqcap \{ \, D_{p \to [t]} \mid p \to [t] \in T \, \} \end{aligned}$$

  where $D_{p\to[y]} \coloneqq \exists R.C_{p\to[y]}$ for $\{R\} = L([x],[y])$,[18] and

$$D_{p\to[b]} \coloneqq \exists R_1.C_{p\to[b]} \sqcap \exists R_2.\{b\} \sqcap \cdots \sqcap \exists R_n.\{b\}$$

  for some arbitrary enumeration $\{R_1, \ldots, R_n\}$ of the label set $L([x],[b])$. Specifically for successors $p \to [b]$ of the second kind, the constructing is depicted below.



  We choose a role R_1 from the label set $L([x],[b])$ and add to C_p the existential restriction $\exists R_1.C_{p\to[b]}$. We further add to C_p the atom $\exists R_i.\{b\}$ for each remaining role R_i in $L([x],[b])$. Since the nominal $\{b\}$ is also a top-level conjunct of $C_{p\to[b]}$, every atom $\exists R_i.\{b\}$ essentially expresses that $C_{p\to[b]}$ is also an R_i-successor of C_p.

[17] Recall that then $[x] = \{x\}$.

[18] Recall that $L([x],[y]) \leq 1$ for variables x and y, due to c-acyclicity. But since $[x]$ is the target of p and $p \to [y]$ is in T, this label set $L([x],[y])$ cannot be empty.

Now, during the expansion of $\exists\{x\}.\{C(x)\}$ by means of the first three rules from Figure 1, we always choose y as the successor variable when the Existential Restriction Rule is applied to an assertion $\exists R.C_{p\to[y]}(u)$ where $y \in X$, i.e., the assertion is replaced by $R(u,y)$ and $C_{p\to[y]}(y)$. This is possible since the above claim holds. In essence, the tree T is recreated during the expansion but with all paths ending with the same individual class $[a]$ identified as one of the individuals in $[a]$. In the end, the identical mapping is the homomorphism from $\exists X.\mathcal{A}$ to the expansion. By means of Proposition 2 we conclude that $\exists\{x\}.\{C(x)\}$ entails $\exists X.\mathcal{A}$.

It remains to show the opposite entailment. By an induction on the tree T, starting at the leafs, it is easy to show that $\exists X.\mathcal{A}$ entails $C_p(t)$ for each $p \in T$ and for each $t \in \mathsf{target}(p)$. It follows that $\exists X.\mathcal{A}$ entails $\exists\{x\}.\{C_{[t_0]}(x)\}$, which equals $\exists\{x\}.\{C(x)\}$. □

The proof of the if-direction is constructive in the sense that it shows how the rewriting can be computed. Thus, if all qABoxes in a given qABox repair request are $\mathcal{ELROI}$ rewritable, then we can reduce qABox repair to $\mathcal{ELROI}$ repair.

Example 23. As an example of a qABox that is not c-acyclic we consider $\exists\{x,y\}.\{r(a,x), s(x,y), s(y,x)\}$. It has a cycle from x to y and then back that does not involve an individual. It is not $\mathcal{ELROI}$ rewritable since an $\mathcal{ELROI}$ concept could only enforce going back from y to the predecessor x if one of them were an individual whose name can be used in the concept. In contrast, the qABox $\exists\{y\}.\{s(a,y), s(y,a)\}$, which is c-acyclic, has the $\mathcal{ELROI}$ rewriting $\{a\} \sqcap \exists s.\exists s.\{a\}$. Note that the qABox $\exists\{x,y\}.\{s(x,y), r(x,y)\}$ is also not c-acyclic. Again, an $\mathcal{ELROI}$ concept cannot enforce that there is a joint s- and r-successor of x. The qABox $\exists\{y\}.\{s(a,y), r(a,y)\}$ has the $\mathcal{ELROI}$ rewriting $\exists r^-.\{a\} \sqcap \exists s^-.\{a\}$.

Example XXXV. The qABox in Example XXXIII is connected and not c-acyclic, but its core is c-acyclic. Thus it is $\mathcal{ELROI}$ rewritable, namely into the concept description $\{a\} \sqcap \exists r.(\{b\} \sqcap B \sqcap \exists s.\{b\})$.

Example XXXVI. The qABox $\exists\{x,y\}.\{A(x),B(y)\}$ is core and c-acyclic but not connected. It has no $\mathcal{ELROI}$ rewriting. If the *universal role* u would be available, which is always interpreted as the full binary relation on the domain of each interpretation, then a rewriting would be $A \sqcap \exists u.B$.

Considering Proposition 22, one might think that non-connectedness of $\mathsf{core}(\exists Z.\mathcal{C})$ for $\exists Z.\mathcal{C} \in \mathcal{P}$ could be an impediment to reducing qABox repair to $\mathcal{ELROI}$ repair. However, this is not the case: it is sufficient that all connected components of $\mathsf{core}(\exists Z.\mathcal{C})$ are $\mathcal{ELROI}$ rewritable. To be more precise, let $\mathfrak{H}(\mathcal{P})$ be the set of all hitting sets of $\{\, \mathsf{CoCo}(\mathsf{core}(\exists Z.\mathcal{C})) \mid \exists Z.\mathcal{C} \in \mathcal{P} \text{ and } \exists X.\mathcal{A} \models^{\mathcal{T},\mathcal{R}} \exists Z.\mathcal{C} \,\}$, where the operator CoCo yields the set of connected components of an input qABox.

Lemma 24. *$\exists Y.\mathcal{B}$ is a repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ iff there is a hitting set $\mathcal{H}$ in $\mathfrak{H}(\mathcal{P})$ such that $\exists Y.\mathcal{B}$ is a repair of $\exists X.\mathcal{A}$ for $\mathcal{H}$ w.r.t. $(\mathcal{T},\mathcal{R})$.*

Proof. The lemma follows easily from the following two observations:

- Each qABox $\exists Z.\mathcal{C}$ in $\mathcal{P}$ is equivalent to its core, and so $\exists Y.\mathcal{B} \models \exists Z.\mathcal{C}$ iff $\exists Y.\mathcal{B} \models \mathsf{core}(\exists Z.\mathcal{C})$.
- $\exists Y.\mathcal{B} \models \mathsf{core}(\exists Z.\mathcal{C})$ iff $\exists Y.\mathcal{B}$ entails all connected components of $\mathsf{core}(\exists Z.\mathcal{C})$. $\square$

According to our previous considerations, we can compute the optimal repairs for such a hitting set $\mathcal{H}$ if each component in $\mathcal{H}$ has an $\mathcal{ELROI}$ rewriting. The elements of $\mathcal{H}$ are connected components of the cores of the elements of $\mathcal{P}$. Since such a core is c-acyclic iff all its connected components are so, it is thus sufficient to require that the cores of the elements of $\mathcal{P}$ are c-acyclic. Under this assumption, the set of all optimal repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ is then obtained by collecting the optimal repairs of $\exists X.\mathcal{A}$ for $\mathcal{H}$ w.r.t. $(\mathcal{T},\mathcal{R})$ for all hitting set $\mathcal{H}$ in $\mathfrak{H}(\mathcal{P})$, and then removing elements from this set that are strictly entailed by other elements.

Theorem 25. *Let $\exists X.\mathcal{A}$ be a qABox, $(\mathcal{T},\mathcal{R})$ a terminating terminology with a regular RBox whose associated automata can effectively be computed, and $\mathcal{P}$ be a qABox repair request. Then the set of all optimal repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ can be effectively computed if $\mathsf{core}(\exists Z.\mathcal{C})$ is c-acyclic for all qABoxes $\exists Z.\mathcal{C}$ in $\mathcal{P}$. In addition, each repair is then entailed by an optimal repair.*

Without restrictions on the qABoxes in the repair request, the set of optimal repairs need not cover all repairs in the sense stated in the theorem, even if the qABox to be repaired is an ABox and the terminology is empty. In fact, it follows from (Nešetřil and Tardif 2000, Corollary 3.5) that the ABox $\{r(a,a)\}$ has no optimal repair for the repair request consisting of the qABox $\exists\{x\}.\{r(x,x)\}$. But the empty ABox is a repair, which is thus not entailed by an optimal one.

To be more precise, (Nešetřil and Tardif 2000, Corollary 3.5) considers relational structures without constants, partially ordered by the homomorphism order: for such structures $\mathbb{A}$ and $\mathbb{B}$, we write $\mathbb{A} \leq \mathbb{B}$ if there is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$, and we write $\mathbb{A} < \mathbb{B}$ if $\mathbb{A} \leq \mathbb{B}$ and $\mathbb{B} \not\leq \mathbb{A}$. Obviously, each qABox without individual names is a relational structure without constants. It is shown that, if $\mathbb{A}$ and $\mathbb{B}$ are such structures, where $\mathbb{A}$ is connected and core but not acyclic and where $\mathbb{B} < \mathbb{A}$, then there exists a structure $\mathbb{C}$ in between, i.e., $\mathbb{B} < \mathbb{C} < \mathbb{A}$.

Translating the latter statement to qABoxes yields the following. If $\exists X.\mathcal{A}$ and $\exists Y.\mathcal{B}$ are qABoxes in which no individual names occur, $\exists X.\mathcal{A}$ is core and connected but not acyclic, and $\exists Y.\mathcal{B}$ is strictly entailed by $\exists X.\mathcal{A}$, then there exists a qABox $\exists Z.\mathcal{C}$ without individuals that is strictly entailed by $\exists X.\mathcal{A}$ and that strictly entails $\exists Y.\mathcal{B}$.

