Technical Report

May 2021

Distribution of Lateral Forces on Reinforced Masonry Bracing Elements Considering Inelastic Material Behavior - Deformation-Based Matrix Method-

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This report has been prepared at the end of the “return phase” spent at the Technical University of Dresden in Germany, by the Experienced Researcher Kenan Michel, as part of CIC-BREL project (Cracked and Inelastic Calculation of BRacing ELements). This project has received funding from the European Union’s Horizon 2020 research and innovation program under agreement No 744332 – CIC – BREL.

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Acknowledgement

This project has been carried out under the guidance of Professor Wolfram Jaeger from the Technical University of Dresden in Germany as a supervisor in the host institute of the return phase, as well as Professor Benson Shing from the University of California in San Diego in the USA as a supervisor in the host institute of the outgoing phase. I highly appreciate their efforts in supporting me during the project and writing this report

A lot of information has been taken from the classes of Prof. Benson Shing in the class Masonry Design and Frame Structure Analysis. Results of shake table tests at the University of California in San Diego has been used to verify the developed analytical method, the tests have been done by Prof. Benson Shing during my stay as a Visiting Scholar at the University in the year 2018.

1. Introduction

The main goal of CIC-BREL project (Cracked and Inelastic Calculation of BRacing Elements) is to develop an analytical method to distribute horizontal forces on bracing elements, in this case reinforced masonry shear walls, of a building considering the cracked and inelastic state of material.

The moment curvature curve of the wall section is created first depending on the section geometry and material properties of both the masonry units and steel reinforcement. This curve will start with an elastic material behavior, then continue in inelastic material behavior where the masonry crushes and the steel start to yield, until the maximum bending moment M_p is reached. Due to reinforced masonry wall ductility, post maximum capacity is also considered assuming a maximum curvature of 0.1%. From the moment curvature curve, the force displacement curve could be extracted depending on the wall height and wall boundary conditions.

Matrix formulation has been developed for both elastic and damaged stiffness matrix, considering different boundary conditions. Fixed-fixed boundary condition which usually exists at the middle stories or last story with strong top diaphragm, fixed-pinned which is the case of the last story that has a relatively soft top diaphragm, and pinned-fixed in the first story case. Other boundary conditions could be considered depending on the degree of fixation on the wall both ends at the top and the bottom.

The matrix formulation combined with the force-displacement curve which considers different material stages (elastic, inelastic, ductile post peak force) is used to define forces in each bracing element even after elastic behavior. After elastic phase of each wall the stiffness of the element will degrade leading to a less portion of the total lateral force; other elastic walls, i.e., stronger walls, will receive more portion of the total force leading to a redistribution of the total force. This process will be iterated until the total force is distributed on each bracing element depending on the wall section state: elastic, inelastic and ductile post-peak capacity. Flowcharts clearly will show this process. Finally, a Fortran code is developed to show examples using this method.

The developed analytical method will be verified by the results of shake table tests held at the University of California in San Diego, USA. Last test performed in the year 2018 uses T-section reinforced masonry walls, subjected to shakings with increased intensity. The total applied force for each shaking could be defined depending on the structural weight and shaking intensity (acceleration). The damage and displacement at each intensity has been recorded and evaluated. Depending on these test results, the results of the analytically developed method will be compared and evaluated. Total system displacement at different lateral load values has been compared for analytical calculations and shake table tests; furthermore, each wall state at increased load has been compared, good agreement could be noticed.

1.1. State of the Art

The bracing system of a building should guarantee the stability - in terms of usability and overall sustainability – of the structure. The decisive loading on the bracing system are horizontal loads due to wind or earthquake ([1] - [4]). Currently, the used method to distribute horizontal forces on the bracing elements assumes an elastic (non-cracked and non-plastic) state, leading to high reserve in the element itself and the structure as a whole, consequently huge unnecessary consumption of building material is inevitable. The distribution occurs only according to the bending stiffness in an uncracked condition. For this distribution the ceilings are assumed to be inflexible (very stiff and don't deform) [5].

The method put all the bracing elements in each story into one single main beam, by using the differential equations of thin wall open cross sections of each single wall [6]. The main beam has the stiffness of the bending stiffnesses of all single walls in the floor plan, it is located in the shear center, it is the point where the applied lateral force causes only displacement, not rotation, of the main beam section [7].

1.2. Elastic Formulae

The formulae for the load distribution of the main beam internal forces on each single wall in the floor plan are simple when the St. Venant torsion of each wall is neglected, and only the bending stiffness of it is considered. The symbols in Figure 1.1 are as follows:

S	center of gravity in floor plan
M	shear center
S_j	center of gravity of wall „j“
$\bar{x}, \bar{y}, \bar{z}$	global coordinate system
x_j, y_j, z_j	local coordinate system of the wall j parallel to x_s, y_s, z_s
x_s, y_s, z_s	coordinate system in the center of gravity
x_M, y_M, z_M	coordinate system in the shear center
$H_{M,i,x}$	horizontal force in point M in x-direction in floor „i“
$H_{M,i,y}$	horizontal force in point M in y-direction in floor „i“

The coordinate system of the shear center, as shown in Figure 1.1

Figure 1.1 Floor plan of the building

$$x_{S,M} = \frac{\sum_{j=1}^n I_{x,j} \cdot x_{S,j}}{\sum_{j=1}^n I_{x,j}} \quad y_{S,M} = \frac{\sum_{j=1}^n I_{y,j} \cdot y_{S,j}}{\sum_{j=1}^n I_{y,j}} \quad (1.1)$$

- $I_{x,j}$ Moment of inertia of wall „j“ around ξ -axis in local coordinate system x_j - y_j - z_j - of each wall parallel to wall and goes through its center of gravity, and
- $I_{y,j}$ Moment of inertia of wall „j“ around η -axis in local coordinate system x_j - y_j - z_j - of each wall parallel to wall and goes through its center of gravity

For a check, the sum of the product of moment of inertia of each wall with its distance to the shear center of the main beam should be zero

$$\sum_{j=1}^n (I_{x,j} \cdot x_{M,j}) = 0 \quad \text{and} \quad \sum_{j=1}^n (I_{y,j} \cdot y_{M,j}) = 0 \quad (1.2)$$

Bending and torsion could be considered independent from each other. For the bending load case, the resulted forces due to horizontal loads $H_{S,i,x}$ and $H_{S,i,y}$ of the floor „i“ in the shear center of the floor. The offset moment $M_{M,z}$ is considered in the torsion load case. For each wall „j“, the translation load is given by:

$$H_{M,i,x} \text{ in } M \rightarrow S_{x,trans,j} = H_{M,i,x} \cdot \frac{E_j \cdot I_{y,j}}{\sum_{j=1}^n (E_j \cdot I_{y,j})} \quad (1.3)$$

$$H_{M,i,y} \text{ in } M \rightarrow S_{y,trans,j} = H_{M,i,y} \cdot \frac{E_j \cdot I_{x,j}}{\sum_{j=1}^n (E_j \cdot I_{x,j})} \quad (1.4)$$

The torsion due to eccentric horizontal load is given by:

$$M_{M,z} = -H_{M,i,x} \cdot y_{S,M} \text{ bzw. } M_{M,z} = -H_{M,i,y} \cdot x_{S,M} \quad (1.5) \text{ a}$$

$$M_{M,z,x} = H_{M,i,x} \cdot (y_{S,M} + Ly/10) \quad M_{M,z,y} = H_{M,i,y} \cdot (y_{S,M} + Lx/10) \quad (1.5) \text{ b}$$

The last equation is when considering horizontal force eccentricity of 10% of the floor plan length according to Euro code 1 [8]-[9].

This torsion causes additional horizontal force which should be carried by the walls. The additional wall loading on the wall „j“ is calculated by:

$$S_{x,rot,j} = -M_{M,z} \cdot \frac{E_j I_{y,j} \cdot y_{M,j}}{\sum_{j=1}^n E_j (I_{x,j} \cdot x_{M,j}^2 + I_{y,j} \cdot y_{M,j}^2)} \quad (1.6)$$

$$S_{y,rot,j} = -M_{M,z} \cdot \frac{E_j I_{x,j} \cdot x_{M,j}}{\sum_{j=1}^n E_j (I_{x,j} \cdot x_{M,j}^2 + I_{y,j} \cdot y_{M,j}^2)} \quad (1.7)$$

The warping resistance C_M of the main beam as written in last two equations is:

$$C_M = \sum_{j=1}^n (I_{x,j} \cdot x_{M,j}^2 + I_{y,j} \cdot y_{M,j}^2) \quad (1.8)$$

The load on each load, from the translational and torsional parts, is given by:

$$S_{x,j} = H_{M,i,x} \cdot \frac{E_j \cdot I_{y,j}}{\sum_{j=1}^n (E_j \cdot I_{y,j})} - M_{M,z} \cdot \frac{E_j I_{y,j} \cdot y_{M,j}}{\sum_{j=1}^n E_j (I_{x,j} \cdot x_{M,j}^2 + I_{y,j} \cdot y_{M,j}^2)} \quad (1.9)$$

$$S_{y,j} = H_{M,i,y} \cdot \frac{E_j \cdot I_{x,j}}{\sum_{j=1}^n (E_j \cdot I_{x,j})} - M_{M,z} \cdot \frac{E_j I_{x,j} \cdot x_{M,j}}{\sum_{j=1}^n E_j (I_{x,j} \cdot x_{M,j}^2 + I_{y,j} \cdot y_{M,j}^2)} \quad (1.10)$$

1.3. Example, Elastic Calculation

The floor plan of a building, as shown in Figures 1.2 and 1.3, will be used as an example to show the elastic calculation [10]. The walls W1 to W6 have the same cross section in all stories from the foundation until the last floor, these walls are assumed to stay elastic uncracked. For simplicity in calculation, the walls W4 to W6 are considered as separate walls, and the joints between them are not considered to transfer shear among different walls.

The point S is the center of gravity for the floor plan, it is considered as point where the lateral horizontal forces (wind or earthquake and eccentricity) are applied. The center of each wall is symbolized by black point which is the middle of the rectangular cross section. All walls have the same building material.

Figure 1.2 Floor plan of the multistory building

Figure 1.3 Vertical section plan of the building

1.3.1. Stiffnesses of the System

To define the stiffness data of the system and to distribute the lateral load on the single walls, the floor plan has been prepared as shown in Figure 1.2. The shear center of the main beam and center of gravity of the floor plan are shown in Figure 1.4

The shear center is calculated from the stiffnesses of each wall and their distances to each other, see Table 1.1.

Wall „j“	$I_{x,j}$ [m ⁴]	$I_{y,j}$ [m ⁴]	$x_{S,j}$ [m]	$y_{S,j}$ [m]	$I_{x,j} \cdot x_{S,j}$ [m ⁵]	$I_{y,j} \cdot y_{S,j}$ [m ⁵]
1	0,0675	0,0000	5,88	-2,75	0,3969	0,0000
2	1,2800	0,0000	5,88	1,50	7,5264	0,0000
3	1,2800	0,0000	1,12	1,50	1,4336	0,0000
4	0,0000	1,5462	-3,63	-3,38	0,0000	-5,2261
5	0,0000	1,5462	-3,63	3,38	0,0000	5,2261
6	6,8600	0,0000	-5,88	0,00	-40,3368	0,0000
	9,4875	3,0924			-30,9799	0,0000

Coordinates of the shear center
(using the original coordinate system)

$$x_{S,M} = -30,9799 / 9,4875 = -3,26 \text{ m}$$

$$y_{S,M} = 0,0000 / 3,0924 = 0,00 \text{ m}$$

Table 1.1: Defining the shear center M

Figure 1.4 Floor plan of the building with shear center M and center of gravity S

1.3.2. Torsion due to Eccentric Lateral Loading

Usually, the center of gravity S, which is the point where the lateral wind load or the load due to misalignment, doesn't coincide with the shear center M, this leads to indispensable torsion around the point M. According to euro code 1 [8]-[9], an eccentricity of the lateral load of $e = \pm b/10$ and $e = \pm d/10$, where a and b are the width and depth of the building, should be taken into account; this will lead to increase of the already existing torsion, Figure 1.5. With this eccentricity, it is necessary to calculate the bracing system three times in each direction (i.e., $e = 0$ und $e = \pm b/10$) in order to consider the worst case on the single walls. This additional eccentricity is not required in the case of misalignment. In this example only one case will be considered, that is $e_y = 3,26 + 1,20 = 4,46$ m und $e_x = 0,00 + 0,70 = 0,70$ m.

Figure 1.5 Torsion due to eccentricity

1.3.3. Distribution of the Lateral Load on Wall “j” and Floor “i”

The stiffness of each wall and the stiffness of whole system are required to distribute the lateral load on each bracing element, Table 1.2.

Wall “j”	I_{xj} [m ⁴]	I_{yj} [m ⁴]	$x_{M,j}$ [m]	$y_{M,j}$ [m]	$I_{xj} \cdot x_{M,j}$ [m ⁵]	$I_{yj} \cdot y_{M,j}$ [m ⁵]	$I_{xj} \cdot x_{M,j}^2$ [m ⁶]	$I_{yj} \cdot y_{M,j}^2$ [m ⁶]
1	0,0675	0	9,14	-2,75	0,6170	0	5,6389	0
2	1,2800	0	9,14	1,50	11,699	0	106,93	0
3	1,2800	0	4,38	1,50	5,606	0	24,556	0
4	0	1,5462	-0,37	-3,38	0	-5,2261	0	17,664
5	0	1,5462	-0,37	3,38	0	5,2261	0	17,664
6	6,8600	0	-2,62	0	-17,973	0	47,090	0
	9,4875 m⁴	3,0924 m⁴			≈0	≈0	184,22	35,33
Kontrolle							C_M=219,5437 m⁶	

Table 1.2: Defining the warping resistance

The shear force in each wall due to translation has been calculated using equations (1.3) and (1.4), and the shear due to rotation has been calculated using equations (1.6) and (1.7). The calculation will be done using a unit load of 1 kN in each direction. As all masonry walls use the same building material, the elasticity modulus could be removed from the formulae. The equations for translational, rotational and the sum of load on each wall “j” and floor “i” in the main directions x and y could be rewritten as:

Translation:

$$S_{x,trans,j} = H_{M,i,x} \cdot \frac{I_{y,j}}{\sum_{j=1}^n I_{y,j}} \quad (1.11)$$

$$S_{y,trans,j} = H_{M,i,y} \cdot \frac{I_{x,j}}{\sum_{j=1}^n I_{x,j}} \quad (1.12)$$

Rotation:

$$S_{y,rot,j} = - M_{M,z} \cdot \frac{I_{y,j} \cdot y_{M,j}}{C_M} \quad (1.13)$$

$$S_{x,rot,j} = - M_{M,z} \cdot \frac{I_{x,j} \cdot x_{M,j}}{C_M} \quad (1.14)$$

sum:

$$S_{x,j} = S_{x,trans,j} + S_{x,rot,j} \quad (1.15)$$

$$S_{y,j} = S_{y,trans,j} + S_{y,rot,j} \quad (1.16)$$

Index i = floor number „i“ and

Index j = wall „j“.

The resulted loading on each wall „j” in the floor “i” from a horizontal unit loading of $H_{M,i,x} = H_{M,i,y} = 1,0$ kN (translation), and the corresponding torsion moments $M_{M,z,y} = 4,46$ kNm and $M_{M,z,x} = 0,70$ kNm (rotation) are shown in Table 1.3, and presented graphically in Figure 1.6.

