Probability and Heat Kernel Estimates for Lévy(-Type) Processes

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Index of notation

Analysis

$\inf \varnothing$	$\inf \varnothing = +\infty$
$x^\top$, $A^\top$	transpose
$x \cdot y$	Euclidean scalar product
$a \vee b$, $a \wedge b$	maximum, minimum
$\arg z$	$\in (-\pi, \pi]$, argument of $z \in \mathbb{C}$
$\operatorname{tr} A$	trace of the matrix A
$\sharp A$	cardinality of the set A
$\mathbb{1}_A$	indicator function of the set A
$\operatorname{spt} f$	support of f
∇f	gradient
$\nabla^2 f$	Hessian matrix
$f(t-)$	left limit $\lim_{s \uparrow t} f(s)$
càdlàg	finite left limits and right-continuous
càglàd	finite right limits and left-continuous
$\\|\cdot\\|_\infty$	uniform norm
$\\|\cdot\\|_{(2)}$	norm on $C_b^2(\mathbb{R}^d)$, p. 7
Γ	Gamma function
$(z)_\alpha$	$= \Gamma(z+\alpha)/\Gamma(z)$, Pochhammer symbol
Log	complex logarithm (principal value)
arctan	arctangent
f^+, f^-	positive part, negative part
$f * g$	convolution
$f \circledast g$	time-space convolution, p. 8
$f \circ g$	composition
$\hat{f}$, $\mathcal{F} f$	Fourier transform
$\check{f}$, $\mathcal{F}^{-1} f$	inverse Fourier transform
$\overline{(A, \mathcal{D})}$	closure of operator $A : \mathcal{D} \to A(\mathcal{D})$

Probability/measure theory

$\perp\!\!\!\perp$	stochastic independence
$\sim$	distributed as
$\stackrel{d}{=}$	equality in distribution
$\stackrel{v}{\to}$	vague convergence
$\stackrel{w}{\to}$	weak convergence
$\mathcal{B}(\mathbb{R}^d)$	Borel σ-algebra on $\mathbb{R}^d$
$\mathcal{A} \otimes \mathcal{B}$	product σ-algebra
δ_x	Dirac measure at x
$\mathbb{P}$ $\mathbb{E}_\mathbb{P}$	probability, expectation with respect to $\mathbb{P}$ ($\mathbb{E} = \mathbb{E}_\mathbb{P}$ for short)
$\mathbb{E}(\cdot \mid \mathcal{F})$	conditional expectation with respect to a σ-algebra $\mathcal{F}$
$\mathbb{P}_X$	$\mathbb{P}(X \in \cdot)$, distribution of a random variable X
$\mathcal{F}_t$	filtration
$\mathcal{F}_t^X$	canonical filtration of stoch. process $(X_t)_{t \geq 0}$
a.s.	almost surely (with respect to $\mathbb{P}$)

Spaces of functions

$\mathcal{B}_b(\mathbb{R}^d)$	Borel measurable, bounded functions $f : \mathbb{R}^d \to \mathbb{R}$
$C(\mathbb{R}^d)$	continuous functions $f : \mathbb{R}^d \to \mathbb{R}$
$C_c(\mathbb{R}^d)$	—, compact support
$C_b(\mathbb{R}^d)$	—, bounded
$C_\infty(\mathbb{R}^d)$	—, vanishing at infinity $\lim_{\|x\| \to \infty} \|f(x)\| = 0$
$C^n(\mathbb{R}^d)$	n-times continuously differentiable functions
$C_b^n(\mathbb{R}^d)$	—, bounded (with all derivatives)
$C_\infty^n(\mathbb{R}^d)$	—, vanishing at infinity (with all derivatives)
$C_c^\infty(\mathbb{R}^d)$	smooth functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact support
$C^{>0}(I)$	p. 160

Sets

A^c	complement of the set A
$A \mathbin{\dot\cup} B$	disjoint union of A and B
$\overline{A}$	closure of the set A
$B(x,r)$	open ball, centre x, radius r
$B[x,r]$	closed ball, centre x, radius r

Markov processes

$(P_t)_{t\geq 0}$	semigroup
$(L, \mathcal{D}(L))$	generator
R_λ	$= (\lambda - L)^{-1}$, resolvent

References

(B1)-(B4)	p. 144
(C1)-(C4)	p. 53
(C5)	p. 57
(C6)	p. 176
(C3'),(C4')	p. 59
(C4")	p. 168
(C5")	p. 169
(D1),(D2)	p. 45
(E1)-(E3)	p. 74
(PMP)	positive maximum principle, p. 11

Abbreviations

NIG	normal inverse Gaussian
NTS	normal tempered stable
TLP	truncated Lévy process

Further notation

$A(m)$	admissible cts. neg. def. fcts., p. 53
$\Omega(m,\vartheta)$	p. 53
$\Lambda(m,R,\vartheta)$	p. 144
$C(\vartheta)$	p. 144
$S(x,\alpha,t)$	p. 56
$F(t,x,y)$	p. 64
$G(t,x,y)$	p. 94
$H^k(t,x,y)$	p. 95
$\Phi(t,x,y)$	p. 97
$g_\gamma(x)$	p. 81
$p_t^\alpha(x)$	p. 63
$p_0(t,x,y)$	parametrix, p. 63
$p_\varepsilon(t,x,y)$	approximate fundamental solution, p. 110
$A^\beta p_t^\alpha$	p. 63
⚠	p. 65

Summary

Lévy processes are stochastic processes with independent and stationary increments. They constitute an important subclass of Markov processes. By the Lévy-Khintchine formula, there is a one-to-one correspondence between Lévy processes and continuous negative definite (in the sense of Schoenberg [8]) functions. In particular, any Lévy process can be uniquely determined by a continuous negative definite function ψ, the so-called characteristic exponent,

$$\psi(\xi) = -ib \cdot \xi + \frac{1}{2}\xi \cdot Q\xi + \int_{\mathbb{R}^d \setminus \{0\}} (1 - e^{iy\cdot\xi} + iy \cdot \xi \mathbb{1}_{(0,1)}(|y|))\, \nu(dy), \quad \xi \in \mathbb{R}^d,$$

where (b, Q, ν) is the Lévy triplet comprising the drift $b \in \mathbb{R}^d$, the diffusion matrix $Q \in \mathbb{R}^{d\times d}$, and the Lévy measure ν. Many distributional properties and path properties of a Lévy process can be described in terms of the characteristic exponent ψ or the Lévy triplet, see e. g. Sato [92, Chapter 4,5], Blumenthal & Getoor [12] and Fristedt [38].

There are other, larger subclasses of Markov processes which can be characterized in terms of a single deterministic function. In this thesis, we focus on, so-called, Feller processes. Roughly speaking, Feller processes behave locally like Lévy processes – that's the reason why they are also called Lévy-type processes – but, in contrast to Lévy processes, Feller processes need not to be homogeneous in space. Typical examples of Feller processes are solutions of Lévy-driven stochastic differential equations (SDEs, for short), affine processes and stable-like processes.

If a Feller process has the additional property that the smooth functions with compact support are contained in the domain of the generator, then we speak of a rich Feller process. Any such rich Feller process can be characterized by its x-dependent symbol

$$q(x,\xi) = -ib(x) \cdot \xi + \frac{1}{2}\xi \cdot Q(x)\xi + \int_{\mathbb{R}^d \setminus \{0\}} (1 - e^{iy\cdot\xi} + iy \cdot \xi \mathbb{1}_{(0,1)}(|y|))\, \nu(x, dy), \quad x, \xi \in \mathbb{R}^d$$

which is the analogue of the characteristic exponent in the Lévy case. Restricted to the smooth functions with compact support, the generator of a rich Feller process is a pseudo-differential operator with symbol q. Among the first to study the connection between rich Feller processes and pseudo-differential operators with negative definite symbols was Jacob [49, 50, 51].

Since Feller processes behave locally like Lévy processes, it is a natural guess that the symbol plays a similar role as the characteristic exponent in the theory of Lévy processes, i. e. that it is a useful tool to describe properties of the process. This has been confirmed by many authors who studied properties of Feller processes in the past years, such as recurrence & transience (Sandric [89]), ergodicity (Sandric [90]), invariant measures (Behme & Schnurr [6]), Hausdorff dimensions (see [19, Section 5.2] and the references therein), the asymptotic growth of the sample paths (Schilling [94], Knopova & Schilling [58]) and Besov regularity ([19, Section 5.5]).

In the first part of this thesis, Chapter 2, we will investigate a distributional property of Feller processes which has barely received any attention so far: existence of generalized moments and moment estimates. For a Lévy process $(L_t)_{t\geq 0}$ and a locally bounded submultiplicative function $f:\mathbb{R}^d\to[0,\infty)$, it is well-known (cf. Sato [92]) that the existence of the generalized moment $\mathbb{E}^x f(L_t)$ can equivalently be characterized in terms of the Lévy measure ν:

$$\mathbb{E}^x f(L_t)<\infty \iff \int_{|y|\geq 1} f(y)\,\nu(dy)<\infty.$$

This implies, in particular, that the existence of generalized moments is a time-independent distributional property, i. e.

$$\exists t>0:\mathbb{E}^x f(L_t)<\infty \iff \forall t>0:\mathbb{E}^x f(L_t)<\infty.$$

In Section 2.1 we will establish similar results for Feller processes. We will show that generalized moments exist backward in time, i. e.

$$\mathbb{E}^x f(X_t)<\infty \implies \forall s\leq t:\mathbb{E}^x f(X_s)<\infty,$$

and that the moments also exist forward in time provided that $\mathbb{E}^x f(X_t-x)$ is bounded in $x\in\mathbb{R}^d$. Furthermore, Theorem 2.4 will give a sufficient condition for the existence of the moment $\mathbb{E}^x f(X_t)$ in terms of the x-dependent Lévy triplet $(b(x),Q(x),\nu(x,\cdot))$: If $f\geq 0$ is comparable to a submultiplicative function $g\geq 0$ which is twice differentiable, then

$$\sup_{x\in K}\int_{|y|\geq 1} f(y)\,\nu(x,dy)<\infty \implies \sup_{x\in K}\sup_{s\leq t}\mathbb{E}^x f(X_{s\wedge\tau_K}-x)<\infty$$

for any compact set $K\subseteq\mathbb{R}^d$ where $\tau_K:=\inf\{t\geq 0; X_t\notin K\}$ denotes the exit time from the set K; if the symbol q of the Feller process $(X_t)_{t\geq 0}$ has bounded coefficients, then $K=\mathbb{R}^d$ is admissible.

In applications it is often useful to have moment estimates, and in the last years there has been a particular interest in estimates of fractional moments $\mathbb{E}|X_t|^\alpha$, e. g. to obtain Harnack inequalities (Deng & Schilling [31]) or to prove the absolute continuity of solutions of Lévy driven SDEs (Fournier & Printems [36]). In our recent publication [67] we have applied different techniques to establish estimates for fractional moments of Feller processes and succeeded in generalizing results for Lévy processes obtained by Luschgy & Pagès [72] and Deng & Schilling [31]. Here, in this thesis, we will first introduce the notion of generalized Blumenthal–Getoor indices (following Schilling [94]) and then derive estimates for fractional moments by combining a maximal inequality for Feller processes (Lemma 1.29) with the identity

$$\mathbb{E}(|X|^\gamma)=\int_{(0,\infty)}\mathbb{P}(|X|\geq r^{1/\gamma})\,dr,\qquad \gamma>0.$$

This is one of the approaches which we have investigated in [67]. There is also the possibility to prove estimates of fractional moments using the Burkholder–Davis–Gundy inequality; we refer to [67] for details.

Finally, as an application of the moment estimates, we will show the absolute continuity of a class of Feller processes with Hölder continuous symbols (Section 2.3).

From Chapter 3 on we will be concerned with questions on the existence of Feller processes. The Lévy–Khintchine formula states that for any continuous negative definite function ψ there is a Lévy process with characteristic exponent ψ. This, however, is no longer true for Feller processes: Given an arbitrary family $(q(x,\cdot))_{x\in\mathbb{R}^d}$ of continuous negative definite functions there does, in general, not exist a Feller process with symbol q (see [19, Example 2.26] for counterexamples). For this reason it is crucial to find sufficient conditions on q or the associated family of triplets $(b(x), Q(x), \nu(x,\cdot))_{x\in\mathbb{R}^d}$ which ensure the existence of a Feller process with given symbol q.

There are different techniques to prove existence results for Feller processes, they range from purely analytic approaches (e. g. via the Hille–Yoshida theorem or a parametrix construction) to probabilistic methods (e. g. Feller processes as solutions of martingale problems or solutions of SDEs). We refer to the monograph [19] for a survey on known results.

Our method of choice is the parametrix construction. Its idea goes back to Levi [70] who obtained the fundamental solution of a parabolic differential equation using a parametrix construction. Feller [34] was one of the first to recognize the possible applications in probability theory. Already in 1936 he showed existence results for diffusions processes and a class of jump processes. Over the last two decades, the parametrix method has become an increasingly popular tool to prove the existence of certain stochastic processes and derive heat kernel estimates, e. g. processes with variable order of differentiation (Kolokoltsov [59, 60] and Chen & Zhang [25]), gradient perturbations of Lévy generators (Bogdan & Jakubowski [13] and Jakubowski & Szczypkowski [55]) and solutions of SDEs with Hölder continuous coefficients (Knopova & Kulik [57] and Huang [45]). Hoh [43] developed a symbolic calculus for pseudo-differential operators with continuous negative definite symbols and used a parametrix construction to obtain rather general existence results for Feller processes. The drawback of his approach is that it requires smoothness of $q(\cdot,\xi)$.

Roughly speaking, there is usually a trade-off between assumptions on the regularity of $x \mapsto q(x,\xi)$ and assumptions on $\xi \mapsto q(x,\xi)$. If $q(x,\cdot)$ is assumed to be of a particular form (typically "stable-like"), then the existence of a Feller process with symbol q can be proved under weak regularity assumptions with respect to the space variable x. In contrast, existence results which are applicable for a broad class of negative definite functions $q(x,\cdot)$ often require smoothness of the symbol with respect to x.

Since we are interested in existence results under weak regularity of $q(\cdot,\xi)$, we have to make some assumptions on the structure of q. We will consider families of continuous negative definite functions $(q(x,\cdot))_{x\in\mathbb{R}^d}$ which can be written in the form

$$q(x,\xi) = \psi_{\alpha(x)}(\xi), \qquad x,\xi \in \mathbb{R}^d,$$

for a Hölder continuous mapping $\alpha : \mathbb{R}^d \to I \subseteq \mathbb{R}^n$ and a family $\psi_\beta : \mathbb{R}^d \to \mathbb{C}$, $\beta \in I$, of continuous negative definite functions. Our main result, Theorem 3.2, states that if

- ψ_β is rotationally invariant, i. e. there exists a mapping $\Psi_\beta : \mathbb{R} \to \mathbb{R}$ such that $\psi_\beta(\xi) = \Psi_\beta(|\xi|)$ for all $\xi \in \mathbb{R}^d$,

- $I \ni \beta \mapsto \Psi_\beta(\xi)$ admits partial derivatives and both Ψ_β and the partial derivatives $\partial_{\beta_j}\Psi_\beta$, $j = 1, \dots, n$, have a holomorphic extension to a certain domain $\Omega \subseteq \mathbb{C}$,

- Ψ_β and $\partial_{\beta_j}\Psi_\beta$ satisfy certain growth conditions on Ω,

then there exists a Feller process $(X_t)_{t\geq 0}$ with symbol

$$q(x,\xi) := \psi_{\alpha(x)}(\xi), \qquad x,\xi \in \mathbb{R}^d.$$

In dimension $d = 1$ we can drop the assumption of rotational invariance (Theorem 3.7). As a by-product of the parametrix construction, we get additional information on the Feller process $(X_t)_{t\geq 0}$:

- The smooth functions with compact support $C_c^\infty(\mathbb{R}^d)$ are a core for the generator L of the Feller process $(X_t)_{t\geq 0}$ (Proposition 3.3).

- The $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed and its unique solution is given by $(X_t)_{t\geq 0}$ (Corollary 3.5).

- The transition probability $\mathbb{P}^x(X_t \in \cdot)$, $t > 0$, has a density $p = p(t,x,y)$ with respect to Lebesgue measure (Theorem 3.2). The density p is the fundamental solution to the Cauchy problem for the operator $\partial_t - L$ (Corollary 3.4).

- We obtain heat kernel estimates for the transition density p and its time derivative (Theorem 3.6). In dimension $d = 1$ we also get heat kernel estimates for the derivative with respect to x (Theorem 3.8).

- In dimension $d = 1$, the Feller process $(X_t)_{t\geq 0}$ is irreducible with respect to Lebesgue measure if $\alpha \in C_b^2(\mathbb{R})$ (Corollary 3.9).

We will prove these results in Chapter 4 using a parametrix construction. The proof has been inspired by the works of Kolokoltsov [60] and Knopova & Kulik [57]. In the first part of the proof, Section 4.1, we will derive heat kernel estimates for a class of rotationally invariant Lévy processes which, we believe, are of independent interest.

Chapter 5 is devoted to applications of the existence result. Because of the assumption of rotational invariance in dimension $d > 1$, it is natural to consider symbols which can be expressed in the form

$$q(x,\xi) = f_{\alpha(x)}(|\xi|^2), \qquad x,\xi \in \mathbb{R}^d,$$

for a family of Bernstein functions $(f_\beta)_{\beta\in I}$ and a Hölder continuous mapping $\alpha : \mathbb{R}^d \to I$. This is a particular case of, so-called, variable order subordination. In Section 5.1 we will present examples of Bernstein functions which satisfy the assumptions of our existence theorem. We will establish, among others, the existence of relativistic stable-like, Lamperti stable-like and normal tempered stable-like Feller processes. Further examples are collected in Table 5.1. Compared to other results in the literature (e. g. Hoh [43]), the novelty is that α need not to be smooth; it suffices to have Hölder continuity. Section 5.2 deals with Feller processes with symbols of varying order,

$$q(x,\xi) = (p(x,\xi))^{\alpha(x)}, \qquad x,\xi \in \mathbb{R}^d,$$

and in Section 5.3 we will obtain existence results for Feller processes of a mixed type. Finally, in the last part of Chapter 5, we will investigate Lévy-driven SDEs with Hölder continuous coefficients b and σ,

$$dX_t = b(X_{t-})\,dt + \sigma(X_{t-})\,dL_t.$$

For a certain class of driving Lévy processes $(L_t)_{t\geq 0}$ we will show the existence of a unique weak solution to the SDE and that the solution is a Feller process (Corollary 5.19). This result was previously known only for isotropic α-stable Lévy processes (see Knopova & Kulik [57] and the references therein).

Further research – not related to the topics presented in this thesis – can be found in the joint work [69] with René Schilling which appeared in *Journal of Theoretical Probability* recently. It is concerned with moderate deviation principles for additive processes and resulted from my diploma thesis [66].

It is my pleasure to thank the many people who have, the one way or the other, contributed to this thesis. My thanks goes to all friends and colleagues who supported me in the past two years, whether it was by providing welcome distraction from mathematics or showing interest in my work (a particular thanks to Björn Böttcher and Victoria Knopova).
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1 Basics

The aim of this chapter is to summarize briefly definitions and results which we will frequently use in this thesis. First, we set up some basic notation and recall standard definitions from probability theory. After introducing (universal) Markov processes in Section 1.2, we define Lévy processes, subordinators and Feller processes and discuss their most important properties. In Section 1.6 we collect some facts on martingale problems. The last part of this chapter, Section 1.7, is devoted to the parametrix method which will play a crucial role later on. Most of the results which we present in this chapter are well-known, and therefore we do not include the proofs of these results, but just gives references.

We consider the Euclidean space $\mathbb{R}^d$ with its scalar product $x \cdot y = \sum_{j=1}^d x_j y_j$, the induced norm $|\cdot|$ and its Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$ and $r > 0$ we use

$$B(x,r) := \{y \in \mathbb{R}^d; |x-y| < r\} \quad \text{and} \quad B[x,r] := \{y \in \mathbb{R}^d; |y-x| \leq r\}$$

to denote the open ball and closed ball, respectively. The real part and imaginary part of a complex number $z \in \mathbb{C}$ are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively, and $\arg z \in (-\pi, \pi]$ is the argument of z. Two functions $f, g : \mathbb{R}^d \to \mathbb{R}$ are said to be comparable if there exists a constant $C > 1$ such that

$$C^{-1} f(x) \leq g(x) \leq C f(x) \qquad \text{for all} \;\; x \in \mathbb{R}^d.$$

We write $C_b(\mathbb{R}^d)$, $C_\infty(\mathbb{R}^d)$ and $C_c(\mathbb{R}^d)$ for the spaces of functions $f : \mathbb{R}^d \to \mathbb{R}$ which are continuous and bounded, continuous and vanishing at infinity, and continuous with compact support, respectively. Superscripts are used to specify the order of differentiability, e. g. $f \in C_b^k(\mathbb{R}^d)$ if, and only if, f and its derivatives up to order k (exist and) are bounded continuous functions. Moreover, $C_c^\infty(\mathbb{R}^d)$ denotes the space of infinitely often differentiable functions with compact support and $\mathcal{B}_b(\mathbb{R}^d)$ is the family of Borel measurable and bounded functions $f : \mathbb{R}^d \to \mathbb{R}$. As usual, we use the shorthand $\partial_{x_j} f$ to denote the partial derivative $\frac{\partial}{\partial x_j} f$, $j \in \{1, \ldots, d\}$, of a function $f : \mathbb{R}^d \to \mathbb{R}$ with respect to x_j and write ∇f and $\nabla^2 f$ for the gradient and Hessian matrix, respectively. If we set

$$\|f\|_{(2)} := \|f\|_\infty + \sum_{j=1}^d \|\partial_{x_j} f\|_\infty + \sum_{i,j=1}^d \|\partial_{x_i} \partial_{x_j} f\|_\infty,$$

then $(C_b^2(\mathbb{R}^d), \|\cdot\|_{(2)})$ is a complete normed space. The support of a function f and a measure μ are denoted by $\operatorname{spt} f$ and $\operatorname{spt}\mu$, respectively.
Let $(X,\mathcal{A})$ be a measurable space and μ a measure on $(X,\mathcal{A})$. For $p \geq 1$ we define by

$$L^p(X,\mathcal{A},\mu) := L^p(\mu) := \left\{ f : X \to \mathbb{R} \text{ measurable}; \int_X |f|^p \, d\mu < \infty \right\}$$
$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}, \qquad f \in L^p(\mu)$$

the normed space $(L^p(\mu), \|\cdot\|_p)$. Following a standard convention, we consider elements in $L^p(\mu)$ as functions (which are determined up to a μ-null set) and not as equivalence classes. If ν is another measure on $(X,\mathcal{A})$, then the convolution

$$(\mu * \nu)(A) := \int_X \mu(A - x) \, \nu(dx), \qquad A \in \mathcal{A}$$

is a measure on $(X,\mathcal{A})$. The n-th convolution power is defined iteratively:

$$\mu^{*n} := \mu * \mu^{*(n-1)} \qquad \mu^{*1} := \mu.$$

The convolution of two functions $f, g : \mathbb{R}^d \to \mathbb{R}$ is given by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x+y) g(y) \, \lambda(dy), \qquad x \in \mathbb{R}^d,$$

whenever the integral on the right-hand side makes sense. Here, and in what follows, we use λ to denote the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Often we will just write "dx" (instead of "$\lambda(dx)$") to denote integration with respect to Lebesgue measure. The time-space convolution of two mappings $f, g : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is defined by

$$(f \circledast g)(t,x,y) := \int_0^t \int_{\mathbb{R}^d} f(t-s,x,z) g(s,z,y) \, dz \, ds, \qquad t \geq 0, x, y \in \mathbb{R}^d, \tag{1.1}$$

whenever the integral is well-defined. Iteratively we introduce the n-th convolution power

$$f^{\circledast n}(t,x,y) := (f \circledast f^{\circledast(n-1)})(t,x,y) \qquad f^{\circledast 1} := f. \tag{1.2}$$

The time-space convolution is associative, i. e. $f \circledast (g \circledast h) = (f \circledast g) \circledast h$. By (1.2) this implies in particular

$$f \circledast g^{\circledast n} = (f \circledast g) \circledast g^{\circledast(n-1)} \qquad \text{for all} \quad n \geq 2. \tag{1.3}$$

For $f \in L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$ we denote by

$$\hat{f}(\xi) := \mathcal{F}f(\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(\xi) \, d\xi, \quad \xi \in \mathbb{R}^d,$$

the Fourier transform of f and by

$$\check{f}(x) := \mathcal{F}^{-1} f(x) := \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(x) \, dx, \quad x \in \mathbb{R}^d,$$

the inverse Fourier transform of f.

1.1 Probability theory & stochastic processes

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For $\mathcal{G} \subseteq \mathcal{A}$ we use $\sigma(\mathcal{G})$ to denote the smallest σ-algebra containing $\mathcal{G}$. A *filtration* $(\mathcal{F}_t)_{t\geq 0}$ is a family of sub-σ-algebras of $\mathcal{A}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$. We set

$$\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s \quad \text{and} \quad \mathcal{F}_\infty := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right),$$

and say that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is *right-continuous* if $\mathcal{F}_{t+} = \mathcal{F}_t$ for all $t \geq 0$. We call $(\mathcal{F}_t)_{t\geq 0}$ *complete* if $\mathcal{F}_0$ contains all subsets of $\mathbb{P}$-null sets, i. e.

$$\{M \subseteq \Omega; \exists N \in \mathcal{A} : M \subseteq N, \mathbb{P}(N) = 0\} \subseteq \mathcal{F}_0.$$

A random variable $\tau : \Omega \to [0,\infty]$ is called a *stopping time* (also: $(\mathcal{F}_t)_{t\geq 0}$-*stopping time*) if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. For a random variable $X : \Omega \to \mathbb{R}^d$ the *distribution of* X is a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ defined by

$$\mathbb{P}_X(B) := \mathbb{P}(X \in B), \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

The distribution is uniquely characterized by the *characteristic function* $\mathbb{E}e^{i\xi\cdot X}$, $\xi \in \mathbb{R}^d$. If two random variables X and Y (possibly defined on two different probability spaces) have the same distribution, we write $X \overset{d}{=} Y$.
Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. We say that μ_n *converges weakly (vaguely)* to a measure μ if

$$\int f(x)\,\mu_n(dx) \xrightarrow{n\to\infty} \int f(x)\,\mu(dx)$$

for all $f \in C_b(\mathbb{R}^d)$ (for all $f \in C_c(\mathbb{R}^d)$). In what follows, we use $\mu_n \overset{w}{\to} \mu$ and $\mu_n \overset{v}{\to} \mu$ to denote weak and vague convergence, respectively. By the portmanteau theorem [10, Theorem 1.2.1] a sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to a probability measure μ if, and only if, $\int f\,d\mu_n \to \int f\,d\mu$ for all bounded, uniformly continuous functions $f : \mathbb{R}^d \to \mathbb{R}$.
A *(d-dimensional real-valued) stochastic process* $(X_t)_{t\geq 0}$ is a family of random variables $X_t : \Omega \to \mathbb{R}^d$, $t \geq 0$.[1] The *canonical filtration* of $(X_t)_{t\geq 0}$ is defined as $\mathcal{F}_t^X := \sigma(X_s; s \leq t)$. Sometimes we will write $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F}_t, X_t; t \geq 0)$ to indicate the underlying probability space and filtration. A stochastic process $(X_t)_{t\geq 0}$ is *adapted* to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if X_t is $\mathcal{F}_t$-measurable for each $t \geq 0$; this is equivalent to saying that $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for all $t \geq 0$. A stochastic process $(X_t)_{t\geq 0}$ has *càdlàg sample paths* if the *sample paths* $[0,\infty) \ni t \mapsto X_t(\omega)$ are right-continuous and have finite left-hand limits for all $\omega \in \Omega$. For a process $(X_t)_{t\geq 0}$ with càdlàg sample paths, we denote by $X_{t-} := \lim_{s\uparrow t} X_s$ the left-hand limit and by $\Delta X_t := X_t - X_{t-}$ the jump height at time t.
If two stochastic processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ satisfy $\mathbb{P}(X_t = Y_t) = 1$ for all $t \geq 0$, then $(Y_t)_{t\geq 0}$ is called a *modification* of $(X_t)_{t\geq 0}$ (and visa versa). Under the additional assumption

[1] This means, in particular, that we only consider *conservative* stochastic processes, i. e. processes satisfying $\mathbb{P}(X_t \in \mathbb{R}^d) = 1$ for all $t \geq 0$.

that $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ have càdlàg sample paths, this implies that $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are *indistinguishable*, i. e.

$$\mathbb{P}(\forall t \geq 0 : X_t = Y_t) = 1.$$

We say that two processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ (possibly defined on different probability spaces) have *the same finite-dimensional distributions*, and write $(X_t)_{t\geq 0} \overset{d}{=} (Y_t)_{t\geq}$, if $(X_{t_1}, \ldots, X_{t_n}) \overset{d}{=} (Y_{t_1}, \ldots, Y_{t_n})$ for any $0 \leq t_1 < \ldots < t_n$, $n \in \mathbb{N}$. Two processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are called *independent*, $(X_t)_{t\geq 0} \perp\!\!\!\perp (Y_t)_{t\geq 0}$, if the σ-algebras $\mathcal{F}_\infty^X$ and $\mathcal{F}_\infty^Y$ are independent.
A stochastic process $(X_t)_{t\geq 0}$ is a *martingale with respect to a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *and a probability measure* $\mathbb{P}$ if $X_t \in L^1(\mathbb{P})$ for all $t \geq 0$, X_t is $\mathcal{F}_t$-measurable and

$$\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s \qquad \text{for all} \quad s \leq t.$$

Unless otherwise mentioned, we always consider the canonical filtration, i. e. $\mathcal{F}_t = \mathcal{F}_t^X$.

1.2 Markov processes

In this section, we introduce Markov processes and some notions which are closely related. Let us remark that there are many different concepts of Markov processes in the literature. We restrict ourselves to, so-called, universal time-homogeneous Markov processes because this is the class of processes which we will encounter later on and which we are interested in. Throughout this section, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes a probability space. The next two definitions are, essentially, taken from the monograph [95] by Schilling.

1.1 Definition A family of mappings $p_t : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \to [0, \infty)$, $t \geq 0$, is called a *transition probability kernel* if

(i) $B \mapsto p_t(x, B)$ is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for all $t \geq 0$, $x \in \mathbb{R}^d$,

(ii) $(t, x) \mapsto p_t(x, B)$ is (Borel-)measurable for all $B \in \mathcal{B}(\mathbb{R}^d)$,

(iii) p_t satisfies the *Chapman–Kolmogorov equation*, i. e.

$$p_{s+t}(x, B) = \int_{\mathbb{R}^d} p_t(y, B)\, p_s(x, dy) \qquad \text{for all} \quad s, t \geq 0, x \in \mathbb{R}^d, B \in \mathcal{B}(\mathbb{R}^d).$$

1.2 Definition A *(universal time-homogeneous) Markov process* is a tuple

$$(\Omega, \mathcal{A}, \mathcal{F}_t, X_t, t \geq 0, \mathbb{P}^x, x \in \mathbb{R}^d)$$

such that $p_t(x, B) := \mathbb{P}^x(X_t \in B)$ defines a transition probability kernel, $\mathbb{P}^x(X_0 = x) = 1$ for all $x \in \mathbb{R}^d$ and the *Markov property*

$$\mathbb{P}^x(X_t \in B \mid \mathcal{F}_s) = p_{t-s}(X_s, B) \qquad \mathbb{P}^x - \text{a.s.} \tag{1.4}$$

is satisfied for all $s \leq t$, $x \in \mathbb{R}^d$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

Because of the Markov property (1.4), the finite-dimensional distributions of a Markov process $(X_t)_{t\geq 0}$ are uniquely determined by the family of one-dimensional distributions $(\mathbb{P}^x(X_t \in \cdot))_{t\geq 0, x\in\mathbb{R}^d}$:

$$\mathbb{P}^x(X_{t_1} \in B_1, \dots, X_{t_n} \in B_n) = \int_{B_1}\int_{B_2} \dots \int_{B_n} p_{t_n - t_{n-1}}(y_{n-1}, dy_n) \dots p_{t_2-t_1}(y_1, dy_2) p_{t_1}(x, dy_1)$$

for any Borel sets $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R}^d)$ and $0 \leq t_1 \leq \dots \leq t_n$, $n \in \mathbb{N}$. The Markov property (1.4) is equivalent to

$$\mathbb{E}^x(f(X_t) \mid \mathcal{F}_s) = \mathbb{E}^{X_s}(f(X_{t-s})) := \int f(y)\, p_{t-s}(X_s, dy) \quad \mathbb{P}^x - \text{a.s. for all } s \leq t, f \in \mathcal{B}_b(\mathbb{R}^d).$$

From this and Definition 1.1 it follows easily that

$$P_t : \mathcal{B}_b(\mathbb{R}^d) \to \mathcal{B}_b(\mathbb{R}^d), f \mapsto \mathbb{E}^\bullet(f(X_t)) := \int_{\mathbb{R}^d} f(y)\, p_t(\bullet, dy), \qquad t \geq 0,$$

defines a family of linear operators which forms a semigroup (i. e. $P_0 = \text{id}$ and $P_{t+s} = P_t \circ P_s$ for all $s, t \geq 0$), the *semigroup associated with (the Markov process)* $(X_t)_{t\geq 0}$. $(P_t)_{t\geq 0}$ has the following properties:

(i) P_t is *contractive*, i. e. $\|P_t f\|_\infty \leq \|f\|_\infty$ for all $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $t \geq 0$.

(ii) P_t has the *sub-Markov property*, i. e. $0 \leq P_t f \leq 1$ for any $0 \leq f \leq 1$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $t \geq 0$. In particular, P_t is *positivity preserving*: $P_t f \geq 0$ for any $f \geq 0$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $t \geq 0$.

(iii) P_t is *conservative*, i. e. $P_t 1 = 1$ for all $t \geq 0$.

We call a family $(P_t)_{t\geq 0}$ of linear operators on $\mathcal{B}_b(\mathbb{R}^d)$ which satisfies (i)-(iv) a *Markov semigroup*. To each Markov semigroup we can associate a generator and a resolvent.

1.3 Definition Let $(X_t)_{t\geq 0}$ be a Markov process with semigroup $(P_t)_{t\geq 0}$. Then the linear operator $(L, \mathcal{D}(L))$ defined by

$$\mathcal{D}(L) := \left\{ f \in C_\infty(\mathbb{R}^d); \exists g \in C_\infty(\mathbb{R}^d) : \lim_{t\to 0} \left\| \frac{P_t f - f}{t} - g \right\|_\infty = 0 \right\}$$
$$Lf := \lim_{t \to 0} \frac{P_t f - f}{t}$$

is the *(infinitesimal) generator of the semigroup* $(P_t)_{t\geq 0}$.[2]

The generator $(L, \mathcal{D}(L))$ of a Markov semigroup satisfies the *positive maximum principle (on $\mathcal{D}(L)$)*, i. e.

$$f(x_0) = \sup_{x\in\mathbb{R}^d} f(x) \implies Lf(x_0) \leq 0 \qquad \text{for all } \; f \in \mathcal{D}(L). \tag{PMP}$$

Indeed: If $f \in \mathcal{D}(L) \subseteq C_\infty(\mathbb{R}^d)$ and f attains its maximum in $x_0 \in \mathbb{R}^d$, then $f(x_0) \geq 0$ and therefore

$$\frac{P_t f(x_0) - f(x_0)}{t} \leq \frac{P_t f^+(x_0) - f(x_0)}{t} \leq \frac{\|f^+\|_\infty - f(x_0)}{t} \leq 0;$$

hence $Lf(x_0) = \lim_{t\to 0} t^{-1}(P_t f(x_0) - f(x_0)) \leq 0$.

[2] $\mathcal{D}(L)$ is contained in the domain of strong continuity and might be empty.