Now if a qABox $\exists X.\mathcal{A}$ is both the input qABox to be repaired and is the only qABox in the repair request, then the repairs are exactly those qABoxes that are strictly entailed by $\exists X.\mathcal{A}$. Thus it follows from the above that, if $\exists X.\mathcal{A}$ is core and connected but contains a cycle, then $\exists X.\mathcal{A}$ has no optimal repairs for the repair request $\{\exists X.\mathcal{A}\}$ w.r.t. the empty terminology, using the fact that then $\exists X.\mathcal{A}$ must strictly entail the empty qABox. An example for such a qABox is $\exists X.\mathcal{A} \coloneqq \exists\{x\}.\{r(x,x)\}$, as above.

To see that also the ABox $\{r(a,a)\}$ has no optimal repair for the repair request $\{\exists\{x\}.\{r(x,x)\}\}$, assume that $\exists Y.\mathcal{B}$ is a repair. Let $\exists Z.\mathcal{C}$ be obtained from $\exists Y.\mathcal{B}$ by replacing each occurrence of the individual name a by a fresh variable x_a. It then follows that $\exists Z.\mathcal{C}$ is a repair of $\exists\{x\}.\{r(x,x)\}$ for $\{\exists\{x\}.\{r(x,x)\}\}$. The above yields a repair $\exists Z'.\mathcal{C}'$ of $\exists\{x\}.\{r(x,x)\}$ for $\{\exists\{x\}.\{r(x,x)\}\}$ that strictly entails $\exists Z.\mathcal{C}$, and so there is a homomorphism h from $\exists Z.\mathcal{C}$ to $\exists Z'.\mathcal{C}'$.

Now let $\exists Y'.\mathcal{B}'$ be obtained from $\exists Z'.\mathcal{C}'$ by replacing each occurrence of $h(x_a)$ with the individual name a. It is obvious that $\{r(a,a)\}$ entails $\exists Y'.\mathcal{B}'$. Furthermore, the matrix $\mathcal{B}'$ cannot contain an r-loop as otherwise $\mathcal{C}'$ would contain an r-loop, which contradicts the fact that $\exists Z'.\mathcal{C}'$ does not entail $\exists\{x\}.\{r(x,x)\}$. Thus $\exists Y'.\mathcal{B}'$ does not entail $\exists\{x\}.\{r(x,x)\}$, and so it is a repair of $\{r(a,a)\}$ for $\{\exists\{x\}.\{r(x,x)\}\}$.

It remains to show that $\exists Y'.\mathcal{B}'$ strictly entails $\exists Y.\mathcal{B}$, which implies that $\exists Y.\mathcal{B}$ is no optimal repair of the ABox $\{r(a,a)\}$ for the repair request $\{\exists\{x\}.\{r(x,x)\}\}$. Since $\exists Y.\mathcal{B}$ is an arbitrary such repair, $\{r(a,a)\}$ has no optimal repairs for $\{\exists\{x\}.\{r(x,x)\}\}$.

1. We show that there is a homomorphism from $\exists Y.\mathcal{B}$ to $\exists Y'.\mathcal{B}'$. Define the mapping $k \coloneqq h$, except $k(a) \coloneqq a$ and $k(t) \coloneqq a$ if $h(t) = h(x_a)$.
   (a) If $r(y,z) \in \mathcal{B}$ where $y \neq a$ and $z \neq a$, then $r(y,z) \in \mathcal{C}$ and $y \neq x_a$ and $z \neq x_a$. We infer $r(h(y),h(z)) \in \mathcal{C}'$.
   - If $h(y) = h(x_a)$ and $h(z) = h(x_a)$, then $\exists Z'.\mathcal{C}'$ would entail $\exists\{x\}.\{r(x,x)\}$, a contradiction.
   - If $h(y) = h(x_a)$ and $h(z) \neq h(x_a)$, then $k(y) = a$, $k(z) = h(z)$, and $\mathcal{B}'$ contains the role assertion $r(a,h(z))$, which equals $r(k(y),k(z))$.
   - The case where $h(y) \neq h(x_a)$ and $h(z) = h(x_a)$ is analogous.
   - If $h(y) \neq h(x_a)$ and $h(z) \neq h(x_a)$, then $k(y) = h(y)$, $k(z) = h(z)$, and $\mathcal{B}'$ contains the role assertion $r(h(y),h(z))$, which equals $r(k(y),k(z))$.

(b) If $r(a,z) \in \mathcal{B}$ where $z \neq a$, then $r(x_a,z) \in \mathcal{C}$ where $z \neq x_a$. We infer $r(h(x_a),h(z)) \in \mathcal{C}'$.
  - If $h(z) = h(x_a)$, then $\exists Z'.\mathcal{C}'$ would entail $\exists\{x\}.\{r(x,x)\}$, a contradiction.
  - If $h(z) \neq h(x_a)$, then $k(a) = a$, $k(z) = h(z)$, and $\mathcal{B}'$ contains the role assertion $r(a,h(z))$, which equals $r(k(a),k(z))$.

(c) Role assertions $r(y,a) \in \mathcal{B}$ where $y \neq a$ can be treated analogously.

(d) The role assertion $r(a,a)$ cannot be in $\mathcal{B}$, as otherwise $\exists Y.\mathcal{B}$ would entail $\exists\{x\}.\{r(x,x)\}$.

2. Assume that there is a homomorphism ℓ from $\exists Y'.\mathcal{B}'$ to $\exists Y.\mathcal{B}$. We show that the mapping $\ell' := \ell$ except where $\ell'(h(x_a)) := a$ would then be a homomorphism from $\exists Z'.\mathcal{C}'$ to $\exists Z.\mathcal{C}$, yielding a contradiction.

(a) If $r(y,z) \in \mathcal{C}'$ where $y \neq h(x_a)$ and $z \neq h(x_a)$, then also $\mathcal{B}'$ contains the role assertion $r(y,z)$. Thus $\mathcal{B}$ contains $r(\ell(y),\ell(z))$, which equals $r(\ell'(y),\ell'(z))$.

(b) If $r(y,z) \in \mathcal{C}'$ where $y = h(x_a)$ and $z \neq h(x_a)$, then $\mathcal{B}'$ contains the role assertion $r(a,z)$. Thus $\mathcal{B}$ contains $r(\ell(a),\ell(z))$, which equals $r(\ell'(y),\ell'(z))$ since $\ell'(y) = \ell'(h(x_a)) = a = \ell(a)$.

(c) Role assertions $r(y,z) \in \mathcal{C}'$ where $y \neq h(x_a)$ and $z = h(x_a)$ can be treated analogously.

(d) $\mathcal{C}'$ cannot contain a role assertion $r(y,z)$ where $y = h(x_a)$ and $z = h(x_a)$, since otherwise $\mathcal{C}'$ would contain an r-loop and so $\exists Z'.\mathcal{C}'$ would entail $\exists\{x\}.\{r(x,x)\}$.

Finally, we conclude that we cannot extend our optimal repair framework to fully support existential self-restrictions $\exists r.\mathsf{Self}$ in the repair request, namely since such a concept is equivalent to the qABox $\exists\{x\}.\{r(x,x)\}$. Specifically, we could still deal with assertions $\exists r.\mathsf{Self}(a)$ as these could be translated to $\exists r.\{a\}(a)$, but such a translation is impossible if an existential self-restriction occurs in a conjunction without nominals.

4.3 Further Extensions

The repair framework developed in this paper can also be used to deal with regular path expressions in the repair request, Horn-$\mathcal{ALCOI}$ TBoxes, and qABoxes that have a static part that must not be changed. The basic idea is to create an $\mathcal{ELROI}$ terminology over an extended signature that is a conservative extension of the input terminology, and in which such extensions can be expressed. Our repair approach is then applied w.r.t. this terminology. However, the repairs obtained this way may still contain names not occurring in the original signature, and thus these additional symbols need to be removed from these repairs appropriately.

We illustrate this for the case of *regular path expressions*, which are regular expressions over the alphabet of all roles. In repair requests, the concepts may now contain such expressions in place of roles. The semantics is defined by interpreting union, concatenation, and Kleene-star in the regular expressions as union, composition, and reflexive-transitive closure of binary relations, and the empty word as the identity relation. For example, the concept assertion $(\exists r^*.\{b\})(a) \in \mathcal{P}$ then says that, in the repair, there should not be an (empty or non-empty) r-path from a to b. To express the regular expression r^*, we introduce a new role name $\lceil r^* \rceil$ and extend the RBox with the RIs $\varepsilon \sqsubseteq \lceil r^* \rceil$, $r \sqsubseteq \lceil r^* \rceil$, and $\lceil r^* \rceil \circ \lceil r^* \rceil \sqsubseteq \lceil r^* \rceil$. In the repair request, we now use $\lceil r^* \rceil$ in place of r^*. A repair computed for this modified request may still contain the new name $\lceil r^* \rceil$, but we can simply remove all assertions containing it to obtain a repair in the original signature.

If a part of the given qABox is known to be correct, one may want to keep this part static when repairing (i.e., the repair should still imply this static part). Since our TBoxes are static and concept and role assertions can be expressed using nominals, the idea is now to move the static part of the qABox to the TBox. However, to express assertions involving variables, the signature needs to be extended by adding these variables as individual names.

More details on how to deal with these two extensions and on how Horn-$\mathcal{ALCOI}$ TBoxes can be expressed will be presented in the following.

First of all, the definition of an inconsistency repair (Definition 18) is changed by additionally taking the static data into account. In particular, we assume that the input qABox, which is to be repaired, is a union of the *refutable part* $\exists X.\mathcal{A}$ and the *static part* $\exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$. The latter must not be changed.

