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A NExpTime-Complete Description Logic Strictly Contained in C^2

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Abstract

We examine the complexity and expressivity of the combination of the Description Logic $\mathcal{ALCQI}$ with a terminological formalism based on cardinality restrictions on concepts. This combination can naturally be embedded into C^2, the two variable fragment of predicate logic with counting quantifiers. We prove that $\mathcal{ALCQI}$ has the same complexity as C^2 but does not reach its expressive power.

1 Introduction

Description Logic (DL) systems can be used in knowledge based systems to represent and reason about taxonomical knowledge of problem domain in a semantically well-defined manner [WS92]. These systems usually consist at least of the following three components: (1) a DL that allows the definition of complex concepts (unary predicates) and roles (binary relations) to be built from atomic ones by the application of a given set of constructors; for example the following concept describes those fathers having at least two daughters:

$$\texttt{Parent} \sqcap \texttt{Male} \sqcap (\geq\ 2\ \texttt{hasChild}\ \texttt{Female})$$

(2) a terminological component (TBox) that allows for the organisation of defined concepts and roles. The TBox formalisms studied in the DL context range from weak ones allowing only for the introduction of abbreviations for complex concepts, over TBoxes capable of expressing various forms of axioms, to cardinality restrictions that can express restrictions on the number of elements a concept may have. Consider the following three expressions:

$$\texttt{BusyParent} = \texttt{Parent} \sqcap (\geq\ 2\ \texttt{hasChild}\ \texttt{Toddler})$$
$$\texttt{Male} \sqcup \texttt{Female} = \texttt{Person} \sqcap (=\ 2\ \texttt{hasChild}^{-1}\ \texttt{Parent})$$
$$(\leq\ 2\ \texttt{Person} \sqcap (\leq\ 0\ \texttt{hasChild}^{-1}\ \texttt{Parent}))$$

The first introduces `BusyParent` as an abbreviation for a more complex concept, the second is an axiom stating that `Male` and `Female` are exactly those persons having two parents, the third is a cardinality restriction expressing that in the domain of discourse there are at most two earliest ancestors.

(3) a reasoning service that performs task like subsumption or consistency test for the knowledge stored in the TBox and ABox.There exist sound and complete algorithms for reasoning in a large number of DLs and different TBox and ABox formalisms that meet the known worst-case complexity these problems (see [DLNN97] for an overview). Generally, reasoning for DLs can be performed in four different ways:

- by structural comparison of syntactical normal forms of concepts [BPS94].
- by tableaux algorithms that are hand-tailored to suit the necessities of the operators used to form the DL and the TBox formalism. Initially, these algorithm were designed to decide inference problems only for the DL without taking into account TBoxes, but it is possible to generalise these algorithms to deal with different TBox formalisms. Most DLs handled this way are at most PSPACE complete but additional complexity may arise from the TBox. The complexity of the tableaux approach usually meets the known worst-case complexity of the problem [SSS91, DLNN97].
- by perceiving the DL as a (fragment of a) modal logic such as PDL [GL96]; many DLs handled in this manner are already EXPTIME-complete, but axioms can be "internalised" [Baa91] into the concepts and hence do not increase the complexity.
- by translation of the problem into a fragment or first order other logic with a decidable decision problem [Bor96, OSH96].

From the fragments of predicate logic that are studied in the second context, only C^2, the two variable fragment of first order predicate logic augmented with counting quantifiers, is capable of dealing with counting expressions that are commonly used in DLs; similarly it is able to express cardinality restrictions. Another thing that comes "for free" when translating DLs into first order logic is the ability to deal with inverse roles.

Combining all these construct into a single DL, one obtains the DL $\mathcal{ALCQI}$—the well-known DL $\mathcal{ALC}$ [SSS91] augmented by qualifying number restrictions ($\mathcal{Q}$) and inverse roles ($\mathcal{I}$). In this work we study both complexity and expressivity of $\mathcal{ALCQI}$ combined with TBoxes based on cardinality restrictions.

Regarding the complexity we show that $\mathcal{ALCQI}$ with cardinality restrictions already is NEXPTIME-hard and hence has the same complexity as C^2 [PST97][1]. To our knowledge this is the first DL for which NEXPTIME-completeness has formally been proved. Since $\mathcal{ALCQI}$ with TBoxes consisting of axioms is still in EXPTIME, this indicates that cardinality restrictions are algorithmically hard to handle.

Despite the fact that both $\mathcal{ALCQI}$ and C^2 have the same worst-case complexity we show that $\mathcal{ALCQI}$ lacks some of the expressive power of C^2. Properties of binary predicates (e.g. reflexivity) that are easily expressible in C^2 can

[1]The NEXPTIME-result is valid only if we assume unary coding of numbers in the counting quantifiers. This is the standard assumption made by most results concerning the complexity of DLs.

not be expressed in $\mathcal{ALCQI}$. We establish our result by giving an Ehrenfeucht-Fraïssé game that exactly captures the expressivity of $\mathcal{ALCQI}$ with cardinality restrictions. This is the first time in the area of DL that a game-theoretic characterisation is used to prove an expressivity result involving TBox formalisms. The game as it is presented here is not only applicable to $\mathcal{ALCQI}$ with cardinality restrictions; straightforward modifications make it applicable to both $\mathcal{ALCQ}$ as well as to weaker TBox formalisms like terminological axioms.

In [Bor96] a DL is presented that has the same expressivity as C^2. This expressivity result is one of the main results of that paper and the DL combines a large number of constructs; the paper does not study the computational complexity of the presented logics. Our motivation is of a different nature: We study the complexity and expressivity of a DL consisting of a only minimal set of constructs that seem sensible when a reduction of that DL to C^2 is to be considered.

2 The Logic $\mathcal{ALCQI}$

In this section we define syntax and semantics of the DL $\mathcal{ALCQI}$.

Definition 2.1 *A* signature *is a pair $\tau = (N_C, N_R)$ where N_C is a finite set of* concepts names *and N_R is a finite set of* role names. Concepts *in $\mathcal{ALCQI}$ are built inductively from these using the following rules:*

All $A \in N_C$ are concepts. If C, C_1, and C_2 are concepts then also

$$\neg C, \ C_1 \sqcap C_2, \ \text{and} \ (\geq \ n \ S \ C)$$

with $n \in \mathbb{N}$, and $S = R$ or $S = R^{-1}$ for some $R \in N_R$ are concepts. We define $C_1 \sqcup C_2$ as an abbreviation for $\neg(\neg C_1 \sqcap \neg C_2)$ and $(\leq \ n \ S \ C)$ as an abbreviation for $\neg(\geq \ (n+1) \ S \ C)$. We also use $(= \ n \ S \ C)$ as an abbreviation for $(\leq \ n \ S \ C) \sqcap (\geq \ n \ S \ C)$.