Wall „j“	$H_{M,i,x} = 1,0 \text{ kN}$, $M_{M,z,x} = 0,70 \text{ kNm}$ (Wind in x-direction)				$H_{M,i,y} = 1,0 \text{ kN}$, $M_{M,z,y} = 4,46 \text{ kNm}$ (Wind in y-direction)			
	$S_{x,trans,j}$ [kN]	$S_{x,rot,j}$ [kN]	$S_{x,j}$ [kN]	$S_{y,rot,j}$ [kN]	$S_{y,trans,j}$ [kN]	$S_{y,rot,j}$ [kN]	$S_{y,j}$ [kN]	$S_{x,rot,j}$ [kN]
1	0,0000	0,0000	0,0000	0,0020	0,0071	0,0125	0,0196	0,0000
2	0,0000	0,0000	0,0000	0,0373	0,1349	0,2377	0,3726	0,0000
3	0,0000	0,0000	0,0000	0,0179	0,1349	0,1139	0,2488	0,0000
4	0,5000	-0,0167	0,4833	0,0000	0,0000	0,0000	0,0000	-0,1062
5	0,5000	0,0167	0,5167	0,0000	0,0000	0,0000	0,0000	0,1062
6	0,0000	0,0000	0,0000	-0,0573	0,7231	-0,3651	0,3579	0,0000
Summe	$\approx 1,000$	$\approx 0,000$	$\approx 1,000$	$\approx 0,000$	$\approx 1,000$	$\approx 0,000$	$\approx 1,000$	$\approx 0,000$

Table 1.3: Loading on the single walls with unit load

Figure 1.6 Resulted wall loadings due to lateral unit load in directions x and y

To check the results, the sum of the forces in x and y directions should be zero as shown in Table 1.4. And sum of moments should fulfill the equilibrium

wall „j“			Wind in x-direction				Wind in y-direction			
	$x_{M,j}$ [m]	$y_{M,j}$ [m]	$S_{x,rot,j}$ [kN]	$S_{y,rot,j}$ [kN]	$S_{x,rot,j} \cdot y_{M,j}$ [kNm]	$S_{y,rot,j} \cdot x_{M,j}$ [kNm]	$S_{y,rot,j}$ [kN]	$S_{x,rot,j}$ [kN]	$S_{y,rot,j} \cdot x_{M,j}$ [kNm]	$S_{x,rot,j} \cdot y_{M,j}$ [kNm]
1	9,14	-2,75	0,00	0,01	0,000	0,018	0,01	0,00	0,115	0,000
2	9,14	1,50	0,00	0,04	0,000	0,341	0,24	0,00	2,172	0,000
3	4,38	1,50	0,00	0,02	0,000	0,078	0,14	0,00	0,500	0,000
4	-0,37	-3,38	-0,02	0,00	0,056	0,000	0,00	-0,11	0,000	0,359
5	-0,37	3,38	0,02	0,00	0,056	0,000	0,00	0,11	0,000	0,359
6	-2,62	0,00	0,00	-0,06	0,000	0,150	-0,36	0,00	0,957	0,000
					0,113	0,587			3,742	0,718
					$\approx 0,70 = m_{x,0,z}$				$\approx 4,46 = m_{x,0,y}$	

Table 1.4: Checking the calculation

All these calculations have been done assuming elastic material behavior of the walls. In the following chapters the formulae for inelastic material will be developed, simple examples will be shown. Finally, the calculations will be compared with shake table tests.

2. Force Displacement Curve of RM Shear Wall

2.1. Introduction

Depending on the material properties of the reinforced masonry wall i.e., masonry and steel, as well as cross-section shape and geometry each wall has a moment-curvature curve. When considering the boundary conditions, the force-displacement curve could be derived.

Figure 2.1: Shape of moment curvature curve of reinforced masonry section

The section has a maximum moment capacity $M_p \approx M_n$, the elastic (M_y) region is approximately 75% to 80% of M_p . In the region 0 - M_e , the section has elastic behavior. At the moment (M_y) the steel starts to yield and the section shows inelastic behavior until the nominal moment capacity (M_n) where the masonry crushes and the steel yields. After (M_p) the section shows inelastic ductile behavior, and the curve shape depends on the steel rate and axial force on the section.

In the following, the force displacement curves will be shown for different wall boundary conditions at each region.

2.2. Cantilever Wall

2.2.1. Cantilever Elastic Wall

When considering a single cantilever wall in elastic regime, i.e.; $M < M_y = 0.8 \times M_p$. The bending moment, curvature and flexural deformation are shown in the Figure 2.2 [13].

Note that the stiffness matrix in elastic region as described in chapter 3.3.1 is calculated considering the shear deformation.

Figure 2.2: Elastic cantilever wall, bending moment, curvature and flexural deformation

The flexural deformation is calculated using the moment area method as:

$$U_f = \frac{1}{2} \left(\frac{F H}{E_m I_n} \right) (H) \left(\frac{2}{3} H \right) = \frac{F H^3}{3 E_m I_n} \quad (2.1)$$

The shear deformation is calculated by:

$$U_s = \frac{1}{2} \left(\frac{F H}{E_m I_n} \right) (H) \left(\frac{2}{3} H \right) = \frac{F H}{K_s G_m A_n} \quad (2.2)$$

Where:

E_m elasticity modulus of masonry. It is given in TMS 402 [11] by:

- $900 f'_m$ for concrete masonry and
- $700 f'_m$ for clay masonry, psi (Mpa)

f'_m specified compressive strength of clay masonry, or of concrete masonry, psi (MPa)

I_n moment of inertia of net cross section.

F horizontal force on the wall.

H wall height

G_m shear modulus of masonry. It is given in TMS 402 [11] by:

- $G_m = 0.4 E_m$, psi (Mpa)

K_s Timoshenko shear factor, in chapter 3 is denoted as κi , it is assumed to be:

- 5/6 for rectangular cross section and
- 1 for T-section.

A_n Net cross section of rectangular section equals gross section. For T-section, net area is assumed to be the area of the web.

I_e effective moment of inertia. Depending on test data, it is equal to:

- $0.15 I_n$ for rectangular cross section, and
- $0.40 I_n$ for T-section.

$(K_s G_m A_n)_e$ effective value is equal to:

- $0.35 K_s G_m A_n$ for rectangular cross section

2.2.2. Cantilever Inelastic Wall

Beyond the elastic limit of 0.8Mp, the wall will start to crack and the steel will start to yield. The bending moment, curvature and flexural deformation are shown in the Figure 2.3. The shear deformation is not considered here because it is small compared to the flexural deformation.

Figure 2.3: Inelastic cantilever wall, bending moment, curvature and flexural deformation

The curvature is the sum of the elastic and plastic curvature: $\phi = \phi_{el} + \phi_{pl}$

$$\phi_{el} = \frac{M}{E_m I_n} \text{ and } \phi_{pl} = \phi - \phi_{el}$$

The flexural deformation is the sum of elastic and plastic deformation as:

$$\begin{aligned} U_f &= U_{el} + U_{pl} = \frac{F H^3}{3 E_m I_n} + \phi_{pl} L_p \left(h - \frac{L_p}{2} \right) \\ &= \frac{F H^3}{3 E_m I_n} + (\phi - \phi_{el}) L_p \left(h - \frac{L_p}{2} \right) \\ &= \frac{F H^3}{3 E_m I_n} + \left(\phi - \frac{F H}{E_m I_n} \right) L_p \left(h - \frac{L_p}{2} \right) \\ U_f &= \frac{F H^3}{3 E_m I_n} + \left(\phi - \frac{F H}{E_m I_n} \right) L_p \left(h - \frac{L_p}{2} \right) \end{aligned} \quad (2.3)$$

Where

L_p is the effective plastic hinge length. It is given by different formulas:

- In Shing [13] it is given as $0.2 H$;
- or by Priestly [12] as $0.04 H + 0.1 L_w + L_{sp}$.
 - $L_{sp} [\text{inch}] = 0.15 f_y [\text{ksi}] db [\text{inch}]$, or
 - $L_{sp} [\text{m}] = 0.15 f_y [\text{MPa}] db [\text{m}]$

F_y yield stress of the reinforcing bars

db bar diameter of the reinforcing bars

$\phi_{pl} = \phi - \frac{F H}{E_m I_n}$ is the curvature corresponds to M_p

The shear deformation is neglected, in inelastic regime the flexural deformation is dominant.

2.2.3. Cantilever Post-Peak Wall

The displacement is calculated in the same way calculated for inelastic region; however, the curvature will be higher. In this study, the curvature at post peak bending moment of 0.75 MP and 0.5 MP will be considered. As shown in Figure 2.4.

Figure 2.4: Shape of moment curvature curve of reinforced masonry section

The flexural displacement is given by:

$$U_f = U_{el} + U_{pl} = \frac{M_\alpha H^2}{3 E_m I_n} + \left(\phi_\alpha - \frac{M_\alpha}{E_m I_n} \right) L_p \left(h - \frac{L_p}{2} \right) \quad (2.4)$$

Where

M_α bending moment at curvature ϕ_α . $M_\alpha = 0.75 M_p$ at ϕ_{75} and $M_\alpha = 0.50 M_p$ at ϕ_{50}

The shear deformation is neglected, in inelastic regime the flexural deformation is dominant

2.3. Fixed-Fixed Wall

2.3.1. Fixed-Fixed Elastic Wall

When considering a single cantilever wall in elastic regime, i.e., $M < M_y = 0.8 M_p$. The bending moment, curvature and flexural deformation are shown in the Figure 2.5. Note that the stiffness matrix in elastic region as described in chapter 3.3 is calculated considering the shear deformation.

Figure 2.5: Elastic fixed-fixed wall, bending moment, curvature and flexural deformation

The flexural deformation is calculated using the moment area method as:

$$U_f = 2 \frac{1}{2} \left(\frac{F H / 2}{E_m I_n} \right) \left(\frac{H}{2} \right) \left(\frac{1}{3} H \right) = \frac{F H^3}{12 E_m I_n} \quad (2.5)$$

The shear deformation is calculated by:

$$U_s = \frac{1}{2} \left(\frac{F H}{E_m I_n} \right) (H) \left(\frac{2}{3} H \right) = \frac{F H}{K_s G_m A_n} \quad (2.6)$$

Where:

E_m elasticity modulus of masonry. It is given in TMS 402 [11] by:

- $900 f'_m$ for concrete masonry and
 - $700 f'_m$ for clay masonry, psi (Mpa)
- f'_m specified compressive strength of clay masonry, or of concrete masonry, psi (MPa)
- I_n moment of inertia of net cross section.
- F horizontal force on the wall.
- H wall height
- G_m shear modulus of masonry. It is given in TMS 402 [11] by:
- $G_m = 0.4 E_m$, psi (Mpa)
- K_s Timoshenko shear factor, in chapter 3 is denoted as κ_i , it is assumed to be:
- 5/6 for rectangular cross section and
 - 1 for T-section.
- A_n Net cross section of rectangular section equals gross section. For T-section, net area is assumed to be the area of the web.
- I_e effective moment of inertia. Depending on test data, it is equal to:
- $0.15 I_n$ for rectangular cross section, and
 - $0.40 I_n$ for T-section.
- $(K_s G_m A_n)_e$ effective value is equal to:
- $0.35 K_s G_m A_n$ for rectangular cross section

2.3.2.Fixed-Fixed Inelastic Wall

Beyond the elastic limit of 0.8 Mp, the wall will start to crack and the steel will start to yield. The bending moment, curvature and flexural deformation are shown in the Figure 2.6. The shear deformation is not considered here because it is small compared to the flexural deformation.

Figure 2.6: Inelastic fixed-fixed wall, bending moment, curvature and flexural deformation

The curvature is the sum of the elastic and plastic curvature:

$$\phi = \phi_{el} + \phi_{pl} \quad \phi_{el} = \frac{M}{E_m I_n} = \frac{F H/2}{E_m I_n} \quad \text{and} \quad \phi_{pl} = \phi - \phi_{el}$$

The flexural deformation is the sum of elastic and plastic deformation as:

$$\begin{aligned} U_f &= U_{el} + U_{pl} = \frac{F H^3}{12 E_m I_n} + 2 \phi_{pl} L_p \left(h/2 - \frac{L_p}{2} \right) \\ &= \frac{F H^3}{12 E_m I_n} + \phi_{pl} L_p (h - L_p) \\ &= \frac{F H^3}{12 E_m I_n} + (\phi - \phi_{el}) L_p (h - L_p) \\ &= \frac{F H^3}{12 E_m I_n} + \left(\phi - \frac{F H}{2 E_m I_n} \right) L_p (h - L_p) \\ U_f &= \frac{F H^3}{12 E_m I_n} + \left(\phi - \frac{F H}{2 E_m I_n} \right) L_p (h - L_p) \end{aligned} \quad (2.7)$$

Where

L_p is the effective plastic hinge length. It is given by different formulas:

- In Shing [13] it is given as $0.2 H$;
- or by Priestly [12] as $0.04 H + 0.1 L_w + L_{sp}$.
 - $L_{sp} [\text{inch}] = 0.15 f_y [\text{ksi}] db [\text{inch}]$, or
 - $L_{sp} [\text{m}] = 0.15 f_y [\text{MPa}] db [\text{m}]$

f_y yield stress of the reinforcing bars

db bar diameter of the reinforcing bars

$\phi_{pl} = \phi - \frac{F H}{E_m I_n}$ is the curvature corresponds to Mp

The shear deformation is neglected, in inelastic regime the flexural deformation is dominant

2.3.3.Fixed-Fixed Post-Peak Wall

The displacement is calculated in the same way calculated for inelastic region; however, the curvature will be higher. In this study, the curvature at post peak bending moment of 0.75 MP and 0.5 Mp will be considered. As shown in Figure 2.4.

$$U_f = U_{el} + U_{pl} = \frac{M_\alpha H^2}{6 E_m I_n} + \left(\phi_\alpha - \frac{M_\alpha}{E_m I_n} \right) L_p (h - L_p) \quad (2.8)$$

Where

M_α bending moment at curvature ϕ_α . $M_\alpha = 0.75 \text{ Mp}$ at ϕ_{75} and $M_\alpha = 0.50 \text{ Mp}$ at ϕ_{50}

The shear deformation is neglected, in inelastic regime the flexural deformation is dominant

2.4. Moment – Curvature Analysis

The assumed formula for stress-strain relationship for concrete and masonry is given by:

$$f_m = f'_m \left[2 \left(\frac{\varepsilon_m}{\varepsilon_p} \right) - \left(\frac{\varepsilon_m}{\varepsilon_p} \right)^2 \right] \geq 0 \quad (2.9)$$

f_m compressive stress in masonry, compression is positive

ε_m compressive strain in masonry, taken as an absolute value

f'_m specified compressive strength of clay masonry or of concrete masonry, psi (MPa)

ε_p compressive strain corresponds to specified compressive strength in masonry.

According to TMS402 [11], the ultimate value of strain is 0.0025 for concrete masonry and 0.0035 for clay masonry

When $f'_m = 2000 \text{ psi}$ and $\varepsilon_p = 0.002$, the stress strain curve looks like Figure 2.7

Figure 2.7: Typical stress strain curve of masonry

The stress strain curve of steel, assuming elastic perfectly plastic, is given by:

$$f_s = E_s \varepsilon_s \leq f_y \quad (2.10)$$

f_y yielding stress in steel

E_s elasticity modulus of steel

When $f_y=60$ ksi and $E_s = 29000$ ksi the curve is shown in Figure 2.8.

Figure 2.8: Typical stress strain curve of steel

For reinforced masonry section $l_w \times b$ as shown in Figure 2.9. the strain diagram along the section is shown, as well as the stresses in steel and in masonry. According to TMS 402 [11], the compression in steel should not be considered when calculating the section capacity.

Figure 2.9: Typical RM section, strain and stress in steel and masonry

The axial compressive force P and bending moment M for each curvature are defined from equilibrium by:

$$P = b \int_0^c f_m dx - \sum_{i=1}^n A_{bi} f_s \quad (2.11)$$

$$M = b \int_0^c x f_m dx + \sum_{i=1}^n A_{bi} f_s (d_i - c) + P \left(\frac{l_w}{2} - c \right) \quad (2.12)$$

With

$f_s = E_s \varepsilon_s$ with $|f_s| \leq f_y$ and $f_s = 0$ if negative according to TMS 402 [11]. f_m compression is positive, f_s tension is positive, P compression is positive. M is about neutral axis.

To create the moment curvature curve in numerical calculation, we divide the section into fibers of thickness Δx , then replace the integral by summation, so the above mentioned axial compressive force P and bending moment M turns into:

$$P = b \sum_0^c f_{mi} \Delta x - \sum_{i=1}^n A_{bi} f_{si} \quad (2.13)$$

$$M = b \Delta x \sum_0^c f_{m(x_i)} x_i + \sum_{i=1}^n A_{bi} f_{si} (d_i - c) + P \left(\frac{l_w}{2} - c \right) \quad (2.14)$$

For each value of curvature, we assume a value of compression zone length c . The calculated value of axial compression force P should be equal to the applied axial load P^* , we change the value of c until $P = P^*$. After defining the value of c , we can calculate the bending moment M that corresponds to the curvature. By increasing the curvature, we can get further values of the bending moment, and we can create the moment curvature curve.

2.5. Example, Rectangle Cross Section, Cantilever

The procedure explained in chapter 2.2 through chapter 2.4 will be used to show the moment curvature and force displacement curves for a simple rectangular reinforced masonry wall shown in Figure 2.10. The wall is fully grouted CMU (concrete masonry unit), with $f'_m = 2000$ psi and steel Grade 60 ($f_y = 60$ ksi). The masonry is assumed to have zero tensile strength, and the compressive stress-strain relation for masonry is given by the following expression

$$f_m = f'_m \left[2 \left(\frac{\varepsilon_m}{\varepsilon_p} \right) - \left(\frac{\varepsilon_m}{\varepsilon_p} \right)^2 \right] \geq 0 \quad (2.9)$$

with $\varepsilon_p = 0.002$.