1.4 Definition Let $(X_t)_{t\geq 0}$ be a Markov process with semigroup $(P_t)_{t\geq 0}$, then we call the family $(R_\lambda)_{\lambda>0}$ of linear operators,

$$R_\lambda f := \int_{(0,\infty)} e^{-\lambda t} P_t f \, \lambda(dt), \qquad f \in C_\infty(\mathbb{R}^d),$$

the *resolvent.*

There is a one-to-one relationship between the semigroup $(P_t)_{t\geq 0}$ and $(R_\lambda)_{\lambda>0}$. Moreover, $R_\lambda = (\lambda - L)^{-1}$ which means, in particular, that $R_\lambda(C_\infty(\mathbb{R}^d)) \subseteq \mathcal{D}(L)$. We refer to [98, Proposition 7.13] for a proof and a discussion of further properties of the resolvent.

Later on we will encounter irreducible Markov processes.

1.5 Definition Let $(X_t)_{t\geq 0}$ be a Markov process and μ a σ-finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. We say that $(X_t)_{t\geq 0}$ is *μ-irreducible* if

$$\int_{(0,\infty)} \mathbb{P}^x(X_t \in B) \, dt > 0$$

for all $x \in \mathbb{R}^d$ and any set $B \in \mathcal{B}(\mathbb{R}^d)$ with $\mu(B) > 0$.

We close this section with some material on hitting times. For a stochastic process $(X_t)_{t\geq 0}$ we define *(first) hitting time* of a set $B \in \mathcal{B}(\mathbb{R}^d)$ by

$$\tau_B := \inf\{t \geq 0; X_t \in B\}, \qquad (\inf \varnothing := \infty).$$

In general it is highly non-trivial to prove that τ_B is a stopping time. In Theorem 1.6 we collect some known results.

1.6 Theorem *Let $(X_t)_{t\geq 0}$ be an $\mathcal{F}_t$-adapted stochastic process.*

(i) If $(X_t)_{t\geq 0}$ has right-continuous sample paths and B is an open set, then τ_B is an $\mathcal{F}_{t+}$-stopping time.

(ii) If $(\mathcal{F}_t)_{t\geq 0}$ is complete and $(X_t)_{t\geq 0}$ a Markov process with càdlàg sample paths, then τ_B is an $\mathcal{F}_{t+}$-stopping time for any closed set B.

(iii) (Début theorem) If $(\mathcal{F}_t)_{t\geq 0}$ is a right-continuous complete filtration and $(X_t)_{t\geq 0}$ is progressively measurably, i. e.

$$([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F}_T) \ni (t,\omega) \mapsto X_t(\omega) \in (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$$

is measurable for all $T > 0$, then τ_B is an $\mathcal{F}_{t+}$-stopping time for any $B \in \mathcal{B}(\mathbb{R}^d)$.

The first statement is easy to prove (see e. g. [98, Lemma 5.7]), and it will be enough for our purposes. Already the proof of (ii) requires much more effort; we refer the reader to Itô [48, Section 2.10]. The idea of the proof is to take a sequence $(U_n)_{n\in\mathbb{N}}$ of open sets decreasing to B and then to show that the $\mathcal{F}_{t+}$-stopping time

$$\tau := \lim_{n\to\infty} \tau_{U_n}$$

satisfies $\mathbb{P}(\tau = \tau_B) = 1$ using the fact that $\mathbb{P}(\{X_\sigma = X_{\sigma-}\} \cap A) = 1$ for any stopping time σ which is accessible on A (cf. [48, Theorem 2, Section 2.9]). The Début theorem is a deep result and therefore hard to prove (see [30] for a proof using capacities and [5] for a rather elementary proof). Since any adapted process with right-continuous sample paths is progressively measurable, it is obvious that (i),(ii) are immediate consequences of the Début theorem if the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and complete.

1.3 Lévy processes

An important subclass of Markov processes (in the sense of Definition 1.2) are Lévy processes. Our main references on Lévy processes are the monographs by Sato [92] and Schilling [95]. Throughout this section, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes a probability space.

1.7 Definition A stochastic process $(L_t)_{t\geq 0}$ is a *Lévy process* if it has the following properties.

(L1) $L_0 = 0$ almost surely.

(L2) $L_t - L_s \perp\!\!\!\perp \mathcal{F}_s^L$ for all $s \leq t$ (*independent increments*).

(L3) $L_t - L_s \overset{d}{=} L_{t-s}$ for all $s \leq t$ (*stationary increments*).

(L4) $t \mapsto L_t(\omega)$ is càdlàg for all $\omega \in \Omega$.

Let us remark that (L4) is equivalent[3] to the regularity assumption

(L4') $\lim_{t\to 0} \mathbb{P}(|L_t - L_0| > \varepsilon) = 0$ for all $\varepsilon > 0$ (continuity in probability).

The implication (L4) $\implies$ (L4') is obvious, but the proof of the converse requires more effort, see e. g. [92, Theorem 11.1] for a proof.

In order to show that any Lévy process is a Markov process, we have to overcome some technical difficulties. A Lévy process $(L_t)_{t\geq 0}$ can be extended to the larger space

$$\tilde{\Omega} := \mathbb{R}^d \times \Omega \qquad \tilde{\mathcal{A}} := \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{A}$$

by setting $L_t(x,\omega) := x + L_t(\omega)$, $x \in \mathbb{R}^d$. For each fixed $x \in \mathbb{R}^d$, we define a probability measure $\mathbb{P}^x$ on $(\tilde{\Omega}, \tilde{\mathcal{A}})$ by $\mathbb{P}^x := \delta_x \otimes \mathbb{P}$. Clearly, the process $(L_t(x,\cdot))_{t\geq 0}$ satisfies (L2)-(L4) (with respect to $\mathbb{P}^x$), and therefore it is called *Lévy process started at x*. Moreover, it is not difficult to see from the stationarity and the independence of the increments that $(\tilde{\Omega}, \tilde{\mathcal{A}}, \mathcal{F}_t^L, L_t, t \geq 0, \mathbb{P}^x, x \in \mathbb{R}^d)$ is a Markov process (in the sense of Definition 1.2) with transition probability function

$$p_t(x,B) = \mathbb{P}^x(L_t \in B) = \mathbb{P}(x + L_t \in B), \qquad x \in \mathbb{R}^d, B \in \mathcal{B}(\mathbb{R}^d), \tag{1.5}$$

[3]in the sense that any stochastic process satisfying (L1)-(L3),(L4') has a modification which satisfies (L1)-(L4)

see e. g. [95] for more details.

Lévy processes are strongly connected with infinitely divisible distributions and continuous negative definite functions.

1.8 Definition A distribution μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is *infinitely divisible* if for each $n \in \mathbb{N}$ there exists a distribution μ_n such that $\mu = \mu_n^{*n}$. Equivalently, a random variable X is called *infinitely divisible* if there exist independent identically distributed variables $X_{n,1}, \dots, X_{n,n}$ such that $X \overset{d}{=} X_{n,1} + \dots + X_{n,n}$ for all $n \in \mathbb{N}$.

It follows from the stationarity and independence of the increments that

$$L_t = \sum_{j=1}^{n} \left(L_{t\frac{j}{n}} - L_{t\frac{j-1}{n}} \right)$$

is infinitely divisible for each $t \geq 0$ for any Lévy process $(L_t)_{t\geq 0}$. Conversely, if μ is an infinitely divisible distribution, then there exists a Lévy process $(L_t)_{t\geq 0}$ such that $\mathbb{P}_{L_1} = \mu$ (cf. [92, Theorem 7.10]). This means that there is a one-to-one correspondence between Lévy processes and infinitely divisible functions. The Lévy-Khintchine formula states that Lévy processes can be uniquely characterized (in the sense of finite-dimensional distributions) by their characteristic exponent, see [92] or [95] for a proof.

1.9 Theorem *Let $(L_t)_{t\geq 0}$ be a (d-dimensional) Lévy process. Then there exists a unique triplet (b, Q, ν) comprising a vector $b \in \mathbb{R}^d$, a positive semi-definite symmetric matrix $Q \in \mathbb{R}^{d\times d}$ and a measure ν on $(\mathbb{R}^d\backslash\{0\}, \mathcal{B}(\mathbb{R}^d\backslash\{0\}))$ with $\int_{\mathbb{R}^d\backslash\{0\}} (|y|^2 \wedge 1)\, \nu(dy) < \infty$ such that*

$$\psi(\xi) := -ib\cdot\xi + \frac{1}{2}\xi\cdot Q\xi + \int_{\mathbb{R}^d\backslash\{0\}} (1 - e^{iy\cdot\xi} + iy\cdot\xi \mathbb{1}_{(0,1)}(|y|))\, \nu(dy), \quad \xi \in \mathbb{R}^d, \tag{1.6}$$

satisfies

$$\mathbb{E}e^{i\xi\cdot L_t} = e^{-t\psi(\xi)} \qquad \text{for all } \xi \in \mathbb{R}^d, t \geq 0. \tag{1.7}$$

Conversely, any ψ of the form (1.6) defines via (1.7) a Lévy process $(L_t)_{t\geq 0}$.

ψ is the *characteristic exponent* of $(L_t)_{t\geq 0}$ and the *Lévy triplet* (b, Q, ν) consists of the *drift* b, the *diffusion matrix* Q and the *Lévy measure* ν.
It is well-known that a function $\psi : \mathbb{R}^d \to \mathbb{C}$ is of the form (1.6) if, and only if, ψ is continuous, $\psi(0) = 0$ and ψ is *negative definite (in the sense of Schoenberg)*, i. e.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} (\psi(\xi_i) + \overline{\psi(\xi_j)} - \psi(\xi_i - \xi_j))c_i\bar{c}_j \geq 0 \qquad \text{for all } (c_1, \dots, c_n) \in \mathbb{C}^n, \xi_1, \dots, \xi_n \in \mathbb{R}^d, n \in \mathbb{N},$$

cf. [8, Definition 7.1, Theorem 10.8]. Therefore, it follows from the Lévy-Khintchine formula that there is a one-to-one correspondence between Lévy processes and continuous negative definite functions. Next we give some classical examples of Lévy processes. We will discuss further examples in Section 5.1.

1.10 Example (i) A *Brownian motion* is a Lévy process with characteristic exponent $\psi(\xi) := |\xi|^2$, $\xi \in \mathbb{R}^d$. It is possible to show that any Lévy process $(L_t)_{t\geq 0}$ with continuous sample paths is of the form

$$L_t = bt + \sqrt{Q}B_t, \qquad t \geq 0,$$

for a Brownian motion $(B_t)_{t\geq 0}$, a positive semidefinite matrix Q and a drift vector b (see e. g. [95, Theorem 8.4] for a proof).

(ii) A stochastic process $(N_t)_{t\geq 0}$ is a *Poisson process (with intensity λ)* if there exists a sequence $(\sigma_j)_{j\in\mathbb{N}}$ of independent identically distributed waiting times, $\sigma_j \sim \mathrm{Exp}(\lambda)$, such that

$$N_t = \sum_{j=1}^{\infty} \mathbb{1}_{[0,t]}(\sigma_1 + \ldots + \sigma_j), \qquad t \geq 0.$$

This means that a Poisson process is a counting process with jumps of height 1 and exponentially distributed waiting times. If we consider, more generally, a process with random jump heights, that is

$$N_t = \sum_{j=1}^{\infty} H_j \mathbb{1}_{[0,t]}(\sigma_1 + \ldots + \sigma_j), \qquad t \geq 0,$$

for a sequence of independent random variables $(H_j)_{j\in\mathbb{N}}$ independent of $(\sigma_j)_{j\in\mathbb{N}}$, $H_j \sim \mu$, then $(N_t)_{t\geq 0}$ is a *compound Poisson process.* Its characteristic exponent is given by

$$\psi(\xi) = \lambda \int_{y\neq 0} (1 - e^{iy\cdot\xi})\, \mu(dy), \qquad \xi \in \mathbb{R}^d,$$

cf. [95, Theorem 3.4].

(iii) A *(symmetric) α-stable* Lévy process is a Lévy process $(L_t)_{t\geq 0}$ with characteristic exponent $\psi(\xi) = |\xi|^\alpha$, $\alpha \in (0,2)$. If $\alpha = 1$ then $(L_t)_{t\geq 0}$ is a *Cauchy process.*

Later on we will often consider rotationally invariant Lévy processes.

1.11 Definition A Lévy process $(L_t)_{t\geq 0}$ with characteristic exponent ψ is *rotationally invariant* if $\psi(\xi) = \Psi(|\xi|)$, $\xi \in \mathbb{R}^d$ for some function Ψ, i. e. $\psi(\xi)$ depends only on $|\xi|$.

Equivalently, $(L_t)_{t\geq 0}$ is rotationally invariant if $L_t \overset{d}{=} RL_t$ for all $t \geq 0$ and any rotation matrix R (cf. [92, E18.3]). Before we state an important representation theorem for Lévy processes, we introduce the *jump counting measure*:

$$N_t(B) := \sharp\{s \in [0,t]; \Delta L_s = L_s - L_{s-} \in B\}, \qquad t \geq 0, B \in \mathcal{B}(\mathbb{R}^d\backslash\{0\}).$$

If $(L_t)_{t\geq 0}$ is a Lévy process with Lévy triplet (b, Q, ν), then $(N_t)_{t\geq 0}$ is a *Poisson random measure with intensity measure ν*, i. e.

(i) $(N_t(B))_{t\geq 0}$ is a Poisson process with intensity $\nu(B)$ for any $B \in \mathcal{B}(\mathbb{R}^d\backslash\{0\})$,

(ii) $(N_t(A))_{t\geq 0} \perp\!\!\!\perp (N_t(B))_{t\geq 0}$ for any two disjoint Borel sets $A, B \in \mathcal{B}(\mathbb{R}^d\backslash\{0\})$.

This allows us to define the stochastic integrals

$$\int_0^t \int f(s,y)\, N(dy,ds) \qquad\qquad \int_0^t \int f(s,y)\, (N(dy,ds) - \nu(dy)ds)$$

with respect to the jump counting measure and compensated jump counting measure, respectively, using the well-known theory for stochastic integrals with respect to random measures. We refer the reader to Ikeda & Watanabe [46] and Schilling [95] for a thorough discussion.

1.12 Theorem (Lévy-Itô decomposition) *Let $(L_t)_{t\geq 0}$ be a Lévy process with Lévy triplet (b,Q,ν) and jump counting measure $(N_t)_{t\geq 0}$. Then there exists a Brownian motion $(B_t)_{t\geq 0}$ such that*

$$L_t = bt + \sqrt{Q}B_t + \int_0^t \int_{|y|<1} y\, (N(dy,ds) - \nu(dy)ds) + \int_0^t \int_{|y|\geq 1} y\, N(dy,ds)$$

for all $t \geq 0$. The four processes on the right-hand side are Lévy processes which are independent.

In Section 1.5 we will encounter a similar representation result for the larger class of Lévy-type processes.

1.4 Subordination

In this section we give a brief introduction to subordination in the sense of Bochner.

1.13 Definition (i) A one-dimensional Lévy process $(S_t)_{t\geq 0}$ with non-decreasing sample paths is a *subordinator*.

(ii) A function $f : (0,\infty) \to \mathbb{R}$ is a *Bernstein function* if there exist constants $a, b \geq 0$ and a measure μ on $((0,\infty), \mathcal{B}((0,\infty)))$ satisfying $\int_{(0,\infty)} \min\{1,r\}\, \mu(dr) < \infty$ such that

$$f(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda r})\, \mu(dr) \qquad \text{for all } \lambda > 0. \tag{1.8}$$

By [92, Theorem 21.5], a one-dimensional Lévy process $(S_t)_{t\geq 0}$ with Lévy triplet (b,Q,ν) is a subordinator if, and only if, $b \geq 0$, $Q = 0$ and the Lévy measure satisfies

$$\nu((-\infty,0)) = 0 \quad \text{and} \quad \int_{0<y\leq 1} y\, \nu(dy) < \infty.$$

We have seen in the previous section that any Lévy process can be uniquely characterized via the Lévy–Khintchine formula (1.7). Since the distribution $\mathbb{P}_{S_t}$ of a subordinator $(S_t)_{t\geq 0}$ is supported in $[0,\infty)$, it is often convenient to use the Laplace transform $\mathbb{E}e^{-\lambda S_t}$, $\lambda \geq 0$, instead of the Fourier transform. There is the following analogue of Theorem 1.9.

1.14 Theorem *Let $(S_t)_{t\geq 0}$ be a subordinator. Then there exists a unique Bernstein function f with $f(0) = 0$ such that*

$$\mathbb{E}e^{-\lambda S_t} = e^{-tf(\lambda)} \qquad \textit{for all } t \geq 0, \lambda > 0. \tag{1.9}$$

Conversely, for any Bernstein function f with $f(0) = 0$ there exists a subordinator $(S_t)_{t\geq 0}$ such that (1.9) *holds true.*

A proof of this statement can be found in [100, Theorem 5.2]. If $(S_t)_{t\geq 0}$ and f are as in Theorem 1.14, we call f the *Laplace exponent* of $(S_t)_{t\geq 0}$. The Laplace exponent f and the characteristic exponent ψ of $(S_t)_{t\geq 0}$ are related through $f(\lambda) = \psi(i\lambda)$, $\lambda > 0$.

1.15 Example The mapping $\lambda \mapsto f(\lambda) := \lambda^\alpha$ is a Bernstein function for any $\alpha \in (0,1]$. *Indeed:* For $\alpha = 1$ this is obvious from the definition. For $\alpha \in (0,1)$ we define a measure μ by $\mu(dr) := r^{-1-\alpha}\mathbb{1}_{(0,\infty)}(r)\, dr$. Then $\int \min\{1,r\}\mu(dr) < \infty$ and, by Tonelli's theorem,

$$\begin{aligned}\int_{(0,\infty)}(1-e^{-\lambda r})\mu(dr) = \lambda\int_{(0,\infty)}\left(\int_0^r e^{-\lambda t}\,dt\right)\frac{1}{r^{1+\alpha}}\,dr &= \lambda\int_{(0,\infty)} e^{-\lambda t}\int_{(t,\infty)}\frac{1}{r^{1+\alpha}}\,dr\,dt \\ &= \frac{\lambda}{\alpha}\int_{(0,\infty)} e^{-\lambda t}t^{-\alpha}\,dt \\ &= \frac{\lambda^\alpha}{\alpha}\Gamma(1-\alpha)\end{aligned}$$

for any $\lambda > 0$, i. e.

$$f(\lambda) = \frac{\alpha}{\Gamma(1-\alpha)}\int_{(0,\infty)}(1-e^{-\lambda r})\,\mu(dr), \qquad \lambda > 0.$$

A subordinator with Laplace exponent $f(\lambda) = \lambda^\alpha$, $\alpha \in (0,1]$, is called an *α-stable subordinator.*

The next result is compiled from [92, Theorem 30.1].

1.16 Theorem *Let $(L_t)_{t\geq 0}$ be a Lévy process with characteristic exponent ψ and let $(S_t)_{t\geq 0}$ be a subordinator with Laplace exponent f. If $(L_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are independent, then the* subordinate process

$$Y_t := L_{S_t}, \qquad t \geq 0,$$

is again a Lévy process and its characteristic exponent equals $f(\psi(\xi))$.

Note that Theorem 1.16 shows in particular that the composition $f \circ \psi$ is a continuous negative definite function for any continuous negative definite function ψ and Bernstein function f.

For further material on subordination and Bernstein functions, we refer to the comprehensive monograph by Schilling, Song & Vondraček [100] and also to Sato [92, Chapter 6].

1.5 Feller processes

Feller processes behave locally like Lévy processes, but the Lévy triplet may depend on the current position of the process – that's why they are also called Lévy-type processes. We will use the terms "Lévy-type process" and "Feller process" synonymously.

1.17 Definition Let $(X_t)_{t\geq 0}$ be a Markov process with semigroup $(P_t)_{t\geq 0}$ and generator $(L, \mathcal{D}(L))$. We say that $(P_t)_{t\geq 0}$

(i) is *strongly continuous* if $\|P_t f - f\|_\infty \xrightarrow{t\to 0} 0$ for any $f \in C_\infty(\mathbb{R}^d)$,

(ii) has the *Feller property* if $P_t f \in C_\infty(\mathbb{R}^d)$ for all $f \in C_\infty(\mathbb{R}^d)$, $t > 0$,

(iii) has the *strong Feller property* if $P_t f \in C_b(\mathbb{R}^d)$ for all $f \in \mathcal{B}_b(\mathbb{R}^d)$, $t > 0$.

$(X_t)_{t\geq 0}$ is called a *Feller process* if $(P_t)_{t\geq 0}$ satisfies (i),(ii). If, additionally, the strong Feller property (iii) holds, then $(X_t)_{t\geq 0}$ is a *strong Feller process.* A *rich Feller process* is a Feller process whose domain of the generator contains $C_c^\infty(\mathbb{R}^d)$.

Using standard martingale techniques, it is possible to show that any Feller process $(X_t)_{t\geq 0}$ has a càdlàg modification $(\tilde{X}_t)_{t\geq 0}$ which is a Feller process (cf. [85, Theorem III.2.7]). Therefore, we assume from now on that all Feller processes which we encounter have càdlàg sample paths. Because of measurability issues, we will sometimes have to assume that $(X_t)_{t\geq 0}$ is a Feller process with respect to a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$; a possible choice is

$$\mathcal{F}_t := \bigcap_{x\in\mathbb{R}^d} \sigma\big(\mathcal{F}_t^X \cup \{M \subseteq \Omega; \exists N \in \mathcal{A}, N \supseteq M : \mathbb{P}^x(N) = 0\}\big)$$

(cf. [19, Theorem 1.20]) where $\mathcal{F}_t^X$ denotes the canonical filtration of $(X_t)_{t\geq 0}$.
It is not difficult to see from the definition and (1.5) that Lévy processes are a subclass of Feller process (cf. [95, Lemma 4.8]). In fact, any Lévy process is a rich Feller process (cf. [95, Lemma 6.3]). There is a result, due to Hawkes, which states that a Lévy process $(L_t)_{t\geq 0}$ is a strong Feller process if, and only if, the distribution $\mathbb{P}_{L_t}$ is absolutely continuous with respect to Lebesgue measure for all $t > 0$ (see e. g. [95, Lemma 4.9] for a proof).
There is the following existence result.

1.18 Theorem *Let $(P_t)_{t\geq 0}$ be a Markov semigroup which is strongly continuous and which has the Feller property. Then there exists a Feller process (with càdlàg sample paths) whose semigroup equals $(P_t)_{t\geq 0}$.*

The proof is based on Riesz' representation theorem and Kolmogorov's extension theorem, see [98, Remark 7.7] for details. Before we discuss the structure of Lévy-type processes, we make the following useful observation (see e. g. [98, Theorem 7.31] for a proof).

1.19 Proposition (Dynkin's formula) *Let $(X_t)_{t\geq 0}$ be a Feller process with infinitesimal generator $(L, \mathcal{D}(L))$. Then*

$$\mathbb{E}^x f(X_\tau) - f(x) = \mathbb{E}^x \left(\int_{[0,\tau)} Lf(X_s)\, ds \right)$$

holds for $f \in \mathcal{D}(L)$ and any stopping time τ such that $\mathbb{E}^x \tau < \infty$.

In order to prove that a function f belongs to the domain of the generator, it has to be shown that the limit $t^{-1}(P_t f - f)$ exists uniformly in $C_\infty(\mathbb{R}^d)$ – and that is usually very hard to verify. The next lemma states that it suffices to check convergence in a pointwise sense provided that the limit is a $C_\infty(\mathbb{R}^d)$-function (see e. g. [98, Theorem 7.15]).

1.20 Proposition *Let $(X_t)_{t\geq 0}$ be a Feller process with semigroup $(P_t)_{t\geq 0}$ and generator $(L, \mathcal{D}(L))$. Then*

$$\mathcal{D}(L) = \left\{ f \in C_\infty(\mathbb{R}^d); \exists g \in C_\infty(\mathbb{R}^d)\ \forall x \in \mathbb{R}^d : \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t} = g(x) \right\}.$$

We have seen in Section 1.3 that a Lévy process can be uniquely characterized by its characteristic exponent. The next part of this section shows that the, so-called, symbol plays a similar role in the theory of Lévy-type processes.
The following statement is due to Courrège and von Waldenfels; for a proof we refer to [19, Theorem 2.21, Corollary 2.23].

1.21 Theorem *Let $(X_t)_{t\geq 0}$ be a rich Feller process with generator $(L, \mathcal{D}(L))$. Then there exists a family $(b(x), Q(x), \nu(x, dy))_{x\in\mathbb{R}^d}$ of Lévy triplets and $c : \mathbb{R}^d \to [0, \infty)$ such that*

$$\begin{aligned} Lf(x) = -c(x)f(x) + b(x)\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}(Q(x)\cdot\nabla^2 f(x)) \\ + \int_{\mathbb{R}^d\setminus\{0\}} (f(x+y) - f(x) - \nabla f(x)\cdot y\mathbb{1}_{(0,1)}(|y|))\,\nu(x, dy) \end{aligned} \tag{1.10}$$

for all $f \in C_c^\infty(\mathbb{R}^d)$. Equivalently,

$$Lf(x) = -\int_{\mathbb{R}^d} q(x,\xi) e^{ix\cdot\xi} \hat{f}(\xi)\,d\xi \tag{1.11}$$

for all $f \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ where

$$q(x,\xi) := c(x) - ib(x)\cdot\xi + \frac{1}{2}\xi\cdot Q(x)\xi + \int_{\mathbb{R}^d\setminus\{0\}} (1 - e^{iy\cdot\xi} + iy\cdot\xi\mathbb{1}_{(0,1)}(|y|))\,\nu(x, dy) \tag{1.12}$$

is the symbol *of the Feller process $(X_t)_{t\geq 0}$.*

Often we will assume that the *killing rate* $c(x) = q(x, 0)$ equals 0. If $q(x, 0) = 0$ then it follows from Dynkin's formula that

$$-q(x,\xi) = \lim_{t\to 0} \frac{\mathbb{E}^x e^{i\xi\cdot(X_{t\wedge\tau_r^x} - x)} - 1}{t} \qquad \text{for all } \ x, \xi \in \mathbb{R}^d$$

where $\tau_r^x := \inf\{t \geq 0; |X_t - x| > r\}$ denotes the exit time from the ball $B[x, r]$. If we use this limit to *define* the symbol, it is possible to introduce the notion of a (probabilistic) symbol not only for rich Feller processes, but for a larger class of stochastic processes, see Schnurr [102] for a thorough discussion.
Theorem 1.21 shows that $L|_{C_c^\infty(\mathbb{R}^d)}$ is a pseudo-differential operator with negative definite symbol.

1.22 Definition Let $p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a function such that $p(x, \cdot)$ is a continuous negative definite function for all $x \in \mathbb{R}^d$. Then

$$Af(x) := -\int_{\mathbb{R}^d} p(x,\xi) e^{ix\cdot\xi} \hat{f}(\xi)\,d\xi, \qquad f \in C_c^\infty(\mathbb{R}^d), x \in \mathbb{R}^d,$$

is a *pseudo-differential operator (with negative definite symbol)* and p is the *symbol* of the operator A.

The pseudo-differential operator A is well-defined since the continuous negative definite function $p(x,\cdot)$ grows at most quadratically for large $|\xi|$ for each fixed $x \in \mathbb{R}^d$ (cf. [8, Corollary 7.16] or [95, Theorem 6.2]). Using the Lévy-Khintchine representation (1.12) for p and standard calculation rules from Fourier analysis, we find that A has a representation of the form (1.10). This implies that the pseudo-differential operator A has an extension to $C_b^2(\mathbb{R}^d)$. In abuse of notation, we denote this extension again by A. It follows easily from (1.10) that A satisfies (PMP) on $C_b^2(\mathbb{R}^d)$.
Before we discuss the properties of the (probabilistic) symbol in more detail, we give examples of Lévy-type processes. The first one is taken from Schilling & Schnurr [99].

1.23 Example (Lévy-driven SDE) Let $(L_t)_{t\geq 0}$ be a d-dimensional Lévy process and let $f : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ be a bounded Lipschitz continuous function. Then the unique (strong) solution to the *Lévy-driven stochastic differential equation* (SDE, for short)

$$dX_t = f(X_{t-})\, dL_t, \qquad X_0 = x,$$

is a rich Feller process. The symbol is given by $q(x,\xi) = \psi(f(x)^T\xi)$, $x,\xi \in \mathbb{R}^n$.

Schilling & Schnurr have shown that the boundedness of f is needed to ensure that the solution is a Feller process (cf. [99, Remark 3.4]). We will prove in Section 5.4 that, for a certain class of Lévy processes $(L_t)_{t\geq 0}$, the regularity assumption on f can be weakened to Hölder continuity (cf. Corollary 5.19).
Let us remark that also solutions to, so-called, (Lévy-driven) *Marcus SDEs*

$$dX_t = f(X_{t-}) \circ dL_t, \qquad X_0 = x, \tag{1.13}$$

are rich Lévy-type processes for "nice" functions f. Marcus SDEs are the analogue of Stratonovich SDEs in the Brownian setting and have been introduced by Marcus [74]; see Kurtz, Pardoux & Protter [64] for a discussion of Marcus SDEs and their properties.

The next example is due to Kolokoltsov [60].

1.24 Example (stable-like process) Let $\alpha : \mathbb{R}^d \to (0,2)$ be a Hölder continuous function such that

$$0 < \inf_{x\in\mathbb{R}^d} \alpha(x) \leq \sup_{x\in\mathbb{R}^d} \alpha(x) < 2.$$

Then there exists a rich Feller process with symbol $q(x,\xi) := |\xi|^{\alpha(x)}$.

In Section 5.1 we will slightly generalize this result by dropping the assumption that α has to be bounded away from 2.

Feller processes with bounded coefficients constitute an important subclass of Feller processes.

1.25 Definition Let $(X_t)_{t\geq 0}$ be a rich Feller process with symbol q. Then $(X_t)_{t\geq 0}$ has *bounded coefficients* if

$$\sup_{x\in\mathbb{R}^d}\left(|q(x,0)|+|b(x)|+|Q(x)|+\int_{\mathbb{R}^d\setminus\{0\}}\min\{|y|^2,1\}\,\nu(x,dy)\right)<\infty.$$

Clearly, any Lévy process is a Feller process with bounded coefficients. By [99, Lemma 6.2] a rich Feller process $(X_t)_{t\geq 0}$ has bounded coefficients if, and only if, there exists a constant $c>0$ such that

$$|q(x,\xi)|\leq c(1+|\xi|^2)\qquad\text{for all}\quad x,\xi\in\mathbb{R}^d.$$

For a rich Feller process with bounded coefficients, it follows easily from (1.10) and Taylor's formula that the generator L has an extension to $C_b^2(\mathbb{R}^d)$ which is continuous:

$$\|Lf\|_\infty\leq C\|f\|_{(2)}\sup_{x\in\mathbb{R}^d}\left[|q(x,0)|+|b(x)|+|Q(x)|+\int_{\mathbb{R}^d\setminus\{0\}}\min\{|y|^2,1\}\,\nu(x,dy)\right].\tag{1.14}$$

The next result shows that $x\mapsto q(x,\xi)$ is continuous whenever $x\mapsto q(x,0)$ is continuous (in particular if $q(x,0)=0$). It is taken from [93, Theorem 4.4].

1.26 Theorem *Let $(X_t)_{t\geq 0}$ be a rich Feller process with generator $(L,\mathcal{D}(L))$ and symbol q with Lévy triplet $(b(x),Q(x),\nu(x,dy))$. Then the following statements are equivalent.*

(i) $x\mapsto q(x,0)$ *is continuous,*

(ii) $x\mapsto q(x,\xi)$ *is continuous for all* $\xi\in\mathbb{R}^d$,

(iii) $\lim_{|\xi|\to 0}\sup_{x\in K}|q(x,\xi)-q(x,0)|=0$ *for all compact sets* $K\subseteq\mathbb{R}^d$,

(iv) $\lim_{r\to\infty}\sup_{x\in K}\nu(x,\mathbb{R}^d\setminus B(0,r))=0$ *for all compact sets* $K\subseteq\mathbb{R}^d$.

Later on we will need the following result which has, to our knowledge, not been discussed in the literature before.

1.27 Theorem *Let $q(x,\xi)$ be a negative definite symbol, $q(x,0)=0$, which is locally bounded and suppose that $x\mapsto q(x,\xi)$ is continuous for all $\xi\in\mathbb{R}^d$. Denote by $(b(x),Q(x),\nu(x,dy))_{x\in\mathbb{R}^d}$ the associated family of Lévy triplets. Then the following statements are equivalent.*

(i) *The pseudo-differential operator A with symbol q satisfies* $A(C_c^\infty(\mathbb{R}^d))\subseteq C_\infty(\mathbb{R}^d)$.

(ii) $\nu(x,K-x)\xrightarrow{|x|\to\infty}0$ *for any compact set* $K\subseteq\mathbb{R}^d$.

(iii) $\nu(x,B[-x,r])\xrightarrow{|x|\to\infty}0$ *for any* $r>0$.

The following conditions imply (i)-(iii):

(a) $(\nu(x,\cdot))_{x\in\mathbb{R}^d}$ *is tight, i. e.* $\lim_{R\to\infty}\sup_{x\in\mathbb{R}^d}\nu(x,\mathbb{R}^d\setminus B[0,R])=0$.

(b) $\limsup_{|\xi|\to 0}\sup_{x\in\mathbb{R}^d}|\operatorname{Re}q(x,\xi)|=0$.

(c) $\lim_{|x|\to\infty} \sup_{|\xi|\le|x|^{-1}} |\operatorname{Re} q(x,\xi)| = 0.$

We remark that Hoh [44, Theorem 3.3] has shown that (a) and (b) are equivalent.

Proof. Without loss of generality, we may assume $b(x) = Q(x) = 0$. To prove that (i)-(iii) are equivalent we show (i) $\implies$ (iii) $\implies$ (ii) $\implies$ (i).

- (i) $\implies$ (iii): Pick $f \in C_c^\infty(\mathbb{R}^d)$, $f \ge 0$, such that $f|_{B[0,r]} = 1$ and $f|_{B[0,2r]^c} = 0$. Then
$$Af(x) = \int_{\mathbb{R}^d\setminus\{0\}} f(x+y)\,\nu(x,dy) \ge \int_{B[-x,r]} f(x+y)\,\nu(x,dy) = \nu(x,B[-x,r])$$
for all $|x| > 2r$. As $Af \in C_\infty(\mathbb{R}^d)$, we get
$$\lim_{|x|\to\infty} \nu(x,B[-x,r]) \le \lim_{|x|\to\infty} Af(x) = 0.$$
- *(iii)* $\implies$ *(ii)*: This is obvious; choose $r > 0$ such that $K \subseteq B[0,r]$.
- *(ii)* $\implies$ *(i)*: Let $f \in C_c^\infty(\mathbb{R}^d)$, then
$$|Af(x)| = \left|\int_{\mathbb{R}^d\setminus\{0\}} f(x+y)\,\nu(x,dy)\right| \le \|f\|_\infty \nu(x, \operatorname{spt} f - x)$$
for all $x \notin \operatorname{spt} f$. Letting $|x| \to \infty$ gives $\lim_{|x|\to\infty} |Af(x)| = 0$. Since $x \mapsto q(x,\xi)$ is continuous for all $\xi \in \mathbb{R}^d$, the continuity of Af follows directly from the definition of A and the dominated convergence theorem using that q is locally bounded.