For technical reasons but w.l.o.g., we assume that each two quantified ABoxes have disjoint sets of variables. Then the *union* $\exists X_1.\mathcal{A}_1 \cup \exists X_2.\mathcal{A}_2$ of two qABoxes is defined as $\exists(X_1 \cup X_2).(\mathcal{A}_1 \cup \mathcal{A}_2)$. Each model of $\exists X_1.\mathcal{A}_1$ and $\exists X_2.\mathcal{A}_2$ is a model of the union, and vice versa.

The terminology $(\mathcal{T},\mathcal{R})$ now consists of a Horn-$\mathcal{ALCOI}$ TBox $\mathcal{T}$ and a regular RBox $\mathcal{R}$. At the same time, we will additionally support *regular path expressions (RPEs)* that can be used in place of roles, but only within the repair request $\mathcal{P}$. We will formally introduce Horn-$\mathcal{ALCOI}$ and RPEs later when we show how they can be translated into $\mathcal{ELROI}$.

Definition XXXVII. An *SI-repair* of $\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ is a quantified ABox $\exists Y.\mathcal{B}$ over Σ such that

(SIRep1) $(\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}) \models^{\mathcal{T},\mathcal{R}} \exists Y.\mathcal{B}$,

(SIRep2) $(\exists Y.\mathcal{B} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s})$ is consistent w.r.t. $(\mathcal{T},\mathcal{R})$,

(SIRep3) $(\exists Y.\mathcal{B} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}) \not\models^{\mathcal{T},\mathcal{R}} C(a)$ for each $C(a) \in \mathcal{P}_{\mathsf{loc}}$, and

(SIRep4) $(\exists Y.\mathcal{B} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}) \not\models^{\mathcal{T},\mathcal{R}} \exists\{x\}.\{D(x)\}$ for each $D \in \mathcal{P}_{\mathsf{glo}}$.

Additionally, $\exists Y.\mathcal{B}$ is *optimal* if there is no other SI-repair $\exists Z.\mathcal{C}$ such that $(\exists Z.\mathcal{C} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}) \models^{\mathcal{T},\mathcal{R}} \exists Y.\mathcal{B}$ but $(\exists Y.\mathcal{B} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}) \not\models^{\mathcal{T},\mathcal{R}} \exists Z.\mathcal{C}$.

In the above definition, we denote by a repair only the modified version of the refutable part $\exists X.\mathcal{A}$. In applications one should obviously return it together with the static part, i.e., return $\exists Y.\mathcal{B} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$.

In the remainder of this section we will explain how the static data $\exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$, the terminology $(\mathcal{T},\mathcal{R})$, and the repair request $\mathcal{P}$ (all over a signature Σ) can be transformed, one by one, into an $\mathcal{ELR}_{\mathsf{reg}}\mathcal{OI}(\bot)$ terminology $(\mathcal{T}^\#,\mathcal{R}^\#)$ and an

$\mathcal{ELROI}$ repair request $\mathcal{P}^{\#}$ (over an extended signature $\Sigma^{\#}$) such that the SI-repairs can be obtained as Σ-restrictions of the inconsistency repairs for $\mathcal{P}^{\#}$ w.r.t. $(\mathcal{T}^{\#}, \mathcal{R}^{\#})$.

All transformations follow a similar pattern, namely the input $\mathfrak{I}$ (over signature $\Sigma_{\mathfrak{I}}$) will be $\Sigma_{\mathfrak{I}}$-inseparable to the output $\mathfrak{O}$ (over the extended signature $\Sigma_{\mathfrak{O}}$) in the following sense:

1. Each model of $\mathfrak{I}$ can be extended to a model of $\mathfrak{O}$.[19]
2. For each model of $\mathfrak{O}$, the $\Sigma_{\mathfrak{I}}$-restriction is a model of $\mathfrak{I}$.

Transforming the TBox A Horn-$\mathcal{ALCOI}$ TBox consists of finitely many concept inclusions of the form $E \sqsubseteq F$ where E and F are built by the following grammar rules.

$$\begin{aligned} E &::= \bot \mid \top \mid A \mid \{a\} \mid E \sqcap E \mid E \sqcup E \mid \exists R.E \\ F &::= \bot \mid \top \mid A \mid \neg A \mid \{a\} \mid \neg\{a\} \\ &\quad\ \mid F \sqcap F \mid \neg E \sqcup F \mid \exists R.F \mid \forall R.F \end{aligned}$$

This extends the definition given in (Jung et al. 2020) with nominals. It is easy to see that each $\mathcal{ELROI}(\bot)$ TBox is also a Horn-$\mathcal{ALCOI}$ TBox, but the converse is not true. However, each Horn-$\mathcal{ALCOI}$ TBox can be transformed into an inseparable $\mathcal{ELROI}(\bot)$ TBox using fresh concept names. As auxiliary signature, let Γ consist of all concept names $\lfloor F \rfloor$ where F is constructed by means of the second grammar rule. The given Horn-$\mathcal{ALCOI}$ TBox $\mathcal{T}$ over Σ is now transformed into an $\mathcal{ELROI}(\bot)$ TBox $\lfloor \mathcal{T} \rfloor$ over $\Sigma \cup \Gamma$ as follows.

Initialize $\lfloor \mathcal{T} \rfloor$ as $\mathcal{T}$. Firstly, we replace each left-hand side by a disjunction of disjunction-free concepts by exhaustively applying the following equivalence-preserving rule.

$$\exists R.(E_1 \sqcup E_2) \quad \rightsquigarrow \quad \exists R.E_1 \sqcup \exists R.E_2$$

Secondly, we split up the so obtained concept inclusions by the following equivalence-preserving rule.

$$E_1 \sqcup \cdots \sqcup E_n \sqsubseteq F \quad \rightsquigarrow \quad E_1 \sqsubseteq F,\ \ldots,\ E_n \sqsubseteq F$$

Thirdly, we transform the right-hand sides as follows.

$$\begin{aligned} E \sqsubseteq F_1 \sqcap \cdots \sqcap F_n &\quad \rightsquigarrow \quad E \sqsubseteq F_1,\ \ldots,\ E \sqsubseteq F_n \\ E \sqsubseteq \neg A &\quad \rightsquigarrow \quad E \sqcap A \sqsubseteq \bot \\ E \sqsubseteq \neg\{a\} &\quad \rightsquigarrow \quad E \sqcap \{a\} \sqsubseteq \bot \\ E_1 \sqsubseteq \neg E_2 \sqcup F &\quad \rightsquigarrow \quad E_1 \sqcap E_2 \sqsubseteq F \\ E \sqsubseteq \exists R.F &\quad \rightsquigarrow \quad E \sqsubseteq \exists R.\lfloor F \rfloor,\ \lfloor F \rfloor \sqsubseteq F \\ E \sqsubseteq \forall R.F &\quad \rightsquigarrow \quad \exists R^-.E \sqsubseteq F \end{aligned}$$

We also need to deconstruct concept inclusions $E \sqsubseteq \exists R.F$ since the filler F need not be an $\mathcal{ELROI}$ concept description. Only for this purpose we introduced the auxiliary concept names $\lfloor F \rfloor$. Of course, one would not need to apply the second-last rule if F is already in $\mathcal{ELROI}$. Finally, note that after the second-last rule has produced a concept inclusion $\lfloor F \rfloor \sqsubseteq F$, then all of the above transformation rules must be applied to it, starting with transforming its left-hand side.

Lemma XXXVIII. *For each quantified ABox $\exists X.\mathcal{A}$ over Σ, the following statements hold:*

1. *Each interpretation over Σ that is a model of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\mathcal{T}$, and $\mathcal{R}$ can be extended to an interpretation over $\Sigma \cup \Gamma$ that is a model of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\lfloor \mathcal{T} \rfloor$, and $\mathcal{R}$.*
2. *For each interpretation $\mathcal{I}$ over $\Sigma \cup \Gamma$ that is a model of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\lfloor \mathcal{T} \rfloor$, and $\mathcal{R}$, the Σ-restriction $\mathcal{I}\restriction_{\Sigma}$ is a model of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\mathcal{T}$, and $\mathcal{R}$.*

Proof. Consider a model $\mathcal{I}$ (over Σ) that is a model of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\mathcal{T}$, and $\mathcal{R}$. We extend it to an interpretation $\mathcal{J}$ over $\Sigma \cup \Gamma$ by additionally defining $\lfloor F \rfloor^{\mathcal{J}} := F^{\mathcal{I}}$ for each $\lfloor F \rfloor \in \Gamma$. Since $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, and $\mathcal{R}$ are defined over Σ, and $\mathcal{I}$ and $\mathcal{J}$ coincide on Σ, also $\mathcal{J}$ must be a model of these. It remains to show that $\mathcal{J}$ is a model of the transformed TBox $\lfloor \mathcal{T} \rfloor$. We do so by an induction along the applications of the above rules. All except the second-last rule are equivalence preserving, i.e., the replaced concept inclusion is equivalent to its replacement. The only interesting case is where a concept inclusion $E \sqsubseteq \exists R.F$ is replaced with $E \sqsubseteq \exists R.\lfloor F \rfloor$ and $\lfloor F \rfloor \sqsubseteq F$. Due to the choice $\lfloor F \rfloor^{\mathcal{J}} = F^{\mathcal{I}}$, it follows that $\mathcal{I} \models E \sqsubseteq \exists R.F$ implies $\mathcal{J} \models E \sqsubseteq \exists R.\lfloor F \rfloor$ and $\mathcal{J} \models \lfloor F \rfloor \sqsubseteq F$, and we are done.