Figure 2.10: Reinforced masonry wall section

a) Moment Curvature Curve

Moment-curvature curves have been created for two axial load scenarios: (i) the axial compressive load $P = 0.05 f'_m A_n = 103.7$ kips, and (ii) $P = 0.20 f'_m A_n = 414.8$ kips. The curves cover a curvature range from zero to a maximum value of 0.001. A_n is the net area of the wall cross section. The two curves are shown in Figure 2.11. When the axial force is small, the section shows less moment capacity, however it shows ductile behavior. When the axial compression force increases to $0.20 f'_m A_n$ the moment capacity

increases too, but it shows more brittle behavior which is not preferred for walls in seismic regions.

Drawing the moment force interaction diagrams shows almost similar values for maximal bending moment at specific axial force, Figure 2.12. For comparison, the moment is also calculated by hand as follows.

$$P_n = 0.05 f'_m A_n = 0.05 \cdot 2 \cdot (136 \cdot 7.625) = 103.7 \text{ kips}$$

$$a = \frac{P_n + A_s F_y}{\gamma f'_m b} = \frac{103.7 + (5 \cdot 1) \cdot 60}{0.8 \cdot 2 \cdot 7.625} = 33.09 \text{ in}$$

$$M_n = \gamma f'_m a b \left(\frac{l_w}{2} - \frac{a}{2} \right) = 0.8 \cdot 2 \cdot 33.09 \cdot 7.625 \cdot \left(\frac{136}{2} - \frac{33.09}{2} \right) = 20772 \text{ kip-in}$$

$$P_n = 0.20 f'_m A_n = 0.20 \cdot 2 \cdot (136 \cdot 7.625) = 414.8 \text{ kips}$$

$$a = \frac{P_n + A_s F_y}{\gamma f'_m b} = \frac{414.8 + (5 \cdot 1) \cdot 60}{0.8 \cdot 2 \cdot 7.625} = 58.59 \text{ in}$$

$$M_n = \gamma f'_m a b \left(\frac{l_w}{2} - \frac{a}{2} \right) = 0.8 \cdot 2 \cdot 58.59 \cdot 7.625 \cdot \left(\frac{136}{2} - \frac{58.59}{2} \right) = 30586 \text{ kip-in}$$

The results are summarized in the table:

	Moment from		
Axial compressive force	moment-curvature curve	moment-force interaction diagram	Hand calculation
$P = 0.05 f'_m A_n = 103.7 \text{ kips}$	23 657 kip-in	21 901 kip-in	20 772 kip-in
$P = 0.20 f'_m A_n = 414.8 \text{ kips}$	35 572 kip-in	30 586 kip-in	30 586 kip-in

Figure 2.11: Moment curvature curves for rectangular RM section for two cases: (i) axial compressive load $P = 0.05 f'_m A_n = 103.7$ kips, and (ii) $P = 0.20 f'_m A_n = 414.8$ kips.

Figure 2.12: Moment axial force interaction diagram, values are shown for: (i) $P = 0.05 f'_m A_n = 103.7$ kips, and (ii) $P = 0.20 f'_m A_n = 414.8$ kips.

b) Force Displacement Curve

The wall section is considered to be a cantilever wall. The wall height is 10 ft. and the wall is subjected to an in-plane horizontal load and an axial compressive load at the top. The effective plastic hinge length for the wall is assumed to be 20% of the wall height when the wall develops inelastic deformation. The two axial load scenarios in part (a) are considered. The self-weight of the wall has been ignored. The load-displacement curve for each of the above two axial load conditions has been calculated and plotted. The curve is piece-wise linear with lines connecting five points: the origin (zero displacement), nominal yield point defined as $0.8M_p$, the point at which the peak load occurs M_p , the last point corresponds to the curvature of 0.001, and a point with a curvature that is midway between the curvature at the peak load and 0.001. The curves are shown in Figure 2.13.

Figure 2.13: Force displacement curves for rectangular RM section, cantilever wall, two axial compressive load cases: (i) $P = 0.05 f'_m A_n = 103.7$ kips, and (ii) $P = 0.20 f'_m A_n = 414.8$ kips.

The force and displacements are calculated as explained in chapter 2.2 for cantilever wall. As with moment curvature curve, the force displacement curve for less axial force shows less resistance for lateral force, however it shows more ductility. With higher axial load, the lateral resistance is higher and shows more brittle behavior.

The ductile force displacement curve shows three distinct regions; elastic region, inelastic pre-peak region and inelastic post-peak region (ductile region). As shown in Figure 2.14

Figure 2.14: Force displacement curve of wall cross section.

2.6. Example, Rectangle Cross Section, Fixed-Fixed

The example before is repeated here, with different boundary conditions to be fixed-fixed wall instead of cantilever wall.

a) Moment Curvature Curve

Moment curvature will be the same as Figure 2.11.

b) Force Displacement Curve

The force displacement will be calculated as explained in chapter 2.3. The force displacement curve is shown in Figure 2.15.

Figure 2.16 shows the effect of boundary conditions on force displacement curves

Figure 2.15: Force displacement curves for rectangular RM section, fixed-fixed wall, two cases of axial compressive load: (i) $P = 0.05 f'_m A_n = 103.7$ kips, and (ii) $P = 0.20 f'_m A_n = 414.8$ kips.

Figure 2.16: Force displacement curves, cantilever & fixed-fixed wall. axial compressive load: (i) $P = 0.05 f'_m A_n = 103.7$ kips, and (ii) $P = 0.20 f'_m A_n = 414.8$ kips.

2.7. Example, T Cross Section, Cantilever

The same procedure explained in chapter 2.2 through chapter 2.4 will be used again to show the moment curvature and force displacement curves for a T-section reinforced masonry wall shown in Figure 2.17. The wall is fully grouted CMU (concrete masonry unit), with $f'_m = 2500$ psi and steel ($f_y = 68$ ksi). The masonry is assumed to have zero tensile strength, and the compressive stress-strain relation for masonry is given by the following expression

$$f_m = f'_m \left[2 \left(\frac{\varepsilon_m}{\varepsilon_p} \right) - \left(\frac{\varepsilon_m}{\varepsilon_p} \right)^2 \right] \geq 0 \quad (2.9)$$

with $\varepsilon_p = 0.002$.

This same section has been used in a building tested by shake table facility at the University of California in San Diego, as will be shown later in chapter 5.

Figure 2.17: Reinforced masonry wall section

a) Moment Curvature Curve

For reinforced masonry T-section as shown in Figure 2.18. the strain diagram along the section is shown, as well as the stresses in steel and in masonry. According to TMS 402 [11], the compression in steel should not be considered when calculating the section capacity.

Figure 2.18: RM T section, strain and stress in steel and masonry, compression in web

The axial compressive force P and bending moment M for each curvature are defined from equilibrium by:

$$P = b \int_0^c f_m dx - \sum_{i=1}^n A_{bi} f_s \quad (2.15)$$

$$M = b \int_0^c x f_m dx + \sum_{i=1}^n A_{bi} f_s (d_i - c) + P \left(\frac{l_w}{2} - c \right) \quad (2.16)$$

With

$f_s = E_s \varepsilon_s$ with $|f_s| \leq f_y$ and $f_s = 0$ if negative according to TMS 402 [11]. f_m compression is positive, f_s tension is positive, P compression is positive. M is about neutral axis.

To create the moment curvature curve in numerical calculation, we divide the section into fibers of thickness Δx , then replace the integral by summation, so the above mentioned axial compressive force P and bending moment M turns into:

$$P = b \sum_0^c f_m \Delta x - \sum_{i=1}^n A_{bi} f_{si} \quad \text{general equation} \quad (2.17)$$

$$P = b \sum_0^c f_m \Delta x - \sum_{i=1}^n A_{bi} f_{si} \quad (2.18)$$

T-section, compressed web, compression within web

$$P = b_f \sum_0^c f_m \Delta x + b \sum_0^c f_m \Delta x - \sum_{i=1}^n A_{bi} f_{si} - A_{bi(f)} f_{si(f)} \quad (2.19)$$

flange web web flange for T-section, compressed web,
compression reaches flange

$$M = b \Delta x \sum_0^c f_m(x_i) x_i + \sum_{i=1}^n A_{bi} f_{si} (d_i - c) + P(\frac{l_w}{2} - c) \quad \text{general equation} \quad (2.20)$$

$$M = \begin{aligned} & b_f \Delta x \sum_0^c f_m(x_i) x_i + A_{bi(f)} f_{si(f)} (d_i - c) + P(\bar{y} - c) \\ & \text{flange} \qquad \qquad \qquad \text{flange} \qquad \qquad \bar{y} \text{ for web compression} \\ & + b \Delta x \sum_0^c f_m(x_i) x_i + \sum_{i=1}^n A_{bi} f_{si} (d_i - c) \end{aligned} \quad (2.21)$$

web web

for T-section, compressed web, compression within web or reaches flange

$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} \text{ center of gravity for the T section, see figure above.}$$

For each value of curvature, we assume a value of compression zone length c . The calculated value of axial compression force P should be equal to the applied axial load P^* , we change the value of c until $P = P^*$. After defining the value of c , we can calculate the bending moment M that corresponds to the curvature. By increasing the curvature, we can get further values of the bending moment, and we can create the moment curvature curve.

Moment-curvature curves have been created for three axial load scenarios: (i) the axial compressive load $P = 0.05 f'_m A_n = 84.23$ kips, and (ii) $P = 0.20 f'_m A_n = 336.93$ kips and (iii) $P = 0.016 f'_m A_n = 26.95$ kips; this last scenario is the case corresponds to the test, so its curves and diagrams will be presented separately later in chapter 5. The curves cover a curvature range from zero to a maximum value of 0.001. A_n is the net area of the wall cross section. The compression stresses in T-section could be in the web or in the flange, both cases will be presented. The three curves are shown in Figure 2.18 and Figure 2.19 for compressed web and flange, consequently. When the axial force is small, the section shows less moment capacity, however it shows more ductile behavior. When the axial compression force increases to $0.20 f'_m A_n$ the moment capacity increases too, but it shows more brittle behavior which is not preferred for walls in seismic regions.

Drawing the moment force interaction diagrams shows almost similar values for maximal bending moment at specific axial force, see Figure 2.20 and Figure 2.21 for compression in web and flange, consequently. The results are summarized in the table below:

	Moment [kip-in.] from			
Axial compressive force	moment-curvature curve		moment-force interaction diagram	
	Compression in web	Compression in flange	Compression in web	Compression in flange
$P = 0.05 f'_m A_n$ = 84.23 kips	6 085	3 972	5 871	4 064
$P = 0.20 f'_m A_n$ = 336.93 kips	11 145	8 128	9 731	7 937
$P = 0.016 f'_m A_n$ = 26.95 kips	4 491	2 923	4 396	3 110

Figure 2.18: Moment curvature curves for T-section RM section for three cases: (i) axial compressive load $P = 0.05 f'_m A_n = 84.23$ kips, (ii) $P = 0.20 f'_m A_n = 336.93$ kips and (iii) $P = 0.016 f'_m A_n = 26.95$ kips. Compression in web.

Figure 2.19: Moment curvature curves for T-section RM section for three cases: (i) axial compressive load $P = 0.05 f'_m A_n = 84.23$ kips, (ii) $P = 0.20 f'_m A_n = 336.93$ kips and (iii) $P = 0.016 f'_m A_n = 26.95$ kips. Compression in flange.

Figure 2.20: Moment axial force interaction diagram, values for (i) $P = 0.05 f'_m A_n = 84.23$ kips, (ii) $P = 0.20 f'_m A_n = 336.93$ kips and (iii) $P = 0.016 f'_m A_n = 26.95$ kips are shown. Compression in web.

Figure 2.21: Moment axial force interaction diagram, values for (i) $P = 0.05 f'_m A_n = 84.23$ kips, (ii) $P = 0.20 f'_m A_n = 336.93$ kips and (iii) $P = 0.016 f'_m A_n = 26.95$ kips are shown. Compression in flange.

b) Force Displacement Curve

The wall section is considered to be a cantilever wall. The wall height is 8 ft. as in the test sample of the shake table and the wall is subjected to an in-plane horizontal load and an axial compressive load at the top. The effective plastic hinge length for the wall is assumed to be 20% of the wall height when the wall develops inelastic deformation. The three axial load scenarios in part (a) are considered. The self-weight of the wall has been ignored. The load-displacement curve for each of the above three axial load conditions has been calculated and plotted for two cases compression in web and compression in flange. Each curve is piece-wise linear with lines connecting five points: the origin (zero displacement), nominal yield point defined as $0.8M_p$, the point at which the peak load occurs M_p , the last point corresponds to the curvature of 0.001, and a point with a curvature that is midway between the curvature at the peak load and 0.001. The curves

for web compression and flange compression are shown in Figures 2.22 and Figure 2.23 respectively.

Figure 2.22: Force displacement curves for T-section RM section, cantilever wall, three cases of axial load: (i) $P = 0.05 f'_m A_n = 84.23$ kips, (ii) $P = 0.20 f'_m A_n = 336.93$ kips, (iii) $P = 0.016 f'_m A_n = 26.95$ kips. Compression in web.

Figure 2.23: Force displacement curves for T-section RM section, cantilever wall, three cases of axial load: (i) $P = 0.05 f'_m A_n = 84.23$ kips, (ii) $P = 0.20 f'_m A_n = 336.93$ kips, (iii) $P = 0.016 f'_m A_n = 26.95$ kips. Compression in flange.

The force and displacements are calculated as explained in chapter 2.2 for cantilever wall. As with moment curvature curve, the force displacement curve for less axial force shows less resistance for lateral force, however it shows more ductility. With higher axial load, the lateral resistance is higher and shows more brittle behavior.

The force displacement curve for the axial force of $P = 0.016 f'_m A_n = 26.95$ kips, that is the case in the test, will be shown in chapter 6.

2.8. Example, T Cross Section, Fixed-Fixed

The example before is repeated here, with different boundary conditions to be fixed-fixed wall instead of cantilever wall.

a) Moment Curvature Curve

Moment curvature will be the same as Figures 2.18 and Figure 2.19.

b) Force Displacement Curve

The force displacement will be calculated as explained in chapter 2.3. The force displacement curves for web compression and flange compression are shown in Figures 2.24 and 2.25.

Figure 2.24: Force displacement curves for T section RM section, fixed-fixed wall, three cases: (i) $P = 0.05 f'_m A_n = 84.23$ kips, (ii) $P = 0.20 f'_m A_n = 336.93$ kips, (iii) $P = 0.016 f'_m A_n = 26.95$ kips. Compression in web.

Figure 2.25: Force displacement curves for T section RM section, fixed-fixed wall, three cases: (i) $P = 0.05 f'_m A_n = 84.23$ kips, (ii) $P = 0.20 f'_m A_n = 336.93$ kips, (iii) $P = 0.016 f'_m A_n = 26.95$ kips. Compression in flange.

Figure 2.26 and Figure 2.27 shows the effect of boundary conditions on force displacement curves

Figure 2.26: Force displacement curves, cantilever & fixed-fixed wall, three cases: (i) $P = 0.05 f'_m A_n = 84.23$ kips, (ii) $P = 0.20 f'_m A_n = 336.93$ kips, (iii) $P = 0.016 f'_m A_n = 26.95$ kips. Compression in web.

Figure 2.27: Force displacement curves, cantilever & fixed-fixed wall, three cases: (i) $P = 0.05 f'_m A_n = 84.23$ kips, (ii) $P = 0.20 f'_m A_n = 336.93$ kips, (iii) $P = 0.016 f'_m A_n = 26.95$ kips. Compression in flange.

3. Matrix Formulation

3.1. Procedure

The general approach in analysis using displacement matrix method is as follows [13]:

- 1- Structure discretization, and modeling each single wall as an element
- 2- Formulate the stiffness matrix $\underline{k}^{(e)}$ for each element, in our case a single shear wall, in its local coordinates (x', y', z')
- 3- Do coordinate transformation to convert $\underline{k}^{(e)}$ in local coordinate to $\underline{k}^{(e)}$ in global coordinate system (x, y, z).

- 4- Assemble the structure stiffness matrix $\underline{K}$

$$\underline{K} = \prod_e \underline{k}^{(e)}$$

$\prod_e \underline{k}^{(e)}$: assemble the stiffness matrices $\underline{k}^{(e)}$ over all elements e .