A cardinality restriction *of $\mathcal{ALCQI}$ is an expression of the form $(\geq n \ C)$ or $(\leq n \ C)$ where C is a concept and $n \in \mathbb{N}$; a* TBox *T of $\mathcal{ALCQI}$ is a finite set of cardinality restrictions.*

The semantics of a concept is defined relative to an interpretation *$\mathcal{I}$, which is a tuple consisting of a domain $\Delta^\mathcal{I}$ and a valuation $(\cdot^\mathcal{I})$ which maps each concept name A to a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$ and each role name R to a subset $R^\mathcal{I}$ of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$. This valuation is extend to arbitrary concept definitions inductively using the following rules, where $\sharp M$ denotes the cardinality of a set M:*

$$\begin{gathered}(\neg C)^\mathcal{I} := \Delta^\mathcal{I} \setminus C^\mathcal{I}, \qquad (C_1 \sqcap C_2)^\mathcal{I} := C_1^\mathcal{I} \cap C_2^\mathcal{I}, \\ (\geq n \ R \ C)^\mathcal{I} := \{a \in \Delta^\mathcal{I} \mid \sharp\{b \in \Delta^\mathcal{I} \mid (a,b) \in R^\mathcal{I} \wedge b \in C^\mathcal{I}\} \geq n\}, \\ (\geq n \ R^{-1} \ C)^\mathcal{I} := \{a \in \Delta^\mathcal{I} \mid \sharp\{b \in \Delta^\mathcal{I} \mid (b,a) \in R^\mathcal{I} \wedge b \in C^\mathcal{I}\} \geq n\}.\end{gathered}$$

An interpretation $\mathcal{I}$ satisfies *a cardinality restriction $(\geq n \ C)$ iff $\sharp(C^\mathcal{I}) \geq n$ and it satisfies $(\leq n \ C)$ iff $\sharp(C^\mathcal{I}) \leq n$. It satisfies a TBox T iff it satisfies all cardinality restrictions in T; in this case, $\mathcal{I}$ is called a* model *of T and we will denote this fact by $\mathcal{I} \models T$. A TBox that has a model is called* consistent.

$$
\begin{array}{lll}
\Psi_x(A) & := & Ax \qquad\qquad \text{for } A \in N_C \\
\Psi_x(\neg C) & := & \neg \Psi_x(C) \\
\Psi_x(C_1 \sqcap C_2) & := & \Psi_x(C_1) \wedge \Psi_x(C_2) \\
\Psi_x(\geq\ n\ R\ C) & := & \exists^{\geq n} y.(Rxy \wedge \Psi_y(C)) \\
\Psi_x(\geq\ n\ R^{-1}\ C) & := & \exists^{\geq n} y.(Ryx \wedge \Psi_y(C)) \\
\Psi(\bowtie\ n\ C) & := & \exists^{\bowtie n} x.\Psi_x(C) \\
\Psi(T) & := & \bigwedge\{\Psi(\bowtie\ n\ C) \mid (\bowtie\ n\ C) \in T\}
\end{array}
$$

Figure 1: The translation from $\mathcal{ALCQI}$ into C^2

With $\mathcal{ALCQ}$ we denote the fragment of $\mathcal{ALCQI}$, that does not contain any inverse roles R^{-1}.

TBox formalisms using cardinality restrictions have first been studied in [BBH96] for the DL $\mathcal{ALCQ}$. Obviously, it is possible to express that two concepts C, D have the same extension in an interpretation by the cardinality restriction $(\leq\ 0\ (C \sqcap \neg D) \sqcup (\neg C \sqcap D))$. This implies that cardinality restrictions can express terminological axioms of the form $C = D$ that are the most expressive TBox formalisms usually studied in the DL context [GL96]. The standard inference service for DL systems is satisfiability of a concept C with respect to a TBox T (i.e., is there an interpretation $\mathcal{I}$ such that $\mathcal{I} \models T$ and $C^{\mathcal{I}} \neq \emptyset$). For a TBox formalism based on cardinality restrictions this is easily reduced to TBox consistency, because obviously C is satisfiable with respect to T iff $T \cup \{(\geq\ 1\ C)\}$ is a consistent TBox. To this the reason we will restrict our attention to TBox consistency.

Until now there does not exists a tableaux based decision procedure for $\mathcal{ALCQI}$ TBox consistency. Nevertheless this problem can be decided with the help of a well-known translation of $\mathcal{ALCQI}$-TBoxes to C^2 [Bor96] given in Fig. 1 where we use $\bowtie$ as a placeholder for both $\leq$ and $\geq$. The logic C^2 is fragment of predicate logic that allows only two variables but is enriched with counting quantifiers of the form $\exists^{\geq l}$. The translation Ψ yields a satisfiable sentence of C^2 if and only if the translated TBox is consistent. Since the translation from $\mathcal{ALCQI}$ to C^2 can be performed in linear time, the NExpTime upper bound [PST97] for satisfiability of C^2 directly carries over to $\mathcal{ALCQI}$-TBox consistency:

Lemma 2.2 *Consistency of an $\mathcal{ALCQI}$-TBox T can be decided in* NExpTime.

Please note that the NExpTime-completeness result from [PST97] is only valid if we assume unary coding of numbers to in the input; this implies that a large number like 1000 may not be stored in logarithmic space in some k-ary representation but consumes 1000 units of storage. This is the standard assumption made by most results concerning the complexity of DLs. We will come back to this issue later in this paper.

3 $\mathcal{ALCQI}$ is NExpTime-complete

To show that NExpTime is also the lower bound for the complexity of TBox consistency we use a bounded version of the domino problem. Domino problems [Wan63, Ber66] have successfully been employed to establish undecidability and complexity results for various description and modal logics [Spa93, BS99].

3.1 Domino Systems

Definition 3.1 *For an $n \in \mathbb{N}$ let $\mathbb{Z}_n$ denote the set $\{0, \ldots, n-1\}$ and $\oplus_n$ denote the addition modulo n. A* domino system *is a triple $\mathcal{D} = (D, H, V)$, where D is a finite set (of tiles) and $H, V \subseteq D \times D$ are relations expressing horizontal and vertical compatibility constraints between the tiles. For $s, t \in \mathbb{N}$ let $U(s,t)$ be the torus $\mathbb{Z}_s \times \mathbb{Z}_t$ and $w = w_0, \ldots, w_{n-1}$ be an n-tuple of tiles (with $n \leq s$). We say that $\mathcal{D}$* tiles $U(s,t)$ with initial condition w *iff there exists a mapping $\tau : U(s,t) \to D$ such that for all $(x,y) \in U(s,t)$:*

- *If $\tau(x,y) = d$ and $\tau(x \oplus_s 1, y) = d'$ then $(d,d') \in H$;*
- *if $\tau(x,y) = d$ and $\tau(x, y \oplus_t 1) = d'$ then $(d,d') \in V$;*
- *$\tau(i,0) = w_i$ for $0 \leq i < n$,*

Bounded domino systems as capable of expressing the computational behaviour of restricted, so called *simple*, Turing Machines (TM). This restriction is non-essential in the following sense: Every language accepted in time $T(n)$ and space $S(n)$ by some one-tape TM is accepted within the same time and space bounds by a simple TM, as long as $S(n), T(n) \geq 2n$.

Theorem 3.2 ([BGG97], Theorem 6.1.2) *Let M be a simple TM with input alphabet Σ. Then there exists a domino system $\mathcal{D} = (D, H, V)$ and a linear time reduction which takes any input $x \in \Sigma^*$ to a word $w \in D^*$ with $|x| = |w|$ such that*

- *If M accepts x in time t_0 with space s_0, then $\mathcal{D}$ tiles $U(s,t)$ with initial condition w for all $s \geq s_0 + 2, t \geq t_0 + 2$;*
- *if M does not accept x, then $\mathcal{D}$ does not tile $U(s,t)$ with initial condition w for any $s, t \geq 2$.*

A simple corollary of this theorem will be used to prove that TBox consistency for $\mathcal{ALCQI}$ is NExpTime-hard:

Corollary 3.3 *Let M be a (w.l.o.g. simple) non-deterministic TM with time- (and hence space-) bound 2^{n^d} (d constant) deciding an arbitrary* NExpTime*-complete language $\mathcal{L}(M)$ over the alphabet Σ. Let $\mathcal{D}$ and trans be the according domino system and reduction from Theorem 3.2. The following is a* NExpTime*-hard problem:*

> *Given an initial condition $w = w_0, \ldots, w_n - 1$ of length n. Does $\mathcal{D}$ tile $U(2^{n^d+1}, 2^{n^d+1})$ with initial condition w?*

Proof. The function *trans* is a linear reduction from $\mathcal{L}(M)$ to the stated problem: For a $v \in \Sigma^*$ with $|v| = n$ it holds that $v \in \mathcal{L}(M)$ iff M accepts v in time and space $2^{|v|^d}$ iff $\mathcal{D}$ tiles $U(2^{n^d+1}, 2^{n^d+1})$ with initial condition $trans(v)$ ∎

3.2 Defining a Torus of Exponential Size

Just as defining infinite grids is the key problem in proving undecidability by reduction from unbounded domino problems, defining a torus of exponential size is the key to obtaining a NExpTime-completeness proof by reduction from bounded domino problems.