Regarding the second statement, let $\mathcal{I}$ be a model (over $\Sigma \cup \Gamma$) of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\lfloor \mathcal{T} \rfloor$, and $\mathcal{R}$. Then the Σ-restriction $\mathcal{I}\restriction_{\Sigma}$ is still a model of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, and $\mathcal{R}$ as these are all defined over Σ. We prove the $\mathcal{I}\restriction_{\Sigma}$ is a model of the original TBox $\mathcal{T}$ by an induction backwards along the applications of the above rules. Again, we only need to take special care of the second-last rule since the others are equivalence preserving. So assume that $E \sqsubseteq \exists R.F$ is replaced by $E \sqsubseteq \exists R.\lfloor F \rfloor$ and $\lfloor F \rfloor \sqsubseteq F$. Of course, if $\mathcal{I} \models E \sqsubseteq \exists R.\lfloor F \rfloor$ and $\mathcal{I} \models \lfloor F \rfloor \sqsubseteq F$, then $\mathcal{I} \models E \sqsubseteq \exists R.F$ holds as well. In the end, $\mathcal{I} \models \lfloor \mathcal{T} \rfloor$ implies $\mathcal{I} \models \mathcal{T}$, and since $\mathcal{T}$ is defined over Σ, it follows that $\mathcal{I}\restriction_{\Sigma} \models \mathcal{T}$. ☐

Transforming the Static qABox Recall that the static data comes in form of a quantified ABox $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$. We simply transform it into a TBox $\mathcal{T}_{\mathsf{s}}$ over the extended signature $\Sigma \cup X_{\mathsf{s}}$ as follows.

$$\begin{aligned} \mathcal{T}_{\mathsf{s}} := &\ \{\, \{u\} \sqsubseteq A \mid A(u) \in \mathcal{A}_{\mathsf{s}} \,\} \\ &\cup \{\, \{u\} \sqsubseteq \exists r.\{v\} \mid r(u,v) \in \mathcal{A}_{\mathsf{s}} \,\} \\ &\cup \{\, \{u\} \sqsubseteq \{v\} \mid u \equiv v \in \mathcal{A}_{\mathsf{s}} \,\} \end{aligned}$$

Lemma XXXIX. *For each quantified ABox $\exists X.\mathcal{A}$ over Σ, the following statements hold:*

1. *Each interpretation over $\Sigma \cup \Gamma$ that is a model of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\lfloor \mathcal{T} \rfloor$, and $\mathcal{R}$ can be extended to an interpretation over $\Sigma \cup \Gamma \cup X_{\mathsf{s}}$ that is a model of $\exists X.\mathcal{A}$, $\lfloor \mathcal{T} \rfloor \cup \mathcal{T}_{\mathsf{s}}$, and $\mathcal{R}$.*
2. *For each interpretation $\mathcal{I}$ over $\Sigma \cup \Gamma \cup X_{\mathsf{s}}$ that is a model of $\exists X.\mathcal{A}$, $\lfloor \mathcal{T} \rfloor \cup \mathcal{T}_{\mathsf{s}}$, and $\mathcal{R}$, the $(\Sigma \cup \Gamma)$-restriction $\mathcal{I}\restriction_{\Sigma \cup \Gamma}$ is a model of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\lfloor \mathcal{T} \rfloor$, and $\mathcal{R}$.*

Proof. Assume that $\mathcal{I}$ is an interpretation over $\Sigma \cup \Gamma$ that is a model of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\lfloor \mathcal{T} \rfloor$, and $\mathcal{R}$. Specifically, there is a variable assignment $\mathcal{Z} \colon X_{\mathsf{s}} \to \mathsf{Dom}(\mathcal{I})$ such that the augmented interpretation $\mathcal{I}[\mathcal{Z}]$ is a model of the matrix $\mathcal{A}_{\mathsf{s}}$. We now extend $\mathcal{I}$ to an interpretation $\mathcal{J}$ over $\Sigma \cup \Gamma \cup X_{\mathsf{s}}$ by additionally defining $x^{\mathcal{J}} := \mathcal{Z}(x)$ for each $x \in X_{\mathsf{s}}$. Since $u^{\mathcal{I}[\mathcal{Z}]} = u^{\mathcal{J}}$ holds for each object $u \in \Sigma_{\mathsf{I}} \cup X_{\mathsf{s}}$, we infer

[19] By an extension we mean an interpretation that only differs in that it defines extensions of the additional symbols.

from $\mathcal{I}[\mathcal{Z}] \models \mathcal{A}_\mathsf{s}$ that $\mathcal{J} \models \mathcal{T}_\mathsf{s}$. Furthermore, $\mathcal{J}$ is a model of $\exists X.\mathcal{A}$, $\lfloor\mathcal{T}\rfloor$, and $\mathcal{R}$ since these are all defined over $\Sigma \cup \Gamma$ and $\mathcal{I}$ and $\mathcal{J}$ coincide on $\Sigma \cup \Gamma$.

Next, we show the second statement. Let $\mathcal{I}$ be a model (over $\Sigma \cup \Gamma \cup X_\mathsf{s}$) of $\exists X.\mathcal{A}$, $\lfloor\mathcal{T}\rfloor \cup \mathcal{T}_\mathsf{s}$, and $\mathcal{R}$. We define the variable assignment $\mathcal{Z}\colon X_\mathsf{s} \to \mathsf{Dom}(\mathcal{I})$ by $\mathcal{Z}(x) := x^\mathcal{I}$ for each $x \in X_\mathsf{s}$. It then holds that $x^{\mathcal{I}\upharpoonright_{\Sigma\cup\Gamma}[\mathcal{Z}]} = x^\mathcal{I}$ for every object $x \in X_\mathsf{s}$, and thus $\mathcal{I} \models \mathcal{T}_\mathsf{s}$ implies $\mathcal{I}\upharpoonright_{\Sigma\cup\Gamma}[\mathcal{Z}] \models \mathcal{A}_\mathsf{s}$, i.e., $\mathcal{I}\upharpoonright_{\Sigma\cup\Gamma} \models \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$. Since $\exists X.\mathcal{A}$, $\lfloor\mathcal{T}\rfloor$, and $\mathcal{R}$ are defined over $\Sigma \cup \Gamma$, it follows that the restriction $\mathcal{I}\upharpoonright_{\Sigma\cup\Gamma}$ is also a model of these. □

Transforming the Regular Path Expressions in the Repair Request A regular path expression is a regular path expression over roles. On the one hand, the $\mathcal{Z}$-family of description logics (Calvanese, Eiter, and Ortiz 2009) allows to use them in place of roles. On the other hand, so-called *conjunctive two-way regular path queries (C2RPQs)* (Ortiz, Rudolph, and Šimkus 2011) extend the formalism of conjunctive queries by allowing binary atoms involving regular path expressions. In order to support the specification of unwanted consequences in a more expressive way, we extend the notion of a repair request accordingly.

A *regular path expression (RPE)* ρ is built with the grammar rule

$$\rho ::= r \mid r^- \mid \varepsilon \mid \rho + \rho \mid \rho \circ \rho \mid \rho^*$$

where r ranges over all role names in Σ. For each interpretation $\mathcal{I}$, the semantics are extended by $(\sigma+\tau)^\mathcal{I} := \sigma^\mathcal{I} \cup \tau^\mathcal{I}$, $(\sigma \circ \tau)^\mathcal{I} := \sigma^\mathcal{I} \circ \tau^\mathcal{I}$, and $(\sigma^*)^\mathcal{I}$ is defined as the reflexive, transitive closure of $\sigma^\mathcal{I}$. Now, $\mathcal{ELROI}(\mathsf{RE})$ concept descriptions are defined like $\mathcal{ELROI}$ concept descriptions but can additionally use regular path expressions in place of roles. The repair request $\mathcal{P}$ is now built from $\mathcal{ELROI}(\mathsf{RE})$ concept descriptions and $\mathcal{ELROI}(\mathsf{RE})$ concept assertions. We will explain in the following how we can deal with it within our repair framework.

Assume that Λ consists of the role names $\lceil\rho\rceil$ for all RPEs ρ occurring in the repair request $\mathcal{P}$ (including subexpressions). We are going to construct an RBox $\mathcal{R}_\mathsf{RE}$ over the extended signature $\Sigma \cup \Lambda$ such that its union with $\mathcal{R}$ is a conservative extension of $\mathcal{R}$. Specifically, $\mathcal{R}_\mathsf{RE}$ consists of the following role inclusions:

- $\varepsilon \sqsubseteq \lceil\varepsilon\rceil$
- $r \sqsubseteq \lceil r\rceil$ and $r^- \sqsubseteq \lceil r^-\rceil$ for each role name r in $\mathcal{P}$
- $\lceil\sigma\rceil \sqsubseteq \lceil\sigma+\tau\rceil$ and $\lceil\tau\rceil \sqsubseteq \lceil\sigma+\tau\rceil$ for each RPE $\sigma+\tau$ in $\mathcal{P}$
- $\lceil\sigma\rceil \circ \lceil\tau\rceil \sqsubseteq \lceil\sigma\circ\tau\rceil$ for each RPE $\sigma\circ\tau$ in $\mathcal{P}$
- $\varepsilon \sqsubseteq \lceil\sigma^*\rceil$, $\lceil\sigma\rceil \sqsubseteq \lceil\sigma^*\rceil$, and $\lceil\sigma^*\rceil \circ \lceil\sigma^*\rceil \sqsubseteq \lceil\sigma^*\rceil$ for each RPE σ^* in $\mathcal{P}$.