Clearly, (a) $\implies$ (ii). Moreover, (b) implies (c) since

$$\begin{aligned}\lim_{|x|\to\infty} \sup_{|\xi|\le|x|^{-1}} |\operatorname{Re} q(x,\xi)| &\le \lim_{R\to\infty} \sup_{|x|\ge R} \sup_{|\xi|\le|x|^{-1}} |\operatorname{Re} q(x,\xi)| \le \lim_{R\to\infty} \sup_{|x|\ge R} \sup_{|\xi|\le R^{-1}} |\operatorname{Re} q(x,\xi)| \\ &\le \lim_{R\to\infty} \sup_{|\xi|\le R^{-1}} \sup_{x\in\mathbb{R}^d} |\operatorname{Re} q(x,\xi)| \\ &= \limsup_{|\xi|\to 0} \sup_{x\in\mathbb{R}^d} |\operatorname{Re} q(x,\xi)|.\end{aligned}$$

Consequently, it suffices to show that (c) implies (iii). We use a similar reasoning as in [93, Theorem 4.4]. Fix $r > 0$. Obviously, by the reverse triangle inequality,

$$\frac{|y|}{|x|} \ge \frac{|x| - |y+x|}{|x|} \ge 1 - \frac{r}{|x|} \ge \frac{1}{2}$$

for any $y \in B[-x,r]$ and $|x| \gg 1$ sufficiently large. Since

$$\inf_{|z|\ge\frac{1}{2}} \frac{|z|^2}{1+|z|^2} = \frac{1}{5} > 0$$

we obtain by applying Tonelli's theorem

$$\frac{1}{5}\nu(x,B[-x,r]) \le \int_{B[-x,r]} \frac{\left(\frac{|y|}{|x|}\right)^2}{\left(\frac{|y|}{|x|}\right)^2 + 1}\,\nu(x,dy) = \int_{B[-x,r]} \int \left(1 - \cos\frac{\eta\cdot y}{|x|}\right) g(\eta)\,d\eta\,\nu(x,dy)$$

$$\leq \int \operatorname{Re} q\left(x, \frac{\eta}{|x|}\right) g(\eta)\, d\eta$$

where the function g is given by

$$g(\eta) := \frac{1}{2} \int_{(0,\infty)} (2\pi r)^{-d/2} e^{-|\eta|^2/2r} e^{-r/2}\, dr.$$

As $\int (1+|\eta|^2) g(\eta)\, d\eta < \infty$, it is not difficult to see from the subadditivity of $\sqrt{|q(x,\cdot)|}$ (cf. [95, Theorem 6.2]) that this implies

$$\nu(x, B[-x,r]) \leq 5C \sup_{|\eta|\leq 1} \operatorname{Re} q\left(x, \frac{\eta}{|x|}\right) \leq 5C \sup_{|\eta|\leq |x|^{-1}} \operatorname{Re} q(x,\eta)$$

for $|x| \gg 1$ sufficiently large (see [93, Lemma 2.3] for details). Letting $|x| \to \infty$ yields (iii). □

The following example shows that neither of the conditions (a)-(c) is equivalent to (i)-(iii).

1.28 Example Choose $b(x) := 0$, $Q(x) := 0$ and $\nu(x,dy) := \delta_x(dy)$, then

$$\operatorname{Re} q(x,\xi) = 1 - \cos(x\cdot\xi),$$

and therefore (c) is clearly not satisfied. On the other hand,

$$Af(x) = \int_{\mathbb{R}^d\setminus\{0\}} (f(x+y) - f(x) - \nabla f(x)\cdot y \mathbb{1}_{|y|<1})\, \nu(x,dy) = f(2x) - f(x)$$

for all $|x| > 1$ which shows $Af \in C_\infty(\mathbb{R}^d)$ for all $f \in C_c^\infty(\mathbb{R}^d)$; hence, (i) holds true. Consequently, (c) is not equivalent to (i)-(iii). Since (a) and (b) are equivalent (cf. Hoh [44, Theorem 3.3]) and (b) implies (c) (see the proof of the previous theorem), we conclude that (a)-(c) are sufficient, but not necessary conditions for (i)-(iii).

It has turned out that the symbol of a Feller process is a very powerful tool to describe and analyse path properties (e. g. growth behaviour, Hausdorff dimensions, regularity) and also to obtain probability estimates. We refer the reader to [19, Chapter 5] for an overview on known results. In Section 2.2 we will use the following maximal inequality to prove estimates of fractional moments in terms of the symbol q.

1.29 Lemma (Maximal inequality) *Let $(X_t)_{t\geq 0}$ be a rich Feller process with symbol q, $q(x,0)=0$, and denote by $\tau_r^x := \inf\{t\geq 0; X_t \notin B[x,r]\}$ the exit time from the closed ball $B[x,r] = \{y\in\mathbb{R}^d; |y-x|\leq r\}$. Then there exists a constant $C>0$ such that*

$$\mathbb{P}^x\left(\sup_{s\leq\sigma} |X_s - x| > r\right) \leq \mathbb{P}^x(\tau_r^x \leq \sigma) \leq C\mathbb{E}^x\left(\int_{[0,\sigma\wedge\tau_r^x)} \sup_{|\xi|\leq r^{-1}} |q(X_s,\xi)|\, ds\right) \tag{1.15}$$

for all $r>0$ and any stopping times σ. In particular,

$$\mathbb{P}^x\left(\sup_{s\leq\sigma} |X_s - x| > r\right) \leq \mathbb{P}^x(\tau_r^x \leq \sigma) \leq C\mathbb{E}^x(\sigma) \sup_{|y-x|\leq r} \sup_{|\xi|\leq r^{-1}} |q(y,\xi)|. \tag{1.16}$$

For the particular case $\sigma := t$ the maximal inequality (1.16) goes back to Schilling [94].

Proof. Since $(X_t)_{t\geq 0}$ has càdlàg sample paths, it follows from the definition of τ_r^x that $\{\sup_{s\leq\sigma}|X_s - x| > r\} \subseteq \{\tau_r^x \leq \sigma\}$, and therefore

$$\mathbb{P}^x\left(\sup_{s\leq\sigma}|X_s - x| > r\right) \leq \mathbb{P}^x(\tau_r^x \leq \sigma)$$

is trivially satisfied. By the truncation inequality, see e. g. [108, Theorem 1.4.8], we have

$$\mathbb{P}^x(\tau_r^x \leq \sigma) \leq \mathbb{P}^x(|X_{\sigma\wedge\tau_r^x} - x| \geq r) \leq 7r^d \int_{[-r^{-1},r^{-1}]^d} \operatorname{Re}(1 - \mathbb{E}^x e^{i\xi(X_{\sigma\wedge\tau_r^x}-x)})\,d\xi.$$

An application of Dynkin's formula yields

$$\mathbb{P}^x(\tau_r^x \leq \sigma) \leq 7r^d \int_{[-r^{-1},r^{-1}]^d} \operatorname{Re}\mathbb{E}^x\left(\int_{[0,\sigma\wedge\tau_r^x)} q(X_s,\xi)e^{i\xi(X_s-x)}\,ds\right)d\xi.$$

Now (1.15) follows from the triangle inequality and Fubini's theorem; (1.16) is a direct consequence of (1.15). □

We close this section with the following representation result which is the analogue of the Lévy-Itô decomposition in the Lévy case. It is compiled from [27, Theorem 3.13], see also [54, Remark III.2.28 3)] and the references therein.

1.30 Theorem *Let $(X_t)_{t\geq 0}$ be a Feller process with triplet $(b(x), Q(x), \nu(x,dy))$. Then there exist a Markov extension*[4] *$(\Omega^\circ, \mathcal{A}^\circ, \mathcal{F}_t^\circ, \mathbb{P}^{\circ,x})$, a Brownian motion $(W_t^\circ)_{t\geq 0}$ and a Cauchy process $(L_t^\circ)_{t\geq 0}$ with jump counting measure N° on $(\Omega^\circ, \mathcal{A}^\circ, \mathcal{F}_t^\circ, \mathbb{P}^{\circ,x})$ such that*

$$X_t - X_0 = X_t^{(1)} + X_t^{(2)}$$

with

$$\begin{aligned} X_t^{(1)} &:= \int_0^t b(X_{s-})\,ds + \int_0^t \sigma(X_{s-})\,dW_s^\circ + \int_0^t\int_{|k|\leq 1} k(X_{s-},z)\,(N^\circ(dz,ds) - \nu^\circ(dz)\,ds) \\ X_t^{(2)} &:= \int_0^t\int_{|k|>1} k(X_{s-},z)\,N^\circ(dz,ds) \end{aligned}$$

for measurable functions $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and $k : \mathbb{R}^d \times (\mathbb{R}\backslash\{0\}) \to \mathbb{R}^d$ satisfying

$$\nu(x,B) = \int_{\mathbb{R}\backslash\{0\}} \mathbb{1}_B(k(x,z))\,\nu^\circ(dz), \qquad B \in \mathcal{B}(\mathbb{R}^d\backslash\{0\}),\ x \in \mathbb{R}^d, \tag{1.17}$$

and $Q(x) = \sigma(x)\sigma(x)^T$; here $\nu^\circ(dz) = \mathbb{1}_{\mathbb{R}\backslash\{0\}}(z)(2\pi)^{-1}|z|^{-2}\,dz$ denotes the Lévy measure of a (one-dimensional) Cauchy process.

[4] A Markov extension is a suitable enlargement of the underlying family of probability spaces $(\Omega, \mathcal{A}, \mathbb{P}^x, x \in \mathbb{R}^d)$; see [54, Section 2e] for the precise definition.

1.6 Martingale problem

In this section we discuss results concerning the existence and uniqueness of solutions to martingale problems. We believe that some of the results are less well known, and therefore we give a self-contained presentation. The main result is Theorem 1.37 which states that the $(L, \mathcal{D})$-martingale problem is well-posed for any Feller generator L with core $\mathcal{D}$.

Throughout this section, $L : \mathcal{D} \to \mathcal{B}_b(\mathbb{R}^d)$ denotes a linear operator with domain $\mathcal{D}$ and $\mathcal{D} \subseteq \mathcal{B}_b(\mathbb{R}^d)$. We write $(L, \mathcal{D})$ to indicate the domain of the operator.

1.31 Definition Let μ be a distribution on $\mathbb{R}^d$ and $(X_t)_{t\geq 0}$ a stochastic process on a probability space $(\Omega, \mathcal{A}, \mathbb{P}^\mu)$ with càdlàg sample paths. Then $(X_t)_{t\geq 0}$ is a *solution to the* $(L, \mathcal{D})$*-martingale problem with initial distribution* μ if $X_0 \overset{d}{=} \mu$ and

$$M_t^f := f(X_t) - \int_0^t Lf(X_s)\, ds, \qquad t \geq 0, \tag{1.18}$$

is a martingale with respect to the canonical filtration $\mathcal{F}_t := \sigma(X_s; s \leq t)$ and $\mathbb{P}^\mu$. We say that $(X_t)_{t\geq 0}$ *solves the* $(L, \mathcal{D})$*-martingale problem* if for every distribution μ there exists a probability measure $\mathbb{P}^\mu$ on $(\Omega, \mathcal{A})$ such that $(X_t)_{t\geq 0}$ is a solution to the $(L, \mathcal{D})$-martingale problem with initial distribution μ on the probability space $(\Omega, \mathcal{A}, \mathbb{P}^\mu)$.

From now on, superscripts are used to specify the initial distribution, i. e. if $(X_t)_{t\geq 0}$ is a stochastic process on $(\Omega, \mathcal{A}, \mathbb{P}^\mu)$, then $\mathbb{P}^\mu(X_0 \in \cdot) = \mu(\cdot)$. Furthermore, we set

$$\mathbb{E}^\mu(Y) := \int_\Omega Y\, d\mathbb{P}^\mu, \qquad Y \in L^1(\mathbb{P}^\mu).$$

1.32 Example Let $(X_t)_{t\geq 0}$ be a Feller process with semigroup $(P_t)_{t\geq 0}$ and generator $(L, \mathcal{D}(L))$. Then $(X_t)_{t\geq 0}$ is a solution to the $(L, \mathcal{D})$-martingale problem for any $\mathcal{D} \subseteq \mathcal{D}(L)$; see e. g. [98, Theorem 7.30] for a detailed proof.

The next result shows that being a solution to a martingale problem is a property of the finite-dimensional distributions.

1.33 Proposition *Let* $(X_t)_{t\geq 0}$ *be a stochastic process with càdlàg sample paths and initial distribution* $X_0 = \mu$. *Then* $(X_t)_{t\geq 0}$ *solves the* $(L, \mathcal{D})$*-martingale problem with initial distribution* μ *if, and only if,*

$$\mathbb{E}^\mu\left(\left[f(X_t) - f(X_s) - \int_s^t Lf(X_r)\, dr\right]\prod_{j=1}^n h_j(X_{s_j})\right) = 0 \tag{1.19}$$

for any choice of $0 \leq s_1 \leq \ldots \leq s_n \leq s \leq t$, $f \in \mathcal{D}$ *and* $h_j \in \mathcal{B}_b(\mathbb{R}^d)$.

Proof. If $(X_t)_{t\geq 0}$ solves the martingale problem, then

$$\mathbb{E}^\mu\left(f(X_t) - f(X_s) - \int_s^t Lf(X_r)\, dr \mid \mathcal{F}_s\right) = 0 \qquad \text{for all} \quad s \leq t, f \in \mathcal{D};$$

therefore (1.19) follows by conditioning w.r.t. $\mathcal{F}_s$. Now assume conversely that (1.19) holds true. For fixed $s \le t$, $s_1 \le \ldots \le s_n \le s$ and $f \in \mathcal{D}$ define

$$\mathcal{S} := \left\{ B \in \mathcal{B}(\mathbb{R}^{nd}); \mathbb{E}^\mu \left(\left[f(X_t) - f(X_s) - \int_s^t Lf(X_r)\, dr \right] \mathbb{1}_B(X_{s_1}, \ldots, X_{s_n}) \right) = 0 \right\}.$$

It is not difficult to see that $\mathcal{S}$ is a Dynkin system. Since, by (1.19), $\mathcal{S}$ contains the $\cap$-stable generator $\mathcal{B}(\mathbb{R}^d) \times \ldots \times \mathcal{B}(\mathbb{R}^d)$, this implies $\mathcal{S} = \mathcal{B}(\mathbb{R}^{nd})$. Hence,

$$\mathbb{E}^\mu \left(\left[f(X_t) - f(X_s) - \int_s^t Lf(X_r)\, dr \right] \mathbb{1}_B(X_{s_1}, \ldots, X_{s_n}) \right) = 0 \qquad \text{for all} \quad B \in \mathcal{B}(\mathbb{R}^{nd}).$$

As

$$\bigcup_{n \in \mathbb{N}} \bigcup_{0 \le s_1 \le \ldots \le s_n \le s} \sigma(X_{s_1}, \ldots, X_{s_n})$$

is a $\cap$-stable generator of $\mathcal{F}_s$, this is equivalent to

$$\mathbb{E}^\mu \left(f(X_t) - f(X_s) - \int_s^t Lf(X_r)\, dr \mid \mathcal{F}_s \right) = 0. \qquad \square$$

1.34 Corollary *Let $(X_t^i)_{t \ge 0}$ be a stochastic process on a probability space $(\Omega_i, \mathcal{A}_i, \mathbb{P}_i)$ for $i = 1, 2$. If $(X_t^1)_{t \ge 0}$ is a solution to the $(L, \mathcal{D})$-martingale problem with initial distribution μ and $(X_t^1)_{t \ge 0} \overset{d}{=} (X_t^2)_{t \ge 0}$, then $(X_t^2)_{t \ge 0}$ is also a solution to the $(L, \mathcal{D})$-martingale problem with initial distribution μ.*

Proof. By approximating the integral with Riemann sums, it follows easily that the distribution of $\int_s^t Lf(X_r^1)\, dr$ only depends on the finite-dimensional distributions of the process $(X_t^1)_{t \ge 0}$. Therefore, Proposition 1.33 shows that the martingale problem is an assertion on the finite-dimensional distributions of a process. $\square$

Corollary 1.34 motivates the following definition.

1.35 Definition We say that *uniqueness* holds for the $(L, \mathcal{D})$-martingale problem (with initial distribution μ) if any two solutions $(X_t^1)_{t \ge 0}$ and $(X_t^2)_{t \ge 0}$ of the martingale problem are unique in the sense of finite-dimensional distributions. The $(L, \mathcal{D})$-martingale problem is *well posed* if for every initial distribution μ the $(L, \mathcal{D})$-martingale problem with initial distribution μ has a solution which is unique.

The next theorem shows that uniqueness of the one-dimensional distributions of solutions to the martingale problem implies uniqueness for the martingale problem. It is taken from [60, Proposition 4.1]; see also [32, Theorem IV.4.2].

1.36 Theorem *Let $L : \mathcal{D} \to \mathcal{B}_b(\mathbb{R}^d)$ be a linear operator with domain $\mathcal{D} \subseteq \mathcal{B}_b(\mathbb{R}^d)$ such that the $(L, \mathcal{D})$-martingale problem has the following property: For any initial distribution μ and any two solutions $(X_t^1)_{t \ge 0}$ and $(X_t^2)_{t \ge 0}$ of the $(L, \mathcal{D})$-martingale problem with initial distribution μ, it holds that*

$$X_t^1 \overset{d}{=} X_t^2 \qquad \text{for all} \quad t \ge 0. \tag{1.20}$$

Then the following statement holds true: If $(X_t^1)_{t \ge 0}$ and $(X_t^2)_{t \ge 0}$ are solutions of the $(L, \mathcal{D})$-martingale problem with initial distribution μ, then $(X_t^1)_{t \ge 0} \overset{d}{=} (X_t^2)_{t \ge 0}$.

Proof. Let $(X_t^i)_{t\geq 0}$ be a stochastic process defined on a probability space $(\Omega_i,\mathcal{A}_i,\mathbb{P}_i)$ for $i=1,2$, and suppose that both processes are solutions for the $(L,\mathcal{D})$-martingale problem with initial distribution μ. Obviously, it suffices to show that[5]

$$\mathbb{E}_{\mathbb{P}_1}\left(\prod_{j=1}^n g_j(X_{t_j}^1)\right)=\mathbb{E}_{\mathbb{P}_2}\left(\prod_{j=1}^n g_j(X_{t_j}^2)\right) \tag{1.21}$$

for all $0\leq t_1\leq\ldots\leq t_n$ and $g_j\in\mathcal{B}_b(\mathbb{R}^d)$, $j=1,\ldots,n$, $n\in\mathbb{N}$. We prove (1.21) by induction.

- basis ($n=1$): For $n=1$ (1.21) is a direct consequence of (1.20).
- induction hypothesis: For any initial distribution μ and any two solutions $(X_t^1)_{t\geq 0}$ and $(X_t^2)_{t\geq 0}$ of the $(L,\mathcal{D})$-martingale problem with initial distribution μ, the identity (1.21) holds for any $0\leq t_1\leq\ldots\leq t_n$ and $g_j\in\mathcal{B}_b(\mathbb{R}^d)$.
- inductive step ($n\rightsquigarrow n+1$): For fixed $0\leq t_1\leq\ldots\leq t_{n+1}$ we have to verify that

$$\mathbb{E}_{\mathbb{P}_1}\left(\prod_{j=1}^{n+1} g_j(X_{t_j}^1)\right)=\mathbb{E}_{\mathbb{P}_2}\left(\prod_{j=1}^{n+1} g_j(X_{t_j}^2)\right) \tag{$\star$}$$

for all $g_j\in\mathcal{B}_b(\mathbb{R}^d)$. Because of the linearity of the integral and the dominated convergence theorem we may assume without loss of generality that there exists some $\varepsilon>0$ such that $g_j>\varepsilon$ for all $j=1,\ldots,n+1$. For $i=1,2$ define a probability measure $\tilde{\mathbb{P}}_i$ on $(\Omega_i,\mathcal{A}_i)$ by

$$\tilde{\mathbb{P}}_i(A):=c\mathbb{E}_{\mathbb{P}_i}\left(\mathbb{1}_A\prod_{j=1}^n g_j(X_{t_j}^i)\right),\qquad A\in\mathcal{A}_i, i\in\{1,2\},$$

where the normalizing constant $c\in(0,\infty)$ is given by

$$c:=\frac{1}{\mathbb{E}_{\mathbb{P}_i}\left(\prod_{j=1}^n g_j(X_{t_j}^i)\right)}.$$

Note that, by our induction hypothesis, the constant c does not depend on i. We claim that $Y_t^i:=X_{t+t_n}^i$, $t\geq 0$, solves the $(L,\mathcal{D})$-martingale problem on the probability space $(\Omega_i,\mathcal{A}_i,\tilde{\mathbb{P}}_i)$. *Indeed:* Let $0\leq s_1\leq\ldots\leq s_m\leq s\leq t$ and $h_j\in\mathcal{B}_b(\mathbb{R}^d)$. Using the definition of $\tilde{\mathbb{P}}_i$ and Proposition 1.33, it follows that

$$\begin{aligned}
&\mathbb{E}_{\tilde{\mathbb{P}}_i}\left(\left[f(Y_t^i)-f(Y_s^i)-\int_s^t Lf(Y_r^i)\,dr\right]\prod_{j=1}^m h_j(Y_{s_j}^1)\right)\\
&\overset{\text{def}}{=} c\mathbb{E}_{\mathbb{P}_i}\left(\left[f(X_{t+t_n}^i)-f(X_{s+t_n}^i)-\int_s^t Lf(X_{r+t_n}^i)\,dr\right]\prod_{j=1}^m h_j(X_{s_j+t_n}^i)\prod_{j=1}^n g_j(X_{t_j}^i)\right)\\
&\overset{\text{P1.33}}{=} 0;
\end{aligned}$$

[5] Here $\mathbb{E}_{\mathbb{P}_i}$ denotes the expectation with respect to $\mathbb{P}_i$.

here we have used in the last step that $(X_t^i)_{t\geq 0}$ solves the martingale problem. Applying Proposition 1.33 another time, we get that $(Y_t^i)_{t\geq 0}$ is indeed a solution to the martingale problem for $i = 1,2$. Furthermore, the induction hypothesis (IH) gives

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{P}}_1}(g(Y_0^1)) = c\mathbb{E}_{\mathbb{P}_1}\left(g(X_{t_n}^1)\prod_{j=1}^n g_j(X_{t_j}^1)\right) &\overset{\text{(IH)}}{=} c\mathbb{E}_{\mathbb{P}_2}\left(g(X_{t_n}^2)\prod_{j=1}^n g_j(X_{t_j}^2)\right) \\ &= \mathbb{E}_{\tilde{\mathbb{P}}_2}(g(Y_0^2))\end{aligned}$$

for any $g \in \mathcal{B}_b(\mathbb{R}^d)$, i.e. $Y_0^1 \overset{d}{=} Y_0^2$. Consequently, we have shown that $(Y_t^1)_{t\geq 0}$ and $(Y_t^2)_{t\geq 0}$ are both solutions to the $(L,\mathcal{D})$-martingale problem with some fixed initial distribution, say ν. It follows from our assumption (1.20) on the martingale problem that $Y_t^1 \overset{d}{=} Y_t^2$ for all $t \geq 0$. Hence,

$$\mathbb{E}_{\tilde{\mathbb{P}}_1}(g_{n+1}(Y_t^1)) = \mathbb{E}_{\tilde{\mathbb{P}}_2}(g_{n+1}(Y_t^2)) \qquad \text{for all} \quad t \geq 0, g_{n+1} \in \mathcal{B}_b(\mathbb{R}^d).$$

Rewriting this identity using the definition of $\tilde{\mathbb{P}}_i$, we get $(\star)$. □

Using a similar reasoning, it is possible to show that, under the assumptions of Theorem 1.21, any solution $(X_t)_{t\geq 0}$ to the martingale problem is a Markov process, see [32, Lemma IV.4.2(i)] for more details.
We close this section with the following result which we will need later on.

1.37 Theorem *Let $(X_t)_{t\geq 0}$ be a Feller process and suppose that $\mathcal{D}$ is a core of the generator $(L,\mathcal{D}(L))$, i.e. for any $f \in \mathcal{D}(L)$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ such that $\|f_n - f\|_\infty + \|Lf_n - Lf\|_\infty \xrightarrow{n\to\infty} 0$. Then the $(L,\mathcal{D})$-martingale problem is well-posed.*

For the proof of Theorem 1.37 we need an auxiliary result. It is taken from [60, Proposition 3.9.3], but we give a slightly different proof.

1.38 Proposition *If $(X_t)_{t\geq 0}$ is a solution to the $(L,\mathcal{D})$-martingale problem with initial distribution μ, then*

$$f(X_t)\alpha(t) - \int_0^t (\alpha'(s)f(X_s) + \alpha(s)Lf(X_s))\,ds$$

is a martingale for any $\alpha \in C_b^1([0,\infty))$ and $f \in \mathcal{D}$.

Remark More generally, the stochastic process defined by

$$\varphi(t,X_t) - \int_0^t \left(\frac{\partial}{\partial t}\varphi(s,X_s) + L_x\varphi(s,X_s)\right) ds$$

is a martingale if $\varphi : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ satisfies certain regularity assumptions. This means that $((t,X_t))_{t\geq 0}$ is a solution to the $(\tilde{L},\tilde{\mathcal{D}})$-martingale problem for

$$\tilde{L}g(t,x) := \frac{\partial}{\partial t}g(t,x) + L_x g(t,x)$$

and a suitable domain $\tilde{\mathcal{D}}$.

Proof of Proposition 1.38. First, we prove the following auxiliary statement: For any càdlàg martingale $(M_t)_{t\geq 0}$ and $\alpha \in C_b^1([0,\infty))$, the process

$$N_t := M_t\alpha(t) - \int_0^t M_r\alpha'(r)\,dr, \qquad t \geq 0,$$

is a martingale. *Indeed:* Fix $s \leq t$. Since $M_t\alpha(t) = (M_t - M_s)\alpha(t) + M_s\alpha(t)$, we have

$$\mathbb{E}^\mu(M_t\alpha(t) \mid \mathcal{F}_s) = M_s\alpha(t).$$

On the other hand, it follows from Fubini's theorem for conditional expectations (cf. [97, Theorem 27.17]) that

$$\begin{aligned}\mathbb{E}^\mu\left(\int_0^t M_r\alpha'(r)\,dr \mid \mathcal{F}_s\right) &= \int_0^s M_r\alpha'(r)\,dr + \int_s^t \mathbb{E}^\mu(M_r \mid \mathcal{F}_s)\alpha'(r)\,dr \\ &= \int_0^s M_r\alpha'(r)\,dr + M_s(\alpha(t) - \alpha(s)).\end{aligned}$$

Subtracting the second identity from the first proves that $(N_t)_{t\geq 0}$ is a martingale.
For fixed $f \in \mathcal{D}$ the integration by part formula shows

$$\int_0^t \alpha(s)Lf(X_s)\,ds = \alpha(t)\int_0^t Lf(X_s)\,ds - \int_0^t \alpha'(s)\left(\int_0^s Lf(X_r)\,dr\right)ds.$$

Consequently,

$$\begin{aligned}f(X_t)\alpha(t) &- \int_0^t (\alpha'(s)f(X_s) + \alpha(s)Lf(X_s))\,ds \\ &= \left(f(X_t) - \int_0^t Lf(X_s)\,ds\right)\alpha(t) - \int_0^t \alpha'(s)\left(f(X_s) - \int_0^s Lf(X_r)\,dr\right)ds \\ &\overset{(1.18)}{=} M_t^f\alpha(t) - \int_0^t \alpha'(s)M_s^f\,ds\end{aligned}$$

which is a martingale by the first part of this proof and (1.18). □

Proof of Theorem 1.37. We follow the proof in [60, Theorem 4.10.3]. Since we know from Example 1.32 that $(X_t)_{t\geq 0}$ is a solution to the $(L,\mathcal{D})$-martingale problem, it just remains to show uniqueness. Fix an initial distribution μ and two stochastic processes $(Y_t^1)_{t\geq 0}$ and $(Y_t^2)_{t\geq 0}$ defined on the probability spaces $(\Omega_1,\mathcal{A}_1,\mathbb{P}_1)$ and $(\Omega_2,\mathcal{A}_2,\mathbb{P}_2)$, respectively. Suppose that both $(Y_t^1)_{t\geq 0}$ and $(Y_t^2)_{t\geq 0}$ solve the $(L,\mathcal{D})$-martingale problem with initial distribution μ.[6] If we can show that the one-dimensional distributions coincide, i. e.

$$\mathbb{P}_1(Y_t^1 \in \cdot) = \mathbb{P}_2(Y_t^2 \in \cdot) \qquad \text{for all} \quad t \geq 0, \tag{$\star$}$$

then uniqueness follows from Proposition 1.36. Applying Proposition 1.38 for $\alpha(t) := e^{-\lambda t}$, $\lambda > 0$, gives

$$f(Y_t^i)e^{-\lambda t} = \mathbb{E}_{\mathbb{P}_i}\left(Y_{t+s}^i e^{-\lambda(t+s)} + \int_t^{t+s} e^{-\lambda r}(\lambda - L)f(Y_r^i)\,dr \mid \mathcal{F}_t\right)$$

[6] Recall that this implies that $(Y_t^1)_{t\geq 0}$ and $(Y_t^2)_{t\geq 0}$ have càdlàg sample paths.

for all $f \in \mathcal{D}$, $s, t \geq 0$ and $i \in \{1, 2\}$. If we multiply both sides by $e^{\lambda t}$, shift the variable of integration and let $s \to \infty$, we get

$$f(Y_t^i) = \mathbb{E}_{\mathbb{P}_i}\left(\int_0^\infty e^{-\lambda r}(\lambda - L)f(Y_{t+r}^i)\,dr \mid \mathcal{F}_t\right) \qquad \text{for all} \quad f \in \mathcal{D}.$$

Since $\mathcal{D}$ is a core, the dominated convergence theorem shows that this identity extends to all functions $f \in \mathcal{D}(L)$. In particular for $f := R_\lambda g \in \mathcal{D}(L)$, $g \in C_\infty(\mathbb{R}^d)$, we find

$$R_\lambda g(Y_t^i) = \mathbb{E}_{\mathbb{P}_i}\left(\int_0^\infty e^{-\lambda r} g(Y_{t+r}^i)\,dr\right) \qquad \text{for all} \quad t \geq 0. \tag{1.22}$$

(Recall that $R_\lambda = (\lambda - L)^{-1}$ denotes the resolvent.) Using this identity for $t = 0$ and the fact that $Y_0^1 \overset{d}{=} Y_0^2 \overset{d}{=} \mu$, we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_1}\left(\int_0^\infty e^{-\lambda r} g(Y_r^1)\,dr\right) &\overset{(1.22)}{=} \mathbb{E}_{\mathbb{P}_1}(R_\lambda g(Y_0^1)) \overset{Y_0^1 \overset{d}{=} Y_0^2}{=} \mathbb{E}_{\mathbb{P}_2}(R_\lambda g(Y_0^2)) \\ &\overset{(1.22)}{=} \mathbb{E}_{\mathbb{P}_2}\left(\int_0^\infty e^{-\lambda r} g(Y_r^2)\,dr\right)\end{aligned}$$

for all $\lambda > 0$ and $g \in C_\infty(\mathbb{R}^d)$. The uniqueness of the Laplace transform, see for instance [100, Proposition 1.2], yields $\mathbb{E}_{\mathbb{P}_1} g(Y_r^1) = \mathbb{E}_{\mathbb{P}_2} g(Y_r^2)$ for (Lebesgue-)almost all $r \geq 0$. Since both $(Y_t^1)_{t\geq 0}$ and $(Y_t^2)_{t\geq 0}$ have càdlàg sample paths, it follows that $r \mapsto \mathbb{E}_{\mathbb{P}_i} g(Y_r^i)$ is càdlàg, and therefore

$$\mathbb{E}_{\mathbb{P}_1} g(Y_r^1) = \mathbb{E}_{\mathbb{P}_2} g(Y_r^2) \qquad \text{for all} \quad r \geq 0.$$

As $g \in C_\infty(\mathbb{R}^d)$ is arbitrary, this proves ($\star$). □

For later reference we make the following observation.

1.39 Remark Let $(X_t)_{t\geq 0}$ be a rich Feller process with bounded coefficients and generator $(L, \mathcal{D}(L))$ and let $\mathcal{C} \subseteq \mathcal{D} \subseteq C_b^2(\mathbb{R}^d)$. If $\mathcal{D} \subseteq \mathcal{D}(L)$ is a core for $(L, \mathcal{D}(L))$ and $\mathcal{C}$ is dense in $\mathcal{D}$ with respect to $\|\cdot\|_{(2)}$, then $\mathcal{C}$ is a core for $(L, \mathcal{D}(L))$; in particular, the $(L, \mathcal{C})$-martingale problem is well posed.
Indeed: By Theorem 1.37 it suffices to show that $\mathcal{C}$ is a core. For fixed $g \in \mathcal{D}$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ such that $\|g - f_n\|_{(2)} \xrightarrow{n\to\infty} 0$. Hence, by (1.14),

$$\|g - f_n\|_\infty + \|Lg - Lf_n\|_\infty \leq C\|g - f_n\|_{(2)} \xrightarrow{n\to\infty} 0,$$

from which we conclude $(g, Lg) \in \overline{(L, \mathcal{C})}^{\|\cdot\|_\infty}$. Thus, $(L, \mathcal{D}) \subseteq \overline{(L, \mathcal{C})}^{\|\cdot\|_\infty}$. Since $\mathcal{D}$ is a core, we get

$$(L, \mathcal{D}(L)) = \overline{(L, \mathcal{D})}^{\|\cdot\|_\infty} \subseteq \overline{(L, \mathcal{C})}^{\|\cdot\|_\infty}.$$

1.7 Parametrix method

The parametrix construction is a method to find (a candidate for) the fundamental solution to the Cauchy problem $\partial_t - L = 0$ for a linear operator L; typically, L is a (pseudo)differential operator. The idea goes back to Levi [70] who constructed the fundamental solution of a parabolic partial differential equation using this approach. Friedman [37] showed, more than 50 years later, that the parametrix method is also applicable for elliptic and parabolic differential operators L with variable (time-dependent) coefficients. Later on, the method was extended to pseudo-differential operators, see e. g. [50, Section 2.7] for a survey on this topic. One of the first who recognized the potential of the parametrix method in probability theory was Feller [34]. He obtained existence and uniqueness results for diffusion processes and even certain jump processes. Since then the parametrix method has become increasingly popular in probability theory; for an overview on recent results see Chapter 3. Our aim in this section is to give the reader an idea how the parametrix construction works, and also to introduce some notation. We start with the following central definition; note that it is consistent with [50, Definition 2.7.12] in case that L is a pseudo-differential operator.

1.40 Definition Let $(L, \mathcal{D}(L))$ be a linear operator. A function $p : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a *fundamental solution to the Cauchy problem for the operator* $\partial_t - L$ if $p(t,\cdot,y) \in \mathcal{D}(L)$ for all $t > 0$ and $y \in \mathbb{R}^d$, $t \mapsto p(t,x,y)$ is differentiable for $x, y \in \mathbb{R}^d$,

$$(\partial_t - L_x)p(t,x,y) = 0 \quad \text{and} \quad p(t,x,\cdot) \xrightarrow[v]{t\to 0} \delta_x. \tag{1.23}$$

It is well-known that, under suitable assumptions on the operator $(L, \mathcal{D}(L))$, the unique solution to the inhomogeneous Cauchy problem

$$\frac{\partial}{\partial t} u(t,x) = L_x u(t,x) + f(t,x), \qquad u(0,x) = g(x) \tag{1.24}$$

is given by

$$u(t,x) = \int g(z)p(t,x,z)\,dz + \int_0^t \int f(s,z)p(t-s,x,z)\,dz\,ds, \qquad t > 0, x \in \mathbb{R}^d \tag{1.25}$$

for $f \in C((0,\infty) \times \mathbb{R}^d)$ and $g \in C_\infty(\mathbb{R}^d)$, see e. g. [60, Theorem 4.1.3].

Let us briefly explain the general idea of the parametrix method before focusing on the particular case we are interested in. We stress that the considerations in the remaining part of this section are purely heuristic.