To be able to apply Corollary 3.3 to TBox consistency for $\mathcal{ALCQI}$ we have to be able to characterise the torus $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ with a TBox of polynomial size. To characterize this torus we will use the $2 \times n$ atomic concepts $X_0, \ldots, X_{n-1}$ and $Y_0, \ldots, Y_{n-1}$; these will be used to code the horizontal and vertical position of an element in the torus as follows:

For a interpretation $\mathcal{I}$ and an element $a \in \Delta^{\mathcal{I}}$ we define $pos(a)$ by

$$pos(a) := (xpos(a), ypos(a)) := \left(\sum_{i=0}^{n-1} x_i \cdot 2^i, \quad \sum_{i=0}^{n-1} y_i \cdot 2^i \right),$$

where

$$x_i = \begin{cases} 0, & \text{if } a \notin X_i^{\mathcal{I}} \\ 1, & \text{otherwise} \end{cases} \qquad y_i = \begin{cases} 0, & \text{if } a \notin Y_i^{\mathcal{I}} \\ 1, & \text{otherwise} \end{cases} .$$

We use a well-known characterisation of binary addition [BGG97] to relate the positions of the elements in the torus:

Lemma 3.4 *Let x, x' be a natural numbers with binary representations*

$$x = \sum_{i=0}^{n-1} x_i \cdot 2^i \quad \text{and} \quad x' = \sum_{i=0}^{n-1} x'_i \cdot 2^i .$$

Then it holds that:

$$\begin{aligned} x' \equiv x + 1 \pmod{2^n} \quad \textit{iff} \quad & \bigwedge_{k=0}^{n-1} (\bigwedge_{j=0}^{k-1} x_j = 1) \rightarrow (x_k = 1 \leftrightarrow x'_k = 0) \\ \wedge & \bigwedge_{k=0}^{n-1} (\bigvee_{j=0}^{k-1} x_j = 0) \rightarrow (x_k = x'_k) \end{aligned}$$

where the empty conjunction and disjunction are interpreted as true and false respectively.

We define the TBox T_n to consist of the following cardinality restrictions:

$$\begin{gathered}(\forall\ (\geq\ 1\ succ_1\ \top)),\quad (\forall\ (\geq\ 1\ succ_2\ \top)),\\ (\forall\ (=1\ succ_1^{-1}.\top)),\quad (\forall\ (=\ 1\ succ_2^{-1}.\top)),\\ (\geq 1\ C_0),\quad (\geq 1\ C_1),\quad (\leq 1\ C_1),\quad (\forall\ D_1 \sqcap D_2).\end{gathered}$$

where we use the following abbreviations:

$(\forall\ C)$ is an abbreviation for the cardinality restriction $(\leq 0\ \neg C)$. $\top$ stands for an arbitrary concept that is satisfied in all interpretations like $A \sqcup \neg A$.

The concept C_0 is satisfied by all elements a of the domain for which $pos(a) = (0,0)$ holds. C_1 is a similar concept, which is satisfied if $pos(a) = (2^n - 1, 2^n - 1)$:

$$C_0 = \prod_{k=0}^{n-1} \neg X_k \sqcap \prod_{k=0}^{n-1} \neg Y_k, \quad C_1 = \prod_{k=0}^{n-1} X_k \sqcap \prod_{k=0}^{n-1} Y_k.$$

The concept D_1 (D_2) enforces that along the role $succ_1$ ($succ_2$) the value of $xpos$ ($ypos$) increases by one while the value of $ypos$ ($xpos$) stays the same. They exactly resemble the formula from Lemma 3.4:

$$\begin{aligned} D_1 = & \prod_{k=0}^{n-1} (\prod_{j=0}^{k-1} X_j) \to ((X_k \to \forall succ_1.\neg X_k) \sqcap (\neg X_k \to \forall succ_1.X_k)) \\ \sqcap & \prod_{k=0}^{n-1} (\bigsqcup_{j=0}^{k-1} \neg X_j) \to ((X_k \to \forall succ_1.X_k) \sqcap (\neg X_k \to \forall succ_1.\neg X_k)) \\ \sqcap & \prod_{k=0}^{n-1} ((Y_k \to \forall succ_1.Y_k) \sqcap (\neg Y_k \to \forall succ_1.\neg Y_k)). \end{aligned}$$

The concept D_2 is similar to D_1 where the roles $succ_1$ and $succ_2$ as well as the variables X_i and Y_i have been swapped.

The following lemma is a simple consequence of the definition of pos and Lemma 3.4.

Lemma 3.5 *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation such that $a, b \in \Delta^{\mathcal{I}}$.*

$$(a,b) \in succ_1^{\mathcal{I}} \text{ and } a \in D_1^{\mathcal{I}} \text{ implies: } \quad \begin{aligned} xpos(b) &\equiv xpos(a) + 1 \pmod{2^n} \\ ypos(b) &= ypos(a) \end{aligned}$$

$$(a,b) \in succ_2^{\mathcal{I}} \text{ and } a \in D_2^{\mathcal{I}} \text{ implies: } \quad \begin{aligned} xpos(b) &= xpos(a) \\ ypos(b) &\equiv ypos(a) + 1 \pmod{2^n} \end{aligned}$$

The TBox T_n defines a torus of exponential size in the following sense:

Lemma 3.6 *Let T_n be the TBox as introduced above. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation such that $\mathcal{I} \models T_n$. This implies*

$$(\Delta^{\mathcal{I}}, succ_1^{\mathcal{I}}, succ_2^{\mathcal{I}}) \cong (U(2^n, 2^n), S_1, S_2)$$

where $U(2^n, 2^n)$ is the torus $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ and S_1, S_2 are the first (resp. second) successor relation on the torus.

Proof. This lemma is established by showing that the function *pos* is an isomorphism from $\Delta^{\mathcal{I}}$ to $U(2^n, 2^n)$.

Firstly we will prove that *pos* is *injective*. We do this by induction on the "Manhattan distance" of the *pos*-value of the elements to then *pos*-value of the upper right corner.

For an element $a \in \Delta^{\mathcal{I}}$ we define $d(a)$ by:

$$d(a) = (2^n - 1 - \mathit{xpos}(a)) + (2^n - 1 - \mathit{ypos}(a))$$

Note that $\mathit{pos}(a) = \mathit{pos}(b)$ implies $d(a) = d(b)$. There is at most one element $a \in \Delta^{\mathcal{I}}$ such that $d(a) = 0$, otherwise $\mathcal{I} \not\models (\leq\ 1\ C_1)$. Hence there is exactly one element a such that $\mathit{pos}(a) = (2^n - 1, 2^n - 1)$. Now assume there are elements $a, b \in \Delta^{\mathcal{I}}$ such that $\mathit{pos}(a) = \mathit{pos}(b)$ and $d(a) = d(b) > 0$. Hence either $\mathit{xpos}(a) < 2^n - 1$ or $\mathit{ypos}(a) < 2^n - 1$. Without loss of generality we may assume $\mathit{xpos}(a) < 2^n - 1$ .From $\mathcal{I} \models T_n$ it follows that $a, b \in (\exists \mathit{succ}_1.\top)^{\mathcal{I}}$. Let a_1, b_1 be elements such that $(a, a_1) \in \mathit{succ}_1^{\mathcal{I}}$ and $(b, b_1) \in \mathit{succ}_1^{\mathcal{I}}$. From Lemma 3.5 it follows that

$$\begin{aligned} \mathit{xpos}(a_1) &\equiv \mathit{xpos}(b_1) \equiv \mathit{xpos}(a) + 1 \pmod{2^n} \\ \mathit{ypos}(a_1) &= \mathit{ypos}(b_1) = \mathit{ypos}(a) \end{aligned}$$

Since $d(a_1) = d(b_1) < d(a)$ we can use the induction hypothesis which yields that $a_1 = b_1$ because $\mathit{pos}(a_1) = \mathit{pos}(b_1)$. This also implies $a = b$, because $a_1 \in (= 1\ \mathit{succ}_1^{-1}.\top)^{\mathcal{I}}$ and $\{(a, a_1), (b, a_1)\} \subseteq \mathit{succ}_1^{\mathcal{I}}$. Hence *pos* is injective.