The RBox $\mathcal{R}_\mathsf{RE}$ is $\prec$-regular as per the definition in (Horrocks, Kutz, and Sattler 2006), and thus it is a regular RBox. Furthermore, for each regular RBox $\mathcal{R}$, the union $\mathcal{R} \cup \mathcal{R}_\mathsf{RE}$ is regular as well.

Lemma XL. *For each quantified ABox $\exists X.\mathcal{A}$ over Σ, the following statements hold:*

1. *Each interpretation $\mathcal{I}$ over $\Sigma \cup \Gamma \cup X_\mathsf{s}$ that is a model of $\exists X.\mathcal{A}$, $\lfloor\mathcal{T}\rfloor \cup \mathcal{T}_\mathsf{s}$, and $\mathcal{R}$ can be extended to an interpretation $\mathcal{J}$ over $\Sigma \cup \Gamma \cup X_\mathsf{s} \cup \Lambda$ that is a model of $\exists X.\mathcal{A}$, $\lfloor\mathcal{T}\rfloor \cup \mathcal{T}_\mathsf{s}$, and $\mathcal{R} \cup \mathcal{R}_\mathsf{RE}$ such that $\rho^\mathcal{I} = \lceil\rho\rceil^\mathcal{J}$ for each RPE ρ occurring in $\mathcal{P}$.*
2. *For each interpretation $\mathcal{I}$ over $\Sigma \cup \Gamma \cup X_\mathsf{s} \cup \Lambda$ that is a model of $\exists X.\mathcal{A}$, $\lfloor\mathcal{T}\rfloor \cup \mathcal{T}_\mathsf{s}$, and $\mathcal{R} \cup \mathcal{R}_\mathsf{RE}$, the $(\Sigma \cup \Gamma \cup X_\mathsf{s})$-restriction $\mathcal{I}\upharpoonright_{\Sigma\cup\Gamma\cup X_\mathsf{s}}$ is a model of $\exists X.\mathcal{A}$, $\lfloor\mathcal{T}\rfloor$, and $\mathcal{R}$.*

Proof. Let $\mathcal{I}$ be an interpretation over $\Sigma \cup \Gamma \cup X_\mathsf{s}$ that is a model of $\exists X.\mathcal{A}$, $\lfloor\mathcal{T}\rfloor \cup \mathcal{T}_\mathsf{s}$, and $\mathcal{R}$. Its extension $\mathcal{J}$ additionally maps each new role name $\lceil\rho\rceil$ in Λ to $\lceil\rho\rceil^\mathcal{J} := \rho^\mathcal{I}$. Since $\mathcal{I}$ and $\mathcal{J}$ coincide on $\Sigma \cup \Gamma \cup X_\mathsf{s}$ and $\exists X.\mathcal{A}$, $\lfloor\mathcal{T}\rfloor \cup \mathcal{T}_\mathsf{s}$, and $\mathcal{R}$ are all defined over $\Sigma \cup \Gamma \cup X_\mathsf{s}$, the extension $\mathcal{J}$ is still a model of these. It is a finger exercise to show that $\mathcal{J}$ is a model of $\mathcal{R}_\mathsf{RE}$, namely by an induction on the RPE ρ.

- The claim is obvious for the base cases where $\rho = \varepsilon$, $\rho = r$, or $\rho = r^-$.
- Now let $\rho = \sigma + \tau$. It holds that
$$\lceil\sigma\rceil^\mathcal{J} \cup \lceil\tau\rceil^\mathcal{J} = \sigma^\mathcal{I} \cup \tau^\mathcal{I} = (\sigma+\tau)^\mathcal{I} = \lceil\sigma+\tau\rceil^\mathcal{J}.$$
So $\mathcal{J}$ is a model of $\lceil\sigma\rceil \sqsubseteq \lceil\sigma+\tau\rceil$ and $\lceil\tau\rceil \sqsubseteq \lceil\sigma+\tau\rceil$.
- Assume that $\rho = \sigma \circ \tau$. We have
$$\lceil\sigma\rceil^\mathcal{J} \circ \lceil\tau\rceil^\mathcal{J} = \sigma^\mathcal{I} \circ \tau^\mathcal{I} = (\sigma\circ\tau)^\mathcal{I} = \lceil\sigma\circ\tau\rceil^\mathcal{J}.$$
It follows that $\mathcal{J}$ is a model of $\lceil\sigma\rceil \circ \lceil\tau\rceil \sqsubseteq \lceil\sigma\circ\tau\rceil$.
- The last case is where $\rho = \sigma^*$. Since $(\sigma^*)^\mathcal{I}$ is reflexive, we obtain
$$\varepsilon^\mathcal{J} = \{\, (\delta,\delta) \mid \delta \in \mathsf{Dom}(\mathcal{I}) \,\} \subseteq (\sigma^*)^\mathcal{I} = \lceil\sigma^*\rceil^\mathcal{J}$$
and thus $\mathcal{J}$ is a model of $\varepsilon \sqsubseteq \lceil\sigma^*\rceil$.
Since $(\sigma^*)^\mathcal{I}$ is a closure of $\sigma^\mathcal{I}$, we infer that
$$\lceil\sigma\rceil^\mathcal{J} = \sigma^\mathcal{I} \subseteq (\sigma^*)^\mathcal{I} = \lceil\sigma^*\rceil^\mathcal{J}$$
and so $\lceil\sigma\rceil \sqsubseteq \lceil\sigma^*\rceil$ is valid in $\mathcal{J}$.
As $(\sigma^*)^\mathcal{I}$ is transitive, it follows that
$$\lceil\sigma^*\rceil^\mathcal{J} \circ \lceil\sigma^*\rceil^\mathcal{J} = (\sigma^*)^\mathcal{I} \circ (\sigma^*)^\mathcal{I} \subseteq (\sigma^*)^\mathcal{I} = \lceil\sigma^*\rceil^\mathcal{J}$$
and hence $\mathcal{J}$ is a model of $\lceil\sigma^*\rceil \circ \lceil\sigma^*\rceil \sqsubseteq \lceil\sigma^*\rceil$.

Regarding the second claim, assume that $\mathcal{I}$ is an interpretation over $\Sigma \cup \Gamma \cup X_\mathsf{s} \cup \Lambda$ that is a model of $\exists X.\mathcal{A}$, $\lfloor\mathcal{T}\rfloor \cup \mathcal{T}_\mathsf{s}$, and $\mathcal{R} \cup \mathcal{R}_\mathsf{RE}$. It immediately follows that the restriction $\mathcal{I}\upharpoonright_{\Sigma\cup\Gamma\cup X_\mathsf{s}}$ is a model of $\exists X.\mathcal{A}$, $\lfloor\mathcal{T}\rfloor \cup \mathcal{T}_\mathsf{s}$, and $\mathcal{R}$ since these are all defined over $\Sigma \cup \Gamma \cup X_\mathsf{s}$. □

The next lemma shows that the semantics of the RPEs is properly encoded within the RBox $\mathcal{R}_\mathsf{RE}$. We will need this fact later.

Lemma XLI. $R_1 \cdots R_n \in L(\rho)$ *iff* $R_1 \circ \cdots \circ R_n \sqsubseteq^{\mathcal{R}_\mathsf{RE}} \lceil\rho\rceil$.

Proof. According to Lemma IV, $R_1 \circ \cdots \circ R_n \sqsubseteq^{\mathcal{R}_\mathsf{RE}} \lceil\rho\rceil$ iff the word $R_1 \cdots R_n$ can be deduced from $\lceil\rho\rceil$ by means of the production rules induced by $\mathcal{R}_\mathsf{RE}$. It is a finger exercise to show by induction on the RPE ρ that $R_1 \cdots R_n$ is deducible from $\lceil\rho\rceil$ iff $R_1 \cdots R_n \in L(\rho)$. For the only-if direction note that inverses of role names $\lceil\sigma\rceil$ will never be produced from $\lceil\rho\rceil$. □

Transforming the repairs Now let $\mathcal{T}^{\#} := \lfloor \mathcal{T} \rfloor \cup \mathcal{T}_{\mathsf{s}}$ and $\mathcal{R}^{\#} := \mathcal{R} \cup \mathcal{R}_{\mathsf{RE}}$. Furthermore, the transformed repair request $\mathcal{P}^{\#}$ is obtained from $\mathcal{P}$ be replacing each occurrence of a RPE ρ by the new role name $\lceil \rho \rceil$,[20] and likewise we define $C^{\#}$ for each $\mathcal{ELROI}(\mathsf{RE})$ concept C. All three are defined over the extended signature $\Sigma^{\#} := \Sigma \cup \Gamma \cup X_{\mathsf{s}} \cup \Lambda$. We obtain the following lemma by combining Lemmas XXXVIII–XL.

Lemma XLII. *For each quantified ABox $\exists X.\mathcal{A}$ over Σ, the following statements hold:*

1. *Each interpretation $\mathcal{I}$ over Σ that is a model of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\mathcal{T}$, and $\mathcal{R}$ can be extended to an interpretation $\mathcal{J}$ over $\Sigma^{\#}$ that is a model of $\exists X.\mathcal{A}$, $\mathcal{T}^{\#}$, and $\mathcal{R}^{\#}$ such that $\delta \in (C^{\#})^{\mathcal{J}}$ implies $\delta \in C^{\mathcal{I}}$ for each $\mathcal{ELROI}(\mathsf{RE})$ concept C.*
2. *For each interpretation $\mathcal{I}$ over $\Sigma^{\#}$ that is a model of $\exists X.\mathcal{A}$, $\mathcal{T}^{\#}$, and $\mathcal{R}^{\#}$, the Σ-restriction $\mathcal{I}\restriction_{\Sigma}$ is a model of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\mathcal{T}$, and $\mathcal{R}$ such that $\delta \in C^{\mathcal{I}\restriction_{\Sigma}}$ implies $\delta \in (C^{\#})^{\mathcal{I}}$ for each $\mathcal{ELROI}(\mathsf{RE})$ concept C.*

Proof. Let $\mathcal{I}$ be a model (over Σ) of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\mathcal{T}$, and $\mathcal{R}$. According to Lemmas XXXVIII–XL, $\mathcal{I}$ can be extended to a model $\mathcal{J}$ (over $\Sigma^{\#}$) of $\exists X.\mathcal{A}$, $\mathcal{T}^{\#}$, and $\mathcal{R}^{\#}$ such that $\rho^{\mathcal{I}} = \lceil \rho \rceil^{\mathcal{J}}$ holds for each RPE occurring in $\mathcal{P}$.

