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Carsten Lutz, Dirk Walther

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Carsten Lutz and Dirk Walther

Institute for Theoretical Computer Science,
TU Dresden, Germany
lutz@tcs.inf.tu-dresden.de and dwalther@gmx.ch

Abstract. Propositional dynamic logic (PDL) is one of the most successful variants of modal logic. To make it even more useful for applications, many extensions of PDL have been considered in the literature. A very natural and useful such extension is with negation of programs. Unfortunately, it is long-known that reasoning with the resulting logic is undecidable. In this paper, we consider the extension of PDL with negation of atomic programs only. We argue that this logic is still useful, e.g. in the context of description logics, and prove that satisfiability is decidable and ExpTime-complete using an approach based on Büchi tree automata.

1 Introduction

Propositional dynamic logic (PDL) is a variant of propositional modal logic that has been developed in the late seventies as a tool for reasoning about programs [1–5]. Since then, PDL was used rather successfully in a large number of application areas such as reasoning about knowledge [6], reasoning about actions [7, 8], description logics [9], and others. Starting almost with its invention, many extensions of PDL have been proposed with the goal to enhance the expressive power and make PDL even more useful for applications; see e.g. [10, 4, 5]. Some of these extensions are tailored toward specific application areas, such as the halt predicate that allows to state termination in the context of reasoning about programs [11]. The majority of proposed extensions, however, is of a general nature and has been employed in many different application areas—for example the very natural extension of PDL with the converse operator [12].

Among the general purpose extensions of PDL, two of the most natural ones are the addition of program intersection “∩” and of program negation “¬” [13, 4, 5]. Since PDL already provides for program union “∪”, the latter is more general than the former: \( \alpha \land \beta \) can simply be expressed as \( \neg(\neg \alpha \cup \neg \beta) \). The main obstacle for using these two extensions in practical applications is that they are problematic w.r.t. their computational properties: first, adding intersection destroys many of the nice model-theoretic properties of PDL. The only known algorithm for reasoning in the resulting logic PDL\( \cap \) is the quite intricate one given in [13]. As of now, it is unknown whether the provided 2-ExpTime upper bound is tight—in contrast to ExpTime-complete reasoning in PDL. Second,
the situation with PDL extended with negation (PDL\(^{-}\)) is even worse; it was observed quite early that reasoning in \(\text{PDL}^{-}\) is undecidable [4].

This undecidability was often regretted [4, 10, 14], in particular since reasoning in \(\text{PDL}^{-}\) would be quite useful for a number of application areas. To illustrate the usefulness of this logic, let us give three examples of its expressive power: first, it was already noted that negation can be used to express intersection. Intersection, in turn, is very useful for reasoning about programs since it allows to capture the parallel execution of programs. Second, program negation allows to express the universal modality \(\Box u \phi\) by writing \([u] \phi \land \neg[q] \phi\), with \(u\) an arbitrary atomic program. The universal modality is a very useful extension of modal logics that comes handy in many applications; see e.g. [15]. Third, program negation can be used to express the window operator \(\mathbf{w}\) [16-18], whose semantics is as follows: \(\mathbf{w} \phi\) holds at a world \(w\) if \(\phi\) holding at a world \(w'\) implies that \(w'\) is \(\alpha\)-accessible from \(w\). In \(\text{PDL}^{-}\), we can thus just write \(\neg [\alpha] \neg \phi\) instead of \(\mathbf{w} \phi\).

The window operator can be viewed as expressing sufficiency in contrast to the standard box operator of modal logic, which expresses necessity. Moreover, the window operator has important applications, e.g. in description logics [19].

Due to the usefulness of program negation, it is natural to attempt the identification of fragments of \(\text{PDL}^{-}\) that still capture some of the useful properties of program negation, but are well-behaved in a computational sense. One candidate for such a fragment is \(\text{PDL}^{-}\). As has already been noted, this fragment is indeed decidable, but has a quite intricate model theory. The purpose of this paper is to explore another natural option: \(\text{PDL}^{-}\), the fragment of \(\text{PDL}^{-}\) that allows the application of program negation to atomic programs only. Indeed, we show that reasoning in \(\text{PDL}^{-}\) is decidable, and \(\text{ExpTime}\)-complete—thus not harder than reasoning in \(\text{PDL}^{-}\) itself. Moreover, \(\text{PDL}^{-}\) has a simpler model theory than \(\text{PDL}\): we are able to use a decision procedure that is an extension of the standard automata-based decision procedure for \(\text{PDL}\) [20], and of the standard automata-based decision procedure for Boolean modal logic [21]. Finally, we claim that \(\text{PDL}^{-}\) is still useful for applications: while intersection cannot be expressed any more, the universal modality and the window operator are still available.

To give some more concrete examples of the usefulness of \(\text{PDL}^{-}\), we take a description logic perspective. Description logics are a family of logics that originated in artificial intelligence as a tool for the representation of conceptual knowledge [22]. It is well-known that many description logics (DLs) are notational variants of modal logics [23, 9]. In particular, the description logic \(\mathcal{ALC}_{\text{reg}}\), which extends the basic DL \(\mathcal{ALC}\) with regular expressions on roles, corresponds to \(\text{PDL}\) [9]. More precisely, DL concepts can be understood as \(\text{PDL}\) formulas, and DL roles as \(\text{PDL}\) programs. Thus, the extension \(\mathcal{ALC}_{\text{reg}}\) of \(\mathcal{ALC}_{\text{reg}}\) with negation of atomic (!) roles is a notational variant of \(\text{PDL}^{-}\). We give two examples of knowledge representation with \(\mathcal{ALC}_{\text{reg}}\). These examples, which use DL syntax rather than \(\text{PDL}\) syntax, illustrate that the combination of regular expressions on roles and of atomic negation of roles is a very useful one.
1. Private universities commonly prefer to admit students whose ancestors donated money to the university. Using $\mathcal{ALC}_{\text{reg}}$, the class of all students having a donating ancestor can be described with the concept $\exists \text{parent}^+ \cdot \text{Donor}$. To describe the set of preferred students, we can now combine this concept with the window operator: the $\mathcal{ALC}_{\text{reg}}^{\omega}$-concept

$$\forall\text{-prefer. } \neg(\exists \text{parent}^+ \cdot \text{Donor})$$

states that all students with donating ancestors are preferred.

2. Suppose that we want to use $\mathcal{ALC}_{\text{reg}}^{\omega}$ to talk about trust and mistrust among negotiating parties. Also assume that we have a very strong notion of trust, namely that it is transitive: if I trust $x$, and $x$ trusts $y$, then I trust $y$ as well. An analogous assumption for mistrust should clearly not be made. Then, we can model mistrust by using an atomic role mistrust and trust by using $(\neg \text{mistrust})^+$ and say, e.g., that I trust some officials, but mistrust every tycoon:

$$\exists (\neg \text{mistrust})^+.$$

Note that reversing the roles of trust and mistrust does not work: first, to achieve transitivity of trust, we'd have to introduce an atomic direct-trust relation. And second, we could then only speak about the negation of direct-trust, but not about the negation of direct-trust", which corresponds to mistrust.

2 PDL with Negation

In this section, we introduce propositional dynamic logic (PDL) with negation of programs. We start with defining full PDL, i.e. PDL extended with negation of (possibly complex) programs. Then, the logics PDL and PDL$^{\omega}$, are defined as fragments of PDL$^\omega$.

**Definition 1 (PDL$^\omega$ Syntax).** Let $\Phi_0$ and $\Pi_0$ be countably infinite and disjoint sets of propositional letters and atomic programs, respectively. Then the set $\Pi^\omega$ of PDL$^\omega$-programs and the set $\Phi^\omega$ of PDL$^\omega$-formulas are defined by simultaneous induction, i.e., they are the smallest sets such that:

- $\Phi_0 \subseteq \Phi^\omega$;
- $\Pi_0 \subseteq \Pi^\omega$;
- if $\phi, \psi \in \Phi^\omega$, then $\{\neg \phi, \phi \land \psi, \phi \lor \psi\} \subseteq \Phi^\omega$;
- if $\pi_1, \pi_2 \in \Pi^\omega$, then $\{\neg \pi_1, \pi_1 \cup \pi_2, \pi_1 \cap \pi_2\} \subseteq \Pi^\omega$;
- if $\pi \in \Pi^\omega$, and $\phi \in \Phi^\omega$, then $\{(\pi), [\pi]\phi\} \subseteq \Phi^\omega$;
- if $\phi \in \Phi^\omega$, then $\phi \in \Pi^\omega$.

We use $\top$ as abbreviation for an arbitrary propositional tautology, and $\bot$ as abbreviation for $\neg \top$. Moreover, for $\pi, \pi' \in \Pi^\omega$ we use $\pi \cap \pi'$ as abbreviation for $\neg (\neg \pi \cup \neg \pi')$.

A formula $\phi \in \Phi^\omega$ is called a PDL$^{\omega}$-formula (PDL-formula) if, in $\phi$, negation occurs only in front of atomic programs and formulas (only in front of formulas).
Throughout this paper, the operator \( \langle \pi \rangle \) is called the diamond operator, \( [\pi] \) is called the box operator, and programs of the form \( \psi \) are called tests. Let us note how formulas of PDL\(^-\) can be converted into concepts of the description logic \( \mathcal{ALC}_r \) mentioned in the introduction: simply replace \( \land, \lor, \langle \pi \rangle \psi \), and \( [\pi] \psi \) with \( \cap, \cup, \exists \pi. \psi \), and \( \forall \pi. \psi \), respectively.