To prove that *pos* is also *surjective* we use a similar technique. This time we use the distance from the lower left corner. For each element $(x, y) \in U(2^n, 2^n)$ we define:

$$d'(x, y) = x + y$$

We will show by induction that for each $(x, y) \in U(2^n, 2^n)$ there is an element $a \in \Delta^{\mathcal{I}}$ such that $\mathit{pos}(a) = (x, y)$. $d'(x, y) = 0$ implies $x = y = 0$. Since $\mathcal{I} \models (\geq 1\ C_0)$ there is an element $a \in \Delta^{\mathcal{I}}$ such that $f(a) = (0, 0)$. Now consider $(x, y) \in U(2^n, 2^n)$ with $d'(x, y) > 0$. Without loss of generality we assume $x > 0$ (if $x = 0$ then $y > 0$ must hold). Hence $(x - 1, y) \in U(2^n, 2^n)$ and $d'(x - 1, y) < d'(x, y)$. From the induction hypothesis it follows that there is an element $a \in \Delta^{\mathcal{I}}$ such that $\mathit{pos}(a) = (x - 1, y)$. Then there must be an element a_1 such that $(a, a_1) \in \mathit{succ}_1^{\mathcal{I}}$ and from Lemma 3.5 it follows that $\mathit{pos}(a_1) = (x, y)$. Hence f is also surjective.

Finally we have to show that $(a, b) \in \mathit{succ}_i^{\mathcal{I}}$ iff $(\mathit{pos}(a), \mathit{pos}(b)) \in S_i$. This is an immediate consequence of Lemma 3.5. ∎

It is interesting to note that we need inverse roles only to guarantee that *pos* is injective. The same can be achieved by adding the cardinality restriction $(\leq (2^n \cdot 2^n)\ \top)$ to T_n, from which the injectivity of *pos* follows from its surjectivity and simple cardinality considerations. Of course the size of this cardinality restriction would only be polynomial in n if we allow binary coding of numbers. Also note that we have made explicit use of the special expressive power of cardinality restrictions by stating that in any model of T_n the extension of C_1 must have *at most* one element. This can not be expressed with a TBox consisting of axioms. Obviously the following holds:

Lemma 3.7 *The size $|T_n|$ (i.e., the number of symbols necessary to write down T_n) is quadratic in n.*

3.3 Reducing Domino Problems to TBox Consistency

Once Lemma 3.6 has been proved it is easy to reduce the bounded domino problem to TBox consistency. We use the standard reduction that, has been applied in the DL context e.g. in [BS99].

Lemma 3.8 *Let $\mathcal{D} = (D, V, H)$ be a domino system. Let $w = w_0, \ldots, w_{n-1} \in D^*$. There is a TBox $T(n, \mathcal{D}, w)$ such that:*

- *$T(n, \mathcal{D}, w)$ is consistent iff $\mathcal{D}$ tiles $U(2^n, 2^n)$ with initial condition w.*
- *$|T(n, \mathcal{D}, w)| = \mathcal{O}(n^2)$ and $T(n, \mathcal{D}, w)$ can be computed in time polynomial in n.*

Proof. We define $T(n, \mathcal{D}, w) := T_n \cup T_{\mathcal{D}} \cup T_w$, where T_n is defined as above, $T_{\mathcal{D}}$ captures the vertical and horizontal compatibility constraints of the domino systems $\mathcal{D}$, and T_w enforces the initial condition. We use an atomic concept C_d for each tile $d \in D$. $T_{\mathcal{D}}$ consists of the following cardinality restrictions:

$$(\forall \bigsqcup_{d \in D} C_d), \quad (\forall \bigsqcap_{d \in D} \bigsqcap_{d' \in D \setminus \{d\}} \neg(C_d \sqcap C_{d'})),$$
$$(\forall \bigsqcap_{d \in D} (D_d \rightarrow (\forall succ_1. \bigsqcup_{(d,d') \in H} C_{d'}))), \quad (\forall \bigsqcap_{d \in D} (D_d \rightarrow (\forall succ_2. \bigsqcup_{(d,d') \in V} C_{d'}))).$$

$\mathcal{T}_w$ consists of the cardinality restrictions

$$(\forall\ (pos = (0,0) \rightarrow C_{w_0})), \ldots, (\forall\ (pos = (n-1, 0) \rightarrow C_{w_{n-1}})$$

where $pos = (x, y)$ is a concept which is satisfied by an element a iff $pos(a) = (x, y)$.

From the definition of $T(n, \mathcal{D}, w)$ and Theorem 3.6 it follows that each model of $T(n, \mathcal{D}, w)$ immediately induces a tiling of $U(2^n, 2^n)$ and vice versa. Also, for a fixed domino system $\mathcal{D}$ the size of $T(n, \mathcal{D}, w)$ is obviously polynomial in n. ∎

The nexttheorem is an immediate consequence of Lemma 3.8 and Corollary 3.3:

Theorem 3.9 *Consistency of $\mathcal{ALCQI}$-TBoxes is* NExpTime-*hard.*

Recalling the note below Lemma 3.8, we see that the same argument also applies to $\mathcal{ALCQ}$ if we allow binary coding of numbers.

Corollary 3.10 *Consistency for $\mathcal{ALCQ}$-TBoxes is* NExpTime-*hard, if binary coding is used to represent numbers in cardinality restrictions.*

Note that for unary coding we needed both inverse roles and cardinality restrictions for the reduction. This is consistent with the fact that satisfiability for $\mathcal{ALCQI}$ concepts with respect to TBoxes consisting of terminological axioms is still in ExpTime, which can be shown by a reduction to Converse-PDL [GM99]. This shows that cardinality restrictions on concepts are an additional source of complexity.

4 Expressiveness of $\mathcal{ALCQI}$

Since reasoning for $\mathcal{ALCQI}$ has the same (worst-case) complexity as for C^2, naturally the question arises how the two logics are related concerning their expressivity. As it turns out, $\mathcal{ALCQI}$ is strictly less expressive then C^2.

4.1 A Definition of Expressiveness

There are different approaches to define the expressivity of DLs [Baa96, Bor96, AdR98], but only the one presented in [Baa96] is capable of handling TBoxes. We will use a definition that is equivalent to the one given in [Baa96] restricted to a special case. It bases the notion of expressivity on the classes of interpretations definable by a sentence (or TBox).