- 5- Assemble the structure nodal vector $\underline{F}$

- 6- Solve the nodal displacement

$$\underline{F} = \underline{K} \cdot \underline{U}$$

- 7- Extract element nodal displacements $\underline{u}^{(e)}$ from the global displacement system $\underline{U}$

- 8- Transform nodal displacements $\underline{u}^{(e)}$ to element local coordinate system $\underline{u}'^{(e)}$

- 9- Compute element nodal forces $\underline{f}'^{(e)}$

$$\underline{f}'^{(e)} = \underline{k}'^{(e)} \cdot \underline{u}'^{(e)}$$

- 10- Compute element internal forces; moment M , shear V and axial N forces from $\underline{f}'^{(e)}$ using equilibrium.

In the following, we will show each of these steps as will be used in Fortran [14] code and solving the forces for a braced building.

3.2. Structure Discretization

In an analogous manner to the finite element method, each wall in the structure will be modelled as a single element. The first-floor wall will be modelled as fixed-fixed, except when there is a basement below it, then it is modeled as pinned-fixed. The repeated floor wall is modeled as fixed-fixed. Last floor wall is modeled as fixed-pinned, except when there is a heavy stiff ceiling, then it is modeled as fixed-fixed. These are shown in Figure 3.1. The ceiling is assumed to be stiff in all stories. There is no transfer of bending moment from the ceiling to the wall perpendicular to the wall level.

Figure 3.1: Wall discretization depending on floor and boundary conditions

3.3. Element, i.e.; Wall, Local Stiffness Matrix

The stiffness matrix $k^{(e)}$ for each element, single shear wall, could be taken from the finite element calculation of a two-dimensional beam element shown in Figure 3.2.

Figure 3.2: Two-dimensional beam element with the considered degrees of freedom

With the coefficient $Q = \frac{12 \cdot EI}{H^2 \cdot G \cdot A_s}$, the shear modulus $G = \frac{E}{2 \cdot (1 + \nu)}$, the shear area $A_s = A / \kappa$, the shear correction factor $\kappa = \frac{6}{5}$ ⁽¹⁾ for rectangular cross section, Poisson ratio ν , beam

¹ shear correction factor $\kappa = 1 / \kappa_i$. κ_i Timoschenko- shear factor is given in different references for different cross sections. For a rectangular section, the formula to calculate κ_i is in Jensen. Eq. 7 [16] $\kappa_i = \frac{1}{A \int \tau_0^2 dA}$ where A section area, τ_0 shear stress due to unit shear force (index 0)

length (wall height) H , moment of inertia I , elasticity modulus E . The four sub-matrices of the local stiffness matrix of a fixed-fixed beam element, considering shear deformation, are given by [15]:

$$\underline{k}^{(e)} = \begin{bmatrix} \underline{k}^{(e)}(ik, ik) & \underline{k}^{(e)}(ik, ki) \\ \underline{k}^{(e)}(ki, ik) & \underline{k}^{(e)}(ki, ki) \end{bmatrix}$$

$$\begin{aligned} \underline{k}^{(e)}(ik, ik) &= \begin{bmatrix} \frac{EA}{H} & 0 & 0 \\ 0 & \frac{12EI}{(1+Q) \cdot H^3} & \frac{6EI}{(1+Q) \cdot H^2} \\ 0 & \frac{6EI}{(1+Q) \cdot H^2} & \frac{(4+Q)EI}{(1+Q) \cdot H} \end{bmatrix} & \underline{k}^{(e)}(ik, ki) &= \begin{bmatrix} \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{(1+Q) \cdot H^3} & \frac{6EI}{(1+Q) \cdot H^2} \\ 0 & \frac{-6EI}{(1+Q) \cdot H^2} & \frac{(2-Q)EI}{(1+Q) \cdot H} \end{bmatrix} \\ \underline{k}^{(e)}(ki, ik) &= \begin{bmatrix} \frac{-EA}{H} & 0 & 0 \\ 0 & \frac{-12EI}{(1+Q) \cdot H^3} & \frac{-6EI}{(1+Q) \cdot H^2} \\ 0 & \frac{6EI}{(1+Q) \cdot H^2} & \frac{(2-Q)EI}{(1+Q) \cdot H} \end{bmatrix} & \underline{k}^{(e)}(ki, ki) &= \begin{bmatrix} \frac{EA}{H} & 0 & 0 \\ 0 & \frac{12EI}{(1+Q) \cdot H^3} & \frac{-6EI}{(1+Q) \cdot H^2} \\ 0 & \frac{-6EI}{(1+Q) \cdot H^2} & \frac{(4+Q)EI}{(1+Q) \cdot H} \end{bmatrix} \end{aligned} \quad (3.1)$$

Shear stress is given by: $\tau = F \cdot \frac{A\bar{y}}{zI}$,

with τ (with $F = 1$) = τ_0 , $A = b(\frac{d}{2} - y)$, $z = b$, $\bar{y} = \frac{1}{2}(\frac{d}{2} + y)$,

$$I = \frac{bd^3}{12}$$

we get the shear stress due to unit force as: $\tau_0 = 1 \cdot \frac{b(\frac{d}{2}-y)\frac{1}{2}(\frac{d}{2}+y)}{b \frac{bd^3}{12}} = \frac{6(\frac{d^2}{4}-y^2)}{bd^3}$

$$\begin{aligned} A \int \tau_0^2 dA &= (b d) \cdot 2b \int_0^{\frac{d}{2}} \left[\frac{6(\frac{d^2}{4}-y^2)}{bd^3} \right]^2 dy &= (b d) \cdot 2b \cdot 36 \int_0^{\frac{d}{2}} \left[\frac{(\frac{d^2}{4}-y^2)}{bd^3} \right]^2 dy \\ &= \frac{2.36}{d^5} \int_0^{\frac{d}{2}} \left(\frac{d^4}{16} + y^4 - \frac{d^2}{2} y^2 \right) dy \\ &= \frac{2.36}{d^5} \left[\frac{d^4 y}{16} + \frac{y^5}{5} - \frac{d^2 y^3}{2 \cdot 3} \right]_0^{\frac{d}{2}} \\ &= \frac{2.36}{d^5} \left[\frac{d^4 d/2}{16} + \frac{d/2^5}{5} - \frac{d^2 d/2^3}{2 \cdot 3} \right]_0^{\frac{d}{2}} \\ &= \frac{2.36}{d^5} \left[\left(\frac{d^5 d/2}{16 \cdot 2} + \frac{d^5}{32 \cdot 5} - \frac{d^2 d^3}{8 \cdot 2 \cdot 3} \right) - 0 \right] = \frac{6}{5} \end{aligned}$$

So $\kappa i = \frac{5}{6}$ (Timoschenko- shear factor) and $\kappa = \frac{6}{5}$ (shear correction factor). Timoschenko- shear factor for circular section is 9/10. For walls with T-section it is assumed to be 1.

The element stiffness matrices for beam element with different boundary conditions are calculated with the formula [15]:

$$\underline{k}_u = \underline{k}'^{(e)} - \underline{\Delta k}_u \quad (3.2)$$

$k'^{(e)}$ is the local element stiffness as calculated above for fixed-fixed beam element, Δk_u is calculated with the formula:

$$\underline{\Delta k}_u = \frac{1}{K_j + k_{jj}} \cdot \underline{k}_j \cdot \underline{k}_j^T \quad (3.3)$$

K_j is the spring stiffness of the connected construction, or $K_j = 0$ when no spring exists (pinned).

k_{jj} j-th main diagonal coefficient of the stiffness matrix

$\underline{k}_j$ j-th column of the stiffness matrix

For the case of fixed-pinned beam we have:

j=6 (rotation at point k is the 6th DOF)

$K_j = K_6 = 0$ (no rotational spring exists).

$$k_{jj} = k_{66} = \frac{(4+Q)EI}{(1+Q) \cdot H}$$

$$\underline{k}_j^T = \underline{k}_6^T = \left(0 \quad \frac{6EI}{(1+Q) \cdot H^2} \quad \frac{(2-Q)EI}{(1+Q) \cdot H} \quad 0 \quad \frac{-6EI}{(1+Q) \cdot H^2} \quad \frac{(4+Q)EI}{(1+Q) \cdot H} \right)$$

$$\underline{\Delta k}_u = \frac{1}{0 + \frac{(4+Q)EI}{(1+Q) \cdot H}} \cdot \begin{pmatrix} 0 \\ \frac{6EI}{(1+Q) \cdot H^2} \\ \frac{(2-Q)EI}{(1+Q) \cdot H} \\ 0 \\ \frac{-6EI}{(1+Q) \cdot H^2} \\ \frac{(4+Q)EI}{(1+Q) \cdot H} \end{pmatrix} \cdot \left(0 \quad \frac{6EI}{(1+Q) \cdot H^2} \quad \frac{(2-Q)EI}{(1+Q) \cdot H} \quad 0 \quad \frac{-6EI}{(1+Q) \cdot H^2} \quad \frac{(4+Q)EI}{(1+Q) \cdot H} \right)$$

$$\underline{\Delta k}_u = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{36EI}{(4+Q)(1+Q)H^3} & \frac{6(2-Q)EI}{(4+Q)(1+Q)H^2} & 0 & \frac{-36EI}{(4+Q)(1+Q)H^3} & \frac{6EI}{(1+Q)H^2} \\ 0 & \frac{6(2-Q)EI}{(4+Q)(1+Q)H^2} & \frac{(2-Q)^2EI}{(4+Q)(1+Q)H^2} & 0 & \frac{-6(2-Q)EI}{(4+Q)(1+Q)H^2} & \frac{(2-Q)EI}{(1+Q)H} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-36EI}{(4+Q)(1+Q)H^3} & \frac{-6(2-Q)EI}{(4+Q)(1+Q)H^2} & 0 & \frac{36EI}{(4+Q)(1+Q)H^3} & \frac{-6EI}{(1+Q)H^2} \\ 0 & \frac{6EI}{(1+Q)H^2} & \frac{(2-Q)EI}{(1+Q)H} & 0 & \frac{-6EI}{(1+Q)H^2} & \frac{(4+Q)EI}{(1+Q)H} \end{bmatrix}$$

And we have already $\underline{k}'^{(e)}$ given by:

$$\underline{k}'^{(e)} = \begin{bmatrix} \frac{EA}{H} & 0 & 0 & \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{(1+Q) \cdot H^3} & \frac{6EI}{(1+Q) \cdot H^2} & 0 & \frac{-12EI}{(1+Q) \cdot H^3} & \frac{6EI}{(1+Q) \cdot H^2} \\ 0 & \frac{6EI}{(1+Q) \cdot H^2} & \frac{(4+Q)EI}{(1+Q) \cdot H} & 0 & \frac{-6EI}{(1+Q) \cdot H^2} & \frac{(2-Q)EI}{(1+Q) \cdot H} \\ \frac{-EA}{H} & 0 & 0 & \frac{EA}{H} & 0 & 0 \\ 0 & \frac{-12EI}{(1+Q) \cdot H^3} & \frac{-6EI}{(1+Q) \cdot H^2} & 0 & \frac{12EI}{(1+Q) \cdot H^3} & \frac{-6EI}{(1+Q) \cdot H^2} \\ 0 & \frac{6EI}{(1+Q) \cdot H^2} & \frac{(2-Q)EI}{(1+Q) \cdot H} & 0 & \frac{-6EI}{(1+Q) \cdot H^2} & \frac{(4+Q)EI}{(1+Q) \cdot H} \end{bmatrix}$$

By applying the equation $\underline{k}_u = \underline{k}'^{(e)} - \Delta \underline{k}_u$ we get the stiffness matrix of fixed-pinned beam as follows:

$$\underline{k}'^{(e)} = \begin{bmatrix} \frac{EA}{H} & 0 & 0 & \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{(4+Q) \cdot H^3} & \frac{12EI}{(4+Q) \cdot H^2} & 0 & \frac{-12EI}{(4+Q) \cdot H^3} & 0 \\ 0 & \frac{12EI}{(4+Q) \cdot H^2} & \frac{12EI}{(4+Q) \cdot H} & 0 & \frac{-12EI}{(4+Q) \cdot H^2} & 0 \\ \frac{-EA}{H} & 0 & 0 & \frac{EA}{H} & 0 & 0 \\ 0 & \frac{-12EI}{(4+Q) \cdot H^3} & \frac{-12EI}{(4+Q) \cdot H^2} & 0 & \frac{12EI}{(4+Q) \cdot H^3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(3.4)

In the same way, the stiffness matrix of a pinned-fixed beam could be calculated. Other way is to use the transformation matrix:

$$\underline{k}'^{(e)}_{new} = \underline{T} \cdot \underline{k}'^{(e)} \cdot \underline{T}^T$$

$$\text{With } \underline{T} = \begin{bmatrix} \underline{T}_0 & 0 \\ 0 & \underline{T}_0 \end{bmatrix}$$

$$\underline{T}_0 = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \alpha = 180^\circ \Rightarrow \underline{T}_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{k}'^{(e)} = \begin{bmatrix} \frac{EA}{H} & 0 & 0 & \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{(4+Q) \cdot H^3} & 0 & 0 & \frac{-12EI}{(4+Q) \cdot H^3} & \frac{12EI}{(4+Q) \cdot H^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-EA}{H} & 0 & 0 & \frac{EA}{H} & 0 & 0 \\ 0 & \frac{-12EI}{(4+Q) \cdot H^3} & 0 & 0 & \frac{12EI}{(4+Q) \cdot H^3} & \frac{-12EI}{(4+Q) \cdot H^2} \\ 0 & \frac{12EI}{(4+Q) \cdot H^2} & 0 & 0 & \frac{-12EI}{(4+Q) \cdot H^2} & \frac{12EI}{(4+Q) \cdot H} \end{bmatrix}$$

(3.5)

3.4. Stiffness Matrix of Fixed-Pinned Beam

3.4.1. Elastic

The element stiffness matrix of a fixed-pinned beam is shown in previous chapter as:

$$\underline{k}^{(e)} = \begin{bmatrix} \underline{k}^{(e)}(ik, ik) & \underline{k}^{(e)}(ik, ki) \\ \underline{k}^{(e)}(ki, ik) & \underline{k}^{(e)}(ki, ki) \end{bmatrix}$$

$$\underline{k}^{(e)}(ik, ik) = \begin{bmatrix} \frac{EA}{H} & 0 & 0 \\ 0 & \frac{12EI}{(4+Q) \cdot H^3} & \frac{12EI}{(4+Q) \cdot H^2} \\ 0 & \frac{12EI}{(4+Q) \cdot H^2} & \frac{12EI}{(4+Q) \cdot H} \end{bmatrix}$$

$$\underline{k}^{(e)}(ik, ki) = \begin{bmatrix} -\frac{EA}{H} & 0 & 0 \\ 0 & \frac{-12EI}{(4+Q) \cdot H^3} & 0 \\ 0 & \frac{-12EI}{(4+Q) \cdot H^2} & 0 \end{bmatrix}$$

$$\underline{k}^{(e)}(ki, ik) = \begin{bmatrix} -\frac{EA}{H} & 0 & 0 \\ 0 & \frac{-12EI}{(4+Q) \cdot H^3} & \frac{-12EI}{(4+Q) \cdot H^2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{k}^{(e)}(ki, ki) = \begin{bmatrix} \frac{EA}{H} & 0 & 0 \\ 0 & \frac{12EI}{(4+Q) \cdot H^3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The stiffness matrix without the normal force part is:

$$\underline{k}^{(e)} = \begin{bmatrix} \frac{12EI}{(4+Q) \cdot H^3} & \frac{12EI}{(4+Q) \cdot H^2} & \frac{-12EI}{(4+Q) \cdot H^3} & 0 \\ \frac{12EI}{(4+Q) \cdot H^2} & \frac{12EI}{(4+Q) \cdot H} & \frac{-12EI}{(4+Q) \cdot H^2} & 0 \\ \frac{-12EI}{(4+Q) \cdot H^3} & \frac{-12EI}{(4+Q) \cdot H^2} & \frac{12EI}{(4+Q) \cdot H^3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The stiffness component k_{ij} is by definition the force at DOF i due to unit DOF at DOF j while all other DOFs are zero. This means the stiffness components could be extracted by imposing unit deformation and getting the forces at the beam nodes.

$$\underline{k}^{(e)} = \left[\begin{array}{c} Q_i \\ M_i \\ Q_k \\ M_k \end{array} \right\} \text{ due to } v_i = 1 \quad \begin{array}{c} Q_i \\ M_i \\ Q_k \\ M_k \end{array} \left\} \text{ due to } \varphi_i = 1 \quad \begin{array}{c} Q_i \\ M_i \\ Q_k \\ M_k \end{array} \left\} \text{ due to } v_k = 1 \quad \begin{array}{c} Q_i \\ M_i \\ Q_k \\ M_k \end{array} \left\} \text{ due to } \varphi_k = 1 \right] \quad \begin{array}{c} Q_i \\ M_i \\ Q_k \\ M_k \end{array}$$

We consider the 3rd column of the stiffness matrix. We apply force Q_k then calculate the displacement v_k depending on Timoshenko beam theory. To get the stiffness at component $k(3,3)$ for example, it is the shear force Q_k when the displacement is unit $v_k = 1$ then $k(3,3) = Q_k$.