**Definition 2 (PDL\(^-\) Semantics).** Let \( M = (W, R, V) \) be a Kripke structure where \( W \) is the set of worlds, \( R \) is a family of accessibility relations for atomic programs \( \{R_\pi \subseteq W^2 \mid \pi \in \Pi_0 \} \), and \( V : \Phi_0 \rightarrow 2^W \) is a valuation function. In the following, we define accessibility relations for compound programs and the satisfaction relation \( \models \) by simultaneous induction, where \( r^* \) denotes the reflexive-transitive closure of the relation \( r \):

\[
R_{\psi \phi} := \{(u, v) \in W^2 \mid M, u \models \psi \land v \models \phi\}
\]

\[
R_{\neg \phi} := W^2 \setminus R_{\phi}
\]

\[
R_{\phi_1 \lor \phi_2} := R_{\phi_1} \cup R_{\phi_2}
\]

\[
R_{\phi_1 \cdot \phi_2} := R_{\phi_1} \circ R_{\phi_2}
\]

\[
R^* := (R^*)^*
\]

\[
M, u \models p \iff u \in V(p) \text{ for any } p \in \Phi
\]

\[
M, u \models \neg \phi \iff M, u \not\models \phi
\]

\[
M, u \models \phi_1 \lor \phi_2 \iff M, u \models \phi_1 \text{ or } M, u \models \phi_2
\]

\[
M, u \models \phi_1 \land \phi_2 \iff M, u \models \phi_1 \text{ and } M, u \models \phi_2
\]

\[
M, u \models \langle \pi \rangle \phi \iff \text{there is a } v \in W \text{ with } (u, v) \in R_\pi \text{ and } M, v \models \phi
\]

\[
M, u \models [\pi] \phi \iff \text{for all } v \in W, (u, v) \in R_\pi \text{ implies } M, v \models \phi
\]

If \( M, u \models \phi \) for some formula \( \phi \in \Phi^- \) and world \( u \in W \), then \( \phi \) is true at \( u \) in \( M \), and \( M \) is called model of \( \phi \). A formula is satisfiable if it has a model. It is called valid if it is true at all worlds in all Kripke structures.

It is well-known that satisfiability and validity of PDL\(^-\)-formulas are undecidable [4]. Since this can be established in a very simple way, we give a proof for illustrative purposes.

The proof is by reduction of the undecidable word-problem for finitely presented semi-groups [?]; given a set of word identities \( \{u_1 = v_1, \ldots, u_k = v_k\} \), the task is to decide whether they imply another word identity \( u = v \). To reduce this problem to PDL\(^-\)-satisfiability, we need to introduce the universal modality \( \Box_U \phi \), which has the following semantics:

\[
M, u \models \Box_U \phi \iff M, v \models \phi \text{ for all } v \in W
\]

Clearly, in PDL\(^-\) we can replace \( \Box_U \phi \) with the equivalent \( [a] \phi \land [\neg a] \phi \), where \( a \in \Pi_0 \) is an arbitrary atomic program. Using the universal modality, the reduction is now easy: we assume that, for every generator of the semi-group, there is an atomic program of the same name, and then note that \( \{u_1 = v_1, \ldots, u_k = v_k\} \) implies \( u = v \) if and only if the following formula is unsatisfiable:

\[
\left( \langle u \land \neg v \rangle T \lor \langle \neg u \land v \rangle T \right) \land \Box_U \left( \bigwedge_{i=1..k} [u_i \land \neg v_i] \land \bigvee [u_i \land \neg u_i] \right).
\]
Here, we assume that the symbols of the words \( u_i \) and \( v_i \) (and of \( u \) and \( v \)) are separated by program composition “;”.

Since PDL\(^(-)\) is a very useful logic for a large number of purposes, this undecidability result is rather disappointing. As has been argued in the introduction, it is thus a natural idea to search for decidable fragments of PDL\(^(-)\) that still extend PDL\(^(-)\) in a useful way. In the remainder of this paper, we will prove that PDL\(^(-)\) is such a fragment. Note that, in PDL\(^(-)\), we can still define the universal modality as described above. Also note that we can use negated atomic programs nested inside other program operators.

3 An Automata-based Variant of PDL\(^(-)\)

Similar to some related results in [20], the decidability proof is based on Büchi-automata on infinite trees. It has turned out that, for such proofs, it is rather convenient to use variants of PDL in which complex programs are described by means of automata on finite words, rather than by regular expressions. Therefore, in this section we define a corresponding variant APDL\(^(-)\) of PDL\(^(-)\).

**Definition 3 (Finite automata).** A (nondeterministic) finite automaton (NFA) \( A \) is a quintuple \( (Q, \Sigma, q_0, \Delta, F) \) where

- \( Q \) is a finite set of states,
- \( \Sigma \) is a finite alphabet,
- \( q_0 \) is an initial state,
- \( \Delta \subseteq Q \times \Sigma \rightarrow 2^Q \) is a transition function, and
- \( F \subseteq Q \) is the set of accepting states.

The function \( \Delta \) can be inductively extended to a function from \( Q \times \Sigma^* \) to \( 2^Q \) in a natural way:

- \( \Delta(q, \varepsilon) := \{ q \} \), where \( \varepsilon \) is the empty word;
- \( \Delta(q, w\alpha) := \{ q' \in Q \mid q'' \in \Delta(q', a) \text{ for some } q' \in \Delta(q, a) \} \).

A sequence \( p_0 \ldots p_n \in Q, n \geq 0 \), is a run of \( A \) on the word \( a_1 \ldots a_n \in \Sigma^* \) if \( p_0 = q_0, p_1 = \Delta(p_{i-1}, a_i) \) for \( 0 < i \leq n \), and \( p_n \in F \). A word \( w \in \Sigma^* \) is accepted by \( A \) if there exists a run of \( A \) on \( w \). The language accepted by \( A \) is the set \( \mathcal{L}(A) := \{ w \in \Sigma^* \mid w \text{ is accepted by } A \} \).

To obtain APDL\(^(-)\) from PDL\(^(-)\), we replace complex programs (i.e. regular expressions) inside boxes and diamonds with automata. For the sake of exactness, we give the complete definition.

**Definition 4 (APDL\(^(-)\) Syntax).** The set \( \Pi_0^{(-)} \) of program literals is defined as \( \{ a, \neg a \mid a \in \Pi_0 \} \). The sets \( \Pi^{(-)} \) of program automata and \( A\Phi^{(-)} \) of APDL\(^(-)\)-formulas are defined by simultaneous induction, i.e., \( \Pi^{(-)} \) and \( A\Phi^{(-)} \) are the smallest sets such that:

- \( \Phi_0 \subseteq A\Phi^{(-)} \);
- if \( \phi, \psi \in A\Phi^{(-)} \), then \( \{ -\phi, \phi \lor \psi, \phi \land \psi \} \subseteq A\Phi^{(-)} \);
- if \( \alpha \in A\Pi^{(-)} \) and \( \phi \in A\Phi^{(-)} \), then \( \{ \alpha \phi, [\alpha] \phi \} \subseteq A\Phi^{(-)} \);
- if \( \alpha \) is a finite automaton with alphabet \( \Sigma \subseteq \Pi_0^{(-)} \cup \{ \psi? | \psi \in A\Phi^{(-)} \} \), then \( \alpha \in A\Pi^{(-)} \).

Note that the alphabet of program automata is composed of atomic programs, of negated atomic programs, and of tests.

**Definition 5 (APDL\(^{-}\) Semantics).** Let \( \mathcal{M} = (W, \mathcal{R}, V) \) be a Kripke structure as in Definition 2. We inductively define a relation \( R \) mapping each program literal, each test, and each program automaton to a binary relation over \( W \). This is done simultaneously with the definition of the satisfaction relation:

\[
R(\alpha) := R_\alpha \text{ for each } \alpha \in \Pi_0
\]
\[
R(-\alpha) := W \setminus R_\alpha \text{ for each } \alpha \in \Pi_0
\]
\[
R(\psi?) := \{ (u, v) \in W^2 \mid M, u \models \psi \}
\]
\[
R(\alpha) := \{ (u, v) \in W^2 \mid \text{there is a word } w = w_1 \cdots w_m \in L(\alpha), \text{ with } \nu \geq 0 \text{ and worlds } u_0, \ldots, u_m \in W \text{ such that } \nu = u_0 R(w_1) u_1 R(w_2) \cdots u_{m-1} R(w_m) u_m = v \}
\]

\( M, u \models \phi \) if and only if \( u \in \mathcal{V}(p) \) for any \( p \in \Phi \).

\( M, u \models \phi \) if and only if \( M, u \not\models \phi \).

\( M, u \models \phi_1 \lor \phi_2 \) if and only if \( M, u \models \phi_1 \) or \( M, u \models \phi_2 \).

\( M, u \models \phi_1 \land \phi_2 \) if and only if \( M, u \models \phi_1 \) and \( M, u \models \phi_2 \).

\( M, u \models [\alpha] \phi \) if and only if \( \exists u' \in W \) such that \( (u, u') \in R(\alpha) \) and \( M, u' \models \phi \).

Truth, satisfiability, and validity are defined as in the PDL\(^{-}\) case.

Since every language defined by a regular expression can also be accepted by a finite automaton and vice versa [25], it is straightforward to verify that PDL\(^{-}\) and APDL\(^{-}\) have the same expressive power. Moreover, upper complexity bounds carry over from APDL\(^{-}\) to PDL\(^{-}\) since conversion of regular expressions to finite automata can be done with at most a polynomial blow-up in size. Interestingly, the converse does not hold true.

It is interesting to note that, in many automata-based decision procedures for variants of PDL, a deterministic version of APDL is used, i.e. a variant of APDL in which there may be at most one successor for each world and each atomic program [20]. In a second step, satisfiability in the non-deterministic APDL-variant is then reduced to satisfiability in the deterministic one. We cannot take this approach here since we cannot w.l.o.g. assume that both atomic programs and their negations are deterministic. Indeed, this would correspond to limiting the size of Kripke structures to only two worlds.

### 4 Hintikka-trees

This section provides a core step toward using Büchi-tree automata for deciding the satisfiability of APDL\(^{-}\)-formulas. The intuition behind this approach is
as follows: to decide the satisfiability of an \( \text{APDL}^{(-)} \)-formula \( \phi \), we translate it into a Büchi-tree automaton \( B_\phi \) such that the trees accepted by the automaton correspond in some way to models of the formula \( \phi \). To decide satisfiability of \( \phi \) it then remains to perform a simple emptiness-test on the automaton \( B_\phi \); the accepted language will be non-empty if and only if \( \phi \) has a model.