Definition 4.1 *Let $\tau = (N_C, N_R)$ be a finite signature. A class $\mathcal{C}$ of τ-interpretations is called* characterisable *by a logic $\mathcal{L}$, if there is a sentence $\varphi_\mathcal{C}$ over τ such that $\mathcal{C} = \{\mathcal{I} \mid \mathcal{I} \models \varphi_\mathcal{C}\}$.*

The class $\mathcal{C}$ is called projectively characterisable, *if there is a sentence $\varphi'_\mathcal{C}$ over a signature $\tau' \supseteq \tau$ such that $\mathcal{C} = \{\mathcal{I}|_\tau \mid \mathcal{I} \models \varphi'_\mathcal{C}\}$, where $\mathcal{I}|_\tau$ denotes the τ-reduct of $\mathcal{I}$.*

A logic $\mathcal{L}_1$ is called as expressive *as another logic $\mathcal{L}_2$ ($\mathcal{L}_1 \geq \mathcal{L}_2$) if for any finite signature τ any $\mathcal{L}_2$-characterisable class $\mathcal{C}$ can be projectively characterised in $\mathcal{L}_1$.*

Since C^2 is usually restricted to a relational signature with relation symbols of arity at most two, this definition is appropriate to relate the expressiveness of $\mathcal{ALCQI}$ and C^2. First of all it is worth noting that clearly $\mathcal{ALCQI}$ is strictly more expressive then $\mathcal{ALCQ}$, because $\mathcal{ALCQ}$ has the finite model property [BBH96], while the following is an $\mathcal{ALCQI}$ TBox that has no finite models:

$$\mathcal{T}_{\text{inf}} = \{(\forall\ (\geq\ 1\ R\ \top)), (\forall\ (\leq\ 1\ R^{-1}\ \top)), (\geq\ 1\ (=\ 0\ R^{-1}\ \top))\}.$$

The first cardinality restriction requires an outgoing R-edge for every element of a model and thus each R-path in the model in infinite. The second and third restriction require the existence of an R-path in the model that contains no cycle, which implies the existence of infinitely many elements in the model. Since $\mathcal{ALCQ}$ has the finite model property, the class $\mathcal{C}_{\text{inf}} := \{\mathcal{I} \mid \mathcal{I} \models \mathcal{T}_{\text{inf}}\}$, which contains only models with infinitely many elements, can not be projectively characterised by an $\mathcal{ALCQ}$-TBox.

The translation Ψ from $\mathcal{ALCQI}$-TBoxes to C^2 sentences given in Fig. 1 not only preserves satisfiability, but the translation also has exactly the same models as the initial TBox. This implies that $\mathcal{ALCQI} \leq C^2$.

4.2 A Game for $\mathcal{ALCQI}$

Usually, the separation of two logics with respect to their expressivity is a hard task and not as easily accomplished as we have just done with $\mathcal{ALCQ}$ and $\mathcal{ALCQI}$. Even for logics of very restricted expressivity, proofs of separation results may become involved and complex [Baa96] and usually require a detailed analysis of the classes of models a logic is able to characterise. A valuable tool for this analysis are Ehrenfeucht-Fraïssé games. In this section we present an Ehrenfeucht-Fraïssé game that exactly captures the expressivity of $\mathcal{ALCQI}$:

Definition 4.2 *For an $\mathcal{ALCQI}$ concept C, the* role depth *$rd(C)$ counts the maximum number of nested cardinality restrictions. Formally we define rd as follows:*

$$\begin{aligned} rd(A) &:= 0 \quad \textit{for } A \in N_C \\ rd(\neg C) &:= rd(C) \\ rd(C_1 \sqcap C_2) &:= max\{rd(C_1), rd(C_2)\} \\ rd(\geq n\ R\ C) &:= 1 + rd(C) \end{aligned}$$

The set $\mathcal{C}^n_m$ is defined to consist of exactly those $\mathcal{ALCQI}$ concepts that have a role depth of at most m, and in which the numbers appearing in number restrictions are bounded by n; the set $\mathcal{L}^n_m$ is defined to consist of all $\mathcal{ALCQI}$-TBoxes T that contain only cardinality restriction of the form $(\bowtie\ k\ C)$ with $k \leq n$ and $C \in \mathcal{C}^n_m$.

*Two interpretations $\mathcal{I}$ and $\mathcal{J}$ are called n-m-*equivalent *($\mathcal{I} \equiv^n_m \mathcal{J}$) iff for all TBoxes T in $\mathcal{L}^n_m$ it holds that $\mathcal{I} \models T$ iff $\mathcal{J} \models T$. Similarly, for $x \in \Delta^{\mathcal{I}}$ and $y \in \Delta^{\mathcal{J}}$ we say that $\mathcal{I}, x$ and $\mathcal{J}, y$ are n-m-*equivalent *($\mathcal{I}, x \equiv^n_m \mathcal{J}, y$) iff for all $C \in \mathcal{C}^n_m$ it holds that $x \in C^{\mathcal{I}}$ iff $y \in C^{\mathcal{J}}$.*

Lemma 4.3 *There are only finitely many pairwise inequivalent definitions in $\mathcal{C}^n_m$.*

Proof. Since we chose N_C, the set of atomic concepts, to be finite, the claim obviously holds for $\mathcal{C}_0^n$, since this set only contains concept definitions which are Boolean combinations of atomic concepts. Since we only have finitely many roles and an upper bound on the numbers which can appear in number restrictions we only get finitely many pairwise inequivalent concept definitions for all $m > 0$. ∎

Corollary 4.4 *For all structures $\mathcal{I}, \mathcal{J}$ and each $x \in \Delta^{\mathcal{I}}$ there is a concept definition C_x such that $y \in C_x^{\mathcal{J}}$ iff $\mathcal{I}, x \equiv_m^n \mathcal{J}, y$.*

Proof. Choose $C_x = \bigsqcap\{C \in \overline{\mathcal{C}_m^n} \mid x \in C^{\mathcal{I}}\}$, where $\overline{\mathcal{C}_m^n}$ only contains one definition from each equivalence class in $\mathcal{C}_m^n$. ∎

We will now define an Ehrenfeucht-Fraisse game for $\mathcal{ALCQI}$ to capture the expressivity of concepts in the classes $\mathcal{C}_m^n$: The game is played by two players. Player I is called the spoiler while Player II is called the duplicator. The spoiler's aim is to prove two structures not to be n-m-equivalent, while Player II tries to prove the contrary. The game consists of a number of rounds in which the players move pebbles on the elements of the two structures.

Definition 4.5 *Let Δ be a nonempty set. Let x be an element of Δ and X a subset of Δ. For any binary relation $\mathcal{R} \subseteq \Delta \times \Delta$ we write $x\mathcal{R}X$ to denote the fact that, for all $x' \in X$ it holds that $(x, x') \in \mathcal{R}$.*

For the set N_R of role names let $\overline{N_R}$ be the union of N_R and $\{R^{-1} \mid R \in N_R\}$.

Let $\mathcal{I}$ and $\mathcal{J}$ be two interpretations. Two elements $x \in \Delta^{\mathcal{I}}$ and $y \in \Delta^{\mathcal{J}}$ are called locally equivalent *$(\mathcal{A}, x \equiv_l \mathcal{B}, y)$, iff for all $A \in N_C$: $x \in A^{\mathcal{I}}$ iff $y \in A^{\mathcal{J}}$.*

A configuration *captures the state of a game in progress. It is of the form $G_m^n(\mathcal{I}, x, \mathcal{J}, y)$, where n is an arbitrary natural number, m denotes the number of moves which still have to be played, and x and y are the elements of $\Delta^{\mathcal{I}}$ resp. $\Delta^{\mathcal{J}}$ on which the pebbles are placed.*

For the configuration $G_m^n(\mathcal{I}, x, \mathcal{J}, y)$ the rules are as follows:

1. *If $\mathcal{I}, x \not\equiv_l \mathcal{J}, y$, then Player II loses; if $m = 0$ and $\mathcal{I}, x \equiv_l \mathcal{J}, y$, then Player II wins.*

2. *If $m > 0$, then Player I selects one of the interpretations; assume this is $\mathcal{I}$ (the case $\mathcal{J}$ is handled dually). He then picks a role $S \in \overline{N_R}$ and a number $l \leq n$. He picks a set $X \subseteq \Delta^{\mathcal{I}}$ such that $xS^{\mathcal{I}}X$ and $\sharp X = l$. The duplicator has to answer with a set $Y \subseteq \Delta^{\mathcal{J}}$ with $yS^{\mathcal{J}}Y$ and $\sharp Y = l$. If there is no such set, then she loses.*

3. *If Player II was able to pick such a set Y, then Player I picks an element $y' \in Y$. Player II has to answer with an element $x' \in X$.*

4. *The game progresses with $G_{m-1}^n(\mathcal{I}, x', \mathcal{J}, y')$.*

We say that Player II has a winning strategy *for $G^n_m(\mathcal{I}, x, \mathcal{J}, y)$ iff she can always reach a winning position no matter which moves Player I plays. We write $\mathcal{I}, x \cong^n_m \mathcal{J}, y$ to denote this fact.*

Theorem 4.6 *For two structures $\mathcal{I}, \mathcal{J}$ and two elements $x \in \Delta^{\mathcal{I}}, y \in \Delta^{\mathcal{J}}$ it holds that $\mathcal{I}, x \cong^{n+1}_m \mathcal{J}, y$ iff $\mathcal{I}, x \equiv^n_m \mathcal{J}, y$.*

Proof. We proof this theorem inductively over m. Concept definitions of role depth 0 can only be (Boolean combinations of) atomic concepts. The claim follows immediately from the definition of local equivalence and the winning condition.