The additional claim, namely that $\delta \in (C^{\#})^{\mathcal{J}}$ implies $\delta \in C^{\mathcal{I}}$, is shown by induction on C. It is only interesting to treat the case $C = \exists \rho.D$. Let $\delta \in (C^{\#})^{\mathcal{J}}$, i.e., there is a domain element γ such that $(\delta, \gamma) \in \lceil \rho \rceil^{\mathcal{J}}$ and $\gamma \in (D^{\#})^{\mathcal{J}}$. The induction hypothesis implies that $\gamma \in D^{\mathcal{I}}$, and we further know that $\lceil \rho \rceil^{\mathcal{J}} = \rho^{\mathcal{I}}$ holds. It follows that $\delta \in C^{\mathcal{I}}$.

Regarding the second statement, consider a model $\mathcal{I}$ (over $\Sigma^{\#}$) of $\exists X.\mathcal{A}$, $\mathcal{T}^{\#}$, and $\mathcal{R}^{\#}$. By Lemmas XXXVIII–XL, the Σ-restriction $\mathcal{I}\restriction_{\Sigma}$ is a model of $\exists X.\mathcal{A}$, $\exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}$, $\mathcal{T}$, and $\mathcal{R}$.

We prove the additional claim, namely that $\delta \in C^{\mathcal{I}\restriction_{\Sigma}}$ implies $\delta \in (C^{\#})^{\mathcal{I}}$, by induction on C. The only interesting case is where $C = \exists \rho.D$ is an existential restriction involving a RPE. The assumption $\delta \in C^{\mathcal{I}\restriction_{\Sigma}}$ yields a domain element γ such that $(\delta, \gamma) \in \rho^{\mathcal{I}\restriction_{\Sigma}}$ and $\gamma \in D^{\mathcal{I}\restriction_{\Sigma}}$. By induction hypothesis we obtain that $\gamma \in (D^{\#})^{\mathcal{I}}$. It remains to show that $(\delta, \gamma) \in \lceil \rho \rceil^{\mathcal{I}}$.

From $(\delta, \gamma) \in \rho^{\mathcal{I}\restriction_{\Sigma}}$ it follows that $(\delta, \gamma) \in \rho^{\mathcal{I}}$, i.e., there are roles $R_1, \ldots, R_n$ such that $R_1 \cdots R_n \in L(\rho)$ and $(\delta, \gamma) \in (R_1 \circ \cdots \circ R_n)^{\mathcal{I}}$. By Lemma XLI we infer that $R_1 \circ \cdots R_n \sqsubseteq^{\mathcal{R}_{\mathsf{RE}}} \lceil \rho \rceil$. Since $\mathcal{I}$ specifically is a model of $\mathcal{R}_{\mathsf{RE}}$, we conclude that $(\delta, \gamma) \in \lceil \rho \rceil^{\mathcal{I}}$. □

The following corollary to Lemma XLII mediates between the two repair notions in Definitions 18 and XXXVII.

Corollary XLIII. *For each quantified ABox $\exists Y.\mathcal{B}$ over Σ, the following statements hold:*

1. *$\exists Y.\mathcal{B}$ satisfies (SIRep1) iff $\exists Y.\mathcal{B}$ satisfies (IRep1):*
   For each qABox $\exists X.\mathcal{A}$ over Σ, it holds that $(\exists X.\mathcal{A} \cup \exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}) \models^{\mathcal{T},\mathcal{R}} \exists Y.\mathcal{B}$ iff $\exists X.\mathcal{A} \models^{\mathcal{T}^{\#},\mathcal{R}^{\#}} \exists Y.\mathcal{B}$.
2. *$\exists Y.\mathcal{B}$ satisfies (SIRep2) iff $\exists Y.\mathcal{B}$ satisfies (IRep2):*
   $(\exists Y.\mathcal{B} \cup \exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}})$ is consistent w.r.t. $(\mathcal{T}, \mathcal{R})$ iff $\exists Y.\mathcal{B}$ is consistent w.r.t. $(\mathcal{T}^{\#}, \mathcal{R}^{\#})$.
3. *$\exists Y.\mathcal{B}$ satisfies (SIRep3) iff $\exists Y.\mathcal{B}$ satisfies (IRep3):*
   For each concept assertion $C(a) \in \mathcal{P}_{\mathit{loc}}$, it holds that $(\exists Y.\mathcal{B} \cup \exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}) \models^{\mathcal{T},\mathcal{R}} C(a)$ iff $\exists Y.\mathcal{B} \models^{\mathcal{T}^{\#},\mathcal{R}^{\#}} C^{\#}(a)$.
4. *$\exists Y.\mathcal{B}$ satisfies (SIRep4) iff $\exists Y.\mathcal{B}$ satisfies (IRep4):*
   For each concept description $D \in \mathcal{P}_{\mathit{glo}}$, it holds that $(\exists Y.\mathcal{B} \cup \exists X_{\mathsf{s}}.\mathcal{A}_{\mathsf{s}}) \models^{\mathcal{T},\mathcal{R}} \exists\{x\}.\{D(x)\}$ iff $\exists Y.\mathcal{B} \models^{\mathcal{T}^{\#},\mathcal{R}^{\#}} \exists\{x\}.\{D^{\#}(x)\}$.

Due to the extended signature $\Sigma^{\#}$, an inconsistency repair for $\mathcal{P}^{\#}$ w.r.t. $(\mathcal{T}^{\#}, \mathcal{R}^{\#})$ can contain symbols that are not in the original signature Σ and that should hence be removed before the final repair is returned. For this purpose, we define the notion of the Σ-restriction of a quantified ABox.

In principle, we only need to remove assertions using concept names or role names not in Σ, and further make every individual in $X_{\mathsf{s}} \subseteq \Sigma^{\#}$ a variable. However, the latter is not so straightforward since we must also resolve equality assertions involving such now variables.

Definition XLIV. Let $\exists Z.\mathcal{C}$ be a quantified ABox over $\Sigma^{\#}$. The restriction $(\exists Z.\mathcal{C})\restriction_{\Sigma} := \exists W.\mathcal{D}$ is obtained as follows:

1. Firstly, let $W := Z \cup X_{\mathsf{s}}$.
2. Secondly, choose a representative $t_{[x]}$ of each equivalence class $[x]_{\exists Z.\mathcal{C}}$ where $x \in X_{\mathsf{s}}$ as follows:
   (a) If $[x]_{\exists Z.\mathcal{C}}$ contains an individual $a \in \Sigma_{\mathsf{I}}$, then let $t_{[x]} := a$.[21]
   (b) Otherwise $[x]_{\exists Z.\mathcal{C}}$ is a subset of X_{s}, and then choose $t_{[x]}$ as an arbitrary element of $[x]_{\exists Z.\mathcal{C}}$.[22]
3. Thirdly, populate the matrix $\mathcal{D}$ as follows:
   (a) Copy over the concept assertions and role assertions from $\mathcal{C}$ to $\mathcal{D}$, but only those involving concept names and role names in Σ and further replace every occurrence of an object $x \in X_{\mathsf{s}}$ with $t_{[x]}$.
   (b) For each individual $a \in \Sigma_{\mathsf{I}}$, let $\{a_1, \ldots, a_n\}$ be an enumeration of $[a]_{\exists Z.\mathcal{C}} \setminus X_{\mathsf{s}}$, and add the equality assertions $a_1 \equiv a_2,\ \ldots,\ a_{n-1} \equiv a_n$ to $\mathcal{D}$.

Next, we show that the Σ-restriction $(\exists Z.\mathcal{C})\restriction_{\Sigma}$ is the most specific qABox that is defined over the signature Σ and is entailed by $\exists Z.\mathcal{C}$.

Lemma XLV. *For each quantified ABox $\exists Z.\mathcal{C}$ over $\Sigma^{\#}$, the following statements hold:*

1. *$(\exists Z.\mathcal{C})\restriction_{\Sigma}$ is entailed by $\exists Z.\mathcal{C}$.*
2. *For each quantified ABox $\exists Y.\mathcal{B}$ over Σ, it holds that $\exists Z.\mathcal{C} \models \exists Y.\mathcal{B}$ iff $(\exists Z.\mathcal{C})\restriction_{\Sigma} \models \exists Y.\mathcal{B}$.*

Proof. 1. The identical mapping on $\Sigma_{\mathsf{I}} \cup X_{\mathsf{s}} \cup Z$ is a homomorphism h from $(\exists Z.\mathcal{C})\restriction_{\Sigma}$ to $\exists Z.\mathcal{C}$, as we have only merged some objects.

[20] We could alternatively support existential restrictions $\exists \rho.C$ in the repair request, where ρ is an RPE, by constructing a finite automation $\mathfrak{A}_{\rho}$ that accepts the language $L(\rho)$ and then replacing $\exists \rho.C$ by $\exists i_{\rho}.C$, where i_{ρ} is the initial state in $\mathfrak{A}_{\rho}$.