Figure 3.3: Fixed-pinned beam, internal forces and nodal forces

From Timoshenko beam theory:

$$M_x = -EI \frac{\partial \varphi}{\partial x} \quad Q_x = \kappa i \cdot AG \left(-\varphi + \frac{\partial v}{\partial x} \right) \quad (3.6)$$

φ , v , κi are rotation, displacement and Timoshenko shear factor.

The bending moment at location x $M_x = -Q_k x \Rightarrow Q_k x = EI \frac{\partial \varphi}{\partial x}$

$$\varphi(x) = \int \partial \varphi = \frac{Q_k x^2}{2 EI} + c_1$$

$$\text{BC.: } \left. \begin{matrix} x = H \\ \varphi = 0 \end{matrix} \right\} \Rightarrow 0 = \frac{Q_k H^2}{2 EI} + c_1 \Rightarrow c_1 = -\frac{Q_k H^2}{2 EI}$$

$$\varphi(x) = \frac{Q_k x^2}{2 EI} - \frac{Q_k H^2}{2 EI} = \frac{Q_k}{2 EI} (x^2 - H^2)$$

$$\text{The shear force } Q_x = -Q_k \Rightarrow -Q_k = \kappa i AG \left(-\varphi + \frac{\partial v}{\partial x} \right) \Rightarrow \frac{\partial v}{\partial x} = \varphi - \frac{Q_k}{\kappa i AG}$$

$$\text{By putting the formula of } \varphi(x) \Rightarrow \frac{\partial v}{\partial x} = -\frac{Q_k}{\kappa i AG} - \frac{Q_k}{2 EI} (H^2 - x^2)$$

$$v(x) = \int \frac{\partial v}{\partial x} = -\frac{Q_k}{\kappa i AG} x - \frac{Q_k}{2 EI} \left(H^2 x - \frac{x^3}{3} + c_2 \right)$$

$$\text{BC.: } \left. \begin{matrix} x = H \\ v = 0 \end{matrix} \right\} \Rightarrow 0 = -\frac{Q_k}{\kappa i AG} H - \frac{Q_k}{2 EI} \left(H^2 H - \frac{H^3}{3} + c_2 \right) \Rightarrow c_2 = -\frac{2}{3} H^3 - \frac{2 EI}{\kappa i AG}$$

$$\Rightarrow v(x) = \frac{Q_k (H-x)}{\kappa i AG} - \frac{Q_k x}{2 EI} \left(H^2 - \frac{x^2}{3} \right) + \frac{Q_k H^3}{3 EI}$$

$$\text{For } x = 0 \quad v_k = \frac{Q_k H}{\kappa i AG} + \frac{Q_k H^3}{3 EI} \Rightarrow Q_k = \frac{v}{\frac{H}{\kappa i AG} + \frac{H^3}{3 EI}}$$

$$\Rightarrow \text{Stiffness } k = \frac{Q_k}{v} = \frac{1}{\frac{H}{\kappa i AG} + \frac{H^3}{3 EI}} \text{ or } Q_k (v_k = 1) = \frac{1}{\frac{H}{\kappa i AG} + \frac{H^3}{3 EI}} \quad \kappa i = 1/\kappa$$

$$\text{This is the same component } k(3,3) = \frac{12EI}{(4+Q) \cdot H^3} = \frac{12EI}{\left(4 + \frac{12 \cdot EI}{H^2 \cdot G \cdot A_s}\right) \cdot H^3} = \frac{1}{\left(\frac{4}{12 \cdot EI} + \frac{1}{H^2 \cdot G \cdot A_s}\right) \cdot H^3} = \frac{1}{\frac{H}{\kappa i AG} + \frac{H^3}{3 \cdot EI}}$$

Other components in column 3 could be defined from equilibrium

$$Q_i(v_k = 1) = -Q_k(v_k = 1) = -\frac{12EI}{(4+Q) \cdot H^3} = k(1,3)$$

$$M_i(v_k = 1) = -Q_k(v_k = 1) \cdot H = -\frac{12EI}{(4+Q) \cdot H^2} = k(2,3) \quad , \quad M_k(v_k = 1) = 0 = k(4,3)$$

Column 1 could be extracted in the same way with noting the different sign:

$$k(1,1) = \frac{12EI}{(4+Q) \cdot H^3} \quad , \quad k(2,1) = \frac{12EI}{(4+Q) \cdot H^2} \quad , \quad k(3,1) = -\frac{12EI}{(4+Q) \cdot H^3} \quad , \quad k(4,1) = 0$$

To calculate column 2, we apply bending moment M_i , this leads to a shear force of $Q_k = -\frac{M_i}{H}$. This means, we may take the column 3 and multiply it with $-\frac{1}{H}$:

$$k(1,2) = \frac{12EI}{(4+Q) \cdot H^2} \quad , \quad k(2,2) = \frac{12EI}{(4+Q) \cdot H} \quad , \quad k(3,2) = -\frac{12EI}{(4+Q) \cdot H^2} \quad , \quad k(4,2) = 0$$

This will give the same stiffness matrix as shown in previous chapter.

3.4.2. Pre-Peak Inelastic

Assuming linear inelastic material behavior after the elastic limit, we may use the same stiffness matrix but with different elasticity modulus.

Figure 3.4: Force displacement curve of wall cross section.

The inclination of the force-displacement curve in the elastic region is given by:

$$k_{el} = \frac{3 E_m I_e}{H^3} = \frac{F_y}{U_y} \quad (3.7)$$

The new elasticity modulus could be extracted from the force displacement curve as:

$$k_{inel} = \left(\frac{3 E_m I_e}{H^3} \right)_{inel} = \frac{F_p}{U_p} = k_{el} * \frac{U_y}{F_y} \cdot \frac{F_p}{U_p} = k_{el} * \frac{U_y}{U_p} \cdot \frac{F_p}{0.8 F_p} = k_{el} * \frac{U_y}{U_p - U_y} \cdot \frac{1}{0.8}$$

$$k_{inel} = k_{el} * \frac{U_y}{U_p - U_y} \frac{1}{0.8} \quad (3.8)$$

3.4.3. Post-Peak Inelastic

We follow the same procedure done with the elastic stiffness, however, here we use the resulted displacement after the maximum bending moment capacity. The displacement depends on the shear force Q_k and the normal force N , so it is written in the form $\delta(N, Q)$. In case we have shear force, normal force and bending moment it will be $\delta(N, Q, M)$.

Figure 3.5: Deformed shape and height of plastic zone

$$\underline{k^{(e)}} = \begin{bmatrix} k(1,1) & k(1,2) & k(1,3) & 0 \\ k(2,1) & k(2,2) & k(2,3) & 0 \\ k(3,1) & k(3,2) & k(3,3) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Starting with column 3

$$k(3,3) = \frac{Q_k}{\delta(N,Q)}$$

$$k(1,3) = -k(3,3) = -\frac{Q_k}{\delta(N,Q)}$$

$$k(2,3) = -k(3,3) \cdot H = -\frac{Q_k \cdot H}{\delta(N,Q)}$$

Column 1 is like column 3 but with different sign:

$$k(1,1) = \frac{Q_k}{\delta(N,Q)}$$

$$k(2,1) = \frac{Q_k \cdot H}{\delta(N,Q)}$$

$$k(3,1) = -\frac{Q_k}{\delta(N,Q)}$$

Column 2 is the result of column 3 multiplied by $-\frac{1}{H}$:

$$k(1,2) = \frac{Q_k \cdot H}{\delta(N,Q)}$$

$$k(2,2) = \frac{Q_k \cdot H^2}{\delta(N,Q)}$$

$$k(3,2) = -\frac{Q_k \cdot H}{\delta(N,Q)}$$

3.4.4. Normal Force Part in the Stiffness Matrix

In elastic case the axial force stiffness $k = \frac{EA}{H}$, in post-peak inelastic state the height of the plastic zone is assumed to be $H_1 = L_p = 0.2 \cdot H$, with effective cross section equal to half the wall cross section (assuming half the wall section is under compression and half in tension). In this case the total longitudinal displacement over the total height $\delta = \frac{N H_1}{E A_1} + \frac{N H_2}{E A_2}$, and the stiffness is

$$k = \frac{N}{\delta} = \frac{N}{\frac{N H_1}{E A_1} + \frac{N H_2}{E A_2}} = \frac{E}{\frac{H_1}{A_1} + \frac{H_2}{A_2}}$$

With $H_1 = L_p = 0.2 \cdot H$, $H_2 = H - H_1 = 0.8 \cdot H$, $A_1 = 0.5 \cdot A$, $A_2 = A$

And the axial stiffness becomes: $k = \frac{E}{\frac{0.2 \cdot H}{0.5 \cdot A} + \frac{0.8 \cdot H}{A}} = \frac{EA}{1.2 \cdot H}$

The total stiffness matrix could be written, without the shear deformation, as:

$$\underline{k'^{(e)}} = \begin{bmatrix} \underline{k'^{(e)}}(ik, ik) & \underline{k'^{(e)}}(ik, ki) \\ \underline{k'^{(e)}}(ki, ik) & \underline{k'^{(e)}}(ki, ki) \end{bmatrix}$$

$$\underline{k'^{(e)}}(ik, ik) = \begin{bmatrix} \frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & \frac{Q_k}{\delta(N,Q)} & \frac{Q_k \cdot H}{\delta(N,Q)} \\ 0 & \frac{Q_k \cdot H}{\delta(N,Q)} & \frac{Q_k \cdot H^2}{\delta(N,Q)} \end{bmatrix}$$

$$\underline{k'^{(e)}}(ik, ki) = \begin{bmatrix} -\frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & -\frac{Q_k}{\delta(N,Q)} & 0 \\ 0 & -\frac{Q_k \cdot H}{\delta(N,Q)} & 0 \end{bmatrix}$$

$$\underline{k'^{(e)}}(ki, ik) = \begin{bmatrix} -\frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & -\frac{Q_k}{\delta(N,Q)} & -\frac{Q_k \cdot H}{\delta(N,Q)} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{k'^{(e)}}(ki, ki) = \begin{bmatrix} \frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & \frac{Q_k}{\delta(N,Q)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.9)$$

The total stiffness matrix could be written, when the shear deformation is considered, as:

$$\underline{k'}^{(e)} = \begin{bmatrix} \underline{k'}^{(e)}(ik, ik) & \underline{k'}^{(e)}(ik, ki) \\ \underline{k'}^{(e)}(ki, ik) & \underline{k'}^{(e)}(ki, ki) \end{bmatrix}$$

$$\begin{aligned} \underline{k'}^{(e)}(ik, ik) &= \begin{bmatrix} \frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & \frac{Q_k}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} & \frac{Q_k \cdot H}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} \\ 0 & \frac{Q_k \cdot H}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} & \frac{Q_k \cdot H^2}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} \end{bmatrix} & \underline{k'}^{(e)}(ik, ki) &= \begin{bmatrix} -\frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & -\frac{Q_k}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} & 0 \\ 0 & -\frac{Q_k \cdot H}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} & 0 \end{bmatrix} \\ \underline{k'}^{(e)}(ki, ik) &= \begin{bmatrix} -\frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & -\frac{Q_k}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} & -\frac{Q_k \cdot H}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} \\ 0 & 0 & 0 \end{bmatrix} & \underline{k'}^{(e)}(ki, ki) &= \begin{bmatrix} \frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & \frac{Q_k}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (3.10)$$

3.5. Stiffness Matrix of Pinned-Fixed Beam

3.5.1. Elastic

In a similar way used in chapter 3.3.1 or as extracted in chapter 3.3, the stiffness matrix for pinned- fixed beam when considering the shear deformation is taken as:

$$\underline{k'}^{(e)} = \begin{bmatrix} \underline{k'}^{(e)}(ik, ik) & \underline{k'}^{(e)}(ik, ki) \\ \underline{k'}^{(e)}(ki, ik) & \underline{k'}^{(e)}(ki, ki) \end{bmatrix}$$

$$\begin{aligned} \underline{k'}^{(e)}(ik, ik) &= \begin{bmatrix} \frac{EA}{H} & 0 & 0 \\ 0 & \frac{12EI}{(4+Q) \cdot H^3} & 0 \\ 0 & 0 & 0 \end{bmatrix} & \underline{k'}^{(e)}(ik, ki) &= \begin{bmatrix} -\frac{EA}{H} & 0 & 0 \\ 0 & \frac{-12EI}{(4+Q) \cdot H^3} & \frac{12EI}{(4+Q) \cdot H^2} \\ 0 & 0 & 0 \end{bmatrix} \\ \underline{k'}^{(e)}(ki, ik) &= \begin{bmatrix} -\frac{EA}{H} & 0 & 0 \\ 0 & \frac{-12EI}{(4+Q) \cdot H^3} & 0 \\ 0 & \frac{12EI}{(4+Q) \cdot H^2} & 0 \end{bmatrix} & \underline{k'}^{(e)}(ki, ki) &= \begin{bmatrix} \frac{EA}{H} & 0 & 0 \\ 0 & \frac{12EI}{(4+Q) \cdot H^3} & \frac{-12EI}{(4+Q) \cdot H^2} \\ 0 & \frac{-12EI}{(4+Q) \cdot H^2} & \frac{12EI}{(4+Q) \cdot H} \end{bmatrix} \end{aligned} \quad (3.11)$$

3.5.2. Post-Peak Inelastic

The stiffness matrix of pinned-fixed beam could be extracted in a similar way explained in chapter 3.4.3:

$$\underline{k}^{(e)} = \begin{bmatrix} \underline{k}^{(e)}(ik, ik) & \underline{k}^{(e)}(ik, ki) \\ \underline{k}^{(e)}(ki, ik) & \underline{k}^{(e)}(ki, ki) \end{bmatrix}$$

$$\begin{aligned} \underline{k}^{(e)}(ik, ik) &= \begin{bmatrix} \frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & \frac{Q_i}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} & 0 \\ 0 & 0 & 0 \end{bmatrix} & \underline{k}^{(e)}(ik, ki) &= \begin{bmatrix} -\frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & -\frac{Q_i}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} & \frac{Q_i \cdot H}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} \\ 0 & 0 & 0 \end{bmatrix} \\ \underline{k}^{(e)}(ki, ik) &= \begin{bmatrix} -\frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & -\frac{Q_i}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} & 0 \\ 0 & \frac{Q_i \cdot H}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} & 0 \end{bmatrix} & \underline{k}^{(e)}(ki, ki) &= \begin{bmatrix} \frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & \frac{Q_i}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} & -\frac{Q_i \cdot H}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} \\ 0 & -\frac{Q_i \cdot H}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} & \frac{Q_i \cdot H^2}{\delta(N, Q)} \cdot \frac{4}{(4+Q)} \end{bmatrix} \end{aligned} \quad (3.12)$$

3.6. Stiffness Matrix of Fixed-Fixed Beam

3.6.1. Elastic

For fixed-fixed beam element, the stiffness matrix is given by:

$$\underline{k}^{(e)} = \begin{bmatrix} \underline{k}^{(e)}(ik, ik) & \underline{k}^{(e)}(ik, ki) \\ \underline{k}^{(e)}(ki, ik) & \underline{k}^{(e)}(ki, ki) \end{bmatrix}$$

$$\begin{aligned} k^{(e)}(ik, ik) &= \begin{bmatrix} \frac{EA}{H} & 0 & 0 \\ 0 & \frac{12EI}{(1+Q) \cdot H^3} & \frac{6EI}{(1+Q) \cdot H^2} \\ 0 & \frac{6EI}{(1+Q) \cdot H^2} & \frac{(4+Q)EI}{(1+Q) \cdot H} \end{bmatrix} & k^{(e)}(ik, ki) &= \begin{bmatrix} -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{(1+Q) \cdot H^3} & \frac{6EI}{(1+Q) \cdot H^2} \\ 0 & \frac{-6EI}{(1+Q) \cdot H^2} & \frac{(2-Q)EI}{(1+Q) \cdot H} \end{bmatrix} \\ k^{(e)}(ki, ik) &= \begin{bmatrix} -\frac{EA}{H} & 0 & 0 \\ 0 & \frac{-12EI}{(1+Q) \cdot H^3} & \frac{-6EI}{(1+Q) \cdot H^2} \\ 0 & \frac{6EI}{(1+Q) \cdot H^2} & \frac{(2-Q)EI}{(1+Q) \cdot H} \end{bmatrix} & k^{(e)}(ki, ki) &= \begin{bmatrix} \frac{EA}{H} & 0 & 0 \\ 0 & \frac{12EI}{(1+Q) \cdot H^3} & \frac{-6EI}{(1+Q) \cdot H^2} \\ 0 & \frac{-6EI}{(1+Q) \cdot H^2} & \frac{(4+Q)EI}{(1+Q) \cdot H} \end{bmatrix} \end{aligned} \quad (3.13)$$