Now let m be greater then 0. We firstly show the implication from left to right. Let $\mathcal{I}, x \cong^{n+1}_m \mathcal{J}, y$ hold. We show $x \in C^{\mathcal{I}}$ iff $y \in C^{\mathcal{J}}$ for all concept definitions $C \in \mathcal{C}^n_m$ by induction over the structure of C. The only interesting cases are when $C = (\geq l\ R\ C')$ or $C = (\leq l\ R\ C')$ with $l \leq n$, $R \in \overline{N_R}$, and $C' \in \mathcal{C}^n_{m-1}$. Firstly we consider the case $C = (\geq l\ R\ C)$.

Assume that $x \in (\geq l\ R\ C')^{\mathcal{I}}$. Then there must be a set $X \subseteq \Delta^{\mathcal{I}}$ with $\sharp X = l$, $xR^{\mathcal{I}}X$, and $X \subseteq (C')^{\mathcal{I}}$. If player I chooses this set, then player II can answer with an according set $Y \subseteq \Delta^{\mathcal{J}}$ which has the properties required by the rules of the game ($\sharp Y = l$ and $yR^{\mathcal{J}}Y$). Player I can now pick an arbitrary element $y' \in Y$ and player II can answer with an appropriate element $x' \in X$. The game progresses with $G^{n+1}_{m-1}(\mathcal{I}, x', \mathcal{J}, y')$ for which player II still has a winning strategy. From the induction hypothesis it follows that $\mathcal{I}, x' \equiv^n_{m-1} \mathcal{J}, y'$, and hence $Y \subseteq (C')^{\mathcal{J}}$ must hold. For symmetry reasons it holds that $y \in C^{\mathcal{J}}$ implies $x \in C^{\mathcal{I}}$.

The case $C = (\leq l\ R\ C)$ can be reduced to the above case. For this we note that $(\leq l\ R\ C) \equiv \neg(\geq (l+1)\ R\ C)$. Since we have considered a game where the spoiler was allowed to pick sets of size at most $n+1$ and the proof only required a set of size at most n we can use this reduction and immediately obtain a proof for the latter case.

To show the implication from right to left we show its contraposition. Let $\mathcal{I}, x \not\cong^{n+1}_m \mathcal{J}, y$ hold. Hence there must be a move of player I, that player II cannot answer appropriately. Let this be the set $X \subseteq \Delta^{\mathcal{I}}$, reachable by role $R \in \overline{N_R}$ with $\sharp X = l, l \leq n+1$, and $xR^{\mathcal{I}}X$. Player II cannot answer with an appropriate set. If player II cannot even pick a reachable set of the needed size, we have that $y \in (\leq (l-1)\ R\ (C \sqcup \neg C))^{\mathcal{J}}$ for any $C \in \mathcal{C}$, while obviously $x \notin (\leq (l-1)\ R\ (C \sqcup \neg C))^{\mathcal{I}}$. Hence it follows that $\mathcal{I}, x \not\equiv^n_m \mathcal{J}, y$. For any set $Y \subseteq \Delta^{\mathcal{J}}$ player II can pick, which satisfies $\sharp Y = l$ and $yR^{\mathcal{J}}Y$ there must be an $y' \in Y$ which player I can pick such that, for all $x' \in X$ player II has no winning strategy for the game $G^{n+1}_{m-1}(\mathcal{I}, x', \mathcal{J}, y')$. Using the induction hypothesis this implies that $\mathcal{I}, x' \not\equiv^n_{m-1} \mathcal{J}, y'$.

We use this fact to construct a concept definition $D \in \mathcal{C}^n_m$ which distinguishes x and y in their respective structures. We simply use $D = (\leq (l-1)\ R\ \bigsqcup\{C_{x'} \mid x' \in X\})$.

CLAIM 1: $x \notin D^{\mathcal{I}}$.

PROOF OF CLAIM 1: For all $x' \in \Delta^{\mathcal{I}}$ we have $x' \in C^{\mathcal{I}}_{x'}$. This yields $X \subseteq \bigsqcup\{C_{x'} \mid x' \in X\}$. Since $xR^{\mathcal{I}}X$ and $\sharp X = l$ claim 1 follows.

Claim 2: $y \in D^{\mathcal{J}}$.

Proof of claim 2: As stated above, in any set $Y \subseteq \Delta^{\mathcal{J}}$ with $yR^{\mathcal{J}}Y$ and $\sharp Y = l$ there is an $y' \in Y$ which is not n-$(m-1)$-equivalent to any $x' \in X$. Hence there are at most $l-1$ elements in $\Delta^{\mathcal{J}}$ which are reachable from y via the relation $R^{\mathcal{J}}$ which are n-$(m-1)$-equivalent to an element of X. This yields $y \in D^{\mathcal{J}}$.

Hence D is a concept which distinguishes the two elements in their structures. Since $D \in \mathcal{C}^n_m$ it follows that $\mathcal{I}, x \not\equiv^n_m \mathcal{J}, y$. ∎

The game as it has been presented so far is suitable only if we have already placed pebbles on the interpretations. To get a game that characterises $\equiv^n_m$ as a relation between interpretations we have to introduce an additional rule that governs the placement of the first pebbles. Since a TBox consists of cardinality restrictions which solely talk about concept membership, we introduce an unconstrained set move as the first move of the game $G^n_m(\mathcal{I}, \mathcal{J})$.

Definition 4.7 *For two interpretations $\mathcal{I}, \mathcal{J}$ the game $G^n_m(\mathcal{I}, \mathcal{J})$ is played as follows:*

1. *Player I picks one of the structures; assume he picks $\mathcal{I}$ (the case $\mathcal{J}$ is handled dually). He then picks a set $X \subseteq \Delta^{\mathcal{I}}$ with $\sharp X = l$ where $l \leq n$. Player II must pick a set $Y \subseteq \Delta^{\mathcal{J}}$ of equal size. If this is impossible then she loses.*

2. *Player I picks a element $y \in Y$, Player II must answer with an $x \in X$.*

3. *The game progresses with $G^n_m(\mathcal{I}, x, \mathcal{I}, y)$.*

Again we say that Player II has a winning strategy for $G^n_m(\mathcal{I}, \mathcal{J})$ iff she can always reach a winning positions no matter which moves Player I chooses. We write $\mathcal{I} \cong^n_m \mathcal{J}$ do denote this fact.