Suppose that p is a fundamental solution to (1.23) for a given linear operator L. The starting point of the parametrix method is a decomposition

$$p(t,x,y) = p_0(t,x,y) + r(t,x,y) \tag{1.26}$$

where p_0 is an approximation of p and $r(t,x,y) := p(t,x,y) - p_0(t,x,y)$ denotes the residue term. Since p is by assumption a fundamental solution, we find that

$$(\partial_t - L_x)r(t,x,y) = F(t,x,y) := -(\partial_t - L_x)p_0(t,x,y),$$

i. e. r solves the inhomogeneous Cauchy problem (1.24) with $f(t,x) := F(t,x,y)$ for each fixed $y \in \mathbb{R}^d$. Assuming that $r(0,x,y) = 0$, it follows from (1.25) that

$$r(t,x,y) \quad = \quad \int_0^t \int p(t-s,x,z)F(s,z,y)\,dz\,ds$$

$$\stackrel{(1.1)}{=} (p \circledast F)(t,x,y) \stackrel{(1.26)}{=} (p_0 \circledast F)(t,x,y) + (r \circledast F)(t,x,y).$$

If we plug in this expression for r on the right-hand side of the equation, we get

$$\begin{aligned} r(t,x,y) &= (p_0 \circledast F)(t,x,y) + ((p_0 \circledast F) \circledast F)(t,x,y) + ((r \circledast F) \circledast F)(t,x,y) \\ &\stackrel{(1.3)}{=} \sum_{i=1}^{2} (p_0 \circledast F^{\circledast i})(t,x,y) + (r \circledast F^{\circledast 2})(t,x,y). \end{aligned}$$

Iterating the procedure, we obtain (formally)

$$r(t,x,y) = \sum_{i=1}^{\infty} (p_0 \circledast F^{\circledast i})(t,x,y);$$

hence, by (1.26),

$$p(t,x,y) = p_0(t,x,y) + \sum_{i=1}^{\infty} (p_0 \circledast F^{\circledast i})(t,x,y). \tag{1.27}$$

So far, we have required the existence of a fundamental solution. Since our argument shows that *any* fundamental solution is of the form (1.27), we can drop this assumption and use the representation (1.27) to *define* a candidate for the fundamental solution. In order to prove that the so-defined function is indeed a fundamental solution to $\partial_t - L = 0$, several properties have to be verified:

- (1.27) makes sense, i. e. $(p_0 \circledast F^{\circledast i})(t,x,y)$ exists for all $i \in \mathbb{N}$ and the sum on the right-hand side of (1.27) converges.
- p is sufficiently smooth: $\partial_t p(t,x,y)$ exists and $p(t,\cdot,y) \in \mathcal{D}(L)$ for all $t > 0$ and $x, y \in \mathbb{R}^d$.
- p satisfies (1.23).

The crucial point of the parametrix method is the choice of the approximation p_0 (sometimes also called *parametrix*). There are two conflicting interests: On the one hand, we need to have a very thorough understanding of p_0 in order to derive suitable bounds (e. g. to prove the convergence of the parametrix series); on the other hand, p_0 has to be a "good" approximation of p.

Now we turn to the particular case that $L|_{C_c^\infty(\mathbb{R}^d)}$ is a pseudo-differential operator:

$$Lf(x) = Af(x) := -\int_{\mathbb{R}^d} q(x,\xi) e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi \qquad \text{for all } \; f \in C_c^\infty(\mathbb{R}^d).$$

The idea is now to freeze the symbol $q(x,\xi)$ at $x = z$ and to consider the family of pseudo-differential operators

$$A^z f(x) := -\int_{\mathbb{R}^d} q(z,\xi) e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi, \qquad x, z \in \mathbb{R}^d, f \in C_c^\infty(\mathbb{R}^d)$$

instead. For each fixed $z \in \mathbb{R}^d$ the fundamental solution to the Cauchy problem $\partial_t - A^z$ is given by the transition density of a Lévy process with characteristic exponent $q(z,\cdot)$, i. e.

$$p^z(t,x,y) := p^z(t,x-y) := \frac{1}{(2\pi)^d} \int e^{-i(x-y)\cdot\xi} e^{-tq(z,\xi)}\, d\xi.$$

For small times $t > 0$ it is reasonable to expect

$$p(t,x,y) \approx p^x(t,x,y) \qquad \text{and} \qquad p(t,x,y) \approx p^y(t,x,y),$$

i. e. $p_0(t,x,y) := p^x(t,x,y)$ and $p_0(t,x,y) := p^y(t,x,y)$ are candidates for the parametrix p_0. It turns out that $p_0(t,x,y) = p^x(t,x,y)$ leads to the fundamental solution for the adjoint L^* of L whereas $p_0(t,x,y) = p^y(t,x,y)$ yields – via the parametrix construction – the fundamental solution to the Cauchy problem for the operator $\partial_t - L$. This corresponds to solving the forward equation

$$\frac{\partial}{\partial t} p(t,x,y) = L_y^* p(t,x,y)$$

and backward equation

$$\frac{\partial}{\partial t} p(t,x,y) = L_x p(t,x,y),$$

respectively. Typically, solving the forward equation requires a higher regularity; we refer to [98] for a discussion of the diffusion case.

2

Moments of Lévy-type processes

In the theory of Lévy processes, the characteristic exponent and the Lévy triplet play a key role. They can not only be used to determine a Lévy process uniquely, but also to characterize many of its distributional and sample path properties (see e. g. Sato [92, Chapter 4,5], Blumenthal & Getoor [12] and Fristedt [38]). Since Lévy-type processes behave locally like Lévy processes, it is a natural guess that the symbol q plays a similar rôle for Lévy-type processes. In fact, many results which are well-known for Lévy processes have successfully been extended from the Lévy case to Lévy-type processes in the past years (see [19] for an overview).
In our recent publication [67] we have studied the existence of moments and derived moment estimates for Lévy-type processes – a topic which had, surprisingly, received only little attention before. A selection of the results will be presented in the first two sections of this chapter. We will give a sufficient condition for the existence of generalized moments in terms of the triplet (Theorem 2.4) and show that generalized moments exist backward in time (Theorem 2.1). Furthermore, we will derive estimates for fractional moments of Feller processes. In the second part, Section 2.3, we combine an idea of Fournier & Printems [36] with our moment estimates to prove the absolute continuity of a class of Lévy-type processes with Hölder continuous symbols.

Throughout this chapter, $(\Omega, \mathcal{A}, \mathcal{F}_t, X_t, t \geq 0, \mathbb{P}^x, x \in \mathbb{R}^d)$ is a (rich) Feller process with respect to a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$, and we assume that its symbol q satisfies $q(x,0) = 0$. Section 2.1 and Section 2.2 are, essentially, taken from our publication [67].

2.1 Existence of moments

Let $f : \mathbb{R}^d \to [0,\infty)$ be a locally bounded function which is submultiplicative, i. e. which satisfies

$$f(x+y) \leq c f(x) f(y) \qquad \text{for all} \quad x, y \in \mathbb{R}^d$$

for some constant $c > 0$. If $(L_t)_{t\geq 0}$ is a Lévy process, then it is well-known that the existence of the generalized moment $\mathbb{E}^x f(L_t)$ is a time-independent distributional property:

$$\exists t > 0 : \mathbb{E}^x f(L_t) < \infty \iff \forall t > 0 : \mathbb{E}^x f(L_t) < \infty; \tag{2.1}$$

cf. [92, Theorem 25.3]. Without assuming submultiplicativity of f this is, in general, not true (see [92, Remark 25.9] for a counterexample). In this section we investigate the question whether this distributional property extends to Lévy-type processes. Under which additional assumptions on the Lévy-type process $(X_t)_{t\geq 0}$ and f does the equivalence (2.1) hold true? First we discuss whether moments exist backward in time; that is

$$\exists t > 0 : \mathbb{E}^x f(X_t) < \infty \implies \forall s \leq t : \mathbb{E}^x f(X_s) < \infty. \tag{2.2}$$

The following theorem is one of the main results in this chapter.

2.1 Theorem *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with bounded coefficients and let $f : \mathbb{R}^d \to (0,\infty)$ be measurable.*

(i) Suppose there exists a bounded measurable function $g : \mathbb{R}^d \to [0,\infty)$, such that $\inf_{|z|\leq r} g(z) > 0$ for $r > 0$ sufficiently small and

$$\inf_{y\in\mathbb{R}^d} \frac{f(z+y)}{f(y)} \geq g(z) \tag{2.3}$$

for all $z \in \mathbb{R}^d$. Then

$$\mathbb{E}^x f(X_t) < \infty \iff \sup_{s\leq t} \mathbb{E}^x f(X_s) < \infty. \tag{2.4}$$

(ii) (2.3), hence (2.4), holds if one of the following conditions is satisfied.

(a) f is submultiplicative and locally bounded.

(b) $\log f$ is Hölder continuous.

(c) f is Hölder continuous and $\inf_{y\in\mathbb{R}^d} f(y) > 0$.

(d) f is differentiable and $\sup_{y\in\mathbb{R}^d} \sup_{|z|\leq r} \frac{|\nabla f(y+z)|}{f(y)} < \infty$ for $r > 0$ sufficiently small.

(e) f is differentiable, $\inf_{y\in\mathbb{R}^d} f(y) > 0$, $\sup_{y\in\mathbb{R}^d} \frac{|\nabla f(y)|}{f(y)} < \infty$ and ∇f is uniformly continuous.

Since any Lévy process is a rich Lévy-type process with bounded coefficients, Theorem 2.1 applies, in particular, if $(X_t)_{t\geq 0}$ is a Lévy process. For the proof of Theorem 2.1 we need an auxiliary lemma.

2.2 Lemma *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with bounded coefficients and let $g \in \mathcal{B}_b(\mathbb{R}^d)$, $g \geq 0$, be such that $\inf_{y\in B[0,r]} g(y) > 0$ for $r > 0$ sufficiently small. Then*

$$\exists \alpha > 0, \delta > 0\ \forall x \in \mathbb{R}^d,\ t \in [0,\delta] : \quad \mathbb{E}^x g(X_t - x) \geq \alpha.$$

Proof. By Theorem 1.6(i), the first exit time $\tau_r^x := \inf\{t > 0; X_t \notin B[x,r]\}$ is a stopping time with respect to the right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. Obviously,

$$\mathbb{E}^x g(X_t - x) = \mathbb{E}^x (g(X_t - x)\mathbb{1}_{\{\tau_r^x > t\}} + g(X_t - x)\mathbb{1}_{\{\tau_r^x \leq t\}})$$

$$\geq \inf_{|y-x|\leq r} g(y-x)(1-\mathbb{P}^x(\tau_r^x \leq t)) - \|g\|_\infty \mathbb{P}^x(\tau_r^x \leq t)$$
$$\geq \inf_{|y|\leq r} g(y) - 2\|g\|_\infty \mathbb{P}^x(\tau_r^x \leq t).$$

Since $(X_t)_{t\geq 0}$ has bounded coefficients, an application of the maximal inequality (1.16) yields

$$\sup_{x\in\mathbb{R}^d} \mathbb{P}^x(\tau_r^x \leq t) \leq Ct\left(1+\frac{1}{r^2}\right)$$

for a constant $C>0$ which does not depend on t and r. Choosing $r>0$ and $\delta>0$ sufficiently small, the claim follows. □

Proof of Theorem 2.1. (i) Obviously, it suffices to prove "⇒". By Lemma 2.2, there exist $\delta>0$, $\alpha\in(0,1)$ such that $\mathbb{E}^y g(X_r-y)\geq\alpha$ for all $y\in\mathbb{R}^d$ and $r\in[0,\delta]$. By the Markov property,

$$\begin{aligned}\mathbb{E}^x f(X_t) &= \mathbb{E}^x\left(\mathbb{E}^{X_s} f(X_{t-s})\right)\\ &= \iint \frac{f((z-y)+y)}{f(y)} f(y)\,\mathbb{P}^y_{X_{t-s}}(dz)\Big|_{y=X_s}\,d\mathbb{P}^x\\ &\geq \iint f(y)g(z-y)\,\mathbb{P}^y_{X_{t-s}}(dz)\Big|_{y=X_s}\,d\mathbb{P}^x\\ &\geq \alpha\mathbb{E}^x f(X_s)\end{aligned}$$

for all $s\in[t-\delta,t]$. Iterating this procedure gives $\mathbb{E}^x f(X_t)\geq\alpha^n\mathbb{E}^x f(X_s)$ for any $s\in[t-n\delta,t]$. If we choose $n\in\mathbb{N}$ sufficiently large, then $\sup_{s\leq t}\mathbb{E}^x f(X_s)\leq\alpha^{-n}\mathbb{E}^x f(X_t)$.

(ii) We check that (2.3) is satisfied for a suitable function g.

(a) By submultiplicativity, $f(y)\leq cf(y+z)f(-z)$, and so

$$\inf_{y\in\mathbb{R}^d}\frac{f(z+y)}{f(y)}\geq\frac{1}{c}\frac{1}{f(-z)}\geq\min\left\{1,\frac{1}{c}\frac{1}{f(-z)}\right\}=:g(z)$$

for all $z\in\mathbb{R}^d$. Since f is locally bounded, we have $\inf_{y\in B[0,r]} g(y)>0$ for small $r>0$.

(b) The Hölder continuity of $\log f$ entails

$$\frac{f(z+y)}{f(y)}=\exp(\log f(z+y)-\log f(y))\geq\exp(-c|z|^\gamma)=:g(z),\qquad z\in\mathbb{R}^d.$$

(c) As $\inf_{y\in\mathbb{R}^d} f(y)>0$, the Hölder continuity of f implies Hölder continuity of $\log f$. Now the claim follows from (ii)(b).

(d) By the gradient theorem,

$$|f(y+z)-f(y)|=\left|\int_0^1\nabla f(y+tz)\cdot z\,dt\right|\leq|z|\sup_{|z|\leq r}|\nabla f(y+z)|$$

for all $|z|\leq r$ and $y\in\mathbb{R}^d$. Applying the Cauchy–Schwarz inequality gives

$$\frac{f(z+y)}{f(y)}\geq\min\left\{1,1-|z|\sup_{y\in\mathbb{R}^d}\sup_{|z|\leq r}\frac{|\nabla f(y+z)|}{f(y)}\right\}=:g(z).$$

(e) This is an immediate consequence of (ii)(d). □

A close look at the proof of Theorem 2.1 reveals that, under the assumptions of Theorem 2.1(i),

$$\sup_{x\in A} \mathbb{E}^x f(X_t - x) < \infty \iff \sup_{x\in A} \sup_{s\leq t} \mathbb{E}^x f(X_s - x) < \infty$$

for any set $A \subseteq \mathbb{R}^d$. The next result shows that the generalized moments also exist forward in time if f is submultiplicative and $\mathbb{E}^x f(X_t - x)$ is bounded in x.

2.3 Corollary *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with bounded coefficients and let $f : \mathbb{R}^d \to (0,\infty)$ be a locally bounded measurable submultiplicative function. Then*

$$\exists t > 0 : \sup_{x\in\mathbb{R}^d} \mathbb{E}^x f(X_t - x) < \infty \implies \forall s \geq 0 : \sup_{r\leq s} \sup_{x\in\mathbb{R}^d} \mathbb{E}^x f(X_r - x) < \infty.$$

Proof. By Theorem 2.1, $M := 1 \vee \sup_{x\in\mathbb{R}^d} \sup_{s\leq t} \mathbb{E}^x f(X_s - x) < \infty$. Using the Markov property of $(X_t)_{t\geq 0}$ and submultiplicativity of f, we find

$$\mathbb{E}^x f(X_r - x) = \mathbb{E}^x \left(\mathbb{E}^y f(X_{r-t} - x)\Big|_{y=X_t} \right) \leq c\mathbb{E}^x \left(\mathbb{E}^y f(X_{r-t} - y) f(y-x)\Big|_{y=X_s} \right) \leq cM^2$$

for all $r \in [t, 2t]$ and $x \in \mathbb{R}^d$. Thus, $\sup_{r\leq s} \sup_{x\in\mathbb{R}^d} \mathbb{E}^x f(X_r - x) < \infty$ for all $s \in [t, 2t]$. The claim follows by iteration. □

Next we derive a sufficient condition for the existence of generalized moments for Lévy-type processes in terms of the triplet. Recall that for a Lévy process $(L_t)_{t\geq 0}$ with Lévy triplet (b, Q, ν) we have

$$\int_{|y|\geq 1} f(y)\, \nu(dy) < \infty \iff \mathbb{E}^x f(L_t) < \infty \text{ for some (all) } t > 0$$

for any locally bounded submultiplicative function $f : \mathbb{R}^d \to (0,\infty)$, cf. [92, Theorem 25.3]. It was shown in [19, Theorem 5.11] that the implication

$$\sup_{x\in\mathbb{R}^d} \int_{|y|\geq 1} f(y)\, \nu(x, dy) < \infty \implies \forall x \in \mathbb{R}^d, t \geq 0 : \mathbb{E}^x f(X_t) < \infty \tag{2.5}$$

holds for $f(y) := \exp(\zeta y)$, $\zeta \in \mathbb{R}^d$, and any Lévy-type process with bounded coefficients. Theorem 2.4 extends (2.5) to non-negative functions f which are (comparable to) a submultiplicative C^2 function.

2.4 Theorem *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with triplet $(b(x), Q(x), \nu(x, dy))$ and $f : \mathbb{R}^d \to (0,\infty)$ a measurable function which is comparable to a submultiplicative twice continuously differentiable function $g : \mathbb{R}^d \to (0,\infty)$. If*

$$\sup_{x\in K} \int_{|y|\geq 1} f(y)\, \nu(x, dy) < \infty$$

for some compact set $K \subseteq \mathbb{R}^d$, then $\sup_{s\leq t} \sup_{x\in K} \mathbb{E}^x f(X_{s\wedge\tau_K} - x) < \infty$ and

$$\mathbb{E}^x f(X_{t\wedge\tau_K}) \leq C f(x) \exp\left(C(M_1 + M_2)t\right); \tag{2.6}$$

here $\tau_K := \inf\{t > 0; X_t \notin K\}$ *denotes the exit time from the set* K, $C = C(K) > 0$ *is a constant (which does not depend on* $(X_t)_{t\geq 0}$ *and* t*) and*

$$M_1 := \sup_{x\in K}\left(|b(x)| + |Q(x)| + \int_{\mathbb{R}^d\setminus\{0\}} (|y|^2 \wedge 1)\, \nu(x,dy)\right) < \infty$$
$$M_2 := \sup_{x\in K}\int_{|y|\geq 1} f(y)\, \nu(x,dy) < \infty.$$

If $(X_t)_{t\geq 0}$ *has bounded coefficients, then* $K = \mathbb{R}^d$ *is admissible.*

Common choices are $f(x) = |x|^\alpha \vee 1$, $\alpha > 0$, $f(x) = \exp(|x|^\beta)$, $\beta \in (0,1]$, or $f(x) = \log(|x| \vee e)$. Let us remark that (the proof of) Theorem 2.4 extends to functions f which are comparable to functions g of the form $g = g_1 \cdot g_2$ where $g_1 \in C^2$ is non-negative submultiplicative and $g_2 \in C^2$ subadditive, $\inf_{y\in\mathbb{R}^d} g_2(y) > 0$.

Proof of Theorem 2.4. To keep notation simple, we give the proof only for $d = 1$. We can assume without loss of generality that $f \in C^2$ is submultiplicative (otherwise replace f by g). Let $(\Omega^\circ, \mathcal{A}^\circ, \mathcal{F}_t^\circ, \mathbb{P}^{\circ,x})$, $(W_t^\circ)_{t\geq 0}$, $(L_t^\circ)_{t\geq 0}$, N° and k, σ be as in Theorem 1.30 and write $X_t - X_0 = X_t^{(1)} + X_t^{(2)}$ with $(X_t^{(1)})_{t\geq 0}$, $(X_t^{(2)})_{t\geq 0}$ defined in Theorem 1.30. For fixed $R > 0$ define an $\mathcal{F}_t^\circ$-stopping time by

$$\tau_R := \inf\left\{t \geq 0; \max\{|X_t^{(1)}|, |X_t^{(2)}|\} > R\right\}$$

and set $\tau := \tau_K \wedge \tau_R$. By the submultiplicativity of f, we have

$$f(X_t - X_0) = f(X_t^{(1)} + X_t^{(2)}) \leq c f(X_t^{(1)}) f(X_t^{(2)})$$

for some constant $c > 0$. Since a submultiplicative function grows at most exponentially, cf. [92, Lemma 25.5], there exist constants $a, b > 0$ such that

$$f(X_t - X_0) \leq a \exp\left(b(\sqrt{(X_t^{(1)})^2 + 1} - 1)\right) f(X_t^{(2)}) =: h(X_t^{(1)}) f(X_t^{(2)}).$$

Moreover, a straightforward calculation shows

$$|h'(x)| + |h''(x)| \leq C_1 h(x), \qquad x \in \mathbb{R}, \tag{2.7}$$

for some constant $C_1 > 0$. By Itô's formula and optional stopping,

$$\begin{aligned}
&\mathbb{E}^{\circ,x}(h(X_{t\wedge\tau}^{(1)}) f(X_{t\wedge\tau}^{(2)})) - a f(0) \\
&= \mathbb{E}^{\circ,x}\left(\int_{[0,t\wedge\tau)} h'(X_{s-}^{(1)}) f(X_{s-}^{(2)}) b(X_{s-})\, ds\right) + \frac{1}{2}\mathbb{E}^{\circ,x}\left(\int_{[0,t\wedge\tau)} h''(X_{s-}^{(1)}) f(X_{s-}^{(2)}) \sigma^2(X_{s-})\, ds\right) \\
&+ \mathbb{E}^{\circ,x}\left(\int_{[0,t\wedge\tau)}\int_{|k|\leq 1} f(X_{s-}^{(2)})\big[h(X_{s-}^{(1)} + k(X_{s-},y)) - h(X_{s-}^{(1)}) - h'(X_{s-}^{(1)}) k(X_{s-},y)\big]\, \nu^\circ(dy)\, ds\right) \\
&+ \mathbb{E}^{\circ,x}\left(\int_{[0,t\wedge\tau)}\int_{|k|>1} h(X_{s-}^{(1)})\big[f(X_{s-}^{(2)} + k(X_{s-},y)) - f(X_{s-}^{(2)})\big]\, \nu^\circ(dy)\, ds\right) \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Recall that ν° denotes the Lévy measure of the Cauchy process $(L_t^\circ)_{t\geq 0}$. We estimate the terms separately. By (2.7) and the definition of M_1, it follows easily that

$$|I_1| + |I_2| \leq \frac{3}{2} C_1 M_1 \mathbb{E}^{\circ,x} \left(\int_{[0,t\wedge\tau)} h(X_{s-}^{(1)}) f(X_{s-}^{(2)}) \, ds \right).$$

For I_4 we note that by the submultiplicativity of f and (1.17),

$$\begin{aligned} |I_4| &\leq c\mathbb{E}^{\circ,x} \left(\int_{[0,t\wedge\tau)} \int_{|k|>1} h(X_{s-}^{(1)}) f(X_{s-}^{(2)})(1 + f(k(X_{s-}, y))) \, \nu^\circ(dy) \, ds \right) \\ &\leq c(M_1 + M_2) \mathbb{E}^{\circ,x} \left(\int_{[0,t\wedge\tau)} h(X_{s-}^{(1)}) f(X_{s-}^{(2)}) \, ds \right) \end{aligned}$$

It remains to estimate I_3. By Taylor's formula, we have

$$|h(x+z) - h(x) - h'(x)z| \leq \frac{1}{2} |h''(\xi)| z^2$$

for some intermediate value $\xi = \xi(x,z) \in (x, x+z)$. Since there exists a constant $C_2 > 0$ such that $|h''(\xi)| \leq C_2 h(x)$ for all $|z| \leq 1$ and $x \in \mathbb{R}$, we get

$$|I_3| \leq C_2 M_1 \mathbb{E}^{\circ,x} \left(\int_{[0,t\wedge\tau)} h(X_{s-}^{(1)}) f(X_{s-}^{(2)}) \, ds \right).$$

Combining all estimates shows that $\varphi(t) := \mathbb{E}^{\circ,x}(h(X_{t\wedge\tau}^{(1)}) f(X_{t\wedge\tau}^2) \mathbb{1}_{\{t<\tau\}})$ satisfies

$$\varphi(t) \leq \mathbb{E}^{\circ,x} \left(h(X_{t\wedge\tau}^{(1)}) f(X_{t\wedge\tau}^{(2)}) \right) \leq af(0) + C_3 \int_0^t \varphi(s) \, ds$$

for some constant $C_3 = C_3(M_1, M_2, f)$. Now it follows from Gronwall's inequality, see e. g. [98, Theorem A.43], that $\varphi(t) \leq af(0)e^{C_3 t}$. Finally, using Fatou's lemma, we can let $R \to \infty$ and obtain

$$\mathbb{E}^{\circ,x} f(X_{t\wedge\tau_K} - x) \leq \mathbb{E}^{\circ,x}(h(X_{t\wedge\tau_K}^{(1)}) f(X_{t\wedge\tau_K}^{(2)})) \leq af(0)e^{C_3 t}.$$

This proves $\sup_{x\in K} \sup_{s\leq t} \mathbb{E}^x f(X_{s\wedge\tau_K} - x) < \infty$; (2.6) follows from $f(X_t) \leq cf(X_t - x)f(x)$ and the previous inequality. □

Theorem 2.4 implies that any rich Lévy-type process $(X_t)_{t\geq 0}$ with uniformly bounded jumps (i. e. $\operatorname{spt} \nu(x, \cdot) \subseteq B(0,R)$ for some absolute constant R) has exponential moments:

$$\sup_{s\leq t} \sup_{x\in\mathbb{R}^d} \mathbb{E}^x e^{\alpha|X_s - x|} < \infty \qquad \text{for all } \ \alpha > 0.$$

The exponential growth of the generalized moments (in t) obtained in Theorem 2.4 is, in general, the best we can expect. For large times t this is because of the (possibly) exponential growth of f; for small times t this is a consequence of $\inf_{y\in B(0,1)} f(y) > 0$.

Remark Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with bounded coefficients. Using Theorem 2.4 it is possible to determine the limits

$$\lim_{t\to 0} \frac{\mathbb{E}^x f(X_t - x)}{t} \quad \text{and} \quad \lim_{t\to 0} \frac{\mathbb{E}^x f(X_t) - f(x)}{t} \tag{2.8}$$

in terms of the triplet $(b(x), Q(x), \nu(x,\cdot))$ for a large class of functions f which need not to be bounded or differentiable, cf. Kühn [68]. By combining this result with the Dynkin–Reuter lemma, Proposition 1.20, it might be possible to show that the domain of the generator contains functions which are locally Hölder continuous with varying Hölder exponent[1]; the main difficulty is to prove that the limit $Af(x) := \lim_{t\to 0} t^{-1}(\mathbb{E}^x f(X_t) - f(x))$ vanishes at ∞. In the Lévy case, the existence of limits (2.8) has been discussed by Jacod [53] and Figueroa-López [35].

In [67] we have also studied how differentiability of $q(x,\cdot)$ (at $\xi = 0$) is related to the existence of moments. For simplicity of notation we state the result only in dimension $d = 1$. We refer to [67, Theorem 4.4] its d-dimensional counterpart and the proof.

2.5 Theorem *Let $(X_t)_{t\geq 0}$ be a one-dimensional rich Lévy-type process with symbol q and $K \subseteq \mathbb{R}$ a compact set. If $\xi \mapsto q(x,\xi)$ is $2n$ times differentiable at $\xi = 0$ for all $x \in K$ and*

$$\left|\frac{\partial^k}{\partial \xi^k} q(x,0)\right| \leq c_k(1+|x|^k), \qquad k = 1,\dots,2n, x \in K,$$

for some constants $c_k > 0$, then there exist $C_1, C_2 > 0$ such that

$$\sup_{x\in K}\sup_{s\leq t} \mathbb{E}^x(|X_s - x|^{2n}) \leq C_1 t e^{C_2 t} \qquad \text{for all } t \geq 0.$$

2.2 Fractional moments

In this section, we present estimates for fractional moments of Lévy-type processes. Recently, Luschgy & Pagès [72] and Deng & Schilling [31] obtained estimates for fractional moments $\mathbb{E}|L_t|^\alpha$ of Lévy processes. Depending on α they used different techniques:

(i) $\alpha \in (0,1]$: bounded variation technique,

(ii) $\alpha \geq 1$: martingale technique based on the Burkholder–Davis–Gunky inequality,

(iii) $\alpha \in (0,2)$: Blumenthal–Getoor indices.

In [67] we have shown how to generalize their ideas to derive moment estimates for Lévy-type processes. Here, we only discuss the third approach. First we recall the notion of generalized Blumenthal–Getoor indices.

2.6 Definition Let $(q(x,\xi))_{x\in\mathbb{R}^d}$ be a family of characteristic exponents. Then we call

$$\begin{aligned}\beta_0^x &:= \sup\left\{\alpha \geq 0; \limsup_{|\xi|\to 0} \frac{1}{|\xi|^\alpha} \sup_{|y-x|\leq|\xi|^{-1}} \sup_{|\eta|\leq|\xi|} |q(y,\eta)| < \infty\right\}, \\ \beta_\infty^x &:= \inf\left\{\alpha \geq 0; \limsup_{|\xi|\to \infty} \frac{1}{|\xi|^\alpha} \sup_{|y-x|\leq|\xi|^{-1}} \sup_{|\eta|\leq|\xi|} |q(y,\eta)| < \infty\right\}\end{aligned} \tag{2.9}$$

[1] i. e. $|f(x) - f(y)| \leq C|x-y|^{\beta(x)}$ for y in a neighbourhood of x

generalized Blumenthal–Getoor index at 0 *and* ∞, respectively. If $q(x,\xi)$ does not depend on x, we write β_0 and β_∞, respectively.

Indices of this type were first introduced for Lévy processes by Blumenthal & Getoor [12] and Pruitt [84] and then generalized to Lévy-type processes by Schilling [94]. They can be used to describe sample path properties, e. g. Hausdorff dimensions [19, Theorem 5.15], asymptotic behaviour & strong variation of sample paths [19, Section 5.3,5.4] and transience & recurrence [89].

2.7 Theorem *Let $(X_t)_{t\geq 0}$ be a Lévy-type process with symbol q and let $x \in \mathbb{R}^d$. Suppose that there exist $\alpha,\beta \in (0,2]$, $\gamma < \beta$ and $C > 0$ such that*

$$\begin{aligned} |q(y,\xi)| &\leq C(1+|y|^\gamma)|\xi|^\beta, && \text{for all } |\xi|\leq 1,\ |y-x|\leq |\xi|^{-1}, \\ |q(y,\xi)| &\leq C(1+|y|^\gamma)|\xi|^\alpha, && \text{for all } |\xi|\geq 1,\ |y-x|\leq |\xi|^{-1}. \end{aligned}$$

Then

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s-x|^\kappa\right) \leq Cf(t)^{\kappa/\gamma} \qquad \text{for all } t\leq 1,\ \kappa\in[0,\gamma],$$

where $C = C(x,\gamma,\alpha,\beta)$ and $f(t) := t^{\frac{\gamma}{\alpha}\wedge 1}$.

Proof. Throughout this proof the constant $C_1 = C_1(\gamma,\alpha,\beta) > 0$ may vary from line to line. Without loss of generality, we may assume that $\gamma \neq \alpha$ and $\kappa \in [0,\gamma]$ (otherwise we choose $\gamma < \beta$ sufficiently large such that these two relations are satisfied). As usual, we denote by $\tau^x(r) := \tau_r^x$ the exit time from $B[x,r]$. Fix $R > 0$. Since

$$\mathbb{E}^x(Y) = \int_{(0,\infty)} \mathbb{P}^x(Y\geq r)\,dr$$

for any non-negative random variable Y, it follows from Lemma 1.29 that

$$\begin{aligned} \mathbb{E}^x\left(\sup_{s\leq t\wedge\tau_R^x}|X_s-x|^\gamma\right) &= \int_0^\infty \mathbb{P}^x\left(\sup_{s\leq t\wedge\tau_R^x}|X_s-x| > r^{1/\gamma}\right)dr \\ &\leq \int_0^\infty \min\left\{1,\ C_1\mathbb{E}^x\left(\int_{[0,t\wedge\tau^x(r^{1/\gamma})\wedge\tau_R^x)}\sup_{|\xi|\leq r^{-1/\gamma}}|q(X_s,\xi)|\,ds\right)\right\}dr. \end{aligned}$$

Using the growth assumptions on q, we get

$$\begin{aligned} \mathbb{E}^x\left(\sup_{s\leq t\wedge\tau_R^x}|X_s-x|^\gamma\right) &\leq \int_0^{t^{\gamma/\alpha}} 1\,dr + C_1\int_{t^{\gamma/\alpha}}^\infty \mathbb{E}^x\left(\int_{[0,t\wedge\tau^x(r^{1/\gamma})\wedge\tau_R^x)}\sup_{|\xi|\leq r^{-1/\gamma}}|q(X_s,\xi)|\,ds\right)dr \\ &\leq t^{\gamma/\alpha} + C_1\left(\int_{t^{\gamma/\alpha}}^1 r^{-\alpha/\gamma}\,dr + \int_1^\infty r^{-\beta/\gamma}\,dr\right)\mathbb{E}^x\left(\int_{[0,t\wedge\tau_R^x)}(1+|X_s|^\gamma)\,ds\right) \\ &\leq f(t) + C_1\int_0^t (1+\mathbb{E}^x(|X_s|^\gamma\mathbb{1}_{\{s<\tau_R^x\}}))\,ds \end{aligned}$$

for all $t\leq 1$. This shows that $\varphi(t) := \mathbb{E}^x\left(\sup_{s\leq t\wedge\tau_R^x}|X_s-x|^\gamma\mathbb{1}_{\{t<\tau_R^x\}}\right)$ satisfies

$$\varphi(t) \leq \mathbb{E}^x\left(\sup_{s\leq t\wedge\tau_R^x}|X_s-x|^\gamma\right) \leq C_1 f(t)(1+|x|^\gamma) + C_1\int_0^t \varphi(s)\,ds.$$

Hence, by Gronwall's inequality,

$$\varphi(t) \le C_1 f(t)(1+|x|^\gamma)\exp(C_1 t).$$

Since the constant C_1 does not depend on R, we can let $R \to \infty$ using Fatou's lemma. Applying Jensen's inequality, we find for $\kappa \in [0,\gamma]$ and $t \le 1$

$$\mathbb{E}^x\left(\sup_{s\le t}|X_s - x|^\kappa\right) \le C f(t)^{\kappa/\gamma}. \qquad \square$$

Theorem 2.7 gives, in particular, bounds for fractional moments of solutions of Lévy-driven SDEs.

2.8 Example Let $(L_t)_{t\ge 0}$ be a Lévy process with symbol ψ and f a function of sublinear growth (i. e. $|f(x)| \le C(1+|x|)^{1-\varepsilon}$ for some $\varepsilon \in (0,1]$). Denote by β_0 and β_∞ the Blumenthal–Getoor index of ψ at 0 and ∞, respectively. If the solution $(X_t)_{t\ge 0}$ to the SDE

$$dX_t = f(X_{t-})\,dL_t, \qquad X_0 = x,$$

is a rich Lévy-type process with symbol $q(x,\xi) = \psi(f(x)^T\xi)$, then

$$\mathbb{E}^x\left(\sup_{s\le t}|X_s - x|^\kappa\right) \le C t^{\kappa/\beta_\infty \wedge 1} \qquad \text{for all} \quad t \le 1, \kappa \in [0,\beta_0).$$

2.9 Corollary *Let $(X_t)_{t\ge 0}$ be a Lévy-type process with symbol q. Assume that*

$$\limsup_{|\xi|\to\infty} \frac{1}{|\xi|^\alpha} \sup_{|y-x|\le|\xi|^{-1}} \sup_{|\eta|\le|\xi|} |q(y,\eta)| < \infty \tag{2.10}$$

for some $\alpha \in (0,2]$. If $\kappa \in [0,\beta_0^x)$, then there exists a constant $C > 0$ such that

$$\mathbb{E}^x\left(\sup_{s\le t}|X_s - x|^\kappa\right) \le C t^{\frac{\kappa}{\alpha}\wedge 1}$$

for all $t \le 1$. Here β_0^x denotes the generalized Blumenthal–Getoor index at 0.