Without the axial force part, the stiffness matrix could be written as:

$$\underline{k}'(e) = \begin{bmatrix} \frac{12EI}{(1+Q) \cdot H^3} & \frac{6EI}{(1+Q) \cdot H^2} & \frac{-12EI}{(1+Q) \cdot H^3} & \frac{6EI}{(1+Q) \cdot H^2} \\ \frac{6EI}{(1+Q) \cdot H^2} & \frac{(4+Q)EI}{(1+Q) \cdot H} & \frac{-6EI}{(1+Q) \cdot H^2} & \frac{(2-Q)EI}{(1+Q) \cdot H} \\ \frac{-12EI}{(1+Q) \cdot H^3} & \frac{-6EI}{(1+Q) \cdot H^2} & \frac{12EI}{(1+Q) \cdot H^3} & \frac{-6EI}{(1+Q) \cdot H^2} \\ \frac{6EI}{(1+Q) \cdot H^2} & \frac{(2-Q)EI}{(1+Q) \cdot H} & \frac{-6EI}{(1+Q) \cdot H^2} & \frac{(4+Q)EI}{(1+Q) \cdot H} \end{bmatrix} \quad (3.14)$$

The stiffness is, by definition, the nodal force due to unit displacement:

$$\underline{k}'(e) = \left[\begin{matrix} Q_i \\ M_i \\ Q_k \\ M_k \end{matrix} \right\} \text{ due to } v_i = 1 \quad \begin{matrix} Q_i \\ M_i \\ Q_k \\ M_k \end{matrix} \left\} \text{ due to } \varphi_i = 1 \quad \boxed{\begin{matrix} Q_i \\ M_i \\ Q_k \\ M_k \end{matrix} \left\} \text{ due to } v_k = 1} \quad \begin{matrix} Q_i \\ M_i \\ Q_k \\ M_k \end{matrix} \left\} \text{ due to } \varphi_k = 1 \right]$$

We repeat the procedure done in chapter 3.4.1. By taking column 3, we apply force Q_k then calculate the displacement v_k considering Timoshenko beam theory. From the force-displacement relationship $Q_k = k(3,3) \cdot v_k$ we can get the stiffness when $v_k = 1$ so $k(3,3) = Q_k$

$$k(3,3) = Q_k(v_k = 1) = \frac{12EI}{(1+Q) \cdot H^3} = \text{Alpha}$$

Other components of column 3 could be defined from equilibrium as:

$$Q_i(v_k = 1) = -Q_k(v_k = 1) = -\text{Alpha} = k(1,3)$$

$$M_i - M_k = 0 \quad \& \quad M_i + M_k = -Q_k(v_k = 1) \cdot H$$

$$\Rightarrow M_i = M_k = -\frac{H}{2} \text{Alpha} = k(2,3) = k(4,3)$$

Column 1 is similar to column 3, but with different sign:

$$k(1,1) = \text{Alpha} \quad , \quad k(2,1) = \frac{L}{2} \text{Alpha} \quad , \quad k(3,1) = -\text{Alpha} \quad , \quad k(4,1) = \frac{L}{2} \text{Alpha}$$

To define the column 2, we apply rotation of $\varphi_i = 1$ to get the nodal forces, without and with considering Timoshenko, as:

$$\begin{aligned} Q_k &= -\frac{6EI}{H^2} \\ Q_k &= -\frac{6EI}{(1+Q) \cdot H^2} \\ Q_i &= \frac{6EI}{H^2} \\ Q_i &= \frac{6EI}{(1+Q) \cdot H^2} \end{aligned}$$

$$\begin{aligned} M_k &= \frac{2EI}{H} \\ M_k &= \frac{(2-Q)EI}{(1+Q) \cdot H} \\ M_i &= \frac{4EI}{H} \\ M_i &= \frac{(4+Q)EI}{(1+Q) \cdot H} \end{aligned}$$

$$k(1,2) = \frac{H}{2} \text{Alpha}$$

$$k(2,2) = \frac{(4+Q) H^2}{12} \text{Alpha}$$

$$k(3,2) = -\frac{H}{2} \text{Alpha}$$

$$k(4,2) = \frac{(2-Q) H^2}{12} \text{Alpha}$$

Column 4 is similar to column 2, with changing the terms related to bending moment as:

$$k(1,4) = \frac{H}{2} \text{Alpha}$$

$$k(2,4) = \frac{(2-Q) H^2}{12} \text{Alpha}$$

$$k(3,4) = \frac{H}{2} \text{Alpha}$$

$$k(4,4) = \frac{(4+Q) H^2}{12} \text{Alpha}$$

And the stiffness matrix without the axial force component:

$$\underline{k'}(e) = \begin{bmatrix} \text{Alpha} & \frac{H}{2} \text{Alpha} & -\text{Alpha} & \frac{H}{2} \text{Alpha} \\ \frac{H}{2} \text{Alpha} & \frac{(4+Q) H^2}{12} \text{Alpha} & -\frac{H}{2} \text{Alpha} & \frac{(2-Q) H^2}{12} \text{Alpha} \\ -\text{Alpha} & -\frac{H}{2} \text{Alpha} & \text{Alpha} & -\frac{H}{2} \text{Alpha} \\ \frac{H}{2} \text{Alpha} & \frac{(2-Q) H^2}{12} \text{Alpha} & -\frac{H}{2} \text{Alpha} & \frac{(4+Q) H^2}{12} \text{Alpha} \end{bmatrix} \text{ with } \frac{12EI}{(1+Q) \cdot H^3} = \text{Alpha} \quad (3.15)$$

3.6.2. Post-Peak Inelastic

The displacement extracted from the force-displacement curve of the wall cross section is used to define the stiffness matrix. The displacement is dependent on the shear force Q_k and the normal force so it is written as $\delta(N, Q)$, or $\delta(N, Q, M)$ when there is bending moment on wall top. By taking the elastic stiffness matrix:

$$\underline{k'}(e) = \begin{bmatrix} \text{Alpha} & \frac{H}{2} \text{Alpha} & -\text{Alpha} & \frac{H}{2} \text{Alpha} \\ \frac{H}{2} \text{Alpha} & \frac{(4+Q) H^2}{12} \text{Alpha} & -\frac{H}{2} \text{Alpha} & \frac{(2-Q) H^2}{12} \text{Alpha} \\ -\text{Alpha} & -\frac{H}{2} \text{Alpha} & \text{Alpha} & -\frac{H}{2} \text{Alpha} \\ \frac{H}{2} \text{Alpha} & \frac{(2-Q) H^2}{12} \text{Alpha} & -\frac{H}{2} \text{Alpha} & \frac{(4+Q) H^2}{12} \text{Alpha} \end{bmatrix}$$

With Alpha (Stiffness) in this case is the shear force Q_k divided by the displacement $\delta(N, Q)$ $\text{Alpha} = \frac{Q_k}{\delta(N, Q)}$. The total stiffness matrix, with the axial force part, could be written as:

$$\underline{k}'(e) = \begin{bmatrix} \frac{EA}{1.2 \cdot H} & 0 & 0 & \frac{-EA}{1.2 \cdot H} & 0 & 0 \\ 0 & \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & 0 & -\frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} \\ 0 & \frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{(4+Q)H^2}{12} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & 0 & -\frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{(2-Q)H^2}{12} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} \\ \frac{-EA}{1.2 \cdot H} & 0 & 0 & \frac{EA}{1.2 \cdot H} & 0 & 0 \\ 0 & -\frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & -\frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & 0 & \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & -\frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} \\ 0 & \frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{(2-Q)H^2}{12} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & 0 & -\frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{(4+Q)H^2}{12} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} \end{bmatrix} \quad (3.16)$$

3.7. Summary of Stiffness Matrices

3.7.1. Fixed-Fixed

Elastic stiffness matrix without the axial force component:

$$\underline{k}'(e) = \begin{bmatrix} \frac{12EI}{(1+Q) \cdot H^3} & \frac{6EI}{(1+Q) \cdot H^2} & \frac{-12EI}{(1+Q) \cdot H^3} & \frac{6EI}{(1+Q) \cdot H^2} \\ \frac{6EI}{(1+Q) \cdot H^2} & \frac{(4+Q)EI}{(1+Q) \cdot H} & \frac{-6EI}{(1+Q) \cdot H^2} & \frac{(2-Q)EI}{(1+Q) \cdot H} \\ \frac{-12EI}{(1+Q) \cdot H^3} & \frac{-6EI}{(1+Q) \cdot H^2} & \frac{12EI}{(1+Q) \cdot H^3} & \frac{-6EI}{(1+Q) \cdot H^2} \\ \frac{6EI}{(1+Q) \cdot H^2} & \frac{(2-Q)EI}{(1+Q) \cdot H} & \frac{-6EI}{(1+Q) \cdot H^2} & \frac{(4+Q)EI}{(1+Q) \cdot H} \end{bmatrix}$$

$$Q = \frac{12 \cdot EI}{H^2 \cdot G \cdot A_s},$$

In simpler form

$$\underline{k}'(e) = \begin{bmatrix} 1 & \frac{H}{2} & -1 & \frac{H}{2} \\ \frac{H}{2} & \frac{(4+Q)H^2}{12} & -\frac{H}{2} & \frac{(2-Q)H^2}{12} \\ -1 & -H/2 & 1 & -\frac{H}{2} \\ \frac{H}{2} & \frac{(2-Q)H^2}{12} & -\frac{H}{2} & \frac{(4+Q)H^2}{12} \end{bmatrix} \frac{12 EI}{(1+Q) \cdot H^3} \quad (3.18)$$

Inelastic stiffness matrix without the axial force component:

$$\underline{k'}(e) = \begin{bmatrix} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & -\frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} \\ \frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{(4+Q)H^2}{12} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & -\frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{(2-Q)H^2}{12} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} \\ -\frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & -\frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & -\frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} \\ \frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{(2-Q)H^2}{12} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & -\frac{H}{2} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} & \frac{(4+Q)H^2}{12} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} \end{bmatrix} \quad (3.19)$$

In simpler form

$$\underline{k'}(e) = \begin{bmatrix} 1 & \frac{H}{2} & -1 & \frac{H}{2} \\ \frac{H}{2} & \frac{(4+Q)H^2}{12} & -\frac{H}{2} & \frac{(2-Q)H^2}{12} \\ -1 & -\frac{H}{2} & 1 & -\frac{H}{2} \\ \frac{H}{2} & \frac{(2-Q)H^2}{12} & -\frac{H}{2} & \frac{(4+Q)H^2}{12} \end{bmatrix} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} \quad (3.20)$$

3.7.2.Fixed-Pinned

Elastic stiffness matrix without the axial force component:

$$\underline{k'}^{(e)} = \begin{bmatrix} \frac{12EI}{(4+Q) \cdot H^3} & \frac{12EI}{(4+Q) \cdot H^2} & \frac{-12EI}{(4+Q) \cdot H^3} & 0 \\ \frac{12EI}{(4+Q) \cdot H^2} & \frac{12EI}{(4+Q) \cdot H} & \frac{-12EI}{(4+Q) \cdot H^2} & 0 \\ \frac{-12EI}{(4+Q) \cdot H^3} & \frac{-12EI}{(4+Q) \cdot H^2} & \frac{12EI}{(4+Q) \cdot H^3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.21)$$

In simpler form

$$\underline{k'}^{(e)} = \begin{bmatrix} 1 & H & -1 & 0 \\ H & H^2 & -H & 0 \\ -1 & -H & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{12EI}{(4+Q) \cdot H^3} \quad (3.22)$$

Inelastic stiffness matrix without the axial force component:

$$\underline{k'}^{(e)} = \begin{bmatrix} \frac{Q_k}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & \frac{Q_k \cdot H}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & -\frac{Q_k}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & 0 \\ \frac{Q_k \cdot H}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & \frac{Q_k \cdot H^2}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & -\frac{Q_k \cdot H}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & 0 \\ -\frac{Q_k}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & -\frac{Q_k \cdot H}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & \frac{Q_k}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.23)$$

In simpler form

$$\underline{k'}^{(e)} = \begin{bmatrix} 1 & H & -1 & 0 \\ H & H^2 & -H & 0 \\ -1 & -H & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{Q_k}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} \quad (3.24)$$

3.7.3. Pinned-Fixed

Elastic stiffness matrix without the axial force component:

$$\underline{k'}^{(e)} = \begin{bmatrix} \frac{12EI}{(4+Q) \cdot H^3} & 0 & \frac{-12EI}{(4+Q) \cdot H^3} & \frac{12EI}{(4+Q) \cdot H^2} \\ 0 & 0 & 0 & 0 \\ \frac{-12EI}{(4+Q) \cdot H^3} & 0 & \frac{12EI}{(4+Q) \cdot H^3} & \frac{-12EI}{(4+Q) \cdot H^2} \\ \frac{12EI}{(4+Q) \cdot H^2} & 0 & \frac{-12EI}{(4+Q) \cdot H^2} & \frac{12EI}{(4+Q) \cdot H} \end{bmatrix} \quad (3.25)$$

In simpler form

$$\underline{k'}^{(e)} = \begin{bmatrix} 1 & 0 & -1 & H \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -H \\ H & 0 & -H & H^2 \end{bmatrix} \frac{12EI}{(4+Q) \cdot H^3} \quad (3.26)$$

Inelastic stiffness matrix without the axial force component:

$$\underline{k'}^{(e)} = \begin{bmatrix} \frac{Q_i}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & 0 & -\frac{Q_i}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & \frac{Q_i \cdot H}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} \\ 0 & 0 & 0 & 0 \\ -\frac{Q_i}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & 0 & \frac{Q_i}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & -\frac{Q_i \cdot H}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} \\ \frac{Q_i \cdot H}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & 0 & -\frac{Q_i \cdot H}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} & \frac{Q_i \cdot H^2}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} \end{bmatrix} \quad (3.27)$$

In simpler form

$$\underline{k'}^{(e)} = \begin{bmatrix} 1 & 0 & -1 & H \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -H \\ H & 0 & -H & H^2 \end{bmatrix} \frac{Q_k}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} \quad (3.28)$$

3.8. Transformation Matrix

The local forces on each wall, element i, are: the axial force $N_{\xi i}$, shear force parallel to the wall $V_{\xi i}$ and bending parallel to the wall $M_{\eta i}$

$$\underline{f'}_i^{(e)} = \begin{bmatrix} N_{\xi i} \\ V_{\xi i} \\ M_{\eta i} \end{bmatrix} \quad (3.29)$$

The local displacements and rotations on each wall, element i , are: the axial displacement $u_{\xi i}$, shear displacement parallel to the wall $u_{\zeta i}$ and rotation parallel to the wall $\varphi_{\eta i}$

$$\underline{u}'_i{}^{(e)} = \begin{bmatrix} u_{\xi i} \\ u_{\zeta i} \\ \varphi_{\eta i} \end{bmatrix} \quad (3.30)$$

As shown in Figure 3.6, the relationship between element local coordinate system and system global coordinate system is given by,

$$\begin{bmatrix} \xi_i \\ \eta_i \\ \zeta_i \end{bmatrix} = \begin{bmatrix} \cos \alpha_i & \sin \alpha_i & 0 \\ -\sin \alpha_i & \cos \alpha_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{x} - \tilde{x}_{s,i} \\ \tilde{y} - \tilde{y}_{s,i} \\ \tilde{z} - \tilde{z}_{s,i} \end{bmatrix} \quad (3.31)$$

Figure 3.6: **a)** Wall with local coordinate system ξ_i - η_i - ζ_i ; **b)** position of the wall in global coordinate system $\tilde{x} - \tilde{y} - \tilde{z}$; **c)** Definition of the local internal forces and deformations

The relationship between internal forces and deformations is:

$$\underline{f}'_i{}^{(e)} = \underline{k}'^{(e)} \cdot \underline{u}'_i{}^{(e)} \quad (3.32)$$