From Theorem 4.6 is can easily be derived that we have a similar statement for the game $G^n_m(\mathcal{I}, \mathcal{J})$:

Theorem 4.8 *For two structures $\mathcal{I}, \mathcal{J}$ it holds that $\mathcal{I} \equiv^n_m \mathcal{J}$ iff $\mathcal{I} \cong^{n+1}_m \mathcal{J}$.*

Proof. Firstly we show the implication from right to left. Let $\mathcal{I} \cong^{n+1}_m \mathcal{J}$ hold. Let $T \in \mathcal{L}^n_m$ a TBox with $\mathcal{I} \models T$. Let $\mathcal{C} \in T$ be a value restriction. From $\mathcal{I} \models T$ is follows that $\mathcal{I} \models \mathcal{C}$. We need to show that this implies $\mathcal{J} \models \mathcal{C}$. We distinguish two cases:

- $\mathcal{C} = (\geq l\ C)$ with $l \leq n$ and $rd(C) \leq m$. If player I choses a set $X \subseteq \Delta^{\mathcal{I}}$ with $X \subseteq C^{\mathcal{I}}$ and $\sharp X = l$ (such a set exists because $\mathcal{I} \models \mathcal{C}$) then player II can answer with an appropriate set $Y \subseteq \Delta^{\mathcal{J}}$. From the rules of the game and the definition of the winning condition we have that for each $y \in Y$ there must be an $x \in X$ such that player two has a winning strategy for the game $G^{n+1}_m(\mathcal{I}, x, \mathcal{J}, y)$. This implies $\mathcal{I}, x \equiv^{n+1}_m \mathcal{J}, y$. It follows that $Y \subseteq C^{\mathcal{J}}$ and hence we have $\mathcal{J} \models \mathcal{C}$. For symmetry reasons we have also that $\mathcal{J} \models \mathcal{C}$ implies $\mathcal{I} \models \mathcal{C}$.

- $\mathcal{C} = (\leq l\ C)$ with $l \leq n$ and $rd(C) \leq m$. Again we can reduce this case to the above one by noting that $\mathcal{I} \models (\leq l\ C)$ if and only if $\mathcal{I} \models \neg(\geq (l{+}1)\ C)$. We have

$$\begin{array}{lll} \mathcal{I} \models (\leq l\ C) & \iff & \\ \mathcal{I} \models \neg(\geq (l+1)\ C) & \iff & \\ \mathcal{I} \not\models (\geq (l+1)\ C) & \iff & (*) \\ \mathcal{J} \not\models (\geq (l+1)\ C) & \iff & \\ \mathcal{J} \models \neg(\geq (l+1)\ C) & \iff & \\ \mathcal{J} \models (\leq l\ C) & & \end{array}$$

  The equivalence marked with (*) is proved as the above case by using the fact that we are allowed to pick sets of size up to $n+1$.

Since we chose $\mathcal{C}$ to be an arbitrary element of T we have $\mathcal{I} \models T$ if and only if $\mathcal{J} \models T$. This concludes the proof of the first implication.

The to show the implication from left to right we show the contraposition. Let $\mathcal{I} \not\cong_m^{n+1}$ hold. If player II has no winning strategy for $G_m^{n+1}(\mathcal{I}, \mathcal{J})$ there must be a move of player I, she cannot answer. Without loss of generality let this be $X \subseteq \Delta^{\mathcal{I}}$ with $\sharp X = l, l \leq n+1$. If player II cannot even pick a set of size l we have that $\mathcal{I} \not\models (\leq (l-1)\ (A \sqcup \neg A))$ while $\mathcal{J} \models (\leq (l-1)\ (A \sqcup \neg A))$ and hence $\mathcal{I} \not\equiv_m^n \mathcal{J}$. If there is a set $Y \subseteq \Delta^{\mathcal{J}}$ with $\sharp Y = l$ there must be a $y \in Y$ player I can pick such that for all $x \in X$ player II has no winning strategy for $G_m^{n+1}(\mathcal{I}, x, \mathcal{J}, y)$. This implies $\mathcal{I}, x \not\equiv_m^n \mathcal{J}, y$ for one y and all $x \in X$. Again we use this fact to construct a cardinality restriction $\mathcal{C} \in \mathcal{L}_m^n$ which distinguishes the two structures.

This is built in a similar fashion as in the previous prove. We choose

$$\mathcal{C} = (\leq (l-1)\ \bigsqcup\{C_x \mid x \in X\}).$$

Claim 1: $\mathcal{I} \not\models \mathcal{C}$.

Proof of claim 1: Since $X \subseteq (\bigsqcup\{C_x \mid x \in X\})^{\mathcal{I}}$ and $\sharp X = l$ this follows immediately by construction.

Claim 2: $\mathcal{J} \models \mathcal{C}$.

Proof of claim 2: In any set $Y \subseteq \Delta^{\mathcal{J}}$ with $\sharp Y = l$ there is at least one element y which is not n-m-equivalent to any $x \in X$. Hence there are at most $l-1$ elements in Y which are n-m-equivalent to an element of X. This yields $\mathcal{J} \models \mathcal{C}$.

Since $\mathcal{C} \in \mathcal{L}_m^n$ we have a TBox which distiguishes the two structures and hence we have $\mathcal{I} \not\equiv_m^n \mathcal{J}$. This concludes the proof. ■

4.3 The Main Result

We will now use this game-theoretic characterisation of the expressivity of $\mathcal{ALCQI}$ to proof the main theorem of this section. Even though we have introduced the powerful tool of Ehrenfeucht-Fraïssé games, the proof is still rather complicated. This is mainly due to the fact that we use a very general definition of expressiveness that allows for the introduction of arbitrary additional role- and concept-names into the signature.

Theorem 4.9 *$\mathcal{ALCQI}$ is not as expressive as C^2.*

Proof. To prove this Theorem we have to show that there is a class $\mathcal{C}$ that is characterisable in C^2 but that cannot be projectively characterised in $\mathcal{ALCQI}$:
CLAIM 1: For an arbitrary $R \in N_R$ the class $\mathcal{C}_{\mathit{refl}} := \{\mathcal{I} \mid R^{\mathcal{I}} \text{ is reflexive}\}$ is not projectively characterisable in $\mathcal{ALCQI}$. Obviously, $\mathcal{C}_{\mathit{refl}}$ is characterisable in C^2.
PROOF OF CLAIM 1: Assume Claim 1 does not hold and that $\mathcal{C}_{\mathit{refl}}$ is projectively characterised by the TBox $\mathcal{T}_{\mathit{refl}} \in \mathcal{L}^n_m$ over an arbitrary (but finite) signature $\tau = (N_C, N_R)$ with $R \in N_R$. We have derived a contradiction once we have shown that there are two τ-interpretations $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} \in \mathcal{C}_{\mathit{refl}}$, $\mathcal{B} \notin \mathcal{C}_{\mathit{refl}}$, but $\mathcal{A} \equiv^n_m \mathcal{B}$. In fact, $\mathcal{A} \equiv^n_m \mathcal{B}$ implies $\mathcal{B} \models \mathcal{T}_{\mathit{refl}}$ and hence $\mathcal{B} \in \mathcal{C}_{\mathit{refl}}$, a contradiction.

In particular, $\mathcal{C}_{\mathit{refl}}$ contains all interpretations $\mathcal{A}$ such that $R^{\mathcal{A}} = \{(x,x) \mid x \in \Delta^{\mathcal{A}}\}$, i.e. interpretations in which R is interpreted as equality. Since $\mathcal{C}^n_m$ contains only finitely many pairwise inequivalent concepts and $\mathcal{C}_{\mathit{refl}}$ contains interpretations of arbitrary size, there is also such an $\mathcal{A}$ such that there are two elements $x_1, x_2 \in \Delta^{\mathcal{A}}$ such that $x_1 \neq x_2$ and $\mathcal{A}, x_1 \equiv^n_m \mathcal{A}, x_2$. We define $\mathcal{B}$ from $\mathcal{A}$ as follows:

$$\begin{aligned}
&\Delta^{\mathcal{B}} := \Delta^{\mathcal{A}},\\
&A^{\mathcal{B}} := A^{\mathcal{A}} \quad \text{for each } A \in N_C,\\
&S^{\mathcal{B}} := S^{\mathcal{A}} \quad \text{for each } S \in N_R \setminus \{R\},\\
&R^{\mathcal{B}} := (R^{\mathcal{A}} \setminus \{(x_1,x_1),(x_2,x_2)\}) \cup \{(x_1,x_2),(x_2,x_1)\}.
\end{aligned}$$

Since $R^{\mathcal{B}}$ is no longer reflexive it holds that $\mathcal{B} \notin \mathcal{C}_{\mathit{refl}}$ as desired. It remains to be shown that $\mathcal{A} \equiv^n_m \mathcal{B}$ holds. We prove this by showing that $\mathcal{A} \cong^{n+1}_m \mathcal{B}$ holds, which is equivalent to $\mathcal{A} \equiv^n_m \mathcal{B}$ by Theorem 4.8.