By the very definition of the Blumenthal–Getoor index (see Definition 2.6), we know that the limit

$$\limsup_{|\xi|\to\infty} \frac{1}{|\xi|^\alpha} \sup_{|y-x|\le|\xi|^{-1}} \sup_{|\eta|\le|\xi|} |q(y,\eta)|$$

is finite (infinite) if $\alpha > \beta_\infty^x$ (if $\alpha < \beta_\infty^x$). Therefore, (2.10) is violated for any $\alpha \in (0,\beta_\infty^x)$ and automatically satisfied for $\alpha \in (\beta_\infty^x, 2]$. The case $\alpha = \beta_\infty^x$ has to be checked individually.

Proof of Corollary 2.9. For $\beta \in (\kappa,\beta_0^x)$ we have by Definition 2.6

$$\limsup_{|\xi|\to 0} \frac{1}{|\xi|^\beta} \sup_{|y-x|\le|\xi|^{-1}} \sup_{|\eta|\le|\xi|} |q(y,\eta)| < \infty.$$

Combining this with the growth condition (2.10), we find that the assumptions of Theorem 2.7 are satisfied for any $\gamma \in (0,\beta)$. Consequently, the assertion is a direct consequence of Theorem 2.7. $\square$

Applying the maximal inequality, Lemma 1.29, it is not difficult to obtain similar estimates for truncated fractional moments:

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s-x|^\kappa\wedge 1\right)\leq\begin{cases}Ct^{\frac{\kappa}{\alpha}\wedge 1}, & \kappa\neq\alpha,\\ Ct|\log t|, & \kappa=\alpha\end{cases},\qquad t\leq 1,\kappa>0.$$

Note that, in contrast to Corollary 2.9, this estimate holds for *any* $\kappa>0$.
In a similar way we can obtain estimates for large times t. The following result extends [31, Theorem 3.3].

2.10 Theorem *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with symbol q and $\beta\in(0,2]$ such that*

$$\limsup_{|\xi|\to 0}\frac{1}{|\xi|^\beta}\sup_{|x-y|\leq|\xi|^{-1}}\sup_{|\eta|\leq|\xi|}|q(x,\eta)|<\infty,$$

then

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s-x|^\kappa\right)\leq Ct^{\kappa/\beta}\qquad \textit{for all } t\geq 1,\ \kappa\in[0,\beta).$$

Proof. An application of the maximal inequality (Lemma 1.29) yields

$$\begin{aligned}\mathbb{E}^x\left(\sup_{s\leq t}|X_s-x|^\kappa\right)&=\int_0^\infty\mathbb{P}^x\left(\sup_{s\leq t}|X_s-x|\geq r^{1/\kappa}\right)dr\\&\leq\int_0^\infty\min\left\{1,Ct\sup_{|y-x|\leq r^{1/\kappa}}\sup_{|\eta|\leq r^{-1/\kappa}}|q(y,\eta)|\right\}dr.\end{aligned}$$

Hence,

$$\mathbb{E}^x\left(\sup_{s\leq t}|X_s-x|^\kappa\right)\leq\int_0^{t^{\kappa/\beta}}1\,dr+C't\int_{t^{\kappa/\beta}}^\infty r^{-\beta/\kappa}\,dr=O(t^{\kappa/\beta}).$$ □

2.3 Absolute continuity of Lévy-type processes with Hölder continuous symbols

Fournier & Printems [36] have shown that the solution of the (one-dimensional) Lévy driven SDE

$$dX_t=b(X_{t-})\,dt+\sigma(X_{t-})\,dL_t,\qquad X_0=x,$$

is absolutely continuous on the set $\{x;\sigma(x)\neq 0\}$ if b, σ are sufficiently regular (in the sense of Hölder continuity) and the characteristic exponent ψ of the Lévy process $(L_t)_{t\geq 0}$ satisfies certain growth conditions. The key idea is to prove that the characteristic function $\hat{\mu}(\xi):=\mathbb{E}e^{i\xi X_t}$ is square integrable with respect to Lebesgue measure (see Lemma 2.11 below). Under weaker assumptions Debussche & Fournier succeeded in showing that the density has a certain Besov regularity. The approach has also been successfully applied in the context of stochastic partial differential equations (cf. [91]) and goes back to Malliavin [73].
We pick up the idea and prove the absolute continuity of a class of Lévy-type processes with Hölder continuous symbols using the moment estimates from the previous section and the following lemma. As usual, $(\Omega,\mathcal{A},\mathbb{P})$ is a probability space.

2.11 Lemma *For any random variable $X : \Omega \to \mathbb{R}^d$,*

$$\int_{|\xi|\geq 1} |\mathbb{E}e^{i\xi\cdot X}|^2 \, d\xi < \infty \iff \int |\mathbb{E}e^{i\xi\cdot X}|^2 \, d\xi < \infty$$
$$\iff \exists f \in L^2(dx) : X \sim f(x)\, dx.$$

If $(L_t)_{t\geq 0}$ is a Lévy process with characteristic exponent ψ, then Lemma 2.11 shows

$$\int_{|\xi|\geq 1} e^{-2t \operatorname{Re}\psi(\xi)} \, d\xi < \infty \iff \exists p_t \in L^2(dx) : L_t \sim p_t(x)\, dx.$$

Proof of Lemma 2.11. Obviously, it suffices to show the second equivalence.We follow the proof in [36]. "$\Leftarrow$" is a direct consequence of Plancherel's theorem. Denote by $\hat{\mu}(\xi) = \mathbb{E}e^{i\xi\cdot X}$ the characteristic function of $\mu := \mathbb{P}(X \in \bullet)$ and set

$$p_t(x) := \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right), \qquad x \in \mathbb{R}^d.$$

The convolution $\mu_n := p_{1/n} * \mu$ has a probability density f_n with respect to Lebesgue measure and its characteristic function is given by

$$\check{\mu}_n(\xi) = \check{p}_{1/n}(\xi)\check{\mu}(\xi) = \exp\left(-\frac{1}{n}\frac{|\xi|^2}{2}\right)\check{\mu}(\xi), \qquad \xi \in \mathbb{R}^d.$$

In particular,

$$\int_{\mathbb{R}^d} |\check{\mu}_n(\xi)|^2 \, d\xi \leq \int_{\mathbb{R}^d} |\check{\mu}(\xi)|^2 \, d\xi < \infty.$$

Applying Plancherel's theorem, we obtain

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |f_n(x)|^2 \, dx = \frac{1}{(2\pi)^d} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |\check{\mu}_n(\xi)|^2 \, d\xi \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\check{\mu}(\xi)|^2 \, d\xi < \infty.$$

Since the unit ball in $L^2(d\xi)$ is weakly (sequentially) compact (Banach–Alaoglu theorem), there exists a subsequence such that $f_{n_k} \to f \in L^2(d\xi)$ in $L^2(d\xi)$. A subsequence of $(f_{n_k})_{k\in\mathbb{N}}$ converges (Lebesgue)almost everywhere to f; thus, $f \geq 0$ a. e. Moreover, it follows easily that $\mu_n(dx) = f_n(x)\, dx$ converges vaguely to $f(x)\, dx$. On the other hand, we have $\check{\mu}_n(\xi) \to \check{\mu}(\xi)$ for all $\xi \in \mathbb{R}^d$, and therefore it follows from Lévy's continuity theorem that $\mu_n \to \mu$ in distribution. This implies in particular $\mu_n \overset{v}{\to} \mu$. Now the uniqueness of vague limits gives $\mu(dx) = f(x)\, dx$, and this proves "$\Rightarrow$". □

2.12 Theorem *Let $(X_t)_{t\geq 0}$ be a rich Lévy-type process with symbol q, $q(x,0) = 0$. Suppose that the following conditions are satisfied.*

(D1) *There exist $\alpha_0 > 0$ and $C_1 > 0$ such that*

$$\operatorname{Re} q(x,\xi) \geq C_1 |\xi|^{\alpha_0} \qquad \text{for all} \;\; x \in \mathbb{R}^d, |\xi| \geq 1.$$

(D2) *There exist $\alpha, \beta \in (0,2]$, $\gamma \in [0,\beta)$ and $C_2 > 0$ such that*

$$|q(x,\xi)| \leq C_2(1 + |x|^\gamma)|\xi|^\beta, \qquad |\xi| \leq 1,$$
$$|q(x,\xi) - q(y,\xi)| \leq C_2 |x-y|^\gamma |\xi|^\alpha, \qquad |\xi| \geq 1$$

for all $x, y \in \mathbb{R}^d$.

If

$$\gamma \geq \alpha, \alpha < -\frac{1}{2} + 2\alpha_0 \quad or \quad \alpha \geq \gamma > \frac{\alpha}{\alpha_0}\left(\alpha - \alpha_0 + \frac{1}{2}\right), \tag{2.11}$$

then $\mathbb{P}^x(X_t \in \bullet)$ *has a square-integrable density with respect to Lebesgue measure for each* $t > 0$ *and* $x \in \mathbb{R}^d$.

We have seen in Theorem 1.30 that Lévy-type processes can be written as solutions of SDEs. However, we cannot apply the results of Fournier & Printems directly in our setting. The reason is that there are, to our best knowledge, no general conditions for the Hölder continuity of the mapping $k(\cdot, y)$ (in terms of the triplet $(b(x), Q(x), \nu(x, dy))$ or the symbol q).

Proof. We only prove the claim for $t = 1$; the proof works analogously for $t \neq 1$. For fixed $\varepsilon > 0$ we choose disjoint sets $(A_j)_{j \in \mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^d)$ such that $\bigcup_{j \in \mathbb{N}} A_j = \mathbb{R}^d$, $A_j \neq \emptyset$ and $\operatorname{diam} A_j < \varepsilon^p$ for some $p = p(\varepsilon)$ (which we will choose later in the proof). Pick $x_j \in A_j$. For each $j \in \mathbb{N}$ let $(L_t^j)_{t \geq 0}$ be a Lévy process[2] with characteristic exponent $\psi_j(\xi) := q(x_j, \xi)$ such that $(L_t^j)_{t \geq 0} \perp\!\!\!\perp \mathcal{F}_\infty^X$. Define an approximation of $(X_{1-\varepsilon+t})_{t \geq 0}$ by

$$Z_t := X_{1-\varepsilon} + \sum_{j \in \mathbb{N}} L_t^j \mathbb{1}_{A_j}(X_{1-\varepsilon}), \quad t \geq 0.$$

Since

$$\int_{|\xi| \geq 1} |\mathbb{E}^x e^{i\xi \cdot X_1}|^2 \, d\xi \leq 2 \int_{|\xi| \geq 1} |\mathbb{E}^x e^{i\xi \cdot Z_\varepsilon}|^2 \, d\xi + 2 \int_{|\xi| \geq 1} |\mathbb{E}^x (e^{i\xi \cdot X_1} - e^{i\xi \cdot Z_\varepsilon})|^2 \, d\xi$$

we find from Lemma 2.11 that it suffices to show that there exists some $\varepsilon = \varepsilon(\xi)$ such that

$$I_1 := \int_{|\xi| \geq 1} |\mathbb{E}^x e^{i\xi \cdot Z_\varepsilon}|^2 \, d\xi < \infty \quad \text{and} \quad I_2 := \int_{|\xi| \geq 1} |\mathbb{E}^x (e^{i\xi \cdot X_1} - e^{i\xi \cdot Z_\varepsilon})|^2 \, d\xi < \infty. \tag{2.12}$$

By the tower property and (D1), we have

$$\begin{aligned} |\mathbb{E}^x(e^{i\xi \cdot Z_\varepsilon})| = \left|\mathbb{E}^x\left(\mathbb{E}^x(e^{i\xi \cdot Z_\varepsilon} \mid \mathcal{F}_{1-\varepsilon})\right)\right| &= \left|\mathbb{E}^x\left(\sum_{j \in \mathbb{N}} \mathbb{1}_{A_j}(X_{1-\varepsilon}) e^{i\xi \cdot X_{1-\varepsilon}} \mathbb{E}^x e^{i\xi \cdot L_\varepsilon^j}\right)\right| \\ &\leq \mathbb{E}^x\left(\sum_{j \in \mathbb{N}} \mathbb{1}_{A_j}(X_{1-\varepsilon}) e^{-\varepsilon \operatorname{Re} \psi_j(\xi)}\right) \\ &\leq e^{-\varepsilon C_1 |\xi|^{\alpha_0}} \end{aligned} \tag{2.13}$$

for all $|\xi| \geq 1$. In order to estimate I_2 we note that

$$e^{i\xi \cdot X_t + t\psi_j(\xi)} - e^{i\xi \cdot X_0} - \int_0^t e^{i\xi \cdot X_s + s\psi_j(\xi)} (-q(X_s, \xi) + \psi_j(\xi)) \, ds$$

is, for each fixed $\xi \in \mathbb{R}^d$, a local martingale. Denote by $(\sigma_n)_{n \in \mathbb{N}}$ a sequence of localizing stopping times. If we set

$$u(t) := |\mathbb{E}^y(e^{i\xi \cdot X_t + t\psi_j(\xi)}) - e^{i\xi \cdot y}|$$

[2] More precisely, $(L_t^j)_{t \geq 0}$ is a Lévy process started at 0 with respect to the (fixed) probability measure $\mathbb{P}^x$; in particular $\mathbb{P}^x(L_0^j = 0) = 1$.

for fixed $y \in \mathbb{R}^d$ and $j \in \mathbb{N}$, then it follows from the martingale property and the dominated convergence theorem that

$$\begin{aligned} u(t) &= \lim_{n\to\infty} |\mathbb{E}^y(e^{i\xi\cdot X_{t\wedge\sigma_n}+(t\wedge\sigma_n)\psi_j(\xi)}) - e^{i\xi\cdot y}| \\ &= \lim_{n\to\infty} \left|\mathbb{E}^y\left(\int_0^{t\wedge\sigma_n} e^{i\xi\cdot X_s+s\psi_j(\xi)}(-q(X_s,\xi)+\psi_j(\xi))\,ds\right)\right|. \end{aligned}$$

Writing

$$e^{i\xi\cdot X_s+s\psi_j(\xi)} = (e^{i\xi\cdot X_s+s\psi_j(\xi)} - e^{i\xi\cdot y}) + e^{i\xi\cdot y}$$

and

$$-q(X_s,\xi)+\psi_j(\xi) = (-q(X_s,\xi)+q(y,\xi)) + (-q(y,\xi)+\psi_j(\xi))$$

and applying the triangle inequality, we find

$$\begin{aligned} u(t) &\le \underbrace{2\left[|q(y,\xi)-\psi_j(\xi)| + \mathbb{E}^y\left(\sup_{s\le t}|q(X_s,\xi)-q(y,\xi)|\right)\right]\int_0^t |e^{s\psi_j(\xi)}|\,ds}_{=:a(t)} \\ &\quad + \underbrace{|-q(y,\xi)+\psi_j(\xi)|}_{=:b}\int_0^t |\mathbb{E}^y(e^{i\xi\cdot X_s+s\psi_j(\xi)}) - e^{i\xi\cdot y}|\,ds \\ &\le a(t) + b\int_0^t u(s)\,ds. \end{aligned}$$

Hence, by Gronwall's lemma,

$$u(t) \le a(t)e^{bt} \qquad \text{for all} \quad t \ge 0. \tag{2.14}$$

For any $y \in A_j$ and $|\xi| \ge 1$, we have by Theorem 2.7

$$\begin{aligned} a(\varepsilon) &\overset{\text{(D2)}}{\le} 2C_2\varepsilon e^{\varepsilon \operatorname{Re}\psi_j(\xi)}\left(\varepsilon^p + \mathbb{E}^y\left(\sup_{s\le\varepsilon}|X_s-y|^\gamma\right)\right)|\xi|^\alpha \\ &\overset{\text{T2.7}}{\le} 4C_2 e^{\varepsilon \operatorname{Re}\psi_j(\xi)}|\xi|^\alpha \varepsilon^{1+\gamma/\alpha\wedge 1} \end{aligned}$$

for $p \ge \gamma/\alpha$. (Here, we use the convention $1+\gamma/\alpha\wedge 1 := 1+[(\gamma/\alpha)\wedge 1]$.) Moreover, (D2) implies

$$b = |-q(y,\xi)+\psi_j(\xi)| \le C_2|y-x_j|^\gamma|\xi|^\alpha \le C_2\varepsilon^{p\gamma}|\xi|^\alpha$$

for any $y \in A_j$ and $|\xi| \ge 1$. Consequently, by (2.14),

$$u(t) \le 4C_2|\xi|^\alpha\varepsilon^{1+\gamma/\alpha\wedge 1}\exp\left(\varepsilon \operatorname{Re}\psi_j(\xi) + C_2\varepsilon^{1+p\gamma}|\xi|^\alpha\right)$$

for $y \in A_j$ and $|\xi| \ge 1$. Using the Markov property of $(X_t)_{t\ge 0}$, we conclude

$$\begin{aligned} |\mathbb{E}^x(e^{i\xi\cdot Z_\varepsilon} - e^{i\xi\cdot X_1})| &= \left|\mathbb{E}^x\left(\mathbb{E}^x\left[e^{i\xi\cdot Z_\varepsilon} - e^{i\xi\cdot X_1} \mid \mathcal{F}_{1-\varepsilon}\right]\right)\right| \\ &= \left|\sum_{j\in\mathbb{N}} \mathbb{E}^x\left(\mathbb{1}_{A_j}(X_{1-\varepsilon})e^{i\xi\cdot X_{1-\varepsilon}}\mathbb{E}(e^{-\xi L_\varepsilon^j}) - \mathbb{1}_{A_j}(X_{1-\varepsilon})\mathbb{E}^{X_{1-\varepsilon}}(e^{i\xi\cdot X_\varepsilon})\right)\right| \\ &= \left|\sum_{j\in\mathbb{N}} \mathbb{E}^x\left(\mathbb{1}_{A_j}(X_{1-\varepsilon})e^{-\varepsilon\psi_j(\xi)}\mathbb{E}^{X_{1-\varepsilon}}[e^{i\xi\cdot X_\varepsilon+\varepsilon\psi_j(\xi)} - e^{i\xi\cdot X_0}]\right)\right| \\ &\le 4C_2|\xi|^\alpha\varepsilon^{1+\gamma/\alpha\wedge 1}\exp\left(C_2\varepsilon^{1+p\gamma}|\xi|^\alpha\right). \end{aligned} \tag{2.15}$$

If we choose $\varepsilon := \varepsilon(\xi) := \delta|\xi|^{-\alpha_0}\log|\xi|$, then it follows from (2.13) that $I_1 < \infty$ for $\delta > 1/C_1$. Choosing p sufficiently large, we find that $\exp(C_2\varepsilon^{1+p\gamma}|\xi|^{\alpha})$ is bounded (in ξ) and therefore, by (2.15),

$$I_2 \le C_2' \int_{|\xi|\ge 1} |\xi|^{2\alpha} \left(\frac{\delta \log|\xi|}{|\xi|^{\alpha_0}}\right)^{2+2(\gamma/\alpha\wedge 1)} d\xi.$$

The integral converges if, and only if,

$$2\alpha - 2\alpha_0\left(1 + \min\left\{\frac{\gamma}{\alpha}, 1\right\}\right) < -1 \iff (2.11). \qquad \square$$

Let us illustrate Theorem 2.12 with some examples. For brevity, we say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is γ-Hölder continuous if f is Hölder continuous with Hölder exponent γ.

2.13 Example Let $(L_t)_{t\ge 0}$ be a one-dimensional symmetric Lévy process with characteristic exponent ψ,

$$\psi(\xi) = \int (1 - \cos(y\xi))\, \nu(dy).$$

Suppose that there exists $\gamma \in (0,1]$ such that $\int |y|^{\gamma}\, \nu(dy) < \infty$ and $\psi(\xi) \ge c|\xi|^{\gamma}$ for $|\xi| \ge 1$. Let $\sigma : \mathbb{R} \to (0,\infty)$ be a β-Hölder-continuous function for some $\beta \in (0,1)$ such that $\inf_{x\in\mathbb{R}} |\sigma(x)| > 0$. If the solution to the SDE

$$dX_t = \sigma(X_{t-})\, dL_t, \qquad X_0 = x,$$

is a rich Lévy-type process with symbol $q(x,\xi) = \psi(\sigma(x)\xi)$, then X_t has an L^2-density for each $t > 0$ if $\beta\gamma > \frac{1}{2}$.

Applying the results of Fournier & Printems [36] and Debussche & Fournier [29] gives the existence of an L^2-density if $\beta\gamma^2 > \frac{1}{2}$; since $\gamma \in (0,1]$ this is a stronger assumption.

Proof of Example 2.13. First, we prove that ψ is γ-Hölder continuous. Fix $\xi, \eta \in \mathbb{R}^d$, $\xi \neq \eta$. From

$$|\cos x - \cos y| \le 2\min\{1, |x-y|\}, \qquad x, y \in \mathbb{R},$$

it follows that

$$\begin{aligned}
|\psi(\xi) - \psi(\eta)| &\le 2 \int \min\{1, |y|\,|\xi-\eta|\}\, \nu(dy) \\
&\le 2|\xi-\eta| \int_{|y|\le \frac{1}{|\xi-\eta|}} |y|^{(1-\gamma)+\gamma}\, \nu(dy) + 2\int_{|y|>\frac{1}{|\xi-\eta|}} \nu(dy) \\
&\le 2|\xi-\eta|^{1-(1-\gamma)} \int |y|^{\gamma}\, \nu(dy) + 2|\xi-\eta|^{\gamma} \int |y|^{\gamma}\, \nu(dy).
\end{aligned}$$

Now we check (D1) and (D2).

(D1) Since the symbol q of $(X_t)_{t\ge 0}$ equals $q(x,\xi) = \psi(\sigma(x)\xi)$, we have

$$q(x,\xi) = \psi(\sigma(x)\xi) \ge c|\sigma(x)\cdot\xi|^{\gamma} \ge c'|\xi|^{\gamma}$$

for all $|\xi| \ge 1$.

(D2) Since ψ is γ-Hölder continuous, we have

$$|q(x,\xi) - q(y,\xi)| \leq C|\xi|^\gamma |\sigma(x) - \sigma(y)|^\gamma \leq C|\xi|^\gamma |x-y|^{\gamma\beta}$$

for all $\xi \in \mathbb{R}^d$. For $y = 0$ this shows $|q(x,\xi)| \leq C|\xi|^\gamma (1 + |x|^{\gamma\beta})$.

As $\beta \in (0,1)$, (2.11) is satisfied iff $\beta\gamma > 1/2$. Applying Theorem 2.12 finishes the proof. □

Theorem 2.12 also applies to Lévy-type processes with symbols of varying order.

2.14 Example Let $\psi \geq 0$ be the symbol of a symmetric one-dimensional Lévy process such that $|\psi(\xi)| \asymp |\xi|$ for $|\xi|$ large and $|\psi(\xi)| \asymp |\xi|^\varrho$ for small ξ. Moreover, let $m : \mathbb{R} \to (0,2)$ be a function such that

$$0 < m^L := \inf_{x \in \mathbb{R}} m(x) \leq \sup_{x \in \mathbb{R}} m(x) =: m^U < 2$$

and

$$|m(x) - m(y)| \leq C|x-y|^\gamma, \qquad x, y \in \mathbb{R},$$

for some $C > 0$ and $\gamma \in (0,1]$. Suppose that the symbol $q(x,\xi) := \psi(\xi)^{m(x)}$ gives rise to a rich Lévy-type process $(X_t)_{t \geq 0}$, then X_t has a square-integrable density whenever

$$m^U < \gamma, m^U < 2m^L - \frac{1}{2} \quad \text{or} \quad \varrho m^L \geq \gamma > \frac{m^U}{m^L}\left(m^U - m^L + \frac{1}{2}\right). \tag{2.16}$$

Proof. Since ψ is symmetric, we have $\psi = \operatorname{Re}\psi \geq 0$, and therefore

$$\operatorname{Re} q(x,\xi) = (\operatorname{Re}\psi(\xi))^{m(x)} = |\psi(\xi)|^{m(x)} \geq c|\xi|^{m^L}$$

for $|\xi| \geq 1$. Moreover, for any $a \geq 1$,

$$\begin{aligned} |a^{m(x)} - a^{m(y)}| = \log a \left| \int_{m(x)}^{m(y)} a^u \, du \right| &\leq a^{m^U} |m(x) - m(y)| \log a \\ &\leq C a^{m^U} |x-y|^\gamma \log a. \end{aligned}$$

If we plug in $a = \psi(\xi)$, this gives the second condition in (D2) for any $\alpha > m^U$. For $|\xi| \leq 1$,

$$|q(x,\xi)| \leq C|\psi(\xi)|^{m^L} \leq C'|\xi|^{m^L \varrho},$$

i. e. the first inequality in (D2) holds with $\beta = m^L \varrho$. Because of (2.16), (2.11) is satisfied, and so the claim follows from Theorem 2.12. □

In Example 2.14 we have to assume the existence of a (rich) Feller process with symbol $q(x,\xi) = \psi(\xi)^{m(x)}$. Since the results in the literature typically require smoothness of α (see e. g. [43, Theorem 7.10]), we are interested in proving the existence of such Feller processes under relaxed regularity assumptions. Using the existence result which we will present in the next chapter we will establish the existence of Feller processes with symbols of varying order for a class of rotationally invariant symbols ψ under the much weaker assumption that $x \mapsto m(x)$ is Hölder continuous.

3

Parametrix construction

By the Lévy-Khintchine formula, there is a one-to-one correspondence between Lévy processes and continuous negative definite functions. For any continuous negative definite function ψ (or equivalently for any Lévy triplet (b, Q, ν)) there exists a Lévy process with characteristic exponent ψ (and triplet (b, Q, ν), respectively). It is natural to ask whether this result can be extended to Feller processes. Given a family $(q(x, \cdot))_{x \in \mathbb{R}^d}$ of continuous negative definite functions, does there exist a Feller process with symbol q? This question was first raised by Jacob [49, 50, 51]. Since the answer is *no* (see e. g. [19, Example 2.26] for counterexamples), it is of importance to find sufficient conditions for the existence; preferably, in terms of the symbol q or the triplet $(b(x), Q(x), \nu(x, \cdot))$. There are several techniques in the literature to tackle this problem. Typically, the first step is to construct a Markov process, e. g.

- using Dirichlet forms
- as a solution to a martingale problem[1]
- as a strong/weak solution to an SDE

and then to establish that the process is, in fact, a Feller process. We refer to the monograph [19] for a concise overview on known results. Many of the results are rather restrictive in the sense that they only apply to symbols/triplets of a very particular form (typically, "stable-like").

A different, purely analytic approach is the parametrix construction and it has become increasingly popular in the last years. One of the first who applied this method to construct Feller processes was Hoh [43]. By developing a symbolic calculus for pseudo-differential operators (with negative definite symbols), he succeeded in generalizing a parametrix construction used by Kumano-go [62] (for operators with classical symbols) and obtained several, rather general, existence results for Feller processes (see [19] or [50] for a survey). Some of his results were extended by Böttcher [15, 16] and Potrykus [81]. The drawback of this approach is that it requires high regularity of the symbol with respect to the space variable x.

[1] see the remark preceding Theorem 1.37

We will use a technique which goes back to Levi [70] and Feller [34]. The idea is to apply the parametrix method described in Section 1.7 to construct (a candidate for) the fundamental solution of the associated Cauchy problem and to show that this fundamental solution is the transition probability of a Feller process. In the last years, several authors have used similar methods to obtain certain classes of stochastic processes and derive heat kernel estimates; for instance, processes with variable order of differentiation (Kolokoltsov [59, 60] and Chen & Zhang [25]), gradient perturbations of Lévy generators (Bogdan & Jakubowski [13] and Jakubowski & Szczypkowski [55]) and solutions of SDEs with Hölder continuous coefficients (Knopova & Kulik [57] and Huang [45]). We have been particularly inspired by the monograph of Kolokoltsov [60] and the article of Knopova & Kulik [57].
Let $q(x,\xi)$ be a symbol which can be written in the form

$$q(x,\xi) = \psi_{\alpha(x)}(\xi), \qquad x,\xi \in \mathbb{R}^d$$

for a family $(\psi_\alpha)_{\alpha\in I}$ of continuous negative definite functions, a Hölder continuous mapping $\alpha : \mathbb{R}^d \to I$ and a set of parameters $I \subseteq \mathbb{R}^n$. Our main result, Theorem 3.2, gives a sufficient condition on $(\psi_\alpha)_{\alpha\in I}$ for the existence of a Feller process with symbol $q(x,\xi) = \psi_{\alpha(x)}(\xi)$. We will show that, under the assumptions of Theorem 3.2, the so defined Feller process is the unique solution to the associated martingale problem and that $C_c^\infty(\mathbb{R}^d)$ is a core for the generator. As a by-product of the parametrix construction, we obtain heat kernel estimates for the transition probability and its time derivative. In dimension $d = 1$, we also get heat kernel estimates for the derivative with respect to the space variable x and, using these estimates, we will deduce that the Feller process is irreducible with respect to Lebesgue measure. Let us mention that we also derive heat kernel estimates for a class of rotationally invariant Lévy processes which, we believe, are of independent interest (see Section 4.1).
We have three different kinds of applications in mind. The class of symbols which we consider includes, in particular, symbols of the form

$$q(x,\xi) = f_{\alpha(x)}(|\xi|^2)$$

where $(f_\alpha)_{\alpha\in I}$ is a family of Bernstein functions. This leads to, so-called, variable order subordination. Applying the results from this chapter, we will obtain many new existence results. We will prove, among others, the existence of relativistic stable-like, normal tempered stable-like and Lamperti stable-like processes and show the existence of Feller processes with symbols of varying order under weak regularity assumptions. Another important application are existence and uniqueness results for Lévy-driven SDEs with Hölder continuous coefficients. Using the parametrix construction, we will succeed in extending known results from the α-stable case to a larger class of driving Lévy processes. Moreover, we will establish the existence of certain Feller processes of mixed type.

The remaining part of this thesis is structured as follows. In Section 3.1, we state our main results. Section 3.2 contains slight extensions of the results presented in Section 3.1, and in Section 3.3 we discuss open problems which might be interesting for future work.

Chapter 4 is devoted to the proof of the main results, and, finally, in Chapter 5, we study applications.

3.1 Main results

Throughout this section, we consider families $(\psi_\alpha)_{\alpha\in I}$ of continuous negative definite functions which satisfy the following set of assumptions.

Assumption Let $I \subseteq \mathbb{R}^n$ be open and convex and $m \geq 0$. We say that a family $(\psi_\alpha)_{\alpha\in I}$ of continuous negative definite functions $\psi_\alpha : \mathbb{R}^d \to \mathbb{C}$ with $\psi_\alpha(0) = 0$ is admissible, and write $(\psi_\alpha)_{\alpha\in I} \in A(m)$, if there exist constants $\vartheta \in (0, \pi/2)$ and $c_1, c_2, c_3 > 0$ such that the following conditions are satisfied.

(C1) For each $\alpha \in I$ there exists a mapping $\Psi_\alpha : \mathbb{R} \to \mathbb{R}$ such that $\psi_\alpha(\xi) = \Psi_\alpha(|\xi|)$, $\xi \in \mathbb{R}^d$. If $m > 0$: Ψ_α is even, i. e. $\Psi_\alpha(r) = \Psi_\alpha(-r)$ for all $r \geq 0$.

(C2) Ψ_α has a holomorphic extension to

$$\Omega := \Omega(m, \vartheta) := \{\zeta \in \mathbb{C}; |\operatorname{Im}\zeta| < m\} \cup \{\zeta \in \mathbb{C}\backslash\{0\}; \arg\zeta \in (-\vartheta, \vartheta) \cup (\pi - \vartheta, \pi + \vartheta)\} \tag{3.1}$$

for all $\alpha \in I$.

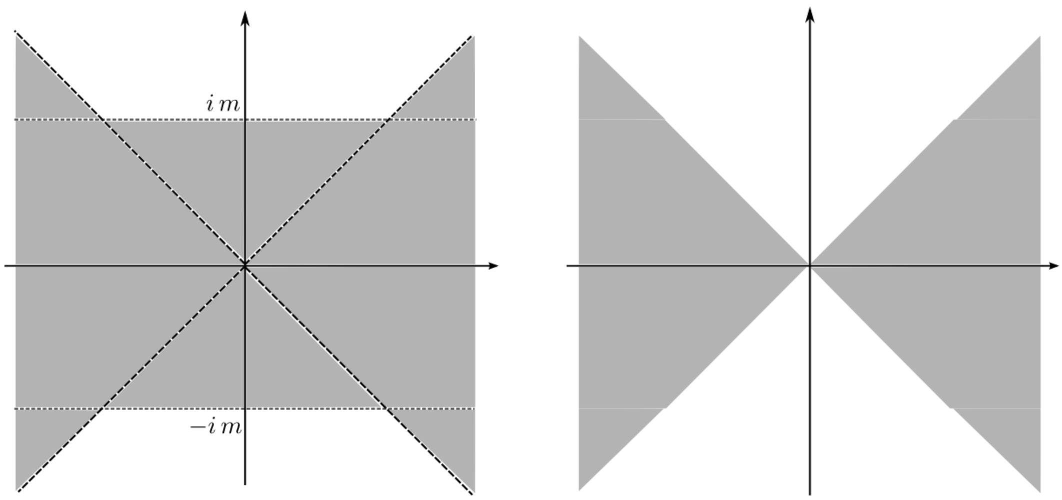

Figure 3.1: The domain $\Omega = \Omega(m)$ for $m > 0$ (left) and $m = 0$ (right).

(C3) There exist measurable mappings $\gamma_0 : I \to (0, 2]$ and $\gamma_\infty : I \to (0, 2]$ such that

$$\operatorname{Re}\Psi_\alpha(\zeta) \geq c_1 |\operatorname{Re}\zeta|^{\gamma_\infty(\alpha)} \qquad \text{for all} \quad \zeta \in \Omega, |\zeta| \geq 1, \tag{C3-1}$$

and

$$|\Psi_\alpha(\zeta)| \leq c_2\big(|\zeta|^{\gamma_0(\alpha)} \mathbb{1}_{\{|\zeta|\leq 1\}} + |\zeta|^{\gamma_\infty(\alpha)} \mathbb{1}_{\{|\zeta|>1\}}\big) \qquad \text{for all} \quad \zeta \in \Omega. \tag{C3-2}$$

Moreover, $\gamma_\infty^L := \inf_{\alpha\in I}\gamma_\infty(\alpha) > 0$, $\gamma_0^L := \inf_{\alpha\in I}\gamma_0(\alpha) > 0$ and $\alpha \mapsto \gamma_\infty(\alpha)$ is Hölder continuous with Hölder exponent $\varrho(\gamma_\infty) \in (0,1]$.[2]

(C4) The partial derivative $\frac{\partial}{\partial\alpha_j}\Psi_\alpha(r)$ exists for all $r\in\mathbb{R}$ and extends holomorphically to Ω for all $j \in \{1,\dots,n\}$ and $\alpha\in I$. There exists an increasing slowly varying (at ∞) function $\ell:(0,\infty)\to(0,\infty)$ such that

$$\left|\frac{\partial}{\partial\alpha_j}\Psi_\alpha(\zeta)\right| \le c_3|\zeta|^{\gamma_\infty(\alpha)}(1+\ell(|\zeta|)), \qquad \zeta\in\Omega, |\zeta|\ge 1, \tag{C4-1}$$

$$\left|\frac{\partial}{\partial\alpha_j}\Psi_\alpha(\zeta)\right| \le c_3|\zeta|^{\gamma_0(\alpha)}(1+\ell(|\zeta|)), \qquad \zeta\in\Omega, |\zeta|\le 1. \tag{C4-2}$$

3.1 Remarks (i) Obviously, (C1) implies that ψ_α is rotationally invariant for all $\alpha\in I$. Moreover, because of (C3), the *sector condition* is satisfied, i. e. there exists a constant $C>0$ such that

$$|\operatorname{Im}\psi_\alpha(\xi)| \le C|\operatorname{Re}\psi_\alpha(\xi)| \qquad \text{for all } \xi\in\mathbb{R}^d. \tag{3.2}$$

Let us remark that Berg & Forst [7, Theorem 3.7] show that the continuous negative definite function ψ_α satisfies the sector condition (3.2) if, and only if,

$$\mathcal{E}(f,g) := \int \psi_\alpha(\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi, \qquad f,g\in C_c^\infty(\mathbb{R}^d),$$

defines a (non-symmetric) Dirichlet form.