In the case of \( \text{APDL}^{(-)} \), one obstacle to this approach is that \( \text{APDL}^{(-)} \) does not enjoy the tree model property (TMP), i.e. there are \( \text{APDL}^{(-)} \)-formulas that are satisfiable only in non-tree models. A simple example is the following formula:

\[
p \land \langle \alpha \rangle [\neg \alpha] \neg p.
\]

Thus, we have to invest some work to obtain tree-shaped representations of (possibly non-tree) models that can then be accepted by Büchi-automata.

As a preliminary, we assume that all \( \text{APDL}^{(-)} \)-formulas are in negation normal form (NNF), i.e. that negation occurs only in front of propositional letters. This assumption can be made without loss of generality since each formula can be converted into an equivalent one in NNF by exhaustively eliminating double negation, applying DeMorgan’s rules, and exploiting the duality between diamonds and boxes.

For the sake of brevity, we introduce the following notational conventions:

- for each \( \text{APDL}^{(-)} \)-formula \( \phi \), \( \neg \phi \) denotes the NNF of \( \neg \phi \);
- for each program literal \( \pi \), \( \overline{\pi} \) denotes \( \neg \pi \) if \( \pi \) is an atomic program and \( a \) if \( \pi = \neg a \) for some atomic program \( a \);
- for each program automaton \( \alpha \), we use \( Q_\alpha, \Sigma_\alpha, q_\alpha, \Delta_\alpha, \) and \( F_\alpha \) to denote the components of \( \alpha = (Q, \Sigma, \phi_0, \Delta, F) \);
- for each program automaton \( \alpha \) and state \( q \in Q_\alpha \), we use \( \alpha_q \) to denote the automaton \( (Q_\alpha, \Sigma_\alpha, q, \Delta_\alpha, F_\alpha) \), i.e. the automaton obtained from \( \alpha \) by using \( q \) as the new initial state.

Before we can develop the tree-shaped abstraction of models, we need to fix a closure, i.e. a set of formulas \( \text{cl}(\phi) \) relevant for deciding the satisfiability of an input formula \( \phi \). This is done analogous to [3, 20]. In the following, when we talk of a subformula \( \psi \) of a formula \( \phi \), we mean that \( \psi \) can be obtained from \( \phi \) by decomposing only formula operators, but not program operators. For example, \( a \) is a subformula of \( \langle b \rangle a \), while \( b \) is not.

**Definition 6 (Closure).** Let \( \phi \) be a \( \text{APDL}^{(-)} \)-formula. The set \( \text{cl}(\phi) \) is the smallest set which is closed under the following conditions:

1. \( \psi \in \text{cl}(\phi) \) (C1)
2. if \( \psi \) is a subformula of \( \psi' \in \text{cl}(\phi) \), then \( \psi \in \text{cl}(\phi) \) (C2)
3. if \( \psi \in \text{cl}(\phi) \), then \( \neg \psi \in \text{cl}(\phi) \) (C3)
4. if \( \langle a \rangle \psi \in \text{cl}(\phi) \), then \( \langle a \rangle \psi' \in \text{cl}(\phi) \) for all \( \psi' \in \Sigma_\alpha \) (C4)
5. if \( \langle a \rangle \psi \in \text{cl}(\phi) \), then \( \langle a \rangle \psi' \in \text{cl}(\phi) \) for all \( q \in Q_\alpha \) (C5)
6. if \( [a] \psi \in \text{cl}(\phi) \), then \( [a] \psi' \in \text{cl}(\phi) \) for all \( \psi' \in \Sigma_\alpha \) (C6)
7. if \( [a] \psi \in \text{cl}(\phi) \), then \( [a] \psi' \in \text{cl}(\phi) \) for all \( q \in Q_\alpha \) (C7)
It is standard to verify that the cardinality of $\text{cl}(\phi)$ is polynomial in the length of $\phi$, see for example [5]. We generally assume the diamond formulas (i.e. formulas of the form $(a)\psi$) in $\text{cl}(\phi)$ to be linearly ordered and use $\epsilon_i$ to denote the $i$-th diamond formula in $\text{cl}(\phi)$, with $\epsilon_1$ being the first one. Note that a changed initial state of an automaton results in a different diamond formula.

To define Hintikka-trees, the tree-shaped abstraction of models underlying our decision procedure, we proceed in three steps. First, we introduce Hintikka-sets that will be used as (parts of) node labels. Intuitively, each node in the tree describes a world of the corresponding model, and its label contains the formulas from the closure of the input formula $\phi$ that are true in this world. Second, we introduce a matching relation that describes the possible “neighborhoods” that we may find in Hintikka-trees, where a neighborhood consists of a labeled node and its labeled successors. And third, we use these ingredients to define Hintikka-trees.

**Definition 7 (Hintikka-set).** Let $\psi \in \Phi^{(-)}$ be an $\text{APDL}^{(-)}$-formula, and $\alpha \in \text{AIL}^{(-)}$ a program automaton. The set $\Psi \subseteq \text{cl}(\phi)$ is a Hintikka-set for $\phi$ if

(H1) if $\psi_1 \land \psi_2 \in \Psi$, then $\psi_1 \in \Psi$ and $\psi_2 \in \Psi$

(H2) if $\psi_1 \lor \psi_2 \in \Psi$, then $\psi_1 \in \Psi$ or $\psi_2 \in \Psi$

(H3) $\psi \in \Psi$ if $\neg \psi \notin \Psi$

(H4) if $[\alpha]\psi \in \Psi$ and $\alpha_q, q \in F$, then $\psi \in \Psi$

(H5) if $[\alpha]\psi \in \Psi$ then, for any state $q \in Q_\alpha$, and test $\theta^? \in \Sigma_\alpha$, $q \in \Delta_\alpha(q_\alpha, \theta^?)$ implies that $\neg \theta \in \Psi$ or $[\alpha_q]\psi \in \Psi$

The set of all Hintikka-sets for $\phi$ is designated by $\mathcal{H}_\phi$.

The conditions (H1) to (H3) are standard, with one exception: (H3) is stronger than usual since it enforces maximality of Hintikka-sets by stating that, for each formula $\psi \in \text{cl}(\phi)$ either $\psi$ or $\neg \psi$ must be in the Hintikka-set. This will be used later on to deal with negated programs. The last two conditions (H4) and (H5) deal with the “local” impact of box formulas. Note that, in (H5), the automaton $\alpha_q$ correspond to the automaton $\alpha$ with new initial state $q$.

Next, we define the matching relation. The purpose of this relation can be understood as follows: in the Hintikka-tree, each node has exactly one successor for every diamond formula in $\text{cl}(\phi)$. The matching relation helps to ensure that all diamond formulas in a node’s label can be satisfied “via” the corresponding successor in the Hintikka-tree, and that none of the box formulas is violated via any successors. We talk of “via” here since going to an immediate successor corresponds to traveling along a single program literal. Since programs in $\text{APDL}^{(-)}$ are automata that may accept words of length greater one, in general we cannot satisfy diamonds by going only to the immediate successor, but rather we must perform a sequence of such moves.

Before we define the matching relation formally, let us fix the structure of node labels of Hintikka-trees. For reasons that will be discussed below, node
labels not only contain a Hintikka-set, but also two additional components. More precisely, if $\phi$ is an APDL$(-)$-formula and $cl(\phi)$ contains $k$ diamond formulas, then we use

- $\Pi_\phi(-)$ to denote the set of all program literals occurring in $\phi$; and
- $\Lambda_\phi$ to abbreviate $\mathbb{H}_\phi \times (\Pi_\phi(-) \cup \{\bot\}) \times \{1, \ldots, k\}$, i.e. the set of triples containing a Hintikka-set for $\phi$, a program literal of $\phi$ or $\bot$, and a number at most $k$.

The elements of $\Lambda_\phi$ will be used as node labels in Hintikka-trees. Intuitively, the first component lists the formulas that are true in a node, the second component fixes the program literal with which the node can be reached from its predecessor (or $\bot$ if this information is not important), and the third component will help to ensure that diamond formulas are eventually satisfied when moving through the tree. For a triple $\lambda \in \Lambda_\phi$, we refer to the first, second and third triple component with $\lambda^1$, $\lambda^2$, and $\lambda^3$, respectively. For the following definition, recall that we use $e_i$ to denote the $i$-th diamond in $cl(\phi)$.

**Definition 8 (Matching).** Let $\phi$ be a formula and $k$ the number of diamond formulas in $cl(\phi)$. A $k + 1$-tuple of $\Lambda_\phi$-triples $(\lambda, \lambda_1, \ldots, \lambda_k)$ is matching if, for $1 \leq i \leq k$ and all automata $\alpha \in \mathcal{APDL}(-)$, the following holds:

(M1) if $e_i = (\alpha)\psi \in \lambda^1$, then there is a word $w = \psi_1 \cdots \psi_n \in \Sigma_\alpha^n$, $n \geq 0$, and a state $q_1 \in Q_\alpha$ such that $\{\psi_1, \ldots, \psi_n\} \subseteq \lambda^1$, $q_1 \in \Delta_\alpha(q_0, w)$, and one of the following holds:

(a) $q_1$ is a final state, $\psi \in \lambda^1$, $\lambda^2 = \bot$, and $\lambda^3 = 0$

(b) there is a program literal $\pi \in \Sigma_\alpha$ and a state $q_2 \in Q_\alpha$ such that $q_2 \in \Delta_\alpha(q_1, \pi)$, $e_j = (\alpha_{q_2})\psi \in \lambda^1$, $\lambda^2 = \pi$, and $\lambda^3 = j$.