[21] This means that all variables in $[x] \cap X_s$ will be identified with the individual a.

[22] This means that all variables in $[x] \subseteq X_s$ will be identified with one of them.

(Hom2) $h(a) = a$ is fulfilled by definition for every individual a.

(Hom1) Let $a \approx_{(\exists Z.\mathcal{C})\restriction_\Sigma} b$. According to Definition XLIV, there must be an equivalence class $[c]_{\exists Z.\mathcal{C}}$ containing a and b. It follows that $a \approx_{\exists Z.\mathcal{C}} b$.

(Hom3) Consider a concept assertion $A(t)$ in the matrix of $(\exists Z.\mathcal{C})\restriction_\Sigma$. Then either $A(t)$ itself is in $\mathcal{C}$, or $A(x)$ is in $\mathcal{C}$ where x has been replaced by t, i.e., $t = t_{[x]}$. In the latter case, we have $t_{[x]} \approx_{\exists Z.\mathcal{C}} x$, and so we are done.

(Hom4) Role assertions can be treated similarly, by case distinction on whether the objects in it have been replaced.

2. The if direction follows from the first statement. Regarding the only-if direction, assume that $\exists Z.\mathcal{C} \models \exists Y.\mathcal{B}$, i.e., there is a homomorphism h from $\exists Y.\mathcal{B}$ to $\exists Z.\mathcal{C}$ by Proposition 2. We define a mapping h' by $h'(a) := a$ for each individual $a \in \Sigma_\mathsf{I}$ and

$$h'(y) := \begin{cases} h(y) & \text{if } h(y) \in \Sigma_\mathsf{I} \cup Z \\ t_{[h(y)]} & \text{otherwise, i.e., if } h(y) \in X_\mathsf{s} \end{cases}$$

for each variable $y \in Y$.

(α) The definition of h' yields $h(t) \approx_{\exists Z.\mathcal{C}} h'(t)$ for each object t.

- If t is an individual, then $h(t) = t = h'(t)$.
- Otherwise, if $h(t) \in \Sigma_\mathsf{I} \cup Z$, then $h(t) = h'(t)$.
- In the remaining case, where $h(t) \in X_\mathsf{s}$, we have $h'(t) = t_{[h(t)]}$ where $t_{[h(t)]} \in [h(t)]_{\exists Z.\mathcal{C}}$. It follows that $h'(t) \approx_{\exists Z.\mathcal{C}} h(t)$.

(β) We further show that, for all objects u, v of $(\exists Z.\mathcal{C})\restriction_\Sigma$, $u \approx_{\exists Z.\mathcal{C}} v$ implies $u \approx_{(\exists Z.\mathcal{C})\restriction_\Sigma} v$. We first get that u and v are in the same equivalence class w.r.t. $\exists Z.\mathcal{C}$.

- If one of them is a variable in Z, then $u = v$ since variables cannot occur in equality assertions. The conclusion $u \approx_{(\exists Z.\mathcal{C})\restriction_\Sigma} v$ is then trivial.
- Now assume that one of them is an object in X_s. Since it occurs in $(\exists Z.\mathcal{C})\restriction_\Sigma$, its equivalence class (w.r.t. $\exists Z.\mathcal{C}$) must be a subset of X_s (otherwise it would have been replaced by an individual in the restriction). But then u and v have been replaced by the same object, which means that they must be equal (as both occur in the restriction). It follows that $u \approx_{(\exists Z.\mathcal{C})\restriction_\Sigma} v$.
- In the remaining case, both are individuals in Σ_I. From $u \approx_{\exists Z.\mathcal{C}} v$ we infer that $u, v \in [a]_{\exists Z.\mathcal{C}} \setminus X_\mathsf{s}$ for an individual a. Definition XLIV ensures that $u \approx_{(\exists Z.\mathcal{C})\restriction_\Sigma} v$.

It is now easy to verify that h' is a homomorphism from $\exists Y.\mathcal{B}$ to $(\exists Z.\mathcal{C})\restriction_\Sigma$.

(Hom2) With h fulfilling (Hom2), we obtain that $h'(a) = h(a) = a$ for every individual a.

(Hom1) Let $a \approx_{\exists Y.\mathcal{B}} b$. Since h satisfies (Hom1), we have $a \approx_{\exists Z.\mathcal{C}} b$. It follows that $a, b \in [a]_{\exists Z.\mathcal{C}} \setminus X_\mathsf{s}$, and according to Definition XLIV we conclude that $a \approx_{(\exists Z.\mathcal{C})\restriction_\Sigma} b$.

(Hom3) Consider a concept assertion $A(t)$ in $\mathcal{B}$. Since $\exists Y.\mathcal{B}$ is defined over Σ, we have $A \in \Sigma$. As h satisfies (Hom3), there is an object v such that $v \approx_{\exists Z.\mathcal{C}} h(t)$ and $A(v) \in \mathcal{C}$. We make a case distinction.

- If $v \in X_\mathsf{s}$, then the restriction $(\exists Z.\mathcal{C})\restriction_\Sigma$ contains $A(t_{[v]})$, and we have $v \approx_{\exists Z.\mathcal{C}} t_{[v]}$ (by choice of $t_{[v]}$ in Definition XLIV). It follows that $t_{[v]} \approx_{\exists Z.\mathcal{C}} h(t)$. We have seen above in Observation α that $h(t) \approx_{\exists Z.\mathcal{C}} h'(t)$ holds, and so we conclude that $t_{[v]} \approx_{\exists Z.\mathcal{C}} h'(t)$. Since $t_{[v]}$ and $h'(t)$ both occur in the restriction, we conclude from the above Observation β that $t_{[v]} \approx_{(\exists Z.\mathcal{C})\restriction_\Sigma} h'(t)$.
- Otherwise, if $v \in \Sigma_\mathsf{I} \cup Z$, then the restriction $(\exists Z.\mathcal{C})\restriction_\Sigma$ also contains $A(v)$. Since $h(t) \approx_{\exists Z.\mathcal{C}} h'(t)$ holds by Observation α, it follows that $v \approx_{\exists Z.\mathcal{C}} h'(t)$. Since v and $h'(t)$ both occur in the restriction, we conclude from the above Observation β that $v \approx_{(\exists Z.\mathcal{C})\restriction_\Sigma} h'(t)$.

(Hom4) Each role assertion in $\mathcal{B}$ can be treated in a similar way, by means of case distinction on both objects in the role assertion that h yields in $\mathcal{C}$ and using the above Observations α and β. □

Using Corollary XLIII and the notion of Σ-restrictions, we can now formulate the following important proposition that shows how the SI-repairs and the inconsistency repairs for the transformed input correspond to each other.

Proposition XLVI. *The following statements hold.*

1. *Each SI-repair of $\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T}, \mathcal{R})$ is an inconsistency repair of $\exists X.\mathcal{A}$ for $\mathcal{P}^\#$ w.r.t. $(\mathcal{T}^\#, \mathcal{R}^\#)$.*
2. *For each inconsistency repair of $\exists X.\mathcal{A}$ for $\mathcal{P}^\#$ w.r.t. $(\mathcal{T}^\#, \mathcal{R}^\#)$, its Σ-restriction is an SI-repair of $\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T}, \mathcal{R})$.*

Proof. Regarding the first claim, let $\exists Y.\mathcal{B}$ be an SI-repair of $\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T}, \mathcal{R})$.

(IRep1) We infer from Condition (SIRep1) that $(\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}) \models^{\mathcal{T},\mathcal{R}} \exists Y.\mathcal{B}$. An application of Corollary XLIII yields $\exists X.\mathcal{A} \models^{\mathcal{T}^\#,\mathcal{R}^\#} \exists Y.\mathcal{B}$.

(IRep2) Since $\exists Y.\mathcal{B}$ fulfills Condition (SIRep2), its union with $\exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ must be consistent w.r.t. $(\mathcal{T}, \mathcal{R})$. With Corollary XLIII we infer that $\exists Y.\mathcal{B}$ is also consistent w.r.t. $(\mathcal{T}^\#, \mathcal{R}^\#)$.

(IRep3) Consider a concept assertion $C^\#(a) \in \mathcal{P}^\#_\mathsf{loc}$, i.e., $C(a) \in \mathcal{P}_\mathsf{loc}$. Due to Condition (SIRep3), it holds that $(\exists Y.\mathcal{B} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}) \not\models^{\mathcal{T},\mathcal{R}} C(a)$. By Corollary XLIII it follows that $\exists Y.\mathcal{B} \not\models^{\mathcal{T}^\#,\mathcal{R}^\#} C^\#(a)$.

(IRep4) Consider a concept description $D^\# \in \mathcal{P}^\#_\mathsf{glo}$, i.e., $D \in \mathcal{P}_\mathsf{glo}$. By Condition (SIRep4) we infer that $(\exists Y.\mathcal{B} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}) \not\models^{\mathcal{T},\mathcal{R}} \exists\{x\}.\{D(x)\}$. By Corollary XLIII it follows that $\exists Y.\mathcal{B} \not\models^{\mathcal{T}^\#,\mathcal{R}^\#} \exists\{x\}.\{D^\#(x)\}$.

We conclude that $\exists Y.\mathcal{B}$ is an inconsistency repair of $\exists X.\mathcal{A}$ for $\mathcal{P}^\#$ w.r.t. $(\mathcal{T}^\#, \mathcal{R}^\#)$.