The relationship between system (roof) global deformations and rotations and the wall local deformation is given by:

$$\underline{u}'_i{}^{(e)} = \underline{T}_i \underline{\tilde{u}} \quad (3.33)$$

where

$$\underline{\tilde{u}}^T = [\tilde{u}_{Mx} \quad \tilde{u}_{My} \quad \tilde{u}_{Sz} \quad \tilde{\theta}_{Sx} \quad \tilde{\theta}_{Sy} \quad \tilde{\theta}_{Mz}] \quad (3.33) \text{ a}$$

$$\underline{T}_i = \begin{bmatrix} 0 & 0 & 1 & \Delta y_{Si} & -\Delta x_{Si} & 0 \\ \cos \alpha_i & \sin \alpha_i & 0 & 0 & 0 & -\Delta \eta_{Mi} \\ 0 & 0 & 0 & -\sin \alpha_i & \cos \alpha_i & 0 \end{bmatrix} \quad (3.33) \text{ b}$$

$$\Delta \eta_{Mi} = (\tilde{y}_{Si} - \tilde{y}_M) \cos \alpha_i - (\tilde{x}_{Si} - \tilde{x}_M) \sin \alpha_i \quad (3.33) \text{ c}$$

$$\Delta y_{Si} = y_{Si} - y_S, \quad \Delta x_{Si} = x_{Si} - x_S \quad (3.33) \text{ d}$$

The transformation matrix $\underline{T}_i$ is extracted from the relationship between ceiling deformation and wall deformation, as shown in Figure 3.7

Figure 3.7: local and global deformation of each wall in floor view [17]

3.9. Assemble the Structure Stiffness Matrix

The element stiffness matrix in the global coordinate system is given by [18]

$$\underline{\tilde{k}}^{(e)} = \sum_i \underline{T}_i^T \cdot \underline{k}'^{(e)} \cdot \underline{T}_i \quad (3.34)$$

The elements of each story are summed into one single element, in the case of 3 stories, as shown in Figure 3.8

Figure 3.8: system with three stories and one beam element

The nodal displacements are $\underline{\tilde{u}}(2)$, $\underline{\tilde{u}}(3)$ and $\underline{\tilde{u}}(4)$. The global forces are $\underline{\tilde{F}}(2)$, $\underline{\tilde{F}}(3)$, $\underline{\tilde{F}}(4)$. The global relationship is given by

$$\begin{pmatrix} \underline{\tilde{F}}(2) \\ \underline{\tilde{F}}(3) \\ \underline{\tilde{F}}(4) \end{pmatrix} = \begin{bmatrix} \underline{\tilde{K}}(21,2) + \underline{\tilde{K}}(23,2) & \underline{\tilde{K}}(23,3) & 0 \\ \underline{\tilde{K}}(32,2) & \underline{\tilde{K}}(32,3) + \underline{\tilde{K}}(34,3) & \underline{\tilde{K}}(34,4) \\ 0 & \underline{\tilde{K}}(43,3) & \underline{\tilde{K}}(43,4) \end{bmatrix} \cdot \begin{pmatrix} \underline{\tilde{u}}(2) \\ \underline{\tilde{u}}(3) \\ \underline{\tilde{u}}(4) \end{pmatrix} \quad (3.35)$$

where

$$\underline{\tilde{K}}(23,2) = \sum_{i=1}^n \underline{T}_i^T \cdot \underline{k}'^{(e)}(23,3) \cdot \underline{T}_i \quad (3.35) \text{ a}$$

3.10. Assemble the Structure Nodal Vector

In the global coordinate system of the building, the forces are the horizontal forces V_{Mx} and V_{My} applied in the shear center of the story M, and the axial forces N_{Sz} applied in the center of gravity S. The moments are M_{Sx} and M_{Sy} applied in the gravity center of the story S, and the moment M_{Mz} applied in the center of shear M.

$$\underline{\tilde{F}}^T = [V_{Mx} \quad V_{My} \quad N_{Sz} \quad M_{Sx} \quad M_{Sy} \quad M_{Mz}] \quad (3.36)$$

This is the global forces on each story. In the example before we have three stories and three nodal forces $\underline{\tilde{F}}(2)$, $\underline{\tilde{F}}(3)$, $\underline{\tilde{F}}(4)$

3.11. Solve, Get Nodal Displacements and Forces

The global equation is

$$\underline{\tilde{C}} \cdot \underline{\tilde{u}} - \underline{\tilde{F}} = 0 \quad (3.37)$$

When the external nodal force known the solution will be

$$\underline{\tilde{u}} = \underline{\tilde{C}}^{-1} \cdot \underline{\tilde{F}} \quad (3.38)$$

This is the global deformation on each story. From it we can go back to the deformation of each wall from the equation

$$\underline{u'_i}^{(e)} = \underline{T_i} \underline{\tilde{u}} \quad (3.39)$$

Then from the local deformation, it is possible to define the local internal forces in each wall. On the top and bottom by

$$\begin{aligned} F_{ik} &= \underline{k'^{(e)}}(ik, ik) \cdot \underline{u_i}(i) + \underline{k'^{(e)}}(ik, ki) \cdot \underline{u_i}(k) \\ F_{ki} &= \underline{k'^{(e)}}(ki, ik) \cdot \underline{u_i}(i) + \underline{k'^{(e)}}(ki, ki) \cdot \underline{u_i}(k) \end{aligned} \quad (3.40)$$

This procedure will be used to get the forces in the bracing elements (single walls) and displacements in the bracing system. In the example shown in Chapter 6, the equations mentioned above will be used in a Fortran [14] code to solve a simple system composed of two T-section walls assumed to be fixed-fixed, and the whole system is subject to lateral seismic loading.

4. Matrix Formulation and Deformation Based Method

4.1. Elastic Method in Distributing Lateral Force

In this chapter both the matrix formulation and the changed stiffness of the wall, i.e., force-displacement curve, will be combined to get the lateral force and status of each wall from the total applied lateral force on the structure. Previous elastic method in distributing the lateral force on bracing elements considers only the elastic state of the wall and does not consider the redistribution of force on the walls due to changed stiffness when the material behavior becomes inelastic with increased load.

The flowchart below demonstrates the classic method. Here we have the geometry, material properties and total lateral force on the structure as inputs. The outputs will be the force in each single wall. When the lateral force increases, the force in each wall increases accordingly linearly. Even if some walls crack or fail the distribution is still linear. In reality, when the wall cracks its stiffness becomes less and consequently receives less portion of the lateral force.

The equations and formulae for this method have been explained thoroughly in Chapter 2. A flowchart is shown in Figure 4.1.

Figure 4.1: Classical elastic method in distributing lateral force on bracing elements

4.2. Elastic and Inelastic Method in Distributing Lateral Force

With the proposed method, the structure displacement will be calculated at the end of the elastic first iteration $k=1$. The force and displacement of each wall will also be defined. The calculation using the matrix form as explained in Chapter 3 will be based on an elastic stiffness matrix of all walls. All this has been done in the well-known elastic calculation method.

At iteration $k=2$, and depending on the wall force resulted from previous elastic calculation, the state and displacement of each wall can be defined depending on the force-displacement curve of this wall, see Chapter 2. The displacement and state of each wall section are used to redefine the stiffness matrix of each wall “j” at iteration “k”. This stiffness could be elastic or inelastic as explained in Chapter 3. With these stiffness matrices the system is calculated again including the displacement of the system Δ_k . At the end of this iteration the percent of $|(\Delta_k - \Delta_{k-1})/\Delta_{k-1}|$ is calculated and compared with a given tolerance for example 5%.

If the percent is less than the tolerance, the calculation is ended and the wall forces, section state and system displacement will be the outputs. If the percent is bigger than the tolerance, a new iteration $k+1$ will be calculated, i.e., each wall force, displacement and state will be defined, then each wall stiffness will be recalculated, after that the whole system will be calculated and its new displacement Δ_{k+1} will be defined. Then again compared with previous system displacement Δ_k .

This process will be repeated until the percent $|(\Delta_k - \Delta_{k-1})/\Delta_{k-1}|$ is less than the given tolerance. This process is shown clearly in Figure 4.2.

Figure 4.2: Proposed elastic and inelastic method in distributing lateral force on bracing elements

This method will be programmed in Fortran code to solve the example shown in Chapter 5. The results and verification are shown in Chapter 6.

5. Shake Table Tests

5.1. Introduction

The proposed analytical method will be verified with the results of shake table tests done in the year 2018 at the outdoor shake table in the NHERI (Natural Hazards Engineering Research Infrastructure) facility at the University of California, San Diego [19]. The test is full scale, single-story, fully grouted, reinforced masonry wall specimen tested until collapse. The specimen has two T-walls as lateral force resisting elements and a stiff roof diaphragm. The specimen and its dimensions are shown in Figure 5.1.

5.2. Design of Test Structure

The specimen has been tested under unidirectional motions. Both T walls has the same dimension and reinforcement and carry the same axial load, Figure 5.2. The

reinforcement details are shown in Figure 5.3, the reinforcing bars have a nominal yield strength of 60 ksi (Grade 60 in ASTM standard [20]). The wall web has 6 vertical bars No. 4 (13mm), spaced at 8 inches (20cm) on center, the flange has 3 vertical bars No. 4 (13mm), spaced at 16 inches (41cm) on center. The horizontal reinforcements are No. 3 (10mm) spaced at 16 inches (41cm) on center. This complies with the prescriptive requirement of TMS 402/602 [11] for special reinforced masonry, except the requirement of maximum spacing of vertical reinforcement in the flange of third wall length.

Figure 5.2: T wall dimensions and reinforcement

The walls are assumed to be fixed-fixed; this is achieved by the stiff roof at the top of 10 inch (25 cm) thick hollow core planks with 5 inch (13 cm) cast in place concrete topping, additionally 10 feet x 10 feet x 14 inch (300x300x36 cm) thick reinforced concrete slabs have been placed above to achieve the required roof mass. At the bottom, the reinforcement extended continuously in the walls to the footing, with 90° hooking according to the requirements of the ACI 318-14 specifications [22] to achieve enough development length, Figure 5.3.

Figure 5.3: T wall reinforcement details [19]

The roof weight, including the extra weight of the additional concrete slabs, is 55.1 kips (245 kN). The axial compressive load ratio, $P / f'_m A_g$, for each wall is 0.016. Where $A_g = 673.9 \text{ inch}^2$ is the gross cross-section area of the T wall. The compressive strength of masonry f'_m has been defined to be 2.5 ksi. The actual seismic weight of the roof including the weight of the upper half wall height is 60 kips (268 kN).

The flexural, diagonal shear and sliding shear strengths of the T wall have been calculated based on the formulae and recommendations of TMS 402/604 [11] assuming fixed-fixed boundary conditions. The calculated flexural strengths of the T-wall for the compressed web and flange are 65 kips and 91 kips, respectively. The average is 78 kips. Diagonal shear strength and sliding shear strength for each wall are 73 kips and 89 kips, respectively.

5.3. Material Properties

The main two material properties are the compression strength of masonry and yield strength of steel. Compression tests were performed on 16 x 8 x 8 inch grouted concrete masonry prisms after 28 days of sample preparation.

The average compression strength of two samples is 2.7 ksi (18 MPa) at 28 days.

The average tensile properties of steel reinforcement are summarized in the Table 5.1.

Bar size	No. of Samples	Yield Stress		Tensile Strength		Yield Strain in./in.	Strain at peak stress in./in.
		MPa	ksi	MPa	ksi		
#3 (10mm)	3	521	75.6	729	105.8	0.00474	0.107
#4 (13mm)	3	462	67.0	671	97.3	0.00342	0.116

Table 5.1: Average tensile properties of reinforcing bars [19]

5.4. Tests and Observations

The specimen has been subjected to 7 dynamic loads and 1 static pull load. The dynamic loads resulted from scaling the Mulholland record to 45%, three times 90%, 120% and 133%. The performed tests and the change in period after each test are listed in Table 5.2.

Test ID	Input motion	Period after test [second]
Initial	N/A	0.072
1	Mul-45%	0.072
2	Mul-45%	0.074
3	Mul-90%	0.090
4	Mul-90%	0.095
5	Mul-90%	0.107
6	Mul-120%	0.166
7	Mul-133%	0.751
8	Static pull	Not measure due to high damage

Table 5.2: Tests IDs and periods after each test [19]

The peak values of roof drift ratios, base shear occurring at the tests ID 5-7 are summarized in Table 5.3. The roof drift ratio is the displacement divided by the height of 8 ft. (96 inch). The base shear is the product of average acceleration along the shaking direction by the mass of the roof including the mass of upper half of the walls.

Test ID	Input motion	Peak roof displacement		Peak roof drift		Peak base shear	
		Pos. [inch]	Neg. [inch]	Pos. %	Neg. %	Pos. [kips]	Neg. [kips]
5	Mul-90%	0.22	0.19	0.23	0.20	105.4	106.1
6	Mul-120%	0.40	0.45	0.42	0.47	134.4	126.1
7	Mul-133%	1.99	2.42	2.07	2.52	139.2	155.1

Table 5.3: Peak values of roof displacement and base shear [19]

5.4.1. Tests up to Mul-90%

After the two tests of Mul-45%, no visible cracks have been observed. The Table 5.2 shows hardly change in the period indicating no damage in the stiffness. However, after the three shakings of Mul-90% the period changed from 0.074 to 0.107 second indicating that some damage has happened. At the end of these shakings, hairline cracks have been observed at the bottom of the two walls.

5.4.2. Tests with Mul-120%

After the Mul-120% shaking, flexural cracks initiate and propagate at the bottom of the flange and web of the walls as shown in Figure 5.4. After this shaking the period changed to 0.166 second indicating cracking of masonry and yielding of steel bars, see Table 5.2. As shown in Table 5.3, the maximum base shear was 134 kips corresponding to maximum drift ratio of 0.42%.

Figure 5.4: Cracks after Mul-120%. A, south view; B, north view [19]

5.4.3. Tests with Mul-133%

After this last dynamic test, severe shear cracks developed. The lateral drift was 2.42 inch (drift ratio of 2.52%), Table 5.3. The period elongated to 0.751 indicating severe damage. Diagonal shear cracks and crushing of the masonry at the bottom of wall webs are observed after this test, Figure 5.5.

(A)

(B)

Figure 5.5: Damage after Mul-133%. A, south view of wall1; B, north view of wall2 [19]

5.5. Deformations

The deformation of each wall is due to flexural, shear and sliding deformation. The analytical formulae for flexural and shear deformations have been explained before in chapter 2 depending on the wall boundary conditions. The flexural deformation is calculated from curvature measured by vertical displacement transducers mounted along the two sides of each wall. The sliding displacement is the sum of sliding at the top and bottom of each wall measured by linear potentiometers. The shear deformation is calculated by subtracting both flexural and sliding deformation from total displacement of the roof, and checked with data from diagonal string potentiometers [19].

Figure 5.6 shows these three displacement components for each wall during Mul-120% earthquake. It can be seen that flexural displacement is the major component, this has been confirmed from the flexural cracks observed during this test, see Figure 5.4.

Figure 5.6: Deformation of Wall1 and Wall 2 during Mul-120% [19]

Figure 5.7 shows also three displacement components for each wall during Mul-133% earthquake. In this case the shear displacement is the major component, this has been seen from opening the diagonal shear cracks during this test, see Figure 5.5.

Figure 5.7: Deformation of Wall1 and Wall 2 during Mul-133% [19]

The results and observations of this test will be used in the next Chapter to verify the proposed analytical calculation method.

6. Verification

In this chapter the specimen presented in the chapter 5 and tested on the shake table will be shown here and will be calculated analytically using the method and formulae presented in the previous chapters. First the moment curvature curves for each wall have been drawn depending on the cross sections dimensions and reinforcement as described in the test in chapter 5, and using the material properties as shown in chapter 5.3. Later, the force displacement curve for each wall has been defined assuming fixed-fixed boundary conditions on the wall. The five main points on the force-displacement curve of each wall has been used to describe the wall stiffness change due to increased force, these points will be input in the Fortran code for each wall. Increased lateral force will be applied on the whole structure, the Fortran code will give, for each force, the lateral force in each wall, the wall status and the total structure displacement. These values will be compared with displacements measured in the real shake table test and the behavior of the walls, i.e.; wall status elastic or inelastic, are compared with that observed in the real test.

6.1. T Cross Section, Dimensions, Reinforcement and Materials

The moment curvature and force displacement curves for a T-section reinforced masonry wall shown in Figure 6.1. The wall is fully grouted CMU (concrete masonry unit) wall has $f'_m = 2500$ psi and steel ($f_y = 68$ ksi). The masonry is assumed to have zero tensile strength, and the compressive stress-strain relation for masonry is given by the following expression

$$f_m = f'_m \left[2 \left(\frac{\varepsilon_m}{\varepsilon_p} \right) - \left(\frac{\varepsilon_m}{\varepsilon_p} \right)^2 \right] \geq 0 \quad (2.9)$$

with $\varepsilon_p = 0.002$.