(M2) if $[\alpha]\psi \in \lambda^1$, $q \in Q_\alpha$, and $\pi \in \Sigma_\alpha$ a program literal such that $q \in \Delta_\alpha(q_0, \pi)$, then $\pi = \lambda^2$ implies $[\alpha_q]\psi \in \lambda^1$.

As already noted, the purpose of the matching relation is to describe the possible neighborhoods in Hintikka-trees. To this end, think of $\lambda$ as the label of a node, and of $\lambda_1, \ldots, \lambda_k$ as the labels of its successors. The purpose of Conditions (M1) and (M2) is to ensure that diamonds are satisfied and that boxes are not violated, respectively. Let us consider only (M1). If a diamond $e_i = (\alpha)\psi$ is in the first component of $\lambda$, it can either be satisfied in the node labeled with $\lambda$ itself (Condition (a)) or we can “delay” its satisfaction to the $i$-th successor node that is reserved specifically for this purpose (Condition (b)). In Case (a), it is not important over which program literal we can reach the $i$-th successor, and thus the second component of $\lambda_1$ can be set to $\bot$. In the second case, we must choose a suitable program literal $\pi$ and a suitable state $q$ of $\alpha$, make sure that the $i$-th successor is reachable over $\pi$ via its second $\lambda_1$-component, and guarantee that the first component of $\lambda_1$ contains the diamond under consideration with the automata $\alpha$ “advanced” to initial state $q$. 
The remaining building block for ensuring that diamonds are satisfied is to enforce that the satisfaction of diamonds is not delayed forever. This is one of the two core parts of the definition of Hintikka-trees, the other being the proper treatment of negation. Before we can discuss the prevention of infinitely delayed diamonds in some more detail, we have to introduce some basic notions.

Let $M$ be a set and $k \in \mathbb{N}$. An (infinite) $k$-ary $M$-tree $T$ is a mapping $T : [k]^* \to M$, where $[k]$ is used (now and in the following) as an abbreviation for the set $\{1, \ldots, k\}$. Intuitively, the node $ai$ is the $i$-th child of $a$. We use $\varepsilon$ to denote the empty word (corresponding to the root of the tree). An infinite path in a $k$-ary $M$-tree is an infinite word $\gamma$ over the alphabet $[k]$. We use $\gamma[n]$, $n \geq 0$, to denote the prefix of $\gamma$ up to the $n$-th element of the sequence (with $\gamma[0]$ yielding the empty sequence).

Now back to the prevention of infinitely delayed diamonds. Given a formula $\phi$ with $k$ diamond formulas in $cl(\phi)$, a Hintikka-tree will be defined as a $k$-ary $A_\phi$-tree in which every neighborhood is matching and some additional conditions are satisfied. To detect infinite delays of diamonds in such trees, it does not suffice to simply look for infinite sequences of nodes that all contain the same diamond: firstly, diamonds are evolving while being “pushed” through the tree since their initial state might be changed. Secondly, such a sequence does not necessarily correspond to an infinite delay of diamond satisfaction: it could as well be the case that the diamond is satisfied an infinite number of times, but always immediately “regenerated” by some other formula. Also note that we cannot use the standard technique from [20] since it only works for deterministic variants of PDL.

Precisely for this purpose, the easy detection of infinitely delayed diamonds, we have introduced the third component of node labels in Hintikka trees: if a diamond was pushed to the current node $x$ from its predecessor, then by (M1) the third component of $x$’s label contains the number of the pushed diamond. Moreover, if the pushed diamond is not satisfied in $x$, we again use the third component of $x$: it contains the number of the successor of $x$ to which the diamond’s satisfaction is (further) delayed. If no diamond was pushed to $x$, its third component is simply zero. Thus, the following definition captures our intuitive notion of infinitely delayed diamonds.

**Definition 9 (Diamond Starvation).** Let $\phi$ be an APDL$^{(-)}$-formula with $k$ diamond formulas in $cl(\phi)$. $T$ a $k$-ary $A_\phi$-tree, $x \in [k]^*$ a node in $T$, and $e_\ell = \langle a \rangle \psi \in T(x)^1$. Then the diamond formula $\langle a \rangle \psi$ is called starving in $x$ if there exists a path $\gamma = \gamma_1 \gamma_2 \cdots \in [k]^\omega$ such that

1. $\gamma_1 = \ell$,
2. $T[x\gamma[i]]^3 = \gamma_{i+1}$ for $i \geq 1$.

We have now gathered all ingredients to define Hintikka-trees formally.

**Definition 10 (Hintikka-tree).** Let $\phi$ be an APDL$^{(-)}$-formula with $k$ diamond formulas in $cl(\phi)$. A $k$-ary $A_\phi$-tree $T$ is a Hintikka-tree for $\phi$ if $T$ satisfies, for
all nodes \(x, y \in [k]^*\), the following conditions:

(T1) \(\phi \in T(\varepsilon)^1\)

(T2) the \(k + 1\)-tuple \((T(x), T(x_1), \ldots, T(x_k))\) is matching

(T3) no diamond formula from \(c\ell(\phi)\) is starring in \(x\)

(T4) if \([x] \psi \in T(x)^1\), \(\pi \in H_\phi^{(-)}\), \(q_\alpha^i \in Q_{\alpha^i}\) and \(q_\beta^j \in Q_{\beta^j}\) such that

\[ q_\alpha^i \in \Delta_\alpha(q_{\alpha^i}, \pi) \text{ and } q_\beta^j \in \Delta_\beta(q_{\beta^j}, \pi) \],

then \([\alpha q_\alpha^i] \psi \notin T(y)^1\) implies \([\beta q_\beta^j] \psi \in T(y)^1\).

Conditions (T1) to (T3) are easily understood. The purpose of Condition (T4) is to deal with negated programs. In particular, for each atomic program \(a\) we have to ensure that any pair of nodes \(x, y\) of a Hintikka-tree \(T\) can be related by one of \(a\) and \(\neg a\) without violating any boxes. This is done by (T4) together with (H3)—indeed, this is the reason for formulating (H3) stronger than usual. Intuitively, the treatment of negation can be understood as follows: suppose that \([a] \psi \in T(x)^1\), let \(q \in \Delta_\alpha(q_{\alpha}, a)\) for some atomic program \(a\), and let \(y\) be a node. By (H3), we have either \([a q_{\alpha}] \psi \in T(y)^1\) or \([\neg a] \psi \in T(y)^1\). In the first case, \(x\) and \(y\) can be related by \(a\). In the second case, (T4) ensures that they can be related by \(\neg a\). This technique is inspired by [21], but generalized to program automata.

We now show that Hintikka-trees are indeed proper abstractions of models. In order to do this, we need two new notions. First, we use \(\Sigma_\phi\) to denote the smallest set containing (i) all program literals occurring in \(\phi\) (i.e. \(H_\phi^{(-)}\)) and (ii) all tests \(\psi\) occurring in \(\phi\). Second, let \(M\) be a Kripke structure and \(u\) a world of \(M\). Then a word \(w = w_1 \cdots w_m \in \Sigma^*\) is said to accomplish a diamond formula \(\langle \alpha \rangle \psi \) at \(u\) if \(w \in L(\alpha)\), and there are worlds \(u_0, \ldots, u_m\) of \(M\) such that

\[ u = u_0 R(w_1) u_1 R(w_2) \cdots u_{m-1} R(w_m) u_m \text{ and } M, u_m \models \psi.\]

**Proposition 1.** An \(APDL^{(-)}\)-formula \(\phi\) is satisfiable iff it has a Hintikka-tree.

**Proof.** Let \(\phi\) be an \(APDL^{(-)}\)-formula and \(k\) the number of diamond formulas in \(c\ell(\phi)\).

\(\Rightarrow.\) Suppose the Kripke structure \(M = (W, R, V)\) is a model of \(\phi\), i.e. there is a world \(u_\phi \in W\) such that \(M, u_\phi \models \phi\). Fix a linear order \(\prec\) on \(\Sigma_\phi^*\) such that

\[- w \prec w' \text{ if } |w| < |w'|,\]

\[- w w' \prec w w'' \text{ if } w' < w''.\]

Then define a partial function \(\ell : c\ell(\phi) \times W \to \mathbb{N}\) as follows: for each \(\langle \alpha \rangle \psi \in c\ell(\phi)\), and \(u \in W\) such that \(M, u \models \langle \alpha \rangle \psi\), \(\ell(\langle \alpha \rangle \psi, u)\) denotes the length of the word \(w \in \Sigma_\phi^*\) that accomplishes \(\langle \alpha \rangle \psi\) at \(u\) and is minimal with this property.