(ii) Because of (C1) it holds that

$$\operatorname{Im}\Psi_\alpha(ir) = 0 \qquad \text{for all } r\in(-m,m), \alpha\in I.$$

If $m=0$ this is trivial; for $m>0$ this follows from the fact that $\Psi_\alpha:\mathbb{R}\to\mathbb{C}$ is an even real-valued function which has a holomorphic extension to $\Omega(m,\vartheta)$ (see Lemma 4.4). We will use this later on in the proof and, in fact, this is the only reason why we assume $\Psi_\alpha|_{\mathbb{R}}$ to be an even function if $m>0$.

(iii) It follows easily from (C3) that there exists a constant $c_4>0$ such that

$$\operatorname{Re}\Psi_\alpha(\zeta) \ge -c_4 \qquad \text{for all } \zeta\in\Omega, \alpha\in I. \tag{3.3}$$

Mind that this condition is not automatically satisfied. Although any symmetric continuous negative definite function $\psi:\mathbb{R}\to\mathbb{C}$, $\psi(0)=0$, satisfies $\operatorname{Re}\psi(\xi)\ge 0$ for all $\xi\in\mathbb{R}$, we can, in general, not expect $\operatorname{Re}\psi(\zeta)\ge 0$ for *complex* ζ.

(iv) If there exists $c>0$ such that

$$\left|\frac{\partial}{\partial\alpha_j}\Psi_\alpha(\zeta)\right| \le c|\Psi_\alpha(\zeta)|, \qquad \zeta\in\Omega, |\zeta|\ge 1, j\in\{1,\dots,n\}$$

then (C4-1) holds true. In symbolic calculus, estimates of this form (for ζ real) are a common assumption to prove the existence of (Lévy-type) processes, see e. g. [43, Theorem 5.24] for a typical result.

[2] We use the following convention: For a Hölder continuous function f we denote by $\varrho(f)$ the Hölder exponent of f.

Beware of the following abuse of notation: From now on, we use α both to denote a canonical element from set of parameters I and to denote a Hölder continuous mapping $\alpha:\mathbb{R}^d \to I$. Whenever confusion might arise, we will write $\alpha(\cdot)$ to indicate that we consider $\alpha:\mathbb{R}^d \to I$ as a mapping.
Theorem 3.2 is one of our main results. It states that for any admissible family $(\psi_\alpha)_{\alpha\in I}$ and any Hölder continuous function $\alpha:\mathbb{R}^d \to I$ there exists a rich Feller process with symbol $q(x,\xi) := \psi_{\alpha(x)}(\xi)$, $x,\xi\in\mathbb{R}^d$. Note that, because of the growth assumption (C3-2), q has bounded coefficients.

3.2 Theorem *Let $I \subseteq \mathbb{R}^n$ be open and convex, $m \geq 0$ and $(\psi_\alpha)_{\alpha\in I} \in A(m)$ a family of continuous negative definite functions $\psi_\alpha:\mathbb{R}^d\to\mathbb{C}$ with $\psi_\alpha(0)=0$. Moreover, let $\alpha:\mathbb{R}^d\to I$ be Hölder continuous with Hölder exponent $\varrho(\alpha)\in(0,1]$. Then the following statements hold true.*

(i) There exists a strong Feller process $(\Omega,\mathcal{A},\mathcal{F}_t,X_t,t\geq 0,\mathbb{P}^x,x\in\mathbb{R}^d)$ with symbol $q(x,\xi):=\psi_{\alpha(x)}(\xi)$.

(ii) The associated semigroup $(P_t)_{t\geq0}$ is a strong Feller semigroup. Moreover, $C_c^\infty(\mathbb{R}^d)$ is contained in the domain $\mathcal{D}(L)$ of the generator L of the semigroup and

$$Lf(x) = -\int e^{ix\cdot\xi}\psi_{\alpha(x)}(\xi)\hat{f}(\xi)\,d\xi \qquad \text{for all } \; x\in\mathbb{R}^d, f\in C_c^\infty(\mathbb{R}^d).$$

(iii) The distribution $\mathbb{P}^x(X_t\in\cdot)$ has a density $p(t,x,\cdot)$ with respect to Lebesgue measure for all $t>0$ and $x\in\mathbb{R}^d$. The mapping $p:(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ is continuous.

Compared to other results in the literature, Theorem 3.2 requires only weak regularity of the symbol with respect to x, and it allows us to deduce existence results for a rather broad class of Feller processes (see Chapter 5). Before we discuss further properties of the Feller process $(X_t)_{t\geq0}$, let us make the following remarks.

Remarks (i) In Theorem 3.2 we make separate assumptions on the regularity of the mappings $I\ni\alpha\mapsto\psi_\alpha(\xi)$ (differentiability) and $\mathbb{R}^d\ni x\mapsto\alpha(x)$ (Hölder continuity). Mind that this is much weaker than assuming that $x\mapsto q(x,\xi)=\psi_{\alpha(x)}(\xi)$ is differentiable. For example, if we consider $\psi_\alpha(\xi)=|\xi|^\alpha$, then the regularity assumptions of Theorem 3.2 are satisfied if $\alpha(\cdot)$ is Hölder continuous; in contrast, differentiability of $x\mapsto q(x,\xi)=|\xi|^{\alpha(x)}$ requires differentiability of $\alpha(\cdot)$.

(ii) The main restrictions of Theorem 3.2 are the rotational invariance of q with respect to the variable ξ and the sector condition (3.2). Theorem 3.7 below shows that we can drop the assumption of rotational invariance in dimension $d=1$. In Section 3.2 we will make a conjecture how to relax this assumption in dimension $d>1$.

Proposition 3.3 states that $(X_t)_{t\geq0}$ is not only a rich Feller process (i. e. $C_c^\infty(\mathbb{R}^d)\subseteq\mathcal{D}(L)$), but that $C_c^\infty(\mathbb{R}^d)\subseteq\mathcal{D}(L)$ is even a core for the generator.

3.3 Proposition *Under the assumptions of Theorem 3.2 it holds that*

(i) $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(L)$ *and*

$$Lf(x) = b(x)\cdot\nabla f(x) + \frac{1}{2}\operatorname{tr}(Q(x)\cdot\nabla^2 f(x)) + \int (f(x+y) - f(x) - \nabla f(x)\cdot y \mathbb{1}_{\{|y|<1\}})\,\nu(x,dy)$$

for all $f \in C_\infty^2(\mathbb{R}^d)$*; here* $(b(x), Q(x), \nu(x,dy))$ *denotes the Lévy triplet associated with* $q(x,\cdot)$*. Moreover, there exists a constant* $C > 0$ *such that*

$$\|Lf\|_\infty \leq C\|f\|_{(2)} \qquad \text{for all} \;\; f \in C_\infty^2(\mathbb{R}^d).$$

(ii) $C_c^\infty(\mathbb{R}^d)$ *is a core for the generator:* $\overline{(L, C_c^\infty(\mathbb{R}^d))}^{\|\cdot\|_\infty} = (L, \mathcal{D}(L))$.

Using Proposition 3.3 we can deduce that the transition probability p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$ in the sense of Definition 1.40.

3.4 Corollary *The function* p*, introduced in Theorem 3.2, is the fundamental solution to the Cauchy problem for the operator* $(\partial_t - L)$*, i. e.* $p(t,\cdot,y) \xrightarrow{t\to 0} \delta_x$ *weakly for all* $x \in \mathbb{R}^d$, $(0,\infty) \ni t \mapsto p(t,x,y)$ *is differentiable for each* $t > 0$ *and* $(\partial_t - L_x)p(t,x,y) = 0$.

Moreover, Proposition 3.3 entails the following result on the well-posedness of the associated martingale problem.

3.5 Corollary *Denote by* $(X_t)_{t\geq 0}$ *the Feller process from Theorem 3.2. Then:*

(i) $(X_t)_{t\geq 0}$ *solves the* $(L, \mathcal{D}(L))$*-martingale problem, i. e. for any* $f \in \mathcal{D}(L)$ *and any probability distribution* μ *on* $\mathbb{R}^d$*, the process*

$$M_t^f := f(X_t) - \int_0^t Lf(X_s)\,ds$$

is a martingale with respect to the canonical filtration $\mathcal{F}_t := \sigma(X_s; s \leq t)$ *and* $\mathbb{P}^\mu$.

(ii) *The* $(L, C_c^\infty(\mathbb{R}^d))$*-martingale problem is well posed; its unique solution*[3] *is given by* $(X_t)_{t\geq 0}$.

In Theorem 3.6 we collect heat kernel estimates for the transition probability p and its time derivative.

3.6 Theorem *Let* $I \subseteq \mathbb{R}^n$ *be open and convex,* $m \geq 0$*,* $\alpha : \mathbb{R}^d \to I$ *be Hölder continuous with Hölder exponent* $\varrho(\alpha) \in (0,1]$ *and suppose that* $(\psi_\alpha)_{\alpha\in I} \in A(m)$ *is, as in Theorem 3.2, a family of admissible continuous negative definite functions. Choose* $\gamma \in (0, 1/\gamma_\infty^U]$ *such that* $\kappa := \min\{\gamma\varrho(\alpha), \gamma(-d+\gamma_\infty^U)+1\} > 0$ *and define*

$$S(x,\alpha,t) := S_m(x,\alpha,t) := \exp\left(-\frac{m}{4}|x|\right)\begin{cases} t^{-d/\gamma_\infty(\alpha)}, & |x| \leq t^{1/\gamma_\infty(\alpha)} \wedge 1, \\ \frac{t}{|x|^{d+\gamma_\infty(\alpha)}}, & t^{1/\gamma_\infty(\alpha)} < |x| \leq 1, \\ \frac{t}{|x|^{d+\gamma_\infty(\alpha)\wedge\gamma_0(\alpha)}}, & |x| > 1 \end{cases} \tag{3.4}$$

[3] in the sense of finite-dimensional distributions

for $x \in \mathbb{R}^d$, $\alpha \in I$ and $t > 0$. Then there exists for any $T > 0$ a constant $C = C(T) > 0$ such that the density function p from Theorem 3.2 satisfies the following estimates.

$$|p(t,x,y) - p_0(t,x,y)| \le Ct^{\kappa}\left(S(x-y,\alpha(y),t) + \frac{1}{1+|x-y|^{d+\gamma_0^L \wedge \gamma_\infty^L}} \exp\left(-\frac{m}{4}|x-y|\right)\right),$$
$$|p(t,x,y)| \le CS(x-y,\alpha(y),t) + Ct^{\kappa}\frac{1}{1+|x-y|^{d+\gamma_0^L \wedge \gamma_\infty^L}} \exp\left(-\frac{m}{4}|x-y|\right),$$
$$\left|\frac{\partial}{\partial t}p(t,x,y)\right| \le Ct^{-1}S(x-y,\alpha(y),t) + Ct^{-1+\kappa}\frac{1}{1+|x-y|^{d+\gamma_0^L \wedge \gamma_\infty^L}} \exp\left(-\frac{m}{4}|x-y|\right)$$

for all $t \in (0,T]$, $x,y \in \mathbb{R}^d$; here p_0 denotes the zero-order approximation of p, see (4.3) *for the definition.*

Remarks (i) The proof of Theorem 3.6 actually shows that we may replace the constant $\frac{m}{4}$ in the above estimates by $m(1-\delta)$ for any $\delta \in (0,1)$. This implies

$$\mathbb{E}^x(e^{m(1-\delta)X_t}) < \infty \qquad \text{for all } \delta \in (0,1), t \ge 0, x \in \mathbb{R}^d,$$

in particular $(X_t)_{t\ge 0}$ has exponential moments of order less than m.

(ii) As mentioned in Section 1.7 it is crucial for the parametrix construction to have suitable bounds for the zero-order approximation p_0. Here, this means that we need heat kernel estimates for Lévy processes whose characteristic exponents ψ satisfy (C1)-(C4). We will derive these heat kernel estimates in Section 4.1. We think that they are of independent interest and refer to Corollary 4.11 and Theorem 4.12 for a summary.

So far, all the results presented in this section are, because of (C1), only applicable to families of continuous negative definite functions ψ_α which are rotationally invariant. Theorem 3.7 shows that we can drop the assumption of rotational invariance in dimension $d = 1$. We believe that this assumption can also be relaxed in dimension $d > 1$, see Conjecture 3.12 in Section 3.2.

3.7 Theorem (Case $d = 1$) *Let $I \subseteq \mathbb{R}^n$ be open and convex, $m \ge 0$ and $(\psi_\alpha)_{\alpha \in I}$ a family of continuous negative definite functions $\psi_\alpha : \mathbb{R} \to \mathbb{C}$ with $\psi_\alpha(0) = 0$ for all $\alpha \in I$. If $\Psi_\alpha(\xi) := \psi_\alpha(\xi)$, $\xi \in \mathbb{R}$, satisfies* (C2)-(C4) *and $\alpha : \mathbb{R} \to I$ is Hölder continuous, then the results of Theorem 3.2 and 3.6, Proposition 3.3, Corollary 3.4 and 3.5 remain valid.*

Finally, we will show that, in dimension $d = 1$, the transition probability p is differentiable with respect to the space variable x provided that $\alpha(\cdot)$ and $\alpha \mapsto \psi_\alpha(\xi)$ are sufficiently smooth.

3.8 Theorem *Let $(\psi_\alpha)_{\alpha \in I}$ be as in Theorem 3.7. Suppose additionally that there exists some constant $c_5 > 0$ such that condition* (C5) *holds.*

(C5) $\frac{\partial^2}{\partial \alpha_j^2}\psi_\alpha(\xi)$ exists for all $\xi \in \mathbb{R}$, $j \in \{1, \dots, n\}$ and has a holomorphic extension to Ω satisfying

$$\left|\frac{\partial^2}{\partial \alpha_j^2}\psi_\alpha(\zeta)\right| \le c_5(1+\ell(|\zeta|))|\zeta|^{\gamma_\infty(\alpha)}, \qquad \zeta \in \Omega, |\zeta| \ge 1,$$
$$\left|\frac{\partial^2}{\partial \alpha_j^2}\psi_\alpha(\zeta)\right| \le c_5(1+\ell(|\zeta|))|\zeta|^{\gamma_0(\alpha)}, \qquad \zeta \in \Omega, |\zeta| \le 1$$

for all $\alpha \in I$; here ℓ denotes the slowly varying function from (C4).

Let $\alpha : \mathbb{R} \to I$ be such that $\alpha \in C_b^2(\mathbb{R})$. Then the transition probability $p(t,x,y)$ is continuously differentiable with respect to x for any $t > 0$ and $y \in \mathbb{R}$. For any $T > 0$ there exists a constant $C = C(T) > 0$ such that

$$\left|\frac{\partial}{\partial x}p(t,x,y)\right| \le Ct^{-1/\gamma_\infty^L}\left[S(x-y,\alpha(y),t) + t^\kappa \frac{1}{1+|x-y|^{d+\gamma_0^L \wedge \gamma_\infty^L}} \exp\left(-\frac{m}{4}|x-y|\right)\right]$$

for all $t \in (0,T]$ and $x, y \in \mathbb{R}$; here $\kappa > 0$ denotes the constant defined in Theorem 3.6.

The following corollary is a simple consequence of Theorem 3.8.

3.9 Corollary *Under the assumptions of Theorem 3.8 the following statements hold true.*

(i) The semigroup $(P_t)_{t\ge 0}$ of the Feller process $(X_t)_{t\ge 0}$ satisfies the gradient estimate

$$\sup_{x\in\mathbb{R}}\left|\frac{\partial}{\partial x}P_t f(x)\right| \le Ct^{-1/\gamma_\infty^L}\|f\|_\infty \qquad \text{for all } t \in (0,T], f \in \mathcal{B}_b(\mathbb{R})$$

for some absolute constant $C = C(T) > 0$.

(ii) Suppose additionally that each $\psi_\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha \in I$, is even and denote by $\kappa > 0$ the constant introduced in Theorem 3.6. Then for any $T > 0$ there exist constants $C_1, C_2, C_3 > 0$ such that

$$p(t,x,y) \ge C_1 t^{-1/\gamma_\infty(\alpha(y))}\left(1 - C_2 t^{-1/\gamma_\infty(\alpha(y))}|x-y| - C_3 t^\kappa\right)^+ \tag{3.5}$$

for all $x, y \in \mathbb{R}$, $t \in (0,T]$.

(iii) If $\psi_\alpha : \mathbb{R} \to \mathbb{R}$ is an even function for all $\alpha \in I$, then $(X_t)_{t\ge 0}$ is λ-irreducible, i. e.

$$\int_{(0,\infty)} \mathbb{P}^x(X_t \in B)\,dt > 0$$

for all $x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$ with $\lambda(B) > 0$.

For further material on irreducibility and ergodicity of Lévy-type processes see Sandric [90] and the references therein.

3.2 Extensions

In this section we discuss extensions of the results presented in Section 3.1. Extension 3.10 and Extension 3.11 show that we can slightly relax the growth conditions (C3) and (C4). Both just require small modifications in the proof.

3.10 Extension (Weakening (C3)) *The results from Section 3.1 remain valid if we replace* (C3) *by*

(C3') There exist measurable mappings $\gamma_0 : I \to (0,2]$ and $\gamma_\infty : I \to (0,2]$ and an increasing slowly varying (at ∞) function $\ell : (0,\infty) \to (0,\infty)$ such that

$$\operatorname{Re}\Psi_\alpha(\zeta) \geq c_1 \frac{1}{\ell(|\zeta|)} |\operatorname{Re}\zeta|^{\gamma_\infty(\alpha)} \qquad \text{for all} \quad \zeta \in \Omega, |\zeta| \geq 1$$

and

$$|\Psi_\alpha(\zeta)| \leq c_2 \ell(|\zeta|)\big(|\zeta|^{\gamma_0(\alpha)} \mathbb{1}_{\{|\zeta|\leq 1\}} + |\zeta|^{\gamma_\infty(\alpha)} \mathbb{1}_{\{|\zeta|>1\}}\big) \qquad \text{for all} \quad \zeta \in \Omega.$$

Moreover, $\gamma_\infty^L := \inf_{\alpha\in I} \gamma_\infty(\alpha) > 0$, $\gamma_0^L := \inf_{\alpha \in I} \gamma_0(\alpha) > 0$ and $\alpha \mapsto \gamma_\infty(\alpha)$ is Hölder continuous with Hölder exponent $\varrho(\gamma_\infty) \in (0,1]$.

Modifications in the proof. Up to some small changes in the proofs of the heat kernel estimates (Section 4.1), the proof works exactly as before. □

3.11 Extension (Weakening (C4)) *Let* $(\psi_\alpha)_{\alpha\in I}$ *be a family of continuous negative definite functions satisfying* (C1),(C2), (C3)[4] *and*

(C4') The partial derivative $\frac{\partial}{\partial \alpha_j}\Psi_\alpha(r)$ exists and extends holomorphically to Ω for all $j \in \{1,\dots,n\}$ and $\alpha \in I$. There exists an increasing function $\ell : (0,\infty) \to (0,\infty)$ which is regularly varying (at ∞) of order $\delta \in [0,1)$ such that

$$\left|\frac{\partial}{\partial \alpha_j}\Psi_\alpha(\zeta)\right| \leq c_3 |\zeta|^{\gamma_\infty(\alpha)}(1+\ell(|\zeta|)), \qquad \zeta \in \Omega, |\zeta| \geq 1,$$

$$\left|\frac{\partial}{\partial \alpha_j}\Psi_\alpha(\zeta)\right| \leq c_3 |\zeta|^{\gamma_0(\alpha)}(1+\ell(|\zeta|)), \qquad \zeta \in \Omega, |\zeta| \leq 1.$$

If $\alpha : \mathbb{R}^d \to I$ *is a Hölder continuous function with Hölder exponent* $\varrho(\alpha) > \delta\gamma_\infty^U/\gamma_\infty^L$, *then the statements from Section 3.1 remain valid.*

Modifications in the proof. As ℓ is regularly varying of order $\delta \in [0,1)$, there exists a slowly varying function f such that

$$\ell(x) = x^\delta f(x), \qquad x > 0,$$

cf. [11, Theorem 1.4.1]. Using this representation, it is not difficult to see that Theorem 4.7 remains valid in this more general setting. This, in turn, implies that the on- and off-diagonal

[4] or (C3')

estimates listed in Corollary 4.11 still hold true. The other necessary modifications concern the proof of Lemma 4.21 and Lemma 4.26. In the proof of Lemma 4.21 we distinguish four different cases; for the first two cases we have to estimate

$$|x-y|^{\varrho(\alpha)}\ell(ct^{-1/\gamma_\infty(\alpha(y))})$$

from above. Under (C4') we get

$$c'|x-y|^{\varrho(\alpha)}t^{-\delta/\gamma_\infty^L}(1+f(ct^{-1/\gamma_\infty^L}))$$

as an upper bound. Using that $\varrho(\alpha) > \delta\gamma_\infty^U/\gamma_\infty^L$, we obtain (following the reasoning in Lemma 4.21) that there exist constants $\kappa_1, \kappa_2 > 0$ and $C > 0$ such that

$$|F(t,x,y)| \leq Ct^{-1+\kappa_1}(1+f(ct^{-1/\gamma_\infty^L}))S(x-y,\alpha(y),t) + Ct^{-1+\kappa_2}g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y),$$

for all $t \in (0,T]$ and $x, y \in \mathbb{R}^d$, i.e. a similar estimate as in Lemma 4.21. In an analogous way we can modify the proof of Lemma 4.26. □

The third extension which we propose is more like a conjecture; it is still work in progress. Let us briefly explain the motivation. If we consider a Lévy-driven SDE, that is an SDE of the form

$$dX_t = b(X_{t-})\,dt + \sigma(X_{t-})\,dL_t, \qquad X_0 = x,$$

for a Lévy process $(L_t)_{t\geq 0}$, then we know from Example 1.23 that the associated (prospective) symbol is of the form $q(x,\xi) = ib(x)\cdot\xi + \psi(\sigma(x)^T\xi)$; here ψ denotes the characteristic exponent of $(L_t)_{t\geq 0}$. Since $q(x,\cdot)$ is not rotationally invariant and q does, in general, not satisfy the sector condition, it violates condition (C1) and condition (C3) (cf. Remark 3.1). Consequently, we cannot apply the results from Section 3.1 to obtain uniqueness and existence results for such SDEs; the only exception is the one-dimensional case which we will discuss in Section 5.4 (see also Remark 5.21). Therefore, it is of interest to prove the following statement which allows us to relax the assumption on the symmetry and also on the sector condition.

3.12 Conjecture *Let $I \subseteq \mathbb{R}^n$, $I \neq \varnothing$, be open and convex and $(\psi_\alpha)_{\alpha\in I} \in A(m)$ a family of admissible continuous negative definite functions $\psi_\alpha : \mathbb{R}^d \to \mathbb{C}$ such that $\psi_\alpha(0) = 0$ for all $\alpha \in I$. Moreover, let $\alpha : \mathbb{R}^d \to I$ and $\beta : \mathbb{R}^d \to \mathbb{R}^d$ be Hölder continuous functions with Hölder exponent $\varrho(\alpha) > 0$ and $\varrho(\beta) > 0$, respectively, such that $\|\beta\|_\infty < \infty$ and*

$$(\varrho(\beta)+1)\cdot\gamma_\infty^L > 1. \tag{3.6}$$

Then there exists a strong Feller process $(X_t)_{t\geq 0}$ with symbol

$$q(x,\xi) := i\beta(x)\cdot\xi + \psi_{\alpha(x)}(\xi), \qquad x,\xi \in \mathbb{R}^d.$$

The process $(X_t)_{t\geq 0}$ has the properties listed in Theorem 3.2, Proposition 3.3, Corollary 3.4, Corollary 3.5 and Theorem 3.6.

We are optimistic that the proof goes through, but we still have to check the details. The idea is to use a similar reasoning as in [57, Case B].

Remarks (i) If the symbol q satisfies the sector condition (or, equivalently, $\gamma_\infty^L \geq 1$), then (3.6) holds automatically. On the other hand, (3.6) implies $\gamma_\infty^L > 1/2$. Condition (3.6) is sometimes called *balance condition*.

(ii) Conjecture 3.12 is a generalization of the results presented in Section 3.1. *Indeed:* If we choose $\beta := 0$, then β is Hölder continuous with Hölder exponent $\varrho(\beta)$ for any $\varrho(\beta) \in (0,\infty)$, and therefore the balance condition (3.6) is equivalent to $\gamma_\infty^L > 0$. Consequently, we recover the results from the previous section.

3.3 Open problems

In order to apply Theorem 3.2, we have to verify that a given continuous negative definite function $\psi : \mathbb{R}^d \to \mathbb{C}$, $\psi(0) = 0$, has a holomorphic extension to the domain

$$\Omega(m,\vartheta) = \underbrace{\{\zeta \in \mathbb{C}; |\operatorname{Im}\zeta| < m\}}_{=:S(m)} \cup \underbrace{\{\zeta \in \mathbb{C}\backslash\{0\}; |\arg\zeta \mod \pi| < \vartheta\}}_{=:C(\vartheta)}$$

(see Figure 3.1) for some $m \geq 0$ and $\vartheta \in (0,\pi/2)$. Therefore, we are interest in sufficient conditions for the existence of such extensions. It is well-known that ψ has a holomorphic extension to the strip $S(m)$ if, and only if, the associated Lévy measure ν satisfies

$$\int_{|y|\geq 1} e^{\tilde{m}y}\,\nu(dy) < \infty \qquad \text{for all } \tilde{m} \in (-m,m), \tag{3.7}$$

but to our best knowledge there aren't any results in the literature which provide conditions for the existence of a holomorphic extension to cones (except the trivial, sufficient, condition that (3.7) holds for all $m > 0$). More generally, one might ask under which assumptions a characteristic function $\chi(\xi) = \mathbb{E}^{i\xi\cdot X}$, $\xi \in \mathbb{R}^d$, has a holomorphic extension to $\Omega(m,\vartheta)$. The existence of extensions to strips $S(m)$, $m > 0$, is well understood (see e. g. [108, Theorem 1.7.4]), but there are only few results for domains of different shapes. A result by Cuppens & Lukasz [28] shows that for any $m > 0$ there exists a characteristic function which is holomorphic on $S(m)$, but cannot be holomorphically extended from $S(m)$ to $\Omega(m,\vartheta)$ for any $\vartheta > 0$. This means that, without any additional assumptions, we cannot expect to extend a characteristic function from $S(m)$ to $\Omega(m,\vartheta)$ in a holomorphic way.

In the next chapter we will present several applications of the parametrix construction, and there we will see that checking the assumptions (C1)-(C4) of Theorem 3.2 is usually not too difficult if the symbol can be written in a closed form. Verifying (C1)-(C4) is more complicated if we are given ψ_α in its Lévy–Khintchine representation, i. e. if we are given, a priori, a family $(b_\alpha, Q_\alpha, \nu_\alpha)_{\alpha\in I}$ of Lévy triplets. For example, suppose that $(\nu_\alpha(dy))_{\alpha\in I}$ is a uniformly bounded family of infinite Lévy measures on $\mathbb{R}$, i. e. that there exists $R > 0$ such that $\operatorname{spt}\nu_\alpha(\cdot) \subseteq B(0,R)$ and $\nu_\alpha(B(0,R)) = \infty$ for all $\alpha \in I$. Then

$$\psi_\alpha(\xi) = \int_{\mathbb{R}} (1 - e^{iy\cdot\xi} + iy\cdot\xi\mathbb{1}_{\{|y|<1\}})\,\nu_\alpha(dy), \qquad \xi \in \mathbb{R},$$

extends holomorphically to $\mathbb{C}$, but we did not succeed in finding reasonable conditions on $(\nu_\alpha)_{\alpha\in I}$ which ensure that the growth condition (C3) is satisfied. Let us mention that Sztonyk [106, Section 5] has derived heat kernel estimates for (symmetric) Lévy processes with bounded jumps by studying the growth of $\psi_\alpha(\xi)$ for *real* $\xi \in \mathbb{R}$.

There are two more open questions:

- One of the main restrictions of Theorem 3.2 is that it only applies to symbols which are rotationally invariant with respect to ξ (at least in dimension $d > 1$). In the previous section, we have already made a conjecture which would allow us to relax this assumption. Another way to weaken the assumption of rotational invariance is to consider "independent sums" of continuous negative definite functions, i. e. families $(\psi_\alpha)_{\alpha\in I}$ of continuous negative definite functions which can be written in the form

$$\psi_\alpha(\xi) = \psi_{\alpha,1}(\xi_1) + \psi_{\alpha,2}(\xi_2) + \ldots + \psi_{\alpha,d}(\xi_d), \qquad \xi \in \mathbb{R}^d \tag{3.8}$$

where each $\psi_{\alpha,j} : \mathbb{R} \to \mathbb{C}$ satisfies (C2)-(C4). Applying Fubini's theorem and using the heat kernel estimates from Section 4.1, we easily obtain heat kernel estimates for the density

$$p_t^\alpha(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t\psi_\alpha(\xi)} \, d\xi, \qquad x \in \mathbb{R}^d, \alpha \in I.$$

This means that we are in a good shape to use a parametrix construction, but it is an open question whether the proof goes through. Note that (3.8) includes, in particular, symbols of the form

$$\psi_\alpha(\xi) := |\xi_1|^{\alpha_1} + \ldots + |\xi_d|^{\alpha_d}, \qquad \alpha = (\alpha_1, \ldots, \alpha_d), \xi \in \mathbb{R}^d.$$

- Because of the growth assumption (C3), it is obvious that any symbol q which satisfies the assumptions of Theorem 3.2 has bounded coefficients. It would be interesting to see whether the parametrix construction can be modified in such a way that it allows for symbols with unbounded coefficients. Since Feller processes with unbounded coefficients are, in general, not conservative, we cannot expect such processes to have all the additional properties which we obtain in the case of bounded coefficients.

4

Parametrix construction: proofs

We have already discussed the general idea of the parametrix construction in Section 1.7. Since the proof is very technical and involved, we split it up in several steps. We choose our notation in such a way that it is consistent with the notation introduced in Section 1.7.

Step 1: Estimates for rotationally invariant Lévy processes (Section 4.1). By assumption ψ_α is a continuous negative definite function with $\psi_\alpha(0) = 0$. Therefore,

$$p_t^\alpha(x) := \frac{1}{(2\pi)^d} \int e^{-ix\cdot\xi} e^{-t\psi_\alpha(\xi)}\, d\xi, \qquad x \in \mathbb{R}^d, t > 0, \tag{4.1}$$

is the density of a Lévy process with characteristic exponent ψ_α for each fixed $\alpha \in I$. In Section 4.1 we derive estimates for p_t^α, the function

$$A^\beta p_t^\alpha(x) := -\frac{1}{(2\pi)^d} \int \psi_\beta(\xi) e^{-ix\cdot\xi} e^{-t\psi_\alpha(\xi)}\, d\xi, \qquad x \in \mathbb{R}^d, \alpha, \beta \in I, \tag{4.2}$$

and certain (partial) derivatives of this function. We refer the reader to Corollary 4.11 for a summary of the estimates. In particular, we show that the function S, defined in (3.4), is an upper bound for p_t^α, i. e. for any $T > 0$ there exists a constant $C = C(T)$ such that

$$p_t^\alpha(x) \leq CS(x, \alpha, t) \qquad \text{for all} \;\; t \in (0, T], x \in \mathbb{R}^d.$$

These estimates will be crucial from Step 3 on.

Step 2: Auxiliary convolution estimates (Section 4.2). Since the parametrix construction involves time-space convolutions, we are interested in the behaviour of the function S under (time-space) convolution. Of particular interest is Lemma 4.17 which states, roughly, that S has, up to an additional "small" term, the subconvolution property. More precisely, we will establish that for any $T > 0$ there exists a constant $C = C(T)$ such that

$$\int_{\mathbb{R}^d} S(x-z, \alpha(z), t-s) S(z-y, \alpha(y), s)\, dz \leq CS(x-y, \alpha(y), t) + Ct \frac{1}{1 + |x-y|^{d+\gamma_0^L \wedge \gamma_\infty^L}}$$

for all $x, y \in \mathbb{R}^d$, $t \in (0, T]$.

Step 3: Construction of (the candidate for) the transition density and its derivative with respect to t (Sections 4.3, 4.4). Define the zero order approximation p_0 by

$$p_0(t, x, y) := p^{\alpha(y)}(t, x, y) := p^{\alpha(y)}(t, x - y) \tag{4.3}$$

and set

$$F(t,x,y) := -(\partial_t - A_x)p_0(t,x,y) = (A_x - A_x^{\alpha(y)})p_0(t,x,y)$$
$$= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(\psi_{\alpha(y)}(\xi) - \psi_{\alpha(x)}(\xi))e^{-i\xi\cdot(x-y)}e^{-t\psi_{\alpha(y)}(\xi)}\,d\xi. \tag{4.4}$$

In Section 4.3 we show that the candidate for the fundamental solution,

$$p(t,x,y) := p_0(t,x,y) + \sum_{i\geq 1}(p_0 \circledast F^{\circledast i})(t,x,y), \tag{4.5}$$

is well-defined and that p is continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. The differentiability of $p(t,x,y)$ with respect to t is established in Section 4.4.

Step 4: Strong continuity and Feller property of the (prospective) semigroup (Section 4.5). We associate with p a family of operators by

$$P_t f(x) := \int p(t,x,y)f(y)\,dy, \qquad f \in \mathcal{B}_b(\mathbb{R}^d), \tag{4.6}$$

and prove that $(P_t)_{t\geq 0}$ is strongly continuous on $C_\infty(\mathbb{R}^d)$. Moreover, we check that $(P_t)_{t\geq 0}$ has both the Feller and the strong Feller property.

Step 5: Properties of the approximate fundamental solution (Section 4.6). Following [57] we define an approximate fundamental solution p_ε by

$$p_\varepsilon(t,x,y) := p_0(t+\varepsilon,x,y) + \sum_{i\geq 1}(p_0(\cdot+\varepsilon,\cdot,\cdot)\circledast F^{\circledast i})(t,x,y).$$

We investigate the properties of the associated family of operators

$$P_{t,\varepsilon}f(x) := \int p_\varepsilon(t,x,y)f(y)\,dy, \qquad t>0, x\in\mathbb{R}^d, f\in\mathcal{B}_b(\mathbb{R}^d),$$

and prove that the name "approximate fundamental solution" is justified, i. e. that $p_\varepsilon \to p$ and $(\partial_t - A_x)p_\varepsilon \to 0$ as $\varepsilon \to 0$. Furthermore, we show that

$$P_{\cdot,\varepsilon}f \in C^1((0,\infty)) \qquad P_{t,\varepsilon})f(\cdot) \in C^2_\infty \qquad \text{for all } \; f\in C_\infty(\mathbb{R}^d),$$

and obtain convergence results for

$$Q_{t,\varepsilon}f(x) := (\partial_t - A_x)P_{t,\varepsilon}f(x), \qquad f \in C_\infty(\mathbb{R}^d),$$

as $\varepsilon \to 0$.

Step 6: $(P_t)_{t\geq 0}$ is a Markov semigroup with generator $(L,\mathcal{D}(L))$ (Section 4.7). We establish that $(P_t)_{t\geq 0}$ is a conservative Markov semigroup using the positive maximum principle and the results from Step 5. Moreover, we show that $C_c^\infty(\mathbb{R}^d)$ is contained in the domain $\mathcal{D}(L)$ of the generator L of the semigroup and that

$$Lf(x) = -\int \psi_{\alpha(x)}(\xi)e^{-ix\cdot\xi}\hat{f}(\xi)\,d\xi, \qquad \text{for all } \; f\in C_c^\infty(\mathbb{R}^d), x\in\mathbb{R}^d.$$

We give the proof of the main results in Section 4.8. Throughout Sections 4.1-4.8 we assume that the assumptions of Theorem 3.2 are satisfied, i. e. that $\alpha: I \to \mathbb{R}^d$ is Hölder continuous with Hölder coefficient $\varrho(\alpha)$ and that $(\psi_\alpha)_{\alpha\in I}\in A(m)$ for some $m\geq 0$.