We proceed with the second claim. Therefore let $\exists Z.\mathcal{C}$ be an inconsistency repair of $\exists X.\mathcal{A}$ for $\mathcal{P}^\#$ w.r.t. $(\mathcal{T}^\#, \mathcal{R}^\#)$. We verify that the Σ-restriction $(\exists Z.\mathcal{C})\restriction_\Sigma$ is an SI-repair of $\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T}, \mathcal{R})$.

(SIRep1) Condition (IRep1) yields that $\exists X.\mathcal{A} \models^{\mathcal{T}^\#,\mathcal{R}^\#} \exists Z.\mathcal{C}$. We further know from Lemma XLV that $\exists Z.\mathcal{C} \models (\exists Z.\mathcal{C})\restriction_\Sigma$, which together with the former implies that $\exists X.\mathcal{A} \models^{\mathcal{T}^\#,\mathcal{R}^\#} (\exists Z.\mathcal{C})\restriction_\Sigma$. With Corollary XLIII we conclude that $(\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}) \models^{\mathcal{T},\mathcal{R}} (\exists Z.\mathcal{C})\restriction_\Sigma$.

(SIRep2) We infer from Condition (IRep2) that $\exists Z.\mathcal{C}$ is consistent w.r.t. $(\mathcal{T}^\#,\mathcal{R}^\#)$. According to Lemma XLV we have $\exists Z.\mathcal{C} \models (\exists Z.\mathcal{C})\restriction_\Sigma$ and so the restriction $(\exists Z.\mathcal{C})\restriction_\Sigma$ is consistent w.r.t. $(\mathcal{T}^\#,\mathcal{R}^\#)$ as well. By Corollary XLIII it follows that $(\exists Z.\mathcal{C})\restriction_\Sigma \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ is consistent w.r.t. $(\mathcal{T},\mathcal{R})$.

(SIRep3) Consider a concept assertion $C(a)$ in the local request $\mathcal{P}_\mathsf{loc}$, i.e., $C^\#(a)$ is in the transformed local request $\mathcal{P}^\#_\mathsf{loc}$. By Condition (IRep3) it holds that $\exists Z.\mathcal{C} \not\models^{\mathcal{T}^\#,\mathcal{R}^\#} C^\#(a)$. Since $\exists Z.\mathcal{C} \models (\exists Z.\mathcal{C})\restriction_\Sigma$ holds by Lemma XLV, we infer that $(\exists Z.\mathcal{C})\restriction_\Sigma \not\models^{\mathcal{T}^\#,\mathcal{R}^\#} C^\#(a)$. By means of Corollary XLIII it follows that $((\exists Z.\mathcal{C})\restriction_\Sigma \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}) \not\models^{\mathcal{T},\mathcal{R}} C(a)$.

(SIRep4) For each concept D in the global request $\mathcal{P}_\mathsf{glo}$, the proof is as above when $C(a)$ is replaced by $\exists\{x\}.\{D(a)\}$ but uses Condition (IRep4). ☐

Proposition XLVII. *Each SI-repair of $\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ is entailed by the Σ-restriction of some optimal inconsistency repair of $\exists X.\mathcal{A}$ for $\mathcal{P}^\#$ w.r.t. $(\mathcal{T}^\#,\mathcal{R}^\#)$.*

Proof. Consider an SI-repair $\exists Y.\mathcal{B}$ of $\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$. According to the first statement of Proposition XLVI, $\exists Y.\mathcal{B}$ is also an inconsistency repair of $\exists X.\mathcal{A}$ for $\mathcal{P}^\#$ w.r.t. $(\mathcal{T}^\#,\mathcal{R}^\#)$. By Theorem 19 and Proposition XXIX there is an optimal inconsistency repair $\exists Y'.\mathcal{B}'$ of $\exists X.\mathcal{A}$ for $\mathcal{P}^\#$ w.r.t. $(\mathcal{T}^\#,\mathcal{R}^\#)$, where $\exists Y'.\mathcal{B}' \models \exists Y.\mathcal{B}$. Now the second statement of Proposition XLVI yields that the Σ-restriction $(\exists Y'.\mathcal{B}')\restriction_\Sigma$ is an SI-repair of $\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$. Since $(\exists Y'.\mathcal{B}')\restriction_\Sigma$ and $\exists Y.\mathcal{B}$ are both defined over Σ, we infer from $\exists Y'.\mathcal{B}' \models \exists Y.\mathcal{B}$ by means of Lemma XLV that also $(\exists Y'.\mathcal{B}')\restriction_\Sigma \models \exists Y.\mathcal{B}$. ☐

Moreover, we obtain the following corollary.

Corollary XLVIII. *Each optimal SI-repair of $\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ is equivalent to the Σ-restriction of some optimal inconsistency repair of $\exists X.\mathcal{A}$ for $\mathcal{P}^\#$ w.r.t. $(\mathcal{T}^\#,\mathcal{R}^\#)$.*

In the end, it follows from Theorems 12 and 19 and Proposition XLVII that the set of all optimal SI-repairs can effectively be computed and covers all SI-repairs, given that the translated terminology terminates and automata recognizing the role languages induced by the translated RBox can effectively be constructed.

Theorem XLIX. *Consider a quantified ABox $\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ where $\exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ is static, an $\mathcal{ELROI}(\mathsf{RE})$ repair request $\mathcal{P}$, and a terminology $(\mathcal{T},\mathcal{R})$ where $\mathcal{T}$ is a Horn-$\mathcal{ALCOI}$ TBox and $\mathcal{R}$ is a regular RBox such that $((\mathcal{T}^\#)_+,\mathcal{R}^\#)$ is terminating and automata for $\mathcal{R}^\#$ can be effectively constructed. Each SI-repair of $\exists X.\mathcal{A} \cup \exists X_\mathsf{s}.\mathcal{A}_\mathsf{s}$ for $\mathcal{P}$ w.r.t. $(\mathcal{T},\mathcal{R})$ is entailed by an optimal SI-repair. Moreover, the set of all optimal SI-repairs can effectively be computed (up to equivalence).*

4.4 Further Expressivity

We have already seen at the end of Section 4.2 that the presence of *existential self-restrictions* $\exists r.\mathsf{Self}$ in the repair request can prevent the existence of optimal repairs, even if the terminology is terminating and the RBox is regular. The following example shows that also *functional roles* can make optimal repair impossible. Thus, it seems to be impossible to extend our optimal repair framework to the full language of Horn-$\mathcal{SROIQ}$.

Example L. Assume that f is a functional role. Further consider the ABox $\{f(a,a)\}$ and the repair request $\{\exists f.\top(a)\}$. For each number $n \geq 0$, the qABox $\exists X_n.\mathcal{A}_n$ defined as

$$\exists\{x_1,\dots,x_n\}.\{f(x_n,x_{n-1}),\dots,f(x_2,x_1),f(x_1,a)\}$$

is then a repair.

Now assume that there is a finite set $\mathfrak{S}$ of repairs that covers all repairs, and let the number n be greater than the number of objects of each repair in $\mathfrak{S}$. So there must be a repair $\exists Y.\mathcal{B}$ in $\mathfrak{S}$ that entails $\exists X_n.\mathcal{A}_n$, i.e., there is a homomorphism h from $\exists X_n.\mathcal{A}_n$ to $\exists Y.\mathcal{B}$. Since n exceeds the number of objects in $\exists Y.\mathcal{B}$, there are indices $i,j \in \{1,\dots,n\}$ where $i < j$ such that $h(x_i) = h(x_j)$. Since the matrix $\mathcal{A}_n$ contains the role assertions $f(x_i,x_{i-1})$ and $f(x_j,x_{j-1})$, where we treat x_0 as the individual name a, it follows that in the other matrix $\mathcal{B}$ the object $h(x_i)$ has f-successors $h(x_{i-1})$ and $h(x_{j-1})$. Since f is functional, we infer $h(x_{i-1}) = h(x_{j-1})$.

By induction, we obtain $h(x_0) = h(x_{j-i})$. Now x_{j-i} still has the f-successor x_{j-i-1}, but $h(x_0) = h(a) = a$ does not have an f-successor because $\exists Y.\mathcal{B}$ is a repair and thus does not entail $\exists f.\top(a)$. This means that, although $\mathcal{A}_n$ contains the role assertion $f(x_{j-i},x_{j-i-1})$, the other matrix $\mathcal{B}$ does not contain the role assertion $f(h(x_{j-i}),h(x_{j-i-1}))$ due to $h(x_{j-i}) = a$, a contradiction.

5 Conclusion

We have shown that the approaches for computing optimal repairs developed in our previous work can be extended to a considerably more expressive DL, which covers most of the DL $\mathcal{EL}^{++}$ underlying the OWL 2 EL profile, but also has inverse roles. Our main result is that, in this setting, optimal repairs can effectively be computed and cover all repairs in the sense that every repair is entailed by an optimal one. In addition, we have demonstrated that this repair approach can deal with several other interesting repair problems.

The paper actually provides two proofs of the main result, one based on showing a small repair property by filtration, and another one based on the construction of canonical repairs. We believe the second approach to be more useful in practice. In fact, when repairing a given quantified ABox w.r.t. an $\mathcal{ELROI}$ terminology, first computing all optimal repairs and then expecting the knowledge engineer to choose an appropriate one among (potentially) exponentially many exponentially large optimal repairs does not appear to be a practically viable repair approach. Since our canonical repairs are determined by polynomially large repair seeds, such a repair can be chosen by making polynomially many decisions regarding certain instance relationships. Once a repair seed is chosen, the induced canonical repair is always exponentially large. However, by adapting the optimized repair approach of (Baader et al. 2021a) to our more expressive language, we can obtain considerably smaller optimized repairs.

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