In the following, we construct a Hintikka-tree for \(\phi\). More precisely, we simultaneously define

\[- a \text{-ary } (2^{c\ell(\phi)} \times H_\phi^{(-)} \cup \{1\} \times \{0, \ldots, k\})\text{-tree } T_\phi,\]

and
— a mapping \( \tau : [k]^* \rightarrow W \)

such that, for all \( x \in [k]^* \), we have

\[
\psi \in T_\phi(x)^1 \text{ iff } \mathcal{M}, \tau(x) \models \psi.
\]

(\*)

The construction of \( \tau \) and \( T_\phi \) is inductive. For the induction base, set

\[
\tau(\varepsilon) := u_\phi, \\
T_\phi(\varepsilon)^1 := \{ \psi \in \mathcal{c}(\phi) \mid \mathcal{M}, u_\phi \models \psi \}, \\
T_\phi(\varepsilon)^2 := \bot, \text{ and} \\
T_\phi(\varepsilon)^3 := 0.
\]

For the induction step, let \( x \in [k]^* \) be a node such that \( \tau(x) \) is already defined, but \( \tau(x_1), \ldots, \tau(x_k) \) are not. For each \( i \in [k] \), we distinguish two cases:

1. \( e_i = \langle \alpha \rangle \psi \in T_\phi(x)^1 \). Then (\*) yields \( \mathcal{M}, \tau(x) \models \langle \alpha \rangle \psi \). By the semantics, there thus exists a word that accomplishes \( \langle \alpha \rangle \psi \) at \( \tau(x) \). Let \( w = w_1 \cdots w_m \in \Sigma_\phi^n \) be the minimal such word w.r.t. the ordering \( < \). Then there are words \( u_0, \ldots, u_m \in W \) such that \( \tau(x) = u_0R(w_1|u_1R(w_2|u_2| \cdots |u_{m-1}R(w_m)u_m \text{ and } \mathcal{M}, u_m \models \psi \). Distiguish two subcases:

   (a) The word \( w \) contains only tests or is empty. This implies that \( \tau(x) = u_m \), and thus \( \mathcal{M}, \tau(x) \models \psi \). By (\*), we get \( \psi \in T_\phi(x)^1 \). Set

   \[
   \tau(x_i) := u_\phi, \\
T_\phi(x_i)^1 := \{ \psi \in \mathcal{c}(\phi) \mid \mathcal{M}, u_\phi \models \psi \}, \\
T_\phi(x_i)^2 := \bot, \text{ and} \\
T_\phi(x_i)^3 := 0.
   \]

   (b) Otherwise, fix an accepting run \( q_0 \cdots q_n \in Q_\phi^n, n \geq 0 \), of \( \alpha \) on \( w \). Take the least \( p \in [m] \) such that \( w_p \) is not a test but a program literal. Let \( j \in [k] \) be such that \( e_j = \langle \alpha_{q_p} \rangle \psi \). Set

   \[
   \tau(x_i) := u_p, \\
T_\phi(x_i)^1 := \{ \psi \in \mathcal{c}(\phi) \mid \mathcal{M}, u_p \models \psi \}, \\
T_\phi(x_i)^2 := w_p, \text{ and} \\
T_\phi(x_i)^3 := j.
   \]

2. \( e_i = \langle \alpha \rangle \psi \notin T_\phi(x)^1 \). Then set

   \[
   \tau(x_i) := u_\phi, \\
T_\phi(x_i)^1 := \{ \psi \in \mathcal{c}(\phi) \mid \mathcal{M}, u_\phi \models \psi \}, \\
T_\phi(x_i)^2 := \bot, \text{ and} \\
T_\phi(x_i)^3 := 0.
   \]
To show that $T_\phi$ is indeed a Hintikka-tree for $\phi$, we first prove the following claim which will be helpful in showing that our Hintikka-tree does not contain starring diamonds.

**Claim.** Let $x, y \in [k]^*$ be nodes such that $x_i = y$ for some $i \in [k]$. If $T_\phi(y)^3 = j \neq 0$, then (i) $e_j \in T_\phi(x)^1$, (ii) $e_j \in T_\phi(y)^1$, and (iii) $\ell(e_j, \tau(y)) < \ell(e_i, \tau(x))$.

**Proof.** Let $x$ and $y$ be as in the claim. Suppose $T_\phi(y)^3 = j \neq 0$. Then $\tau(y)$ and $T_\phi(y)$ have been set in Case 1b of the induction step. Hence $e_i \in T_\phi(x)^1$ and thus Point (i) is proved. Let $w = w_1 \cdots w_m$, $q_0 \cdots q_m$, and $p$ be as in Case 1b. Clearly, $w \in L(\alpha)$ implies $w_{p+1} \cdots w_m \in L(\alpha_q)$. Since

$$u_{p+1}R(w_1)u_{p+2}R(w_2)u_{p+3} \cdots u_{m-1}R(w_m)u_m \tag{1}$$

and $M, u_m \models \psi$, we have $M, \tau(y) \models [\alpha_{o_0}]\psi = e_j$, which shows Point (ii). As shown by $w_{p+1} \cdots w_m \in L(\alpha_q)$, (1), and $M, u_m \models \psi$, $w_{p+1} \cdots w_m$ accomplishes $e_j$ at $\tau(y)$. Clearly $w_{p+1} \cdots w_m$ is minimal with this property; under the assumption of non-minimality, it is easy to derive a contradiction to the minimality of the word $w$ for accomplishing $e_i$ at $x$. Hence $\ell(e_j, \tau(y)) = |w_{p+1} \cdots w_m| < m = |w| = \ell(e_i, \tau(x))$ finishing the proof of the claim.

We now prove that $T_\phi$ is a Hintikka-tree for $\phi$. First, it is shown that, for each node $x \in [k]^*$, $T_\phi(x)^1$ is a Hintikka-set. Observe that Conditions (H1), (H2), and (H3) easily follow from the semantics and the definition of the closure $cl$. Now consider the Condition (H4). Suppose that $[\alpha] \psi \in T_\phi(x)^1$, and $q_{\alpha} \in F_{o_0}$. We get $M, x \models [\alpha] \psi$ (by (1)), $(\tau(x), \tau(x)) \in R(\alpha)$, and thus also $M, \tau(x) \models \psi$. Hence $\psi \in T_\phi(x)^1$ by (1). Finally, look at the remaining Condition (H5). Suppose that $[\alpha] \psi \in T_\phi(x)^1$, and let $q \in Q_{\alpha}$ be a state, and $\theta^? \in \Sigma_{\alpha}$ a test such that $q \in \Delta_{\alpha}(q_{\alpha}, \theta^?)$. By (H3), $\neg \theta \in T_\phi(x)^1$ or $\theta \in T_\phi(x)^1$. In the former case, (H4) follows immediately. Consider the latter case. By (1), it holds that $M, \tau(x) \models [\alpha] \psi$, and $M, \tau(x) \models \theta$. From the semantics it follows that $M, \tau(x) \models [\alpha]\psi$. Thus, $[\alpha] \psi \in T_\phi(x)^1$ by (1).

Since the first triple components of the node labels in tree $T_\phi$ are Hintikka-sets, $T_\phi$ is a $k$-ary $A_k$-tree. It remains to show that $T_\phi$ additionally satisfies the conditions for Hintikka trees (T1) to (T4). Let $x, y \in [k]^*$ be nodes.

**(T1)** Holds by definition of $T_\phi$, see induction start.

**(T2)** It is to show that the $k + 1$-tuple $(T_\phi(x), T_\phi(x_1), \ldots, T_\phi(x_k))$ is matching. We first consider the matching condition (M1). Suppose $e_i \models [\alpha] \psi \in T_\phi(x)^1$. By (1), $M, \tau(x) \models [\alpha] \psi$. Thus there exists a word that accomplishes $e_i$ at $\tau(x)$. Let $w = w_1 \cdots w_m$ be the minimal such word w.r.t. $<$. If $w$ is comprised only of tests, then $M, \tau(x) \models \psi$ and $\tau(x)$ and $T_\phi(x_i)$ are defined in Case 1a of the induction step. It is thus readily checked that Case (a) of (M1) is satisfied; since $w \in L(\alpha)$, we find a final state $q_1$ as required. Since $M, \tau(x) \models \psi$ and we are in Case 1a, we obtain $\psi \in T_\phi(x)^1$, $T_\phi(x)^2 = \perp$, and $T_\phi(x)^3 = 0$. Now assume that $w$ does contain a program literal and let $p \in [m]$ be minimal such that $w_p$ is a program literal. Then $\tau(x)$ and $T_\phi(x)$ are defined in Case 1b of the induction step. As in the proof of the claim, it
follows that $\epsilon_j = \langle \alpha \rangle \psi \in T_\phi(x_1)$, where $q \in \Delta(q_0, w_1 \cdots w_n)$. Moreover, we have $T_{\phi}(x_i) = w_j$ and $T_{\phi}(x_1) = j$. Thus, Case (b) of (M1) is fulfilled.

Consider the remaining Condition (M2). Suppose $[\alpha] \psi \in T_\phi(x_1)$. By (\ast), it holds that $M, \tau(x) \models [\alpha] \psi$. Let $q \in Q_0$ be a state of $\alpha$ and $\pi \in \Sigma_\alpha$ a program literal such that $q \in \Delta_\alpha(q_0, \pi)$. Assume $\pi = T_\phi(x_1)^2$ for some $i \in [k]$. Then $\tau(x_i)$ and $T_\phi(x_i)$ have been set in Case (1b) of the induction step. Thus $(\tau(x), \tau(x_i)) \in R(\pi)$. Together with the fact that $M, \tau(x) \models [\alpha] \psi$ it follows that $M, \tau(x_i) \models [\alpha] \psi$. Consequently, $[\alpha] \psi \in T_\phi(x_1)$.

(T3) Suppose by contradiction that diamond formula $\epsilon_i = \langle \alpha \rangle \psi \in T_\phi(x_1)$ is starring in node $x$, i.e., there is a path $\gamma = \gamma_1 \gamma_2 \cdots \in \{k\}^+$ such that $\gamma_1 = i$, and $T_\phi(x\gamma[n])^1 = \gamma_{n+1}$ for $n \geq 1$. Label $x$ with the natural number $\ell(\epsilon_{x_1}, \tau(x))$, and the nodes $x\gamma[n]$, $n \geq 1$, with $\ell(\epsilon_{x_1}, \tau(x\gamma[n]))$. By the above claim, these numbers are strictly decreasing along the path $\gamma$: a contradiction to the fact that the range of the function $\ell$ is $\mathbb{N}$.

(T4) Suppose that $[\alpha] \psi, [\beta] \theta \in T_\phi(x_1)$. Let $\pi \in \Pi_\phi^{(\ast)}$ be a program literal, and $q_\alpha' \in Q_0$, $q_\beta' \in Q_0$ be states such that $q_\alpha' \in \Delta_\alpha(q_0, \pi)$, and $q_\beta' \in \Delta_\beta(q_0, \pi)$. Further suppose that $[\alpha] \psi \notin T(\pi)^1$. We show that $(\tau(x), \tau(\pi)) \in R(\pi)$. Since $M, \tau(x) \models [\alpha] \psi$ by (\ast), the semantics yields, $M, \tau(\pi) \models [\alpha] \psi$. Then (\ast) yields $[\alpha] \psi \in T(\pi)^1$, a contradiction. Consequently, $(\tau(x), \tau(\pi)) \in R(\pi)$. Since $M, \tau(x) \models [\beta] \theta$ by (\ast), the semantics yields, $M, \tau(\pi) \models [\beta] \theta$. Then, again by (\ast), we have $[\beta] \theta \in T(\pi)^1$ as required by (T4).