For the readers' convenience we mark the passage of a proof by a "watch out"-symbol if the rotational invariance of ψ_α (i. e. (C1)) is needed. We will see that the rotational invariance is only used in Section 4.1. Therefore, in order to prove Theorem 3.7, it suffices to show that the results of Section 4.1 remain valid in dimension $d = 1$ even if ψ_α is not necessarily rotationally invariant. Finally, in Section 4.9, we prove the differentiability of $x \mapsto p(t,x,y)$ in dimension $d = 1$.

4.1 Heat kernel estimates for rotationally invariant Lévy processes

Let $(\psi_\alpha)_{\alpha\in I} \in A(m)$ be a family of admissible continuous negative definite functions. In this section we derive heat kernel estimates for the transition probability

$$p_t^\alpha(x,y) := p_t^\alpha(x-y) = \frac{1}{(2\pi)^d}\int e^{-i(x-y)\cdot\xi} e^{-t\psi_\alpha(\xi)}\,d\xi, \qquad x,y\in\mathbb{R}^d, t>0$$

of a Lévy process with characteristic exponent ψ_α. Furthermore, we obtain estimates for the function

$$A^\beta p_t^\alpha(x) = -\frac{1}{(2\pi)^d}\int \psi_\beta(\xi) e^{-ix\cdot\xi} e^{-t\psi_\alpha(\xi)}\,d\xi, \qquad x\in\mathbb{R}^d, t>0,$$

and its (partial) derivatives with respect to β and t. As $A^\alpha p_t^\alpha = \partial_t p_t^\alpha$ for each fixed $\alpha\in I$, this gives in particular an upper bound for $\partial_t p_t^\alpha$. We will need these estimates later in the proof e. g. to show that certain series are convergent. For a summary of the estimates which we will prove in this section we refer the reader to Corollary 4.11. We think that the results are of independent interest. Our approach relies on Cauchy's theorem and a classical result from Fourier analysis.

Unless otherwise mentioned, we assume throughout this section that $(\psi_\alpha)_{\alpha\in I}$ is a family of continuous negative definite functions such that $(\psi_\alpha)_{\alpha\in I}\in A(m)$ for some $m\geq 0$. We pick up the notation from (C1)-(C4). Recall that $\Omega = \Omega(m)\subseteq\mathbb{C}$ was defined in (C1) and that the slowly varying function ℓ was introduced in (C4). We will frequently use the fact that

$$\min\left\{t^{-d/\kappa}, \frac{t}{|x|^{d+\kappa}}\right\} = \begin{cases} t^{-d/\kappa}, & |x|\leq t^{1/\kappa}, \\ \frac{t}{|x|^{d+\kappa}}, & |x|>t^{1/\kappa} \end{cases} \tag{4.7}$$

which is a direct consequence of the monotonicity of the mapping $r\mapsto \frac{t}{r^{d+\kappa}}$ for fixed $t>0$.

Theorem 4.1 provides heat kernel estimates for a class of rotationally invariant Lévy processes; it is one of the main results in this section. Let us mention that Zabczyk [113, p. 245] has shown that any rotationally invariant Lévy process $(L_t)_{t\geq 0}$ is either a compound Poisson process *or* the distribution of each L_t, $t>0$, is absolutely continuous with respect to Lebesgue measure.

4.1 Theorem *Let $\psi_\alpha : \mathbb{R}^d \to \mathbb{C}$, $\alpha \in I \subseteq \mathbb{R}^n$, be a family of continuous negative definite functions such that $(\psi_\alpha)_{\alpha\in I} \in A(m)$ for some $m \geq 0$. Then there exists some constant $C = C(T) > 0$ such that*

$$p_t^\alpha(x) \leq C \min\left\{t^{-d/\gamma_\infty(\alpha)}, \frac{t}{|x|^{d-2}} \frac{1}{1+|x|^{2+\gamma_0(\alpha)}} + \frac{t}{|x|^{d+\gamma_\infty(\alpha)}}\right\} \exp\left(-\frac{m}{4}|x|\right) \tag{4.8}$$

for all $x \in \mathbb{R}^d$, $t \in (0,T]$, $\alpha \in I$.

We do not need condition (C4) for the proof of the theorem; so, in fact, the theorem holds for any family of continuous negative definite functions satisfying (C1)-(C3). If we define

$$S(x,\alpha,t) := S_m(x,\alpha,t) := \exp\left(-\frac{m}{4}|x|\right) \begin{cases} t^{-d/\gamma_\infty(\alpha)}, & |x| \leq t^{1/\gamma_\infty(\alpha)} \wedge 1, \\ \frac{t}{|x|^{d+\gamma_\infty(\alpha)}}, & t^{1/\gamma_\infty(\alpha)} < |x| \leq 1, \\ \frac{t}{|x|^{d+\gamma_\infty(\alpha)\wedge\gamma_0(\alpha)}}, & |x| > 1, \end{cases} \tag{4.9}$$

then Theorem 3.1 reads

$$p_t^\alpha(x) \leq CS(x,\alpha,t). \tag{4.10}$$

Note that

$$|S(x,\alpha,t)| \leq \max\{t^{-d/\gamma_\infty(\alpha)}, t\}. \tag{4.11}$$

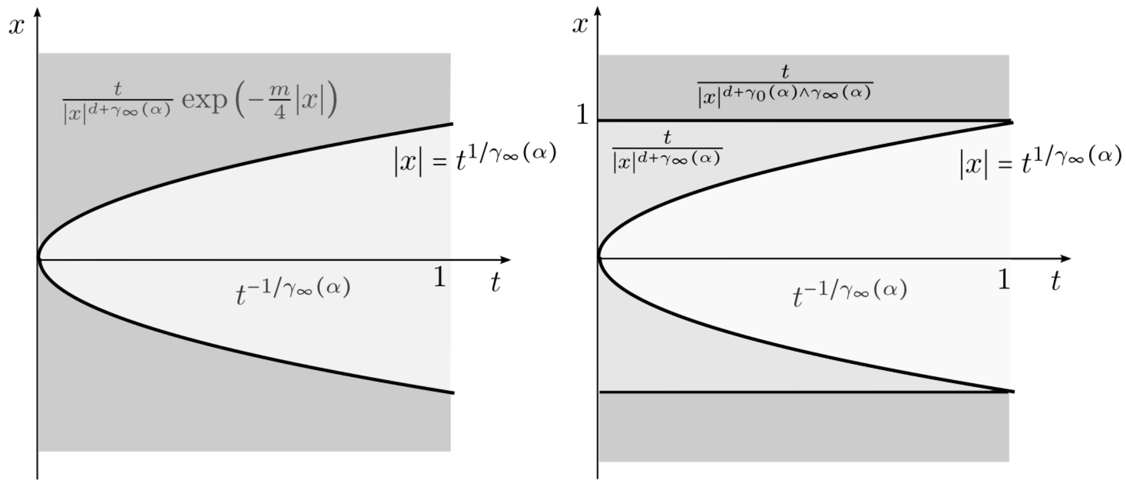

Figure 4.1: The upper bounds for the density p_t^α in the exponential setting (left) and the non-exponential setting (right) in dimension $d = 1$.

Before we start to prove the theorem we give some remarks on the exponential case, i. e. if $(\psi_\alpha)_{\alpha\in I} \in A(m)$ for some $m > 0$.

4.2 Remark (Exponential case) In the exponential case the estimate (4.8) boils down to

$$p_t^\alpha(x) \leq C \min\left\{t^{-d/\gamma_\infty}(\alpha), \frac{t}{|x|^{d+\gamma_\infty(\alpha)}}\right\} \exp\left(-\frac{m}{4}|x|\right).$$

The proof of Theorem 4.1 actually shows that for any $\varepsilon \in (0,1)$ and $T > 0$ there exists a constant $C = C(T,\varepsilon) > 0$ such that

$$p_t^\alpha(x) \le C \min\left\{t^{-d/\gamma_\infty(\alpha)}, \frac{t}{|x|^{d+\gamma_\infty(\alpha)}}\right\} \exp\left(-m(1-\varepsilon)|x|\right) \tag{4.12}$$

for all $x \in \mathbb{R}^d$ and $t \in (0,T]$. The reason why we only prove (4.8) is that we want to keep notation as simple as possible.
Now denote by $(X_t)_{t\ge 0}$ a Lévy process with characteristic exponent ψ_α for fixed $\alpha \in I$. By assumption, the characteristic function of X_t

$$\mathbb{E}e^{i\xi\cdot X_t^\alpha} = e^{-t\psi_\alpha(\xi)}, \qquad \xi \in \mathbb{R}^d,$$

is analytic on the strip $\{z \in \mathbb{C}; |\operatorname{Im} z| < m\}$. It is well-known (see e. g. [108, Theorem 1.7.1]) that this is equivalent to $\mathbb{E}e^{m(1-\varepsilon)|X_t|} < \infty$ for any $\varepsilon \in (0,1)$. This means that (4.12) is optimal in the sense that we can, in general, not expect a faster exponential decay.

For the proof of Theorem 4.1 we need several ingredients. The first one concerns the Fourier transform of a rotationally symmetric function; see e. g. [111, Theorem 5.26] or [96, Beispiel 20.16] for a proof.

4.3 Lemma *Let $u \in L^1(\mathbb{R}^d, d\xi)$ be of the form $u(\xi) = f(|\xi|)$ for a function $f : [0,\infty) \to \mathbb{R}$. Then*

$$\int_{\mathbb{R}^d} u(\xi) e^{-ix\cdot\xi}\, d\xi = \frac{(2\pi)^{d/2}}{|x|^{d/2-1}} \int_{(0,\infty)} f(r) r^{d/2} J_{d/2-1}(r|x|)\, dr;$$

here J_n denotes the Bessel function of the first kind.

For a comprehensive study of Bessel functions we refer the reader to Watson [110] and Whittaker & Watson [112]. In the proof of Theorem 4.1 we will use that the Bessel function of the first kind J_n and the *Whittaker function* $W_{0,n}$ satisfy the relation

$$J_n(r) = 2\operatorname{Re}\left(\frac{1}{\sqrt{2\pi r}} \exp\left(\frac{i\pi}{2}\frac{n+1}{2}\right) W_{0,n}(2ir)\right), \qquad r > 0, n \in (\tfrac{1}{2}\mathbb{Z})\backslash(-2\mathbb{N}), \tag{4.13}$$

cf. [112, Section 17.212]. Moreover, it is known that

$$|W_{0,n}(z)| \le C|z|^{\frac{1}{2}-n} \mathbb{1}_{\{|z|\le 1\}} + C|e^{-z/2}| \mathbb{1}_{\{|z|>1\}}, \tag{4.14}$$

for some absolute constant $C = C(n)$, cf. [112, Section 16.3] or [78, 13.14 (iii),(13.14.21)]. We also need the following auxiliary result.

4.4 Lemma *Let $f : \mathbb{R} \to \mathbb{R}$ be an even function admitting a holomorphic extension to $\Omega(m,\vartheta)$ (defined in* (C2)*) for some $m > 0$ and $\vartheta \in (0,\pi/2)$. Then*

$$\operatorname{Im} f(ix) = 0 \qquad \textit{for all} \;\; x \in (-m,m).$$

Proof. Since $f:\mathbb{R}\to\mathbb{R}$ is even and real-valued, we have $f(y)=f(-y)=\overline{f(y)}$ for all $y\in\mathbb{R}$. Therefore, both

$$f_1(z):=\overline{f(\overline{z})} \qquad \text{and} \qquad f_2(z):=f(-z), \qquad z\in\Omega,$$

coincide with f on the real line. Moreover, f_1 and f_2 are holomorphic on Ω; for f_1 this follows e. g. from the Cauchy–Riemann equations. By the identity theorem,

$$f(z)=\overline{f(\overline{z})}=f(-z) \qquad \text{for all } \ z\in\Omega;$$

hence,

$$f(\bar{z})=\overline{f(-z)} \qquad \text{for all } \ z\in\Omega. \tag{4.15}$$

Suppose that $\operatorname{Im} f(ix)>0$ for some $x\in(-m,m)$. Since f is continuous, there exists a neighbourhood $U\subseteq\Omega$ of ix such that $\operatorname{Im} f(z)>0$ for all $z\in U$. Pick $y>0$ such that $ix\pm y\in U$. If we set $z:=-ix+y=-(ix-y)$, then

$$\operatorname{Im} f(\overline{z})=\operatorname{Im} f(ix+y)>0 \qquad \text{and} \qquad \operatorname{Im} f(-z)=\operatorname{Im} f(ix-y)>0$$

in contradiction to (4.15). The same argument gives a contradiction if $\operatorname{Im} f(ix)<0$ for some $x\in(-m,m)$. This finishes the proof. □

Before we start to prove Theorem 4.1, we state another, very elementary, result which we will apply several times.

4.5 Lemma

$$|e^{tz}-1|\le t|z|\max\{1,e^{t\operatorname{Re} z}\} \qquad \textit{for all } \ t\ge 0, z\in\mathbb{C}.$$

Proof. If we set $\sigma(r):=rtz$, $r\in[0,1]$, then by the fundamental theorem of calculus

$$|e^{tz}-1|=\left|\int_\sigma e^r\,dr\right|=t|z|\left|\int_0^1 e^{trz}\,dr\right|\le t|z|\max\{1,e^{t\operatorname{Re} z}\}.$$ □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We start with the on-diagonal estimate. Because of the growth assumptions (3.3) and (C3), we have

$$\begin{aligned} p_t^\alpha(x) &\le (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-t\operatorname{Re}\psi_\alpha(\xi)}\,d\xi \\ &\le (2\pi)^{-d}\int_{|\xi|<1} e^{c_4T}\,d\xi+(2\pi)^{-d}\int_{|\xi|\ge 1} e^{-c_1 t|\xi|^{\gamma_\infty(\alpha)}}\,d\xi \end{aligned}$$

for all $x\in\mathbb{R}^d$. Hence,

$$\begin{aligned} p_t^\alpha(x) &\le (2\pi)^{-d}e^{c_4T}\lambda^d(B(0,1))+(2\pi)^{-d}t^{-d/\gamma_\infty(\alpha)}\int_{|\eta|\ge t^{1/\gamma_\infty(\alpha)}} e^{-c_1|\eta|^{\gamma_\infty(\alpha)}}\,d\eta \\ &\le (2\pi)^{-d}\left(e^{c_4T}\lambda^d(B(0,1))T^{d/\gamma_\infty^L}+\lambda^d(B(0,1))+\int_{|\eta|\ge 1} e^{-c_1|\eta|^{\gamma_\infty^L}}\,d\eta\right)t^{-d/\gamma_\infty(\alpha)} \end{aligned}$$

for all $t \le T$. (Recall that $\gamma_\infty^L := \inf_{\alpha \in I} \gamma_\infty(\alpha) > 0$.) It remains to prove the off-diagonal estimate. Fix $x \neq 0$. An application of Lemma 4.3 and the dominated convergence theorem yield

$$\begin{aligned} p_t^\alpha(x) &= (2\pi)^{-d} \lim_{R \to \infty} \int_{R^{-1} \le |\xi| \le R} e^{-ix\cdot\xi} e^{-t\psi_\alpha(\xi)} \, d\xi \\ &= (2\pi)^{-d} \lim_{R \to \infty} \int_{R^{-1} \le |\xi| \le R} e^{-ix\cdot\xi} e^{-t\Psi_\alpha(|\xi|)} \, d\xi \\ &\overset{\text{L4.3}}{=} \frac{1}{(2\pi)^{d/2} |x|^{(d-1)/2}} \lim_{R \to \infty} \int_{R^{-1}}^{R} e^{-t\Psi_\alpha(r)} r^{d/2} J_{d/2-1}(r|x|) \, dr. \end{aligned}$$

By (4.13) and a change of variables, we get

$$p_t^\alpha(x) = \frac{c_d}{|x|^{(d-1)/2}} \lim_{R\to\infty} \operatorname{Re}\left(\int_{R^{-1}}^{R} \underbrace{e^{-t\Psi_\alpha(r)} (ir)^{(d-1)/2} W_{0,d/2-1}(2ir|x|)}_{=:u(r,|x|)} \, dr \right) \tag{4.16}$$

$$= \frac{c_d}{|x|^d} \lim_{R\to\infty} \operatorname{Re}\left(\int_{R^{-1}|x|}^{R|x|} \underbrace{e^{-t\Psi_\alpha(r/|x|)} (ir)^{(d-1)/2} W_{0,d/2-1}(2ir)}_{=:v(r,|x|)} \, dr \right) \tag{4.17}$$

for some constant $c_d > 0$. In order to estimate the integrals on the right-hand side, we are going to apply Cauchy's theorem.

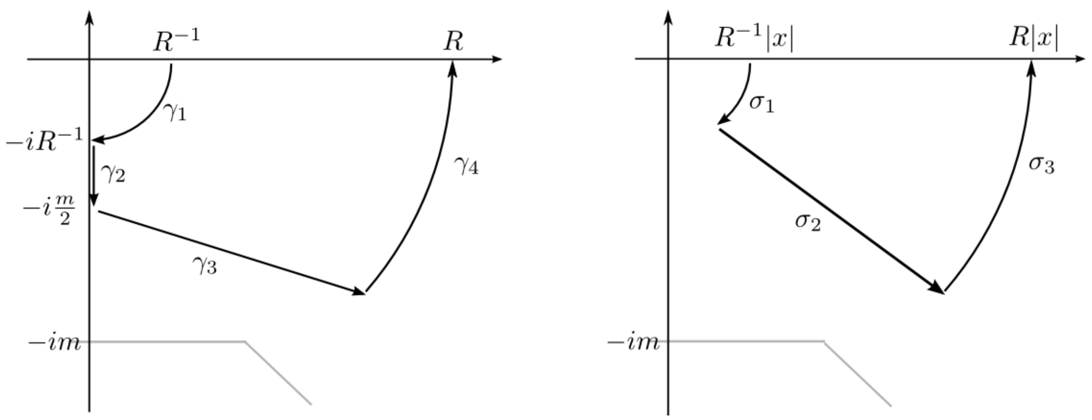

Figure 4.2: It depends on $x \in \mathbb{R}^d$ and $m \ge 0$ which contour of integration we have to use for the proof of Theorem 3.1. If $m > 0$ and $|x| > 1$ we choose the contour of integration on the left-hand side; otherwise we use the contour of integration on the right.

We consider two cases separately:

(i) $m = 0$: Set

$$\begin{aligned} \sigma_1(\theta) &:= R^{-1}|x| e^{-i\theta}, & \theta &\in [0, \Theta], \\ \sigma_2(r) &:= r e^{-i\Theta}, & r &\in (R^{-1}|x|, R|x|), \\ \sigma_3(\theta) &:= R|x| e^{i\theta}, & \theta &\in [2\pi - \Theta, 2\pi] \end{aligned} \tag{4.18}$$

where $\Theta \in (0, \frac{\pi}{2})$ is chosen such that $re^{-i\Theta} \in \Omega$ for all $r \geq 0$. By Cauchy's theorem,[1]

$$\operatorname{Re}\left(\int_{R^{-1}|x|}^{R|x|} v(r,|x|)\,dr\right) = \sum_{i=1}^{3} \operatorname{Re}\left(\int_{\sigma_i} v(r,|x|)\,dr\right) \tag{4.19}$$

with v defined in (4.17). By Lemma 4.6 below, we have

$$\lim_{R\to\infty} \left|\operatorname{Re}\left(\int_{\sigma_1} v(r,|x|)\,dr + \int_{\sigma_3} v(r,|x|)\,dr\right)\right| = 0.$$

In order to estimate the remaining integral we note that, by (4.14),

$$\begin{aligned} |\sigma_2(r)|^{(d-1)/2} |W_{0,d/2-1}(2i\sigma_2(r))| &\leq C_1 r^{(d-1)/2} (r^{(3-d)/2} \mathbb{1}_{\{r\leq 1\}} + e^{-r\sin\Theta} \mathbb{1}_{\{r>1\}}) \\ &\leq C_2 r e^{-\sin(\Theta)r/2}. \end{aligned} \tag{4.20}$$

Moreover, it follows from Lemma 4.5 and (3.3) that there exists a constant $c_4 > 0$ such that

$$|e^{-t\Psi_\alpha(\sigma_2(r)/|x|)} - 1| \leq t|\Psi_\alpha(\sigma_2(r)/|x|)| \max\{1, e^{t\operatorname{Re}\Psi_\alpha(\sigma_2(r)/|x|)}\} \leq e^{c_4 T} t |\Psi_\alpha(\sigma_2(r)/|x|)| \tag{4.21}$$

for all $t \leq T$, $r \geq 0$ and $x \in \mathbb{R}^d \backslash \{0\}$. If we set

$$\varepsilon(R) := \left|\operatorname{Re}\left(\int_{\sigma_2} (ir)^{(d-1)/2} W_{0,d/2-1}(2ir)\,dr\right)\right|,$$

then by Lemma 4.5 and (4.20),

$$\begin{aligned} &\left|\operatorname{Re}\left(\int_{\sigma_2} v(r,|x|)\,dr\right)\right| \\ &\overset{(4.17)}{\leq} \varepsilon(R) + \int_{R^{-1}|x|}^{R|x|} |e^{-t\Psi_\alpha(\sigma_2(r)/|x|)} - 1| \, |\sigma_2(r)|^{(d-1)/2} |W_{0,d/2-1}(2i\sigma_2(r))|\,dr \\ &\overset{(4.21)}{\leq} \varepsilon(R) + C_2 t e^{c_4 T} \int_0^R |\Psi_\alpha(\sigma_2(r)/|x|)|\, e^{-t\operatorname{Re}\Psi_\alpha(\sigma_2(r)/|x|)} r e^{-\sin(\Theta)r/2}\,dr. \end{aligned}$$

Because (C3), we have

$$\left|\Psi_\alpha\left(\frac{\sigma_2(r)}{|x|}\right)\right| \leq c_2 \left(\frac{r}{|x|}\right)^{\gamma_0(\alpha)} \mathbb{1}_{\{r\leq|x|\}} + c_2 \left(\frac{r}{|x|}\right)^{\gamma_\infty(\alpha)} \mathbb{1}_{\{r>|x|\}}.$$

Consequently,

$$\begin{aligned} \left|\operatorname{Re}\left(\int_{\sigma_2} v(r,|x|)\,dr\right)\right| &\leq \varepsilon(R) + C_2 c_2 e^{c_4 T} t \frac{1}{|x|^{\gamma_0(\alpha)}} \int_0^{|x|} r^{1+\gamma_0(\alpha)} e^{-\sin(\Theta)r/2}\,dr \\ &\quad + C_2 c_2 e^{c_4 T} t \frac{1}{|x|^{\gamma_\infty(\alpha)}} \int_{|x|}^{\infty} r^{1+\gamma_\infty(\alpha)} e^{-\sin(\Theta)r/2}\,dr \\ &\leq \varepsilon(R) + C_3 t \left(|x|^2 \frac{1}{1+|x|^{2+\gamma_0(\alpha)}} + \frac{1}{|x|^{\gamma_\infty(\alpha)}}\right) \end{aligned}$$

[1] Because $W_{0,d/2-1}$ has a singularity at $z = 0$, cf. (4.14), we cut out the origin. Strictly speaking, this is not necessary since the (smoothed) mapping $z \mapsto z^{(d-1)/2} W_{0,d/2-1}(z)$ is holomorphic.

where we have used that

$$\frac{1}{|x|^{\gamma_0(\alpha)}}\int_0^{|x|} r^{1+\gamma_0(\alpha)}e^{-\sin(\Theta)r/2}\,dr \le \frac{C}{|x|^{\gamma_0(\alpha)}}\min\{1,|x|^{2+\gamma_0(\alpha)}\} \asymp |x|^2\frac{1}{1+|x|^{2+\gamma_0(\alpha)}}.$$

Yet another application of Cauchy's theorem shows $\varepsilon(R)\xrightarrow{R\to\infty}0$, cf. Lemma A.4(ii). Combining (4.17) with (4.19) and the above estimates, the assertion follows.

(ii) $m>0$: As $(\psi_\alpha)_{\alpha\in I}\in A(m)\subseteq A(0)$, we know from the first part of this proof that

$$p_t^\alpha(x)\le C\min\left\{t^{-d/\gamma_\infty(\alpha)},\frac{t}{|x|^{d-2}}\frac{1}{1+|x|^{2+\gamma_0(\alpha)}}+\frac{t}{|x|^{d+\gamma_\infty(\alpha)}}\right\}.$$

Obviously, this implies (4.8) for all $|x|\le 1$. Since

$$\exp\left(-\frac{m}{2}|x|\right)\le C_1\left(\frac{1}{|x|^2}\frac{1}{1+|x|^{d-2+\gamma_0(\alpha)}}+\frac{1}{|x|^{d+\gamma_\infty(\alpha)}}\right)\exp\left(-\frac{m}{4}|x|\right)$$

it therefore suffices to prove that for any $T>0$ there exists a constant $C_2=C_2(T)$ such that

$$p_t^\alpha(x)\le C_2 t\exp\left(-\frac{m}{2}|x|\right)\qquad\text{for all } |x|>1, t\in(0,T], \alpha\in I. \tag{$\star$}$$

Fix $x\in\mathbb{R}^d$, $|x|>1$. By (C2) there exists $z_0\in\mathbb{C}$, $|z_0|=1$, such that $\operatorname{Re} z_0>0$, $\operatorname{Im} z_0>0$ and $-i\frac{m}{2}-rz_0\in\Omega$ for all $r\ge 0$. We define a contour of integration as follows, see Figure 4.2,

$$\begin{aligned}
\gamma_1(\theta)&:=R^{-1}e^{-i\theta}, & \theta&\in\left[0,\frac{\pi}{2}\right],\\
\gamma_2(r)&:=-ir, & r&\in\left[R^{-1},\frac{m}{2}\right],\\
\gamma_3(r)&:=-i\frac{m}{2}-rz_0, & r&\in[0,f(R)],\\
\gamma_4(\theta)&:=Re^{i\theta}, & \theta&\in[g(R),2\pi];
\end{aligned}\tag{4.22}$$

here $f(R)\ge 0$ and $g(R)\in(\frac{3}{2}\pi,2\pi)$ are chosen in such a way that $|\gamma_3(f(R))|=R$ and $\gamma_4(g(R))=\gamma_3(f(R))$. Clearly, $f(R)\asymp R$ for $R\gg 1$. Applying Cauchy's theorem yields

$$\operatorname{Re}\left(\int_{[R^{-1},R]}u(r,|x|)\,dr\right)=\operatorname{Re}\left(\sum_{i=1}^4\int_{\gamma_i}u(r,|x|)\,dr\right).\tag{4.23}$$

By Lemma 4.6 below, applied for the constant function $h:=1$, we have

$$\lim_{R\to\infty}\operatorname{Re}\left[\left(\int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_4}\right)u(r,|x|)\,dr\right]=0.\tag{4.24}$$

Therefore it just remains to estimate $\operatorname{Re}\left(\int_{\gamma_3}u(r,|x|)\,dr\right)$. Since $|x\gamma_3(r)|\ge\frac{m}{2}>0$ for all $r\ge 0$ and $|x|\ge 1$, there exists by (4.14) some absolute constant $C_1=C_1(m)$ such that

$$|W_{0,d/2-1}(2i|x|\gamma_3(r))|\le C_1e^{|x|\operatorname{Im}\gamma_3(r)}=C_1\exp\left(-\frac{m}{2}|x|-r|x|\operatorname{Im} z_0\right).\tag{$\star\star$}$$

If we set

$$\varepsilon(R) := \left|\mathrm{Re}\left(\int_{\gamma_3} (ir)^{(d-1)/2} W_{0,d/2-1}(2ir|x|)\,dr\right)\right|,$$

then it follows from Lemma 4.5, (3.3) and ($\star\star$) that

$$\begin{aligned}
&\left|\mathrm{Re}\left(\int_{\gamma_3} u(r,|x|)\,dr\right)\right| \\
&\leq \varepsilon(R) + \left|\mathrm{Re}\int_{\gamma_3} (e^{-t\Psi_\alpha(r)}-1)(ir)^{(d-1)/2} W_{0,d/2-1}(2ir|x|)\,dr\right| \\
&\leq \varepsilon(R) + te^{c_4 T}\int_0^{f(R)} |\Psi_\alpha(\gamma_3(r))|\,|\gamma_3(r)|^{(d-1)/2}|W_{0,d/2-1}(2i|x|\gamma_3(r))\,dr \\
&\leq \varepsilon(R) + C_1 t \exp\left(c_4 T - \frac{m}{2}|x|\right)\int_0^{f(R)} |\gamma_3(r)|^{\gamma_\infty(\alpha)+(d-1)/2} e^{-r|x|\,\mathrm{Im}\,z_0}\,dr.
\end{aligned}$$

From the definition of γ_3 it is obvious that $|\gamma_3(r)| \leq c(1+r)$ for some constant $c>0$. Hence,

$$\left|\mathrm{Re}\left(\int_{\gamma_3} u(r,|x|)\,dr\right)\right| \leq \varepsilon(R) + C_2 t \exp\left(-\frac{m}{2}|x|\right)\int_0^{f(R)} (1+r)^{\gamma_\infty(\alpha)+(d-1)/2} e^{-r|x|\,\mathrm{Im}\,z_0}\,dr.$$

As $|x| \geq 1$ and $\gamma_\infty(\alpha) \in (0,2]$, this implies

$$\left|\mathrm{Re}\left(\int_{\gamma_3} u(r,|x|)\,dr\right)\right| \leq \varepsilon(R) + C_2 t \exp\left(-\frac{m}{2}|x|\right)\int_0^{\infty} (1+r)^{2+(d-1)/2} e^{-r\,\mathrm{Im}\,z_0}\,dr.$$

Moreover, applying Cauchy's theorem we find $\varepsilon(R) \to 0$ as $R \to \infty$, cf. Lemma A.4(i). Combining (4.16) with (4.23) and (4.24), we get ($\star$). □

For the proof of Theorem 4.1 we have used the following lemma. Since we need similar convergence results in the remaining part of this section, we state it in a more general form than needed in the above proof.

4.6 Lemma *Let $(\psi_\alpha)_{\alpha\in I} \in A(m)$ be a family of continuous negative definite functions for some $m \geq 0$, and denote by Ω the domain defined in* (C2). *Let $h_j : \Omega \to \mathbb{C}$, $j \in J$, be a family of measurable functions which satisfies the following assumptions.*

- *(a) There exists $\varrho > 0$ such that $|h_j(\zeta)| \leq |\zeta|^\varrho$ for all $\zeta \in \Omega$, $|\zeta| \geq 1$, $j \in J$,*
- *(b) If $m>0$: $h_j|_{\mathbb{R}}$ is even, i. e. $h_j(r) = h_j(-r)$ for all $r \geq 0$,*
- *(c) $C := \sup_{j\in J}\sup_{\substack{\zeta\in\Omega \\ |\zeta|\leq 1}} |h_j(\zeta)| < \infty$.*

Then:

- *(i) If $m>0$, then*

$$\lim_{R\to\infty}\sup_{j\in J}\left|\mathrm{Re}\left[\left(\int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_4}\right) h_j(r)u(r,|x|)\,dr\right]\right| = 0 \tag{4.25}$$

for all $|x| \geq 1$.

(ii) If $m \geq 0$, then

$$\lim_{R\to\infty} \sup_{j\in J} \left| \mathrm{Re} \left[\left(\int_{\sigma_1} + \int_{\sigma_3} \right) h_j(r/|x|) v(r,|x|)\, dr \right] \right| = 0 \tag{4.26}$$

for all $x \in \mathbb{R}^d$.

Here, u and v denote the functions defined in (4.16) and (4.17), respectively, and $\gamma_i = \gamma_i(R)$ and $\sigma_i = \sigma_i(R)$ the curves from (4.22) and (4.18), respectively.

Proof. First we prove (4.25). Fix $m > 0$, $x \in \mathbb{R}^d$, $|x| > 1$ and $R \geq 1$. Recall that

$$u(r,|x|) = e^{-t\Psi_\alpha(r)} (ir)^{(d-1)/2} W_{0,d/2-1}(2ir|x|).$$

Since $|\gamma_1(\theta)| = R^{-1} \leq 1$, we have

$$\begin{aligned} I_1 := \left| \int_{\gamma_1} h_j(r) u(r,|x|)\, dr \right| &= R^{-1} \left| \int_0^{\pi/2} h_j(\gamma_1(\theta)) u(\gamma_1(\theta),|x|)\, d\theta \right| \\ &\leq CR^{-1} \int_0^{\pi/2} |u(\gamma_1(\theta),|x|)|\, d\theta \\ &\overset{(3.3)}{\leq} Ce^{c_4 T} R^{-1+(1-d)/2} \int_0^{2\pi} |W_{0,d/2-1}(2i\gamma_1(\theta)|x|)|\, d\theta. \end{aligned}$$

If $R \gg 1$ then (4.14) gives

$$|W_{0,d/2-1}(2i\gamma_1(\theta)|x|) \leq C_1 \big|\gamma_1(\theta)\,|x|\big|^{(3-d)/2} = C_1 R^{(d-3)/2} |x|^{(3-d)/2}$$

for some absolute constant C_1. Consequently, $I_1 \leq C_2 R^{-2} \xrightarrow{R\to\infty} 0$ uniformly in $j \in J$. Similarly, we find using $|h_j(\mu_4(\theta))| \leq |\mu_4(\theta)|^\varrho = R^\varrho$ that

$$\begin{aligned} I_4 := \left| \int_{\gamma_4} h_j(r) u(r,|x|)\, dr \right| &\leq R^{1+\varrho} \int_{g(R)}^{2\pi} |u(\mu_4(\theta),|x|)|\, d\theta \\ &= R^{1+\varrho+(d-1)/2} \int_{g(R)}^{2\pi} e^{-t\,\mathrm{Re}\,\Psi_\alpha(\gamma_4(\theta))} |W_{0,d/2-1}(2i|x|\gamma_4(\theta))|\, d\theta. \end{aligned}$$

Since (4.14) implies

$$|W_{0,d/2-1}(2iz)| \leq C_3 |e^{-iz}| = C_3 e^{\mathrm{Im}\, z} \qquad \text{for all } |z| \geq 1 \tag{4.27}$$

for some absolute constant C_3, we have for $R \gg 1$

$$\begin{aligned} I_4 &\overset{(4.27)}{\leq} C_3 R^{1+\varrho+(d-1)/2} \int_{\theta_0}^{2\pi} e^{-t\,\mathrm{Re}\,\Psi_\alpha(\gamma_4(\theta))} \underbrace{e^{|x|R\sin\theta}}_{\leq 1}\, d\theta \\ &\overset{(C3)}{\leq} C_3 R^{1+\varrho+(d-1)/2} \int_{\theta_0}^{2\pi} e^{-c_1 t|R\cos\theta|^{\gamma_\infty(\alpha)}}\, d\theta \\ &\leq C_3 R^{1+\varrho+(d-1)/2} 2\pi e^{-c_1 t|R\cos\theta_0|^{\gamma_\infty^L}} \end{aligned}$$

where $\theta_0 \in (\frac{3}{2}\pi, 2\pi)$ is chosen such that $[g(R), 2\pi] \subseteq [\theta_0, 2\pi]$ for $R \gg 1$. As $\cos\theta_0 > 0$, it follows that $I_4 \xrightarrow{R\to\infty} 0$. For the remaining integral we recall that, by Lemma 4.4, we have $\mathrm{Im}\,\Psi_\alpha(-ir) = 0$ and $\mathrm{Im}\,h_j(-ir) = 0$ for all $r \in (-m,m)$. Thus,

$$\mathrm{Re}\left(\int_{\gamma_2} h_j(r) u(r,|x|)\, dr \right) = \mathrm{Re}\left(-i \int_{R^{-1}}^{m/2} h_j(-ir) e^{-t\Psi_\alpha(-ir)} r^{(d-1)/2} W_{0,d/2-1}(-2r|x|)\, dr \right)$$

$$= \int_{R^{-1}}^{m/2} h_j(-ir) e^{-t \operatorname{Re}\Psi_\alpha(-r)} \sin(t \underbrace{\operatorname{Im}\Psi_\alpha(ir)}_{0}) W_{0,d/2-1}(-2r|x|)\, dr$$

$$= 0.$$

This proves (4.25). Since the proof of (4.26) is very similar, we just sketch it. In order to estimate $\left|\int_{\sigma_1} h_j(r/|x|) v(r,|x|)\, dr\right|$ we use the (uniform) boundedness of h_j on $\Omega \cap B(0,1)$ and

$$|W_{0,d/2-1}(2i\sigma_1(\theta))| \le C_4 |\sigma_1(\theta)|^{(3-d)/2} = C' R^{(d-3)/2} |x|^{(3-d)/2};$$

this is a direct consequence of (4.14) and the fact that $|\sigma_1(\theta)| = R^{-1}|x| \le 1$ for $R \gg 1$. The second integral converges to 0 because of the exponential decay of the Whittaker function, cf. (4.27), and the at most polynomial growth of h_j. □

To make the parametrix construction work, we need on- and off-diagonal estimates for

$$A^\beta p_t^\alpha(x) = -\frac{1}{(2\pi)^d} \int \psi_\beta(\xi) e^{-ix\cdot\xi} e^{-t\psi_\alpha(\xi)}\, d\xi$$

and certain derivatives of this function with respect to β and t. To this end, we first establish a more general statement which will allow us to deduce the required estimates easily.