Figure 6.1: Reinforced masonry wall section

6.2. Moment Curvature Curve

Moment-curvature curves have been created for axial load $P = 0.016 f'_m A_n = 26.95$ kips. The curves cover a curvature range from zero to a maximum value of 0.001. A_n is the net area of the wall cross section. The compression stresses in T-section could be in the web or in the flange, both cases will be presented. The curves are shown in Figure 6.2 and Figure 6.3 for compressed web and flange, consequently. When the axial force is small, the section shows less moment capacity, however it shows more ductile behavior.

Drawing the moment force interaction diagrams shows almost similar values for maximal bending moment at specific axial force, see Figure 6.4 and Figure 6.5 for compression in web and flange, consequently. The results are summarized in the table 6.1:

Axial compressive force	Moment [kip-in.] from			
	moment-curvature curve		moment-force interaction diagram	
	Compression in web	Compression in flange	Compression in web	Compression in flange
$P = 0.016 f'_m A_n$ $= 26.95$ kips	4 491	2 923	4 396	3 110

Table 6.1: T Section capacity using different methods

Figure 6.2: Moment curvature curves for T-section RM section for $P = 0.016 f'_m A_n = 26.95$ kips. Compression in web.

Figure 6.3: Moment curvature curves for T-section RM section for $P = 0.016 f'_m A_n = 26.95$ kips. Compression in flange.

Figure 6.4: Moment axial force interaction diagram, values for (i) $P = 0.05 f'_m A_n = 84.23$ kips, (ii) $P = 0.20 f'_m A_n = 336.93$ kips and (iii) $P = 0.016 f'_m A_n = 26.95$ kips are shown. Compression in web.

Figure 6.5: Moment axial force interaction diagram, values for (i) $P = 0.05 f'_m A_n = 84.23$ kips, (ii) $P = 0.20 f'_m A_n = 336.93$ kips and (iii) $P = 0.016 f'_m A_n = 26.95$ kips are shown. Compression in flange.

6.3. Force Displacement Curve

The wall section is considered to be a fixed-fixed wall. The wall height is 8 ft. and the wall is subjected to an in-plane horizontal load and an axial compressive load at the top. The effective plastic hinge length for the wall is assumed to be 20% of the wall height at the top and 20% at the bottom when the wall develops inelastic deformation. The axial load $P = 0.016 f'_m A_n = 26.95$ kips is considered. The self-weight of the wall has been ignored. The load-displacement curve has been calculated and plotted for two cases compression in web and compression in flange. Each curve is piece-wise linear with lines connecting

five points: the origin (zero displacement), nominal yield point defined as $0.8M_p$, the point at which the peak load occurs M_p , the last point corresponds to the curvature of 0.001, and a point with a curvature that is midway between the curvature at the peak load and 0.001. The curves for web compression and flange compression are shown in Figures 6.6 and Figure 6.7, respectively.

Figure 6.6: Force displacement curves for T-section RM section, fixed-fixed wall, $P = 0.016 f'_m A_n = 26.95$ kips. Compression in web.

Figure 6.7: Force displacement curves for T-section RM section, fixed-fixed wall, $P = 0.016 f'_m A_n = 26.95$ kips. Compression in flange.

For programming in Fortran, the ultimate bending moment capacity M_p will be used. The elastic bending moment will be assumed to be $0.8M_p$; this has been noticed in test results and could be seen as well in the moment curvature curves shown above. The corresponding force, as shown before in chapter 3, is M_p/h for the case of cantilever wall, and $2 M_p/h$ in case of fixed-fixed wall, where h is the height of the wall. Beyond the ultimate load, the wall deforms either brittle when the axial load is high or ductile when the axial load is small, Figure 2.18. The example shown in chapter 2.7 showed brittle behavior when the axial load was $P = 0.20 f'_m A_n$ and ductile behavior when it was $P = 0.05 f'_m A_n$. In the shake table test the axial load was $P = 0.016 f'_m A_n$ that is smaller than $0.05 f'_m A_n$ and consequently will behave ductile. The force-displacement curve will be defined, in addition to the points corresponding to M_p and $0.8M_p$, at curvatures of 0.001 and 0.00075 as pot-peak values.

In summary, we have five points at the force displacement curve:

- 1- Zero -zero.
- 2- Elastic limit at $0.8 M_p$, (F_{el}, δ_{el}) .
- 3- Plastic limit at M_p , $(F_p \text{ or } F_y, \delta_y)$.
- 4- Post peak point at $2/3M_p$, corresponds to curvature of 0.00075 (F_4, δ_{75}) .
- 5- Post peak point at $1/2M_p$, corresponds to curvature of 0.001 (F_5, δ_{100}) .

These points are for wall 1 (compressed web) as follows, Figure 6.8:

$M_p=4491$ kip-inch (from moment curvature curve or calculated ultimate moment value)

$M_e=0.8M_p=3521$ kip-inch

$V_p=2 M_p/h=2 \cdot 4491/96 = 94$ kips, $\delta_y=0.518$ inch

$V_e=2 M_e/h=2 \cdot 3521/96 = 75$ kips, $\delta_{el}=0.030$ inch

$F_4=2/3 V_p=2/3 \cdot 94 = 62$ kips, $\delta_{75}=0.959$ inch

$F_5=1/2 V_p=1/2 \cdot 94 = 44$ kips, $\delta_{100}=1.475$ inch

And for wall 2 (compressed flange) as follows, 6.9:

$M_p=2923$ kip-inch (from moment curvature curve or calculated ultimate moment value)

$M_e=0.8M_p=2339$ kip-inch

$V_p=2 M_p/h=2 \cdot 2923/96 = 61$ kips, $\delta_y=0.517$ inch

$V_e=2 M_e/h=2 \cdot 2339/96 = 50$ kips, $\delta_{el}=0.020$ inch

$F_4=2/3 V_p=2/3 \cdot 61 = 41$ kips, $\delta_{75}=0.959$ inch

$F_5=1/2 V_p=1/2 \cdot 61 = 31$ kips, $\delta_{100}=29$ inch

In the force displacement curve, the force stays constant at 61 kips.

Figure 6.8: Force displacement curves for T-section RM section, Wall1

Figure 6.9: Force displacement curves for T-section RM section, Wall2

6.4. Force Displacement Curve of the Structure

The force displacement curve of each wall has been used as input in the Fortran code, the results should show the force displacement curve of the whole structure as well as the status of each wall by increased loading.

Figure 6.10 shows good agreement between calculated force displacement curve and the curve drawn from the shake table test in elastic region, as well as in post-peak inelastic region. In the inelastic pre-peak region, there is discrepancy between the measured

displacement and the calculated displacement; the measured displacements were little bigger. This could be attributed to the damage already happened to the structure in previous tests.

Figure 6.10: Force displacement curves for walls and the structure, test vs. calculation

The behavior of each wall and its portion of the load could also be seen during the calculation as shown in Table 6.2. The wall behavior in calculation agrees with the observed cracks on the walls during testing. For the case of Mul-120%, the load during test was in both directions for this reason both walls were inelastic and showed flexural cracking, while in the calculation the load has been applied in one direction.

Point	Structure Calculation		Wall1, calculation		Wall1, test	Wall2, calculation		Wall2, test	Remarks
	U [inch]	F [kips]	F1 [kips]	behavior	behavior	F2 [kips]	behavior	behavior	
1	0	0	0	-		0	-		
2	0,015	50	25	elastic		25	elastic		
3	0,029	96	48	elastic		48	elastic		
4	0,037	105	62,9	elastic	elastic	42,1	elastic	elastic	Mul-90%
5	0,039	110	65,7	elastic		44,3	elastic		
6	0,041	115	68,8	elastic		46,2	elastic		
7	0,043	120	71,8	elastic		48,3	inelastic		
8	0,045	125	74,7	elastic	inelastic	50,3	inelastic	inelastic	Mul-120%
9	0,149	130	77,7	inelastic		52,3	inelastic		
10	0,227	135	80,7	inelastic		54,3	inelastic		
11	0,345	140	83,7	inelastic	inelastic	56,3	inelastic	inelastic	Mul-133%
12	1,126	145	86,7	inelastic		58,3	inelastic		
13	1,66	146	87,3	inelastic		58,7	inelastic		
14		147	not converged						

Table **6.2**: Forces in each wall with increased lateral force on the structure, test vs. calculation

The stiffer the wall, i.e., the division of force/displacement, the more it receives from the lateral load. At a small lateral load, both walls have the same stiffness and receive the same amount of lateral load. With increased lateral load, wall 1, which is stiffer, receives more lateral load and both walls stay elastic. When the lateral load further increases, the load in wall2 increases too, but then it enters the inelastic region; wall 2 becomes less stiff and then it receives less lateral load. At the last steps, both walls become inelastic and receive load depending on the stiffness of each wall. Note that wall1 reaches its elastic limit at 75kips, while wall2 reaches it at 49 kips.

7. Conclusions and Suggestions

This study started with showing the formulae and equations used for elastic calculation of general bracing system, supported with an example in the first Chapter.

In the second Chapter, the moment-curvature curves, and force displacement curves have been developed for walls with different boundary conditions. At the end of this Chapter explanatory examples for rectangular section and T-section walls with either fixed-pinned or fixed-fixed boundary conditions have been added.

The matrix formulation for calculating frame structures has been explained and shown in Chapter three. The stiffness matrices for different wall boundary conditions have been developed. The stiffness matrix for both elastic and inelastic material has been shown.

Flowcharts showing first the classical analytical method to calculate the bracing system have been shown in Chapter four. Then combining the matrix formulation and changed stiffness influence on the calculation has been shown also in a flowchart.

The shake table test conducted on a reinforced masonry building at the University of California in San Diego in the year 2018 has been shown, with the main features like test setup, forces and displacement results, observations and failure of the building are shown in Chapter five, these results will be used to verify the analytical method.

The displacement capacity of a building has been evaluated depending on the displacement capacity of each single wall in the building. This displacement capacity is important in the case of extreme earthquake for life safety and collapse prevention. Depending on the section geometry, material properties and boundary conditions of the wall, moment-curvature curves and subsequently the force-displacement curves have been drawn. The later has been used to describe the change of the wall stiffness with increased force. This includes, in addition to elastic behavior, inelastic behavior pre and post peak value. The change of the wall stiffness will lead to change of distributed force on each wall, which is different from load distribution in case of considering only elastic wall behavior.

In this study, the lateral capacity and force-displacement of each single wall was an input in a Fortran code, the output was the force-displacement curve of the whole structure as well as its lateral force capacity. This code considers the redistribution of lateral force on each wall due to changed stiffness of the wall with increased load.

Possible future improvement could be to distinguish between shear, flexural and sliding displacement when calculating the single walls or the whole system. Furthermore, current verification has been done on a simple reinforced masonry structure of two T-section walls in one story, verification of a multi-story building of other materials like concrete or unreinforced masonry could also be done in the future.

8. References

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Appendix 1, Timoshenko Beam

Here the stiffness matrices according to Timoshenko beam theory as shown in Chapter 3 will be put together in a simple form.

• Fixed-Fixed

The elastic stiffness, without the axial stiffness, is:

$$\underline{k'}(e) = \begin{bmatrix} 1 & \frac{H}{2} & -1 & \frac{H}{2} \\ \frac{H}{2} & \frac{(4+Q)H^2}{12} & -\frac{H}{2} & \frac{(2-Q)H^2}{12} \\ -1 & -\frac{H}{2} & 1 & -\frac{H}{2} \\ \frac{H}{2} & \frac{(2-Q)H^2}{12} & -\frac{H}{2} & \frac{(4+Q)H^2}{12} \end{bmatrix} \frac{12EI}{(1+Q)H^3} \quad Q = \frac{12EI}{H^2 \cdot G \cdot A_s} \quad (3.18)$$

Inelastic stiffness matrix without the axial force component:

$$\underline{k'}(e) = \begin{bmatrix} 1 & \frac{H}{2} & -1 & \frac{H}{2} \\ \frac{H}{2} & \frac{(4+Q)H^2}{12} & -\frac{H}{2} & \frac{(2-Q)H^2}{12} \\ -1 & -\frac{H}{2} & 1 & -\frac{H}{2} \\ \frac{H}{2} & \frac{(2-Q)H^2}{12} & -\frac{H}{2} & \frac{(4+Q)H^2}{12} \end{bmatrix} \frac{Q_k}{\delta(N,Q)} \cdot \frac{1}{(1+Q)} \quad (3.20)$$

• Fixed-Pinned

Elastic stiffness matrix, without the axial force component:

$$\underline{k'}(e) = \begin{bmatrix} 1 & H & -1 & 0 \\ H & H^2 & -H & 0 \\ -1 & -H & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{12EI}{(4+Q)H^3} \quad Q = \frac{12EI}{H^2 \cdot G \cdot A_s} \quad (3.22)$$

Inelastic stiffness matrix without the axial force component:

$$\underline{k'}(e) = \begin{bmatrix} 1 & H & -1 & 0 \\ H & H^2 & -H & 0 \\ -1 & -H & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{Q_k}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} \quad (3.24)$$

- **Pinned-Fixed**

Elastic stiffness matrix, without the axial force component:

$$\underline{k'(e)} = \begin{bmatrix} 1 & 0 & -1 & H \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -H \\ H & 0 & -H & H^2 \end{bmatrix} \frac{12EI}{(4+Q) \cdot H^3} \quad Q = \frac{12 \cdot EI}{H^2 \cdot G \cdot A_s} \quad (3.26)$$

Inelastic stiffness matrix without the axial force component:

$$\underline{k'(e)} = \begin{bmatrix} 1 & 0 & -1 & H \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -H \\ H & 0 & -H & H^2 \end{bmatrix} \frac{Q_k}{\delta(N,Q)} \cdot \frac{4}{(4+Q)} \quad (3.28)$$

Appendix 2, Bernoulli Beam

Stiffness Matrix according to Bernoulli is a special case from Timoshenko stiffnesses when the shear part $Q = \frac{12 \cdot EI}{H^2 \cdot G \cdot A_s} = 0$. The stiffness matrices according to Bernoulli beam theory will be put here in a simple form.

• Fixed-Fixed

The elastic stiffness, without the axial stiffness, is:

$$\underline{k'}(e) = \begin{bmatrix} 1 & \frac{H}{2} & -1 & \frac{H}{2} \\ \frac{H}{2} & \frac{H^2}{3} & -\frac{H}{2} & \frac{H^2}{6} \\ -1 & -\frac{H}{2} & 1 & -\frac{H}{2} \\ \frac{H}{2} & \frac{H^2}{6} & -\frac{H}{2} & \frac{H^2}{3} \end{bmatrix} \frac{12 EI}{H^3} \quad (3.18) \text{ a}$$

Inelastic stiffness matrix without the axial force component:

$$\underline{k'}(e) = \begin{bmatrix} 1 & \frac{H}{2} & -1 & \frac{H}{2} \\ \frac{H}{2} & \frac{H^2}{3} & -\frac{H}{2} & \frac{H^2}{6} \\ -1 & -\frac{H}{2} & 1 & -\frac{H}{2} \\ \frac{H}{2} & \frac{H^2}{6} & -\frac{H}{2} & \frac{H^2}{3} \end{bmatrix} \frac{Q_k}{\delta(N,Q)} \quad (3.20) \text{ a}$$

• Fixed-Pinned

Elastic stiffness matrix, without the axial force component:

$$\underline{k'}(e) = \begin{bmatrix} 1 & H & -1 & 0 \\ H & H^2 & -H & 0 \\ -1 & -H & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{3EI}{H^3} \quad (3.22) \text{ a}$$

Inelastic stiffness matrix without the axial force component:

$$\underline{k'}(e) = \begin{bmatrix} 1 & H & -1 & 0 \\ H & H^2 & -H & 0 \\ -1 & -H & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{Q_k}{\delta(N,Q)} \quad (3.24) \text{ a}$$

- **Pinned-Fixed**

Elastic stiffness matrix, without the axial force component:

$$\underline{k'}(e) = \begin{bmatrix} 1 & 0 & -1 & H \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -H \\ H & 0 & -H & H^2 \end{bmatrix} \frac{3EI}{H^3} \quad (3.26) \text{ a}$$

Inelastic stiffness matrix without the axial force component:

$$\underline{k'}(e) = \begin{bmatrix} 1 & 0 & -1 & H \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -H \\ H & 0 & -H & H^2 \end{bmatrix} \frac{Q_k}{\delta(N,Q)} \quad (3.28) \text{ a}$$