\textit{“$\Rightarrow$”}. Suppose $T$ is a Hintikka-tree for $\phi$. Construct a Kripke Structure $M = (W, R, V)$ in the following way:

- $W := [k]^*$;
- For each atomic program $a$, set
  $$ R_\phi(a) := \{ (x, y) \in W^2 \mid y = xi \text{ for some } i \in [k], \text{ and } T(y)^2 = a \}, \text{ and }$$
  $$ R(\pi) := \{ (x, y) \in W^2 \mid \text{there is a } [\alpha] \psi \in T(x)^1 \text{ and a } q \in \Delta_\alpha(q_0, \neg a) \text{ such that } [\alpha] \psi \notin T(y)^1 \}$$

Now we can define the accessibility relation for $a$: $R(a) := R_\phi(a) \cup R(\pi)$;
- The valuation function $V(p)$ for propositional variables $p$ is defined as
  $$ V(p) := \{ x \in W \mid p \in T(x)^1 \}.$$  

To show that $M$ is a model of $\phi$, we prove two claims. The first is concerned with the relational structure:

Claim 1. For each literal $\pi \in \Pi_\phi^{(\ast)}$, node $x \in [k]^*$, and $i \in [k]$, we have that $T(x)^2 = \pi$ implies $(x, xi) \in R(\pi)$.

Proof: If $\pi$ is an atomic program, then the claim is an immediate consequence of the definition of $R(\pi)$. Thus suppose that $\pi = \neg c$ is a negated atomic program. Assume that $T(x)^2 = \neg a$ and $(x, xi) \notin R(\neg a)$. By the semantics, we obtain...
\[(x, x_i) \in R(a).\] Since \(T(x)^2 = -a\), in particular we have \((x, x_i) \in R_2(a)\). Thus there is a \([a] \psi \in T(x)^1\) and a \(q \in \Delta_a(q_o, -a)\) such that \([a] \psi \notin T(y)^1\). Since \(T(x)^2 = -a\), however, this is in contradiction to \((M2)\).

The second claim is concerned with the truth of formulas.

**Claim 2.** For each \(\psi \in \text{cl}(\phi)\) and \(x \in [k]^*\), \(\psi \in T(x)^1\) implies \(\mathcal{M}, x \models \psi\).

**Proof:** Let \(\psi\) and \(x\) be as in the claim. Inductively define the norm \(\| \cdot \|\) of \(APDL^{(-)}\)-formulas in NNF as follows:

\[
\begin{align*}
\| p \| & := \| -p \| := 0 \text{ for } p \in \Phi_0 \\
\| \psi_1 \land \psi_2 \| & := \| \psi_1 \lor \psi_2 \| := 1 + \| \psi_1 \| + \| \psi_2 \| \\
\| [a] \psi \| & := 1 + \| \psi \| + \sum_{\theta \in \Sigma_a} \| \theta \|
\end{align*}
\]

The proof is by induction on the norm \(\| \cdot \|\). The induction base has two cases:

- \(\psi\) is a propositional variable. Immediate by definition of \(\mathcal{M}\).
- \(\psi = \neg p\) We have \(p \in \Phi_0\) due to NNF. With \((H3)\), \(\psi \in T(x)^1\) thus implies that \(p \notin T(x)^1\). By definition of \(\mathcal{M}\), \(x \notin V(p)\). Hence, \(\mathcal{M}, x \models \neg p\).

The induction step consists of several cases:

- \(\psi = \psi_1 \lor \psi_2\) or \(\psi = \psi_1 \land \psi_2\). Since \(T(x)^1\) is a Hintikka-set, these cases are straightforward by Conditions \((H1)\) and \((H2)\) and the induction hypothesis.
- \(\psi = \epsilon_i = \langle a \rangle \theta\). Inductively define a (potentially infinite) path \(\gamma = \gamma_1 \gamma_2 \cdots \in [k]^\omega\) as follows:
  - \(\gamma_1 := i\)
  - if \(T(x \gamma [n])^3 \neq 0\), then \(\gamma_{n+1} := T(x \gamma [n])^3\); otherwise, \(\gamma_n\) is the last node in the path.

By \((T3)\), the diamond formula \(\langle a \rangle \theta\) is not starving at \(x\). It is thus readily checked that the path \(\gamma\) is finite, i.e. \(\gamma = \gamma_1 \cdots \gamma_m\) and \(T(x \gamma)^3 = 0\). By \((M1)\), there exist two sequences of states \(q_0, \ldots, q_m \in Q_a\) and \(q_0', \ldots, q_m' \in Q_a\), a sequence of program literals \(\pi_1, \ldots, \pi_m \in \Sigma_a\), and a sequence of sequences of tests \(t_0, \ldots, t_m \in \Sigma_a\) such that, for \(n \leq m\), we have

1. \(q_0 = q_o\);
2. if \(t_n = \psi_1 ? \cdots \psi_l ?\), then \(\psi_1 ? \cdots \psi_l ? \in T(x \gamma [n])^1\);
3. \(q_n^o \in \Delta_a(q_o, t_n)\);
4. \(T(x \gamma [n] - 1)^3 = \pi_n\) if \(n > 0\);
5. \(q_{n+1} \in \Delta_a(q_n^o, \pi_n)\) if \(n < m\);
6. \(q'_m\) is a final state;
7. \(\theta \in T(x \gamma)^1\).

By induction hypothesis, we obtain the following from Point 2 and 7: for \(n \leq m\), we have

8. if \(t_n = \psi_1 ? \cdots \psi_l ?\), then \(\mathcal{M}, x \gamma [n] \models \psi_j ?\) for \(1 \leq j \leq l\);
9. \(\mathcal{M}, x \gamma \models \theta\).

Together with Claim 1, we obtain from Point 4:
10. for \(n \in [m]\), we have \((x \gamma [n - 1], x \gamma [n]) \in R(x)\).
Using Points 1, 3, 5, 6, and 6, it is readily checked that \((x, x\gamma) \in R(\alpha)\). Together with Point 9, we thus obtain \(M, x \models \alpha \theta\) as required.

\(- \psi = [\alpha] \theta\). By the semantics, it needs to be shown that for any world \(y \in W\), \((x, y) \in R(\alpha)\) implies \(M, y \models \theta\). Thus suppose \((x, y) \in R(\alpha)\) for some \(y \in W\), i.e., there is a world \(w = w_1 \cdots w_m \in L(\alpha), m \geq 0\), and worlds \(x_0, \ldots, x_m \in W\) with \(x = x_0 R(w_1) x_1 R(w_2) \cdots x_{m-1} R(w_m) x_m = y\). Fix an accepting run \(q_0 \cdots q_m \in Q^a\) of the program automaton \(\alpha\) on \(w\). In the following, it is shown that \([\alpha_{q_0}] \theta \in T(x_i)^1\) for \(i \leq m\). We proceed by induction on \(i\). The induction start is immediate. For the step, assume that \([\alpha_{q_0}] \theta \in T(x_i)^1\) has already been shown. We distinguish the following cases:

1. \(w_i = \delta\) is a test. From \((x_i, x_{i+1}) \in R(\delta)\) it follows by the semantics that \(x_i = x_{i+1}\), and \(M, x_i \models \delta\). Suppose by contradiction that \(\dot{\delta} \notin T(x_i)^1\). The induction hypothesis yields \(M, x_i \models \neg \delta\), a contradiction to \(M, x_i \models \delta\). Thus \(\dot{\delta} \notin T(x_i)^1\), and by (H5) we obtain \([\alpha_{q_{i+1}}] \theta \in T(x_{i+1})^1 = T(x_i)^1\).

2. \(w_i\) is an atomic program \(a\) in \(\mathcal{R}_\varphi^{(\sim)}\). By definition of \(R(\alpha)\), \((x_i, x_{i+1}) \in R_\varphi(a) \cup R_\varphi(\alpha)\). Firstly, suppose that \((x_i, x_{i+1}) \in R_\varphi(\alpha)\). The definition of \(R_\varphi(\alpha)\) yields \(x_{i+1} = x_{i+j}\) for some \(j \in \{k\}\), and \(T(x_{i+1})^2 = a\). Consequently, by Condition (M2), \([\alpha_{q_{i+1}}] \theta \in T(x_{i+1})^1\). Secondly, suppose that \((x_i, x_{i+1}) \in R_\varphi(\alpha)\). By definition of \(R_\varphi(\alpha)\), there is \([\beta] \delta \in T(x_i)^1\), and \([\beta_q] \delta \notin T(x_{i+1})^1\) for some \(q \in \Delta_\beta(q_j, \alpha)\). Since \([\alpha_{q_0}] \theta \in T(x_i)^1\) by assumption, the Condition (T4) yields \([\alpha_{q_{i+1}}] \theta \in T(x_{i+1})^1\).

3. \(w_i\) is a negated atomic program \(\neg a\) in \(\mathcal{R}_\varphi^{(\sim)}\). By the semantics, we have \(\mathcal{R}(\neg a) = W^2 \setminus R(\alpha)\). By definition of \(R(\alpha)\), this yields \((x_i, x_{i+1}) \in W^2 \setminus (R_\varphi(a) \cup R_\varphi(\alpha))\). The fact that \((x_i, x_{i+1}) \notin R_\varphi(\alpha)\) implies that, for any box formula \([\beta] \delta\) and state \(q \in Q_\beta\) with \(q \in \Delta_\beta(q_j, \alpha)\), \([\beta] \delta \in T(x_i)^1\) implies \([\beta_q] \delta \in T(x_{i+1})^1\). Consequently, \([\alpha_{q_{i+1}}] \theta \in T(x_{i+1})^1\).