Any opening move of Player I can be answered by Player II in a way that leads to the configuration $G^{n+1}_m(\mathcal{A}, x, \mathcal{B}, x)$, where x depends on the choices of Player I. We have to show that for any configuration of this type, Player II has a winning strategy. Since certainly $\mathcal{A}, x \cong^{n+1}_m \mathcal{A}, x$ this follows from Claim 2:
CLAIM 2: For all $k \leq m$ it holds that $\mathcal{A}, x \cong^{n+1}_k \mathcal{A}, y$ implies $\mathcal{A}, x \cong^{n+1}_k \mathcal{B}, y$.
PROOF OF CLAIM 2: We show Claim 2 by induction over k. Denote Player II's strategy for the configuration $G^{n+1}_k(\mathcal{A}, x, \mathcal{A}, y)$ by $\mathcal{S}$.

For the case $k = 0$, Claim 2 follows immediately from the construction of $\mathcal{B}$: $\mathcal{A}, x \cong^{n+1}_0 \mathcal{A}, y$ implies $\mathcal{A}, x \equiv_l \mathcal{A}, y$ and $\mathcal{A}, y \equiv_l \mathcal{B}, y$ since $\mathcal{B}$ agrees with $\mathcal{A}$ on the interpretation of all atomic concepts. It follows that $\mathcal{A}, x \equiv_l \mathcal{B}, y$ which means that Player II wins the game $G^{n+1}_0(\mathcal{A}, x, \mathcal{B}, y)$. For $0 < k \leq m$ assume that Player I selects an arbitrary structure and a legal subset of the respective domain. Player II tries to answer that move according to $\mathcal{S}$ which provides her with a move on the interpretation $\mathcal{A}$ since $\mathcal{S}$ is a winning strategy for the game $G^{n+1}_k(\mathcal{A}, x, \mathcal{A}, y)$. There are two possibilities:

- The move provided by $\mathcal{S}$ is a valid move also for the game $G_k^{n+1}(\mathcal{A}, x, \mathcal{B}, y)$: Player II can answer the choice of Player I according to $\mathcal{S}$ without violating the rules, which yields a configuration $G_{k-1}^{n+1}(\mathcal{A}, x', \mathcal{B}, y')$ such that for x', y' it holds that $\mathcal{A}, x' \cong_{k-1}^{n+1} \mathcal{A}, y'$ (because Player II moved according to $\mathcal{S}$). From the induction hypothesis it follows that $\mathcal{A}, x' \cong_{k-1}^{n+1} \mathcal{B}, y'$.

- The move provided by $\mathcal{S}$ is not a valid move for the game $G_k^{n+1}(\mathcal{A}, x, \mathcal{B}, y)$ This requires more detailed analysis: Assume Player I has chosen to move in $\mathcal{A}$ and has chosen an $S \in \overline{N_R}$ and a set X of size $l \leq n+1$ such that $xS^{\mathcal{A}}X$. Let Y be the set that Player II would choose according $\mathcal{S}$. This implies that Y has also l elements and that $yS^{\mathcal{A}}Y$. That this choice is not valid in the game $G_k^{n+1}(\mathcal{A}, x, \mathcal{B}, y)$ implies that there is an element $z \in Y$ such that $(y, z) \notin S^{\mathcal{B}}$. This implies $y \in \{x_1, x_2\}$ and $S \in \{R, R^{-1}\}$, because these are the only elements and relations that are different in $\mathcal{A}$ and $\mathcal{B}$. W.l.o.g. assume $y = x_1$ and $S = R$. Then also $z = x_1$ must hold, because this is the only element such that $(x_1, z) \in R^{\mathcal{A}}$ and $(x_1, z) \notin R^{\mathcal{B}}$. Thus, the choice $Y' := (Y \setminus \{x_1\}) \cup \{x_2\}$ is a valid one for Player II in the game $G_m^{n+1}(\mathcal{A}, x, \mathcal{B}, y)$: $x_1 R^{\mathcal{B}} Y'$ and $|Y'| = l$ because $(x_1, x_2) \notin R^{\mathcal{A}}$.

  There are two possibilities for Player I to choose an element $y' \in Y'$:

  1. $y' \neq x_2$: Player II chooses $x' \in X$ according to $\mathcal{S}$. This yields a configuration $G_{k-1}^{n+1}(\mathcal{A}, x', \mathcal{B}, y')$ such that $\mathcal{A}, x' \cong_{k-1}^{n+1} \mathcal{A}, y'$.
  2. $y' = x_2$: Player II answers with the $x' \in X$ that is the answer to the move x_1 of Player I according to $\mathcal{S}$. For the obtained configuration $G_{k-1}^{n+1}(\mathcal{A}, x', \mathcal{B}, y')$ it also holds that $\mathcal{A}, x' \cong_{k-1}^{n+1} \mathcal{A}, y'$: By the choice of x_1, x_2 it holds that $\mathcal{A}, x_1 \equiv_m^n \mathcal{A}, x_2$ and since $k-1 < m$ also $\mathcal{A}, x_1 \equiv_{k-1}^n \mathcal{A}, x_2$ holds which implies $\mathcal{A}, x_1 \cong_{k-1}^{n+1} \mathcal{A}, x_2$ by Theorem 4.8. Since Player II chose x' according to $\mathcal{S}$ it holds that $\mathcal{A}, x' \cong_{k-1}^{n+1} \mathcal{A}, x_1$ and hence $\mathcal{A}, x' \cong_{k-1}^{n+1} \mathcal{A}, x_2$ since $\cong_{k-1}^{n+1}$ is transitive.

  In both cases we can apply the induction hypothesis; this yields $\mathcal{A}, x' \cong_{k-1}^{n+1} \mathcal{B}, y'$ and hence Player II has a winning strategy for $G_k^{n+1}(\mathcal{A}, x, \mathcal{B}, y)$. The case that Player I chooses from $\mathcal{B}$ instead of $\mathcal{A}$ can be handled similarly.

■

By adding constructs to $\mathcal{ALCQI}$ that allow to form more complex role expressions one can obtain a DL that has the same expressive power as C^2, such a DL is presented in [Bor96]. It has the ability to express a universal role that makes it possible to internalise both TBoxes based on terminological axioms and cardinality restrictions on concepts.

5 Conclusion

We have shown that with a rather limited set of constructors one can define a DL whose reasoning problems as as hard as that of C^2 without reaching the expressive power of the latter. This shows that cardinality restrictions, although interesting for knowledge representation, are inherently hard to handle algorithmically. At a first glance, this makes $\mathcal{ALCQI}$ with cardinality restrictions on concepts obsolete for knowledge representation, because C^2 delivers more expressive power at the same computational price. Yet, is is likely that a dedicated algorithm for $\mathcal{ALCQI}$ will have better average complexity than the C^2 algorithm; such an algorithm has yet to be developed. An interesting question lies in the coding of numbers: If we allow binary coding of numbers, the translation approach together with the result from [PST97] leads to a 2-NExpTime algorithm. It is an open question whether this additional exponential blow-up is necessary. A positive answer would settle the same question for C^2 while a proof of the negative answer might give hints how the result for C^2 might be improved.

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