It thus holds that \([\alpha_{q_{m}}] \theta \in T(x)^1\). Since \(q_m \in F\), by condition (H4) it follows that \(\theta \in T(x)^1\) as required.

This finishes the proof of Claim 2. By condition (T1), we have \(\phi \in \mathcal{T}^1\). Thus, it follows directly from Claim 2 that \(M\) is a model of \(\phi\).

## 5 Büchi Automata for Hintikka-trees

In this section, we show that it is possible to construct, for every \(\text{APDL}^{(\sim)}\)-formula \(\phi\), a Büchi tree automaton \(B_\phi\) that accepts exactly the Hintikka-trees for \(\phi\). By Proposition 1, since the size of \(B_\phi\) is at most exponential in the length of \(\phi\), and since the emptiness of Büchi-tree automata can be verified in quadratic time [20], this yields an \(\text{ExpTime}\) decision procedure for the satisfiability of \(\text{APDL}^{(\sim)}\)-formulas. We start with introducing Büchi tree automata.

**Definition 11 (Büchi Tree Automaton).** A Büchi tree automaton \(B\) is a quintuple \((Q, M, I, \Delta, F)\), where
- $Q$ is a finite set of states,
- $M$ is a finite alphabet,
- $I \subseteq Q$ is the set of initial states,
- $\Delta \subseteq Q \times M \times Q$ is the transition relation, and
- $F \subseteq Q$ is the set of accepting states.

Let $M$ be a set of labels, and $T$ a $k$-ary $M$-tree. Then, a run of $B$ on $T$ is a $k$-ary $M$-tree $r$ such that

1. $r(x) \in I$, and
2. $(r(x), T(x), r(x_1), \ldots, r(x_k)) \in \Delta$ for all nodes $x \in [k]^*$.

Let $\gamma \in [k]^\omega$ be a path. The set $inf_r(\gamma)$ contains the states in $Q$ that occur infinitely often in run $r$ along branch $\gamma$. A run $r$ of $B$ on $T$ is accepting if, for each path $\gamma \in [k]^\omega$, we have $inf_r(\gamma) \cap F \neq \emptyset$. The language accepted by $B$ is the set $\mathcal{L}(B) = \{ T \mid \text{there is an accepting run of } B \text{ on } T \}$.

Given a Büchi automaton $B$, the problem whether its language is empty, i.e., whether it holds that $\mathcal{L}(B) = \emptyset$, is called the emptiness problem. This problem is solvable in time quadratic in the size of the automaton [20].

We now give the translation of APDL metr-formulas $\phi$ into Büchi-automata $B_\phi$. To simplify notation, we write $P_\Psi(\phi)$ to denote the set of sets $\{ \{ \alpha \psi, [\beta] \theta \} \mid \alpha \psi, [\beta] \theta \in cl(\phi) \}$. We first introduce our automata formally and then explain the intuition.

**Definition 12.** Let $\phi$ be an APDL metr-formula with $cl(\phi)$ containing $k$ diamond formulas. The Büchi tree automaton $B_\phi = (Q, \Lambda_\phi, I, \Delta, F)$ is defined as follows:

- $Q$ contains those triples

\[
((\Psi, \pi, \ell), P, d) \in \Lambda_\phi \times 2^{P_\Psi(\phi)} \times \{ \emptyset, \uparrow \}
\]

that satisfy the following conditions:

1. If $\{ \alpha \psi, [\beta] \theta \} \subseteq \Psi$, then $\{ \alpha \psi, [\beta] \theta \} \in P$.
2. If $\{ \alpha \psi, [\beta] \theta \} \in P, \pi \in \Pi^\omega$, $q_0 \in \Delta_\alpha(q_{00}, \pi)$, $q_\beta \in \Delta_\beta(q_{00}, \pi)$, and $[\alpha_\psi, [\beta] \theta] \notin \Psi$, then $[\beta q_\beta] \not\in \Psi$.

- $I := \{ ((\Psi, \pi, \ell), P, d) \in Q \mid \phi \in \Psi, \text{ and } d = \emptyset \}$.

- $(\lambda_0, P_0, d_0), (\Psi, \pi, \ell), (\lambda_1, P_1, d_1), \ldots, (\lambda_k, P_k, d_k)) \in \Delta$ if and only if, for each $i \in [k]$, the following holds:

1. $\lambda_0 = (\Psi, \pi, \ell)$,
2. $P_0 = P$,
3. the tuple $(\lambda_0, \ldots, \lambda_k)$ is matching,
4. $d_i = \left\{ \begin{array}{l l} \uparrow & \text{if } d_z = \emptyset, \ell_i \neq \emptyset \text{ and } \ell_i \in \Psi \\ \emptyset & \text{if } d_0 = \uparrow, \ell_0 = i, \text{ and } \ell_i \neq \emptyset \end{array} \right.$

- The set $F$ of accepting states is $F := \{ (\lambda, P, d) \in Q \mid d = \emptyset \}$.
While it is not hard to see how the set of initial states enforces (T1) of Hintikka-trees and how the transition relation enforces (T2). Conditions (T3) and (T4) are more challenging. In the following, we discuss them in detail.

Condition (T3) is enforced with the help of the third component of states, which may take the values “⇒” and “↑”. Intuitively, the fourth point in the definition of ∆ ensures that, whenever the satisfaction of a diamond is delayed in a node x and r is a run, then r assigns states with third component ↑ to all nodes on the path that “tracks” the diamond delay. Note that, for this purpose, the definition of ∆ refers to the third component of Λk-tuples, which is “controlled” by (M1) in the appropriate way. All nodes that do not appear on delayed diamond paths are labeled with ∅. Then, the set of accepting states ensures that there is no path that, from some point on, is constantly labeled with ↑. Thus, we enforce that no diamonds are delayed infinitely in trees accepted by our automata. i.e. no starvation occurs.

There is one special case that should be mentioned. Assume that a node x contains a diamond e_j = ⟨α⟩ψ that is not satisfied “within this node” (Condition 1 of (M4) does not apply). Then there is a potential starvation path for e_j that starts at x and goes through the node xi; (M4) “advances” the automaton α to αq, and ensures that e_j = ⟨αq⟩ψ ∈ T(xi) and that \( T(xi)^j = j \). Now suppose that T(xi)^j contains another diamond e_k = ⟨β⟩θ with e_j ≠ e_k. If e_k is not satisfied within xi, there is a potential starvation path for e_k starting at xi and going through xik. Since the starvation path for e_j and the starvation path for e_k are for different diamonds, we must be careful to separate them—failure in doing this would result in some starvation-free Hintikka-trees to be rejected. Thus, the definition of ∆ ensures that runs label xik with ∅, and the constant ↑-labeling of the starvation path for e_k is delayed by one node: it starts only at the successor of xik on the starvation path for e_k.

Now for Condition (T4). In contrast to Conditions (T1) and (T2), this condition has a global flavor in the sense that it does not only concern a node and its successors. Thus, we need to employ a special technique to enforce that (T4) is satisfied: we use the second component of states as a “bookkeeping component” that allows to propagate global information. More precisely, Point (1) of the definition of Q and Point (1) of the definition of ∆ ensure that, whenever two boxes appear in a Hintikka-set labeling a node x in a Hintikka-tree T, then this joint occurrence is recorded in the second component of the state that runs assign to x. Via the definition of the transition relation (second point), we further ensure that all states appearing in a run share the same second component. Thus, we may use Point (2) of the definition of Q and Point (1) of the definition of ∆ to ensure that any node y satisfies the property stated by Condition (T4).

The following proposition shows that the Büchi tree automaton B_φ indeed accepts precisely the Hintikka-trees for APDL(¬)-formula φ.

**Proposition 2.** Let φ be an APDL(¬)-formula and T a k-ary Λφ-tree. Then T is a Hintikka-tree for φ iff \( T \in \mathcal{L}(B_\phi) \).
Proof. Let \( \phi \) be a APDL\(^{(-)}\)-formula and \( k \) the number of diamond formulas in \( c(\phi) \).

\( \Rightarrow \). Suppose \( T \) is a Hintikka-tree for \( \phi \). In the following, we show that \( T \in \mathcal{L}(B_\phi) \), i.e. that there is an accepting run of the Büchi automaton \( B_\phi \) on \( T \). Set

\[ P_2(T) := \{ [x] \psi \in [\beta] \theta \mid \text{there is an } x \in [k]^* \text{ such that } [x] \psi \in [\beta] \theta \in T(x)^1 \}. \]

Inductively define a \( A_\phi \times \{ P_2(T) \} \times \{ \emptyset, \uparrow \} \)-tree \( r \) as follows:

- set \( r(\varepsilon) := (T(\varepsilon), P_2(T), \emptyset) \);
- let \( x \in [k]^* \) such that \( r(x) = (T(x), P_2(T), d_x) \) is already defined, and let \( i \in [k] \). Set \( r(xi) := (T(xi), P_2(T), d_{xi}) \) where

\[ d_{xi} := \begin{cases} 
\uparrow & \text{if } d_x = \emptyset, \text{ } T(x)i \neq 0, \text{ and } \varepsilon_i \in T(x)^1 \\
\uparrow & \text{if } d_x = \uparrow, \text{ } T(x)i = i, \text{ and } T(x)i \neq 0 \\
\emptyset & \text{otherwise}
\end{cases} \]

We claim that the tree \( r \) is an accepting run of \( B_\phi \) on \( T \). We start with showing that \( r \) is a \( Q \)-tree. To this end, let \( x \in [k]^* \) be a node. We have to show that \( r(x) \in Q \). Since \( T \) is a Hintikka-tree for \( \phi \), we have \( T(x) \in A_\phi \). Moreover \( P_2(T) \subseteq P_\alpha \), and thus \( r(x) = (T(x), P_2(T), d_x) \in A_\phi \times 2^{P_\alpha (\psi)} \times \{ \emptyset, \uparrow \} \). We still have to show that \( r(x) \) satisfies Properties (1) and (2) of the definition of \( Q \).

1. Holds by definition of \( P_2(T) \) and since \( r(x)^1 = T(x) \);
2. Suppose that \( \{ [x] \psi \in [\beta] \theta \in P_2(T) \}, \text{ and } [x] \psi \not\in T(x)^1 \) where \( q_x \in \Delta_0(q_0, \pi) \) for some program literal \( \pi \in \Pi^{(-)} \). Let \( q_x' \in \Delta_3(q_x, \pi) \). By definition of \( P_2(T) \), there is a node \( y \in [k]^* \) with \( \{ [x] \psi \in [\beta] \theta \} \subseteq T(y)^1 \). Then, the Condition (T4) yields \( [\beta] \theta \in T(x)^1 \).

Thus, \( r \) is a \( Q \)-tree. To show that \( r \) is a run of \( B_\phi \) on \( T \), it remains to verify the two conditions from Definition 11.

- \( r(\varepsilon) \in I \). Condition (T1) yields \( \phi \in T(\varepsilon)^1 \). Hence, \( r(\varepsilon) = (T(\varepsilon), P_2(T), \emptyset) \in I \).

- \( r(x), T(x), r(x) \ldots, r(xk) \in \Delta \) for all nodes \( x \in [k]^* \). Let \( r(x) = (T(x), P_2(T), d_x) \).

According to the definition of \( \Delta \), there are four conditions that need to be checked. Conditions 1 and 2 are trivial by definition of \( r \). By (T2), the tuple \( (T(x), T(x1), \ldots, T(xk)) \) is matching. Thus, Condition 3 is satisfied. Finally, using the definition of \( r \) it is easily checked that Condition 4 is also satisfied.

It remains to show that the run \( r \) is accepting. Suppose that it is not, i.e. that there is a path \( \gamma = \gamma_1 \gamma_2 \ldots \in [k]^* \) such that \( in_f(\gamma) \cap F = \emptyset \). By definition of \( F \), the set \( in_f(\gamma) \) contains only states \( q \) with third component \( \uparrow \). Consequently, there is a position \( p \) in the sequence \( \gamma \) such that, for all \( m \geq p \), the third component of \( r(\gamma[m]) \) is \( \uparrow \). Let \( p \) be the minimal such position. By definition of \( r(\varepsilon) \), we know that \( p > 0 \). Minimality of \( p \) thus yields that the third component
of \( r(\gamma[p - 1]) \) is \( \varnothing \). Together with the fact that the third component of \( r(\gamma[p]) \) is \( \uparrow \) and by definition of \( r \) (the third component), this implies that the diamond formula \( \epsilon_\ell \) with \( \ell = \gamma_p \) is in \( T(\gamma[p - 1]) \). We show that the diamond \( \epsilon_\ell \) is starving in \( \gamma[p - 1] \), in this way obtaining a contradiction to the fact that \( T \) is a Hintikka-tree satisfying (T3).

To show that \( \epsilon_\ell \) is starving in \( \gamma[p - 1] \): we show that the path \( \gamma_p \gamma_{p+1} \cdots \) satisfies Properties 1 and 2 from Definition 9:

- \( \ell = \gamma_p \): Satisfied by choice of \( \ell \).
- for \( m \geq \ell \) \( T(\gamma[m]) \uparrow = \gamma_{m+1} \). Fix an \( m \geq p \). Since the third component of \( r(\gamma[m]) \) and of \( r(\gamma[m + 1]) \) is \( \uparrow \), we obtain \( T(\gamma[m]) \uparrow = \gamma_{m+1} \) by definition of \( r \) (third component).

“\( \Rightarrow \)”: Suppose \( T \in \mathcal{L}(B^\phi) \), i.e., there is an accepting run \( r \) of \( B^\phi \) on \( T \). We show that \( T \) is a Hintikka tree for \( \phi \). By definition of the \( Q \) and \( \Delta \) components of \( B^\phi \), \( T \) is a \( k \)-ary \( A^\phi \)-tree. Let \( x, y \in [k]^e \) be a nodes in \( T \) such that \( r(x) = (T_x, P_y, d'_x) \) and \( r(y) = (T'_y, P_y, d'_y) \). In the following, we check that \( T \) fulfills the conditions for Hintikka trees (T1) to (T4).

(T1) Let \( r(\varepsilon) = (T(\varepsilon), P, d') \). By definition of a run, it holds that \( r(\varepsilon) \in I \). Thus, \( \phi \in T(\varepsilon)^1 \) by definition of \( I \).

(T2) By definition of a run, \( r(x), T(x), r(x1), \ldots, r(xk) \in \Delta \). Thus, it follows by definition of \( \Delta \) that the tuple \( (T(x), T(x1), \ldots, T(xk)) \) is matching.

(T3) Assume, to the contrary of what is to be shown, that there is a diamond formula \( \epsilon_\ell \in T(x)^1 \) that is starving in \( x \). Then there exists a path \( \gamma = \gamma_1 \gamma_2 \cdots \in [k]^e \) satisfying Conditions 1 and 2 from Definition 9. We show that the third component of \( r(x \gamma[i]) \) is \( \uparrow \) for all \( i \geq 2 \). This is done by induction on \( i \).

- \( i = 2 \). We distinguish two cases:
  * The third component of \( r(x \gamma_1) \) is \( \varnothing \).
    By Condition 1 of starvation, \( \gamma_1 = \ell \). Thus, we have \( \epsilon_{\gamma_1} \in T(x)^1 \).
    Condition 2 of starvation yields \( T(x \gamma_1)^1 = \gamma_2 \). Since we have already proved (T2), we may use (M1) to derive \( \epsilon_{\gamma_2} \in T(x \gamma_1)^1 \). Together with the fact that the third component of \( r(x \gamma_1) \) is \( \varnothing \), this yields that the third component of \( r(x \gamma_1 \gamma_2) \) is \( \uparrow \) by Condition 4 of the definition of \( \Delta \).
  * The third component of \( r(x \gamma_1) \) is \( \uparrow \).
    By Condition 2 of starvation, \( T(x \gamma_1) = \gamma_2 \neq 0 \). Thus, the third component of \( r(x \gamma_1 \gamma_2) \) is \( \uparrow \) by Condition 4 of the definition of \( \Delta \).

- \( i > 2 \). By induction hypothesis, the third component of \( r(x \gamma[i - 1]) \) is \( \uparrow \). Due to Condition 2 of starvation, we have \( T(x \gamma[i - 1])^3 = \gamma_i \) and \( T(x \gamma[i])^3 = \gamma_{i+1} \neq 0 \). Thus, the definition of \( \Delta \) implies that the third component of \( r(x \gamma[i]) \) is \( \uparrow \).
(T4) Suppose \([\alpha]\psi, [\beta]\theta \in T(x)^1\). Let \(\pi \in \Pi^{(\sim)}_1\) be a program literal and \(q'_\alpha \in Q_\alpha, q'_\beta \in Q_\beta\) states such that \(q'_\alpha \in \Delta_\alpha(q_\alpha, \pi)\), and \(q'_\beta \in \Delta_\beta(q_\beta, \pi)\). Suppose
\([\alpha]_z\psi \neq T(y)^1\) for some node \(y \in [k]^\ast\). By Condition (1) in the definition of \(Q\), we have \([\alpha]z\psi, [\beta]\theta \in P_x\). By the definition of \(\Delta\) this implies \([\alpha]_z\psi, [\beta]\theta \in P_x\). Thus, Condition (2) in the definition of \(Q\) yields
\([\alpha]_z\psi \neq T(y)^1\).

Putting together Propositions 1 and 2, it is now easy to establish decidability and ExPTIME-complexity of \(\text{APDL}^{(\sim)}\) and thus also of \(\text{PDL}^{(\sim)}\).

**Theorem 1.** Satisfiability and validity of \(\text{PDL}^{(\sim)}\)-formulas are \(\text{ExPTIME}\)-complete.

**Proof.** From Propositions 1 and 2, it follows that an \(\text{APDL}^{(\sim)}\)-formula \(\phi\) is satisfiable if and only if \(L(B_\phi) \neq \emptyset\). The emptiness problem for Büchi automata is decidable in time quadratic in the size of the automaton [20]. To show that \(\text{APDL}^{(\sim)}\)-formula satisfiability is in ExPTIME, it thus remains to show that the size of \(B_\phi = (Q, A_\phi, I, \Delta, F)\) is at most exponential in \(\phi\).

Let \(n\) be the length of \(\phi\). Since the cardinality of \(cl(\phi)\) is polynomial in \(n\), the cardinality of \(H_\phi\) (the set of Hintikka-sets for \(\phi\)) is at most exponential in \(n\). Thus, it is readily checked that the same holds for \(A_\phi\) and \(Q\). The exponential upper bound on the cardinalities of \(I\) and \(F\) is trivial. It remains to determine the size of \(\Delta\); since the size of \(Q\) is exponential in \(n\) and the out-degree of trees accepted by automata is polynomial in \(n\), we obtain an exponential bound.

Thus, \(\text{APDL}^{(\sim)}\)-formula satisfiability and hence also \(\text{PDL}^{(\sim)}\)-formula satisfiability are in ExPTIME. For the lower bound, it suffices to recall that PDL-formula satisfiability is already ExPTIME-hard [3].

6 Conclusion

This paper introduces the propositional dynamic logic \(\text{PDL}^{(\sim)}\), which extends standard PDL with negation of atomic programs. We were able to show that this logic extends PDL in an interesting and useful way, yet retaining its appealing computational properties. There are some natural directions for future work. For example, it should be simple to further extend \(\text{PDL}^{(\sim)}\) with the converse operator without destroying the ExPTIME upper bound. It would be more interesting, however, to investigate the interplay between (full) negation and PDL’s program operators in some more detail. For example, to the best of our knowledge it is unknown whether the fragment of PDL that has only the program operators \(\sim\) and \(\cdot\) is decidable.

References