4.7 Theorem *Let $m \ge 0$, $T > 0$, J an arbitrary index set and $(\psi_\alpha)_{\alpha\in I} \in A(m)$, $I \subseteq \mathbb{R}^n$, be a family of continuous negative definite functions $\psi_\alpha : \mathbb{R}^d \to \mathbb{C}$. Let $(h_j)_{j\in J}$ be a family of rotationally invariant measurable functions $h_j(\xi) = H_j(|\xi|)$, $\xi \in \mathbb{R}^d$, such that the following conditions hold.*

(E1) *H_j has a holomorphic extension to Ω. If $m > 0$: $H_j|_{\mathbb{R}}$ is even.*

(E2) *There exists a constant $C_1 > 0$ such that that for any $j \in J$ there exists $\varrho_0(j) > 0$ such that $|H_j(\zeta)| \le C_1 |\zeta|^{\varrho_0(j)}$ for all $\zeta \in \Omega$, $|\zeta| \le 1$.*

(E3) *There exist an increasing slowly varying function ℓ and $C_2 > 0$ such that for any $j \in J$ there exists $\varrho_\infty(j) > 0$ such that $|H_j(\zeta)| \le C_2(1 + \ell(|\zeta|))|\zeta|^{\varrho_\infty(j)}$ for all $\zeta \in \Omega$, $|\zeta| \ge 1$. Moreover, $\varrho^U := \sup_{j\in J} \varrho_\infty(j) < \infty$.*

If we set

$$G_j(x) := G_{j,\alpha}(t,x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} h_j(\xi) e^{-t\Psi_\alpha(\xi)} e^{-ix\cdot\xi}\, d\xi,$$

then there exists a constant $C = C(T) > 0$ and $c = c(T,\ell) > 0$ such that

$$|G_j(x)|$$
$$\le C \exp\left(-\frac{m}{4}|x|\right) \min\left\{ (1 + \ell(ct^{-1/\gamma_\infty(\alpha)})) t^{-(d+\varrho_\infty(j))/\gamma_\infty(\alpha)}, \frac{1}{|x|^{d-2}} \frac{1}{1 + |x|^{2+\varrho_0(j)}} + \frac{1 + \ell(c|x|^{-1})}{|x|^{d+\varrho_\infty(j)}} \right\}$$

for all $x \in \mathbb{R}^d$, $t \in (0,T]$, $j \in J$ and $\alpha \in I$.

For the proof of the on-diagonal estimate we use the following auxiliary result.

4.8 Lemma *Let $\psi_\alpha(\xi) = \Psi_\alpha(|\xi|)$ be a family of Borel measurable functions such that $\operatorname{Re}\Psi_\alpha(r) \geq c_1 r^{\gamma_\infty(\alpha)}$ for all $\alpha \in I$ and $r \geq 1$. Let $(h_j)_{j\in J}$ be as in Theorem 4.7. Then there exists a constant $C = C(T) > 0$ such that*

$$\int_{|\xi|\geq 1} \left| h_j(\xi) e^{-t\psi_\alpha(\xi)} \right| d\xi \leq C(1 + \ell((c_1 t)^{-1/\gamma_\infty(\alpha)}))t^{-(d+\varrho_\infty(j))/\gamma_\infty(\alpha)} \tag{4.28}$$

for all $\alpha \in I$, $j \in J$ and $t \in (0, T]$.

Proof. By the growth assumptions on $\operatorname{Re}\Psi_\alpha$ and H_j, we have

$$\begin{aligned}\int_{|\xi|\geq 1} \left| h_j(\xi) e^{-t\psi_\alpha(\xi)} \right| d\xi &\leq C_2 \int_{|\xi|\geq 1} |\xi|^{\varrho_\infty(j)} (1 + \ell(|\xi|)) e^{-tc_1 |\xi|^{\gamma_\infty(\alpha)}} \, d\xi \\ &\leq C_2 d\omega_d \int_{(0,\infty)} r^{\varrho_\infty(j) + (d-1)} (1 + \ell(r)) e^{-tc_1 r^{\gamma_\infty(\alpha)}} \, dr.\end{aligned}$$

Applying the Karamata–Tauberian theorem, Theorem A.3, we find $C = C(T, \ell) > 0$ such that

$$\int_{|\xi|\geq 1} \left| h_j(\xi) e^{-t\psi_\alpha(\xi)} \right| d\xi \leq C(1 + \ell((c_1 t)^{-1/\gamma_\infty(\alpha)}))t^{-(d+\varrho_\infty(j))/\gamma_\infty(\alpha)}$$

for all $t \in (0, T]$. □

Proof of Theorem 4.7. The on-diagonal estimate is a direct consequence of Lemma 4.8:

$$\begin{aligned}|G_j(x)| &\leq \frac{1}{(2\pi)^d} \int_{|\xi|<1} |h_j(\xi)| e^{-t \operatorname{Re}\psi_\alpha(\xi)} \, d\xi + \frac{1}{(2\pi)^d} \int_{|\xi|\geq 1} |h_j(\xi) e^{-t\psi_\alpha(\xi)}| \, d\xi \\ &\overset{\text{(E2)}}{\underset{(3.3)}{\leq}} \frac{C_1}{(2\pi)^d} e^{c_4 T} \lambda^d(B(0,1)) + \frac{1}{(2\pi)^d} \int_{|\xi|\geq 1} |h_j(\xi) e^{-t\psi_\alpha(\xi)}| \, d\xi \\ &\overset{\text{L4.8}}{\leq} C t^{-(d+\varrho_\infty(j))/\gamma_\infty(\alpha)} (1 + \ell((c_1 t)^{-1/\gamma_\infty(\alpha)}))\end{aligned}$$

for all $t \leq T$. To prove the off-diagonal estimate, we apply the dominated convergence theorem and Lemma 4.3 as in the proof of Theorem 3.1 to obtain

$$G_j(x) = \frac{c'_d}{|x|^{(d-1)/2}} \lim_{R\to\infty} \operatorname{Re} \int_{R^{-1}}^{R} H_j(r) \underbrace{e^{-t\Psi_\alpha(r)} r^{(d-1)/2} W_{0,d/2-1}(2ir|x|)}_{=u(r,|x|)} \, dr, \tag{4.29}$$

$$= \frac{c'_d}{|x|^d} \lim_{R\to\infty} \operatorname{Re} \int_{R^{-1}|x|}^{R|x|} H_j(r/|x|) \underbrace{e^{-t\Psi_\alpha(r/|x|)} r^{(d-1)/2} W_{0,d/2-1}(2ir)}_{=v(r,|x|)} \, dr. \tag{4.30}$$

Note that the functions u and v already appeared in the proof of Theorem 4.1, cf. (4.16) and (4.17). Through the remaining part of the proof we pick up the notation from the proof of Theorem 4.1; in particular the curves defined in (4.22) and (4.18), cf. Figure 4.2.

(i) $m = 0$: Fix $x \in \mathbb{R}^d \backslash \{0\}$. It is a direct consequence of Lemma 4.6 that

$$\left| \operatorname{Re}\left[\left(\int_{\sigma_1} + \int_{\sigma_3} \right) H_j(r/|x|) v(r, |x|) \, dr \right] \right| \xrightarrow{R\to\infty} 0.$$

Recall that (4.14) implies that there exists a constant $C_3 > 0$ such that

$$|\sigma_2(r)|^{(d-1)/2}|W_{0,d/2-1}(2i\sigma_2(r))| \le C_3 r e^{-r/2\sin\Theta}$$

for all $r \ge 0$, cf. (4.20). If we combine this with (3.3), we find

$$|v(\sigma_2(r),|x|)| \le C_3 e^{c_4 T} r e^{-\sin(\Theta)r/2}$$

for all $r \ge 0$. On the other hand, as $|\gamma_2(r)| = r$, we get from (E2) and (E3)

$$H_j(\sigma_2(r)/|x|) \le C_1 \left(\frac{r}{|x|}\right)^{\varrho_0(j)} \mathbb{1}_{\{r\le|x|\}} + C_2 \left(\frac{r}{|x|}\right)^{\varrho_\infty(j)} (1+\ell(r/|x|))\mathbb{1}_{\{r>|x|\}}.$$

Consequently,

$$\left|\int_{\sigma_2} H_j(r/|x|)v(r,|x|)\,dr\right| \le C_4(I_1+I_2)$$

where

$$\begin{aligned} I_1 &:= \frac{1}{|x|^{\varrho_0(j)}} \int_0^{|x|} r^{1+\varrho_0(j)} e^{-\sin(\Theta)r/2}\,dr \\ &\le \frac{1}{|x|^{\varrho_0(j)}} \min\left\{\int_0^\infty \max\{r, r^{1+\varrho^U}\} e^{-\sin(\Theta)r/2}\,dr, |x|^{2+\varrho_0(j)}\right\} \\ &\le C_5 \frac{1}{|x|^{\varrho_0(j)}} \frac{|x|^2}{1+|x|^{2+\varrho_0(j)}} \end{aligned}$$

and, by Karamata's theorem (Theorem A.3),

$$I_2 := \frac{1}{|x|^{\varrho_\infty(j)}} \int_0^\infty (1+\ell(r/|x|)) r^{1+\varrho_\infty(j)} e^{-\sin(\Theta)r/2}\,dr \le \frac{C_6}{|x|^{\varrho_\infty(j)}}(1+\ell(2(|x|\sin\Theta)^{-1})).$$

Adding all up, proves the assertion.

(ii) $m > 0$: Since $(\psi_\alpha)_{\alpha\in I} \in A(m) \subseteq A(0)$ the first part of this proof shows that the claimed estimate holds for all $|x| \le 1$. As in the proof of Theorem 4.1, we are therefore done if we can show that

$$|G_j(x)| \le C\exp\left(-\frac{m}{2}|x|\right) \qquad \text{for all } |x| > 1. \tag{$\star\star$}$$

Fix $x \in \mathbb{R}^d$ with $|x| > 1$. By Cauchy's theorem,

$$\operatorname{Re}\left(\int_{R^{-1}}^R H_j(r)u(r,|x|)\,dr\right) = \operatorname{Re}\left(\sum_{i=1}^4 \int_{\gamma_i} H_j(r)u(r,|x|)\,dr\right) \tag{4.31}$$

with $\gamma_i = \gamma_i(R)$ defined in (4.22). By Lemma 4.6,

$$\lim_{R\to\infty}\left|\operatorname{Re}\left[\left(\int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_4}\right) H_j(r)u(r,|x|)\,dr\right]\right| = 0.$$

It remains to estimate $\int_{\gamma_3} H_j(r)u(r,|x|)\,dr$. Using (E3), (3.3) and (4.27), we get

$$\left|\int_{\gamma_3} H_j(r)u(r,|x|)\,dr\right|$$

$$\leq C \int_0^{f(R)} |\gamma_3(r)|^{\varrho_\infty(j)+(d-1)/2}(1+\ell(|\gamma_3(r)|))e^{-t\operatorname{Re}\Psi_\alpha(\gamma_3(r))}e^{|x|\operatorname{Im}\gamma_3(r)}\,dr$$
$$\leq Ce^{c_4T}\exp\left(-\frac{m}{2}|x|\right)\int_0^{f(R)} |\gamma_3(r)|^{\varrho_\infty(j)+(d-1)/2}(1+\ell(|\gamma_3(r)|))e^{-r|x|\operatorname{Im} z_0}\,dr.$$

Choose $c>0$ such that $|\gamma_3(r)|\leq c(1+r)$. As $|x|\geq 1$ and $\varrho_\infty(j)\leq \varrho^U=\sup_j \varrho_\infty(j)<\infty$, we obtain

$$\left|\int_{\gamma_3} H_j(r)u(r,|x|)\,dr\right|$$
$$\leq C'\exp\left(-\frac{m}{2}|x|\right)\int_0^\infty (1+r)^{\varrho^U+(d-1)/2}(1+\ell(c(1+r)))e^{-r\operatorname{Im} z_0}\,dr.$$

Combining these estimates with (4.31) and (4.29) gives

$$G_j(x)\leq C''\exp\left(-\frac{m}{2}|x|\right)\qquad\text{for all } |x|>1. \qquad \square$$

4.9 Corollary *Let $\psi_\alpha:\mathbb{R}^d\to\mathbb{C}$, $\alpha\in I\subseteq\mathbb{R}^n$, be a family of continuous negative definite functions such that $(\psi_\alpha)_{\alpha\in I}\in A(m)$ for some $m\geq 0$. Then there exists for any $T>0$ a constant $C=C(T)$ such that*

$$|A^\beta p_t^\alpha(x)|\leq C_T\exp\left(-\frac{m}{4}|x|\right)\min\left\{t^{-(d+\gamma_\infty(\beta))/\gamma_\infty(\alpha)},\frac{1}{|x|^{d-2}}\frac{1}{1+|x|^{2+\gamma_0(\beta)}}+\frac{1}{|x|^{d+\gamma_\infty(\beta)}}\right\}$$

and

$$|\partial_t A^\beta p_t^\alpha(x)|\leq C_T\min\exp\left(-\frac{m}{4}|x|\right)\left\{t^{-1-(d+\gamma_\infty(\beta))/\gamma_\infty(\alpha)},\right.$$
$$\left.\frac{1}{|x|^{d-2}}\frac{1}{1+|x|^{2+\gamma_0(\beta)+\gamma_0(\alpha)}}+\frac{1}{|x|^{d+\gamma_\infty(\beta)+\gamma_\infty(\alpha)}}\right\}$$

for all $x\in\mathbb{R}^d$, $\alpha,\beta\in I$ and $t\leq T$.

Proof. If we set $h_\beta(\xi):=\psi_\beta(\xi)$, $\xi\in\mathbb{R}^d$, $j\in I$, then it follows from (C1)-(C4) that the family $(h_\beta)_{\beta\in I}$ satisfies the assumptions of Theorem 4.7 (with $\varrho_0(\beta):=\gamma_0(\beta)$, $\varrho_\infty(\beta):=\gamma_\infty(\beta)$ and $\ell:=1$). Therefore, the first estimate is a direct consequence of Theorem 4.7.
It remains to prove the estimate for $\partial_t A^\beta p_t^\alpha$. Because of the growth assumption (C3), we may apply the differentiation lemma for parametrized integrals:

$$\partial_t A^\beta p_t^\alpha(x)=-\int \psi_\alpha(\xi)\psi_\beta(\xi)e^{-ix\cdot\xi}e^{-t\psi_\alpha(\xi)}\,d\xi.$$

If we use Theorem 4.7 for the family $(\psi_\alpha\cdot\psi_\beta)_{\alpha,\beta\in I}$ (with $\varrho_\infty(\alpha,\beta)=\gamma_\infty(\alpha)+\gamma_\infty(\beta)$ and $\ell:=1$), then the assertion follows. $\square$

4.10 Corollary *Let $(\psi_\alpha)_{\alpha\in I}\in A(m)$ be a family of continuous negative definite functions with $\psi_\alpha(0)=0$. Then there exist constants $C=C(T)>0$ and $c=c(T)>0$ such that*

$$\left|\frac{\partial}{\partial\beta_j}A^\beta p_t^\alpha(x)\right|\leq C_T\exp\left(-\frac{m}{4}|x|\right)\min\left\{t^{-(d+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}(1+\ell(ct^{-1/\gamma_\infty(\alpha)})),\right.$$

$$\frac{1}{|x|^{d-2}}\frac{1}{1+|x|^{2+\gamma_0(\beta)}}+\frac{1+\ell(c|x|^{-1})}{|x|^{d+\gamma_\infty(\beta)}}\Bigg\}$$

and

$$\left|\partial_t\frac{\partial}{\partial\beta_j}A^\beta p_t^\alpha(x)\right|\leq C_T\exp\left(-\frac{m}{4}|x|\right)\min\Bigg\{t^{-1-(d+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}(1+\ell(ct^{-1/\gamma_\infty(\alpha)})),$$
$$\frac{1}{|x|^{d-2}}\frac{1}{1+|x|^{2+\gamma_0(\beta)+\gamma_0(\alpha)}}+\frac{1+\ell(c|x|^{-1})}{|x|^{d+\gamma_\infty(\beta)+\gamma_\infty(\alpha)}}\Bigg\}$$

for all $x\in\mathbb{R}^d$, $t\in(0,T]$, $\alpha,\beta\in I$ *and* $j\in\{1,\ldots,n\}$; *here* $\partial_{\beta_j}\psi_\beta(\xi):=\partial_{\beta_j}\Psi_\beta(|\xi|)$.

Proof. This is a direct consequence of the differentiation lemma for parametrized integrals and Theorem 4.7 applied for the families $(\partial_{\beta_j}\psi_\beta)_{\beta\in I}$ and $(\psi_\alpha\cdot\partial_{\beta_j}\psi_{\beta_j})_{\alpha,\beta\in I}$, respectively. □

The following corollary summarizes the estimates obtained in this section.

4.11 Corollary *Let* $(\psi_\alpha)_{\alpha\in I}\in A(m)$, $I\subseteq\mathbb{R}^n$, *be a family of continuous negative definite functions* $\psi_\alpha:\mathbb{R}^d\to\mathbb{C}$ *and* $T>0$. *Then there exist* $c=c(T)>0$ *and* $C=C(T)>0$ *such that*

$$p_t^\alpha(x)\leq C\exp\left(-\frac{m}{4}|x|\right)\begin{cases}t^{-d/\gamma_\infty(\alpha)}, & |x|\leq t^{1/\gamma_\infty(\alpha)},\\ \frac{t}{|x|^{d+\gamma_\infty(\alpha)}}, & t^{1/\gamma_\infty(\alpha)}<|x|\leq 1,\\ \frac{t}{|x|^{d+\gamma_\infty(\alpha)\wedge\gamma_0(\alpha)}}, & |x|>1,\end{cases}$$

$$|A^\beta p_t^\alpha(x)|\leq C\exp\left(-\frac{m}{4}|x|\right)\begin{cases}t^{-(d+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}, & |x|\leq t^{1/\gamma_\infty(\alpha)},\\ \frac{1}{|x|^{d+\gamma_\infty(\beta)}}, & t^{1/\gamma_\infty(\alpha)}<|x|\leq 1,\\ \frac{1}{|x|^{d+\gamma_\infty(\beta)\wedge\gamma_0(\beta)}}, & |x|>1,\end{cases}$$

$$|\partial_{\beta_j}A^\beta p_t^\alpha(x)|\leq C\exp\left(-\frac{m}{4}|x|\right)(1+\ell(ct^{-1/\gamma_\infty(\alpha)}))\begin{cases}t^{-(d+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}, & |x|\leq t^{1/\gamma_\infty(\alpha)},\\ \frac{1}{|x|^{d+\gamma_\infty(\beta)}}, & t^{1/\gamma_\infty(\alpha)}<|x|\leq 1,\\ \frac{1}{|x|^{d+\gamma_\infty(\beta)\wedge\gamma_0(\beta)}}, & |x|>1,\end{cases}$$

$$|\partial_t A^\beta p_t^\alpha(x)|\leq C\exp\left(-\frac{m}{4}|x|\right)\begin{cases}t^{-1-(d+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}, & |x|\leq t^{1/\gamma_\infty(\alpha)},\\ \frac{1}{|x|^{d+\gamma_\infty(\beta)+\gamma_\infty(\alpha)}}, & t^{1/\gamma_\infty(\alpha)}<|x|\leq 1,\\ \frac{1}{|x|^{d+(\gamma_\infty(\beta)+\gamma_\infty(\alpha))\wedge(\gamma_0(\beta)+\gamma_0(\alpha)}}, & |x|>1,\end{cases}$$

and

$$|\partial_t\partial_{\beta_j}A^\beta p_t^\alpha(x)|$$
$$\leq C\exp\left(-\frac{m}{4}|x|\right)(1+\ell(ct^{-1/\gamma_\infty(\alpha)}))\begin{cases}t^{-1-(d+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}, & |x|\leq t^{1/\gamma_\infty(\alpha)},\\ \frac{1}{|x|^{d+\gamma_\infty(\beta)+\gamma_\infty(\alpha)}}, & t^{1/\gamma_\infty(\alpha)}<|x|\leq 1,\\ \frac{1}{|x|^{d+(\gamma_\infty(\beta)+\gamma_\infty(\alpha))\wedge(\gamma_0(\beta)+\gamma_0(\alpha))}}, & |x|>1,\end{cases}$$

for all $t\in(0,T]$, $\alpha,\beta\in I$ *and* $x\in\mathbb{R}^d$.

Proof. This follows by combining Theorem 4.1, Corollary 4.9 and Corollary 4.10 with (4.7) and the elementary estimate

$$\frac{1}{|x|^{d-2}}\frac{1}{1+|x|^{2+\varrho}}+\frac{1}{|x|^{d+\kappa}}\le\begin{cases}\frac{2}{|x|^{d+\kappa}}, & |x|\le 1,\\ \frac{2}{|x|^{d+\kappa\wedge\varrho}}, & |x|>1.\end{cases}\qquad\square$$

Note that, since $A_\alpha p_t^\alpha = \partial_t p_t^\alpha$, Corollary 4.11, gives, in particular, a bound for $\partial_t p_t^\alpha$. For some applications it is of interest to have upper bounds for the partial derivatives $\partial_{x_j} p_t^\alpha(x)$, $j = 1,\dots,d$. Since p_t^α is rotationally symmetric, i. e. $p_t^\alpha(x) = p_t^\alpha(|x|)$, it is also natural to consider the derivative of the radial part of p_t^α

$$\frac{d}{dr}p_t^\alpha(r), \qquad r>0.$$

Since p_t^α is defined as the inverse Fourier transform of the function $e^{-t\psi_\alpha(\cdot)}$, it follows easily from the differentiation lemma for parametrized integrals, the Riemann–Lebesgue lemma and the growth assumption (C3) that $p_t^\alpha \in C_\infty^2(\mathbb{R}^d)$. Theorem 4.12 provides an off-diagonal estimate which we will need later in the proof.

4.12 Theorem *Let $(\psi_\alpha)_{\alpha\in I}\in A(m)$, $I\subseteq\mathbb{R}^n$, be a family of continuous negative definite functions $\psi_\alpha:\mathbb{R}^d\to\mathbb{C}$. Then there exists a constant $C = C(T)>0$ such that*

$$\left|\frac{\partial}{\partial x_j}p_t^\alpha(x)\right|\le Ct^{-1/\gamma_\infty(\alpha)}S(x,\alpha,t)$$
$$\left|\frac{\partial^2}{\partial x_i\partial x_j}p_t^\alpha(x)\right|\le Ct^{-2/\gamma_\infty(\alpha)}S(x,\alpha,t)$$

for all $t\in(0,T]$, $x\in\mathbb{R}^d$, $\alpha\in I$ and $i,j\in\{1,\dots,d\}$.

For the proof of Theorem 4.12 we use the following result; it has recently been rediscovered by Grafakos & Teschl [40, Theorem 1.1], but actually goes back to Matheron [76, Section 1.4] ("montée et descente en clavier isotrope").

4.13 Lemma *Let $f:[0,\infty)\to\mathbb{R}$ be such that $f(|\cdot|)\in L^1(\mathbb{R}^d,dx)\cap L^1(\mathbb{R}^{d+2},dx)$ for some $d\ge 1$ and set $u_d(x):=f(|x|)$, $x\in\mathbb{R}^d$. Then*

$$F_{d+2}f(r) = -\frac{1}{2\pi}\frac{1}{r}\frac{d}{dr}F_d f(r), \qquad r>0,$$

where $F_k f$ denotes the unique rotationally invariant function such that $\mathcal{F}_k u_k(\xi) = F_k f(|\xi|)$, $\xi\in\mathbb{R}^k$, for $k\in\{d,d+2\}$.

Proof of Theorem 4.12. Because of (C3),

$$p_t^{\alpha,k}(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^k} e^{-ix\cdot\xi}e^{-t\Psi_\alpha(|\xi|)}\,d\xi, \quad x\in\mathbb{R}^k,$$

defines a rotationally invariant real-valued function for any $k\in\mathbb{N}$. Since

$$p_t^{\alpha,k}(x) = F_k(e^{-t\Psi_\alpha})(|x|) =: p_t^{\alpha,k}(|x|), \qquad x\in\mathbb{R}^k,$$

it follows from Lemma 4.13 that

$$\frac{d}{dr} p_t^{\alpha,k}(r) = -2\pi r p_t^{\alpha,k+2}(r) \qquad \text{for all } \; r > 0.$$

If we set $g_t^{\alpha,k}(r) := p_t^{\alpha,k}(\sqrt{r})$, then we find

$$\frac{d}{dr} g_t^{\alpha,k}(r) = \frac{1}{2} \frac{1}{\sqrt{r}} \frac{d}{ds} p_t^{\alpha,k}(s) \bigg|_{s=\sqrt{r}} = -\pi g_t^{\alpha,k+2}(r).$$

Hence, by the chain rule,

$$\frac{\partial}{\partial x_j} p_t^{\alpha,d}(x) = \frac{\partial}{\partial x_j} g_t^{\alpha,d}(|x|^2) = -2\pi x_j p_t^{\alpha,d+2}(x).$$

Combining this identity with Corollary 4.11, we get the estimate for $\partial_{x_j} p_t^{\alpha}$.[2] Using the same reasoning another time, we also obtain the estimates for the derivatives of order 2. □

Applying Theorem 4.12, we can derive a lower bound for the density p_t^{α}. We will use it for the proof of Corollary 3.9.

4.14 Corollary *Let $(\psi_\alpha)_{\alpha \in I} \in A(m)$, $I \subseteq \mathbb{R}^n$, be a family of continuous negative definite functions $\psi_\alpha : \mathbb{R}^d \to \mathbb{C}$. Then for any $T > 0$ there exist constants $C_1, C_2 > 0$ such that*

$$p_t^{\alpha}(x) \geq C_1 t^{-d/\gamma_\infty(\alpha)} (1 - C_2 t^{-1/\gamma_\infty(\alpha)} |x|)^+$$

for all $x \in \mathbb{R}^d$, $\alpha \in I$, $t \in (0, T]$.

Proof. Fix $T > 0$. It follows from the growth assumption (C3) that there exists a constant $c > 0$ such that

$$p_t^{\alpha}(0) \geq c t^{-d/\gamma_\infty(\alpha)} \qquad \text{for all } \; t \in (0, T]. \tag{4.32}$$

Indeed: By (C3), we have

$$|\operatorname{Re} \psi_\alpha(\xi)| \leq |\psi_\alpha(\xi)| \leq c_2 |\xi|^{\gamma_\infty(\alpha)} \qquad \text{for all } \; |\xi| \geq 1, \alpha \in I.$$

It follows from the rotational invariance of ψ_α that $\psi_\alpha = \operatorname{Re} \psi_\alpha$, and therefore

$$\begin{aligned} p_t(0) = \operatorname{Re} p_t(0) = \frac{1}{(2\pi)^d} \int e^{-t \operatorname{Re} \psi_\alpha(\xi)} \, d\xi &\geq \frac{1}{(2\pi)^d} \int_{|\xi| \geq 1} e^{-t \operatorname{Re} \psi_\alpha(\xi)} \, d\xi \\ &\geq \frac{1}{(2\pi)^d} \int_{|\xi| \geq 1} e^{-c_2 t |\xi|^{\gamma_\infty(\alpha)}} \, d\xi. \end{aligned}$$

Now a simple change of variables shows

$$p_t(0) \geq \frac{1}{(2\pi)^d} t^{-d/\gamma_\infty(\alpha)} \int_{|\eta| \geq t^{1/\gamma_\infty(\alpha)}} e^{-c_2 |\eta|^{\gamma_\infty(\alpha)}} \, d\eta.$$

[2] Note that $\mathbb{R}^k \ni \xi \mapsto \psi_\alpha(\xi) = \Psi_\alpha(|\xi|^2)$ might not be a negative definite function for $k \neq d$. Since we haven't used that ψ_α is negative definite for the proof of Corollary 4.11, the estimates hold true for any dimension $k \geq 1$.

As $\gamma_\infty(\alpha) \le 2$, it follows easily that there exists a constant $c = c(T) > 0$ such that (4.32) holds. Now we are ready to prove the lower bound for p_t^α. Applying Taylor's formula, we find

$$p_t^\alpha(x) = p_t^\alpha(0) + \nabla p_t^\alpha(\xi) \cdot x$$

for some $\xi = \xi(x) \in \mathbb{R}^d$. By Theorem 4.12, we have $\|\nabla p_t^\alpha\|_\infty \le C t^{-(d+1)/\gamma_\infty(\alpha)}$, and therefore

$$\begin{aligned} |p_t^\alpha(x)| \ge |p_t(0)| - \|\nabla p_t^\alpha\|_\infty |x| &\ge c t^{-d/\gamma_\infty(\alpha)} - C t^{-(d+1)/\gamma_\infty(\alpha)}|x| \\ &= c t^{-d/\gamma_\infty(\alpha)} \left(1 - \frac{C}{c} t^{-1/\gamma_\infty(\alpha)}|x|\right) \end{aligned}$$

for all $t \in (0,T]$ and $x \in \mathbb{R}^d$. Since the density p_t^α is non-negative, this proves the assertion. □

4.2 Auxiliary convolution estimates

We have seen in the previous section that the function S, defined in (4.9), is an upper bound for the density p_t^α. We want to use this estimate to prove the convergence of the parametrix series (4.5). Since the series (4.5) involves time-space convolutions of p_t^α, we have to investigate the behaviour of S under convolution. The main result of this section is Lemma 4.17 which shows that S has – up to a perturbation term – the subconvolution property. More precisely, it states that for any $T > 0$ there exists a constant $C = C(T) > 0$ such that

$$\int S(x-z, \alpha(z), t-s) S(z-y, \alpha(y), s)\, dz \le C S(x-y, \alpha(y), t) + C t g_{\gamma_0^L \wedge \gamma_\infty^L}(x-y),$$

for all $x, y \in \mathbb{R}^d$ and $0 < s < t \in (0,T]$ where

$$g_\beta(y) := g_{\beta,m}(y) := \frac{1}{1+|y|^{d+\beta}} \exp\left(-\frac{m}{4}|y|\right), \qquad y \in \mathbb{R}^d, \beta > 0. \tag{4.33}$$

We will prove this result in the first part of this section. Afterwards we discuss the properties of the perturbation term g under convolution (Lemma 4.18 and Lemma 4.19). Finally, in the last part, we formulate the results in terms of time-space convolutions, cf. Corollary 4.20.

In this section we follow Kolokoltsov [60, Section 7.4] who showed the convolution estimates for the particular case $\gamma_0(\alpha) = \gamma_\infty(\alpha) = \alpha$ and $m = 0$. In order to cover our more general framework, we slightly modify the proofs. Throughout this section, we consider, for simplicity of notation, the finite time horizon $(0,1]$, i. e. we set $T = 1$.

First we state some elementary properties of S which follow from the definition of S by direct calculations.

4.15 Lemma *(i)* $\sup_{x\in\mathbb{R}^d} S(x,\alpha,t) \le t^{-d/\gamma_\infty(\alpha)}$ *for all* $t \le 1$.

(ii)

$$\sup_{x\in\mathbb{R}^d} \sup_{t\in(0,1]} \int S(x-z,\alpha,t)\, dz \le C\left(\frac{1}{\gamma_\infty(\alpha)} + \frac{1}{\gamma_0(\alpha)}\right).$$

(iii) For any fixed constants $A_1, A_2 > 0$, we have

$$S(x,\alpha,t) \asymp \exp\left(-\frac{m}{4}|x|\right) \cdot \begin{cases} t^{-d/\gamma_\infty(\alpha)}, & |x| \leq A_1 t^{1/\gamma_\infty(\alpha)}, \\ \frac{t}{|x|^{d+\gamma_\infty(\alpha)}}, & A_1 t^{1/\gamma_\infty(\alpha)} < |x| \leq A_2, \\ \frac{t}{|x|^{d+\gamma_0(\alpha)\wedge\gamma_\infty(\alpha)}}, & |x| > A_2. \end{cases} \tag{4.34}$$

The next lemma is one of the main tools to derive the convolution estimates we are interested in. It is essentially taken from [60, Lemma 7.4.1], with minor modifications.

4.16 Lemma *Let $I \subseteq \mathbb{R}^n$ and $\gamma_0, \gamma_\infty : I \to (0,2]$ be measurable mappings such that*

$$0 < \gamma_0^L := \inf_{\alpha\in I} \gamma_0(\alpha) \leq \sup_{\alpha \in I} \gamma_0(\alpha) =: \gamma_0^U \leq 2$$
$$0 < \gamma_\infty^L := \inf_{\alpha\in I} \gamma_\infty(\alpha) \leq \sup_{\alpha \in I} \gamma_\infty(\alpha) =: \gamma_\infty^U \leq 2.$$

Moreover, let $\alpha : \mathbb{R}^d \to I$ be a measurable mapping such that $\gamma_\infty(\alpha(\cdot))$ is Hölder continuous with exponent $\varrho \in (0,1]$. Then there exists a constant $C > 0$ such that

$$\int S(x-z,\alpha(z),t)\,dz \leq C \qquad \text{for all} \quad x \in \mathbb{R}^d, t \in (0,1].$$

Proof. Since $\exp(-\frac{m}{4}|x|) \leq 1$ for any $m \geq 0$ and $x \in \mathbb{R}^d$, it suffices to consider the case $m = 0$. We split up the integral,

$$\int S(x-z,\alpha(z),t)\,dz = \left(\int_{|x-z|>1} + \int_{t^{1/\gamma_\infty^U} < |x-z| \leq 1} + \int_{|x-z|\leq t^{1/\gamma_\infty^U}}\right) S(x-z,\alpha(z),t)\,dz$$
$$=: J_1 + J_2 + J_3,$$

and estimate the terms separately. If $|x-z| > 1$ then by (4.9)

$$S(x-z,\alpha(z),t) \leq \frac{t}{|x-z|^{d+\gamma_0(\alpha(z))\wedge\gamma_\infty(\alpha(z))}} \overset{|x-z|>1}{\leq} \frac{t}{|x-z|^{d+\gamma_0^L \wedge \gamma_\infty^L}}.$$

Introducing spherical coordinates, we find

$$J_1 \leq t \int_{|x-z|>1} \frac{1}{|x-z|^{d+\gamma_0^L\wedge\gamma_\infty^L}}\,dz \leq d\cdot\omega_d \int_1^\infty \frac{1}{r^{1+\gamma_0^L\wedge\gamma_\infty^L}}\,dr =: C_1$$

for $t \leq 1$; as usual, ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. If $1 \geq |x-z| > t^{1/\gamma_\infty^U}$, then, as $\gamma_\infty^U \geq \gamma_\infty(\alpha(z)))$,

$$|x-z| \geq t^{1/\gamma_\infty^U} \geq t^{1/\gamma_\infty(\alpha(z))} \qquad \text{for all} \quad t \leq 1.$$

Consequently, by the definition of S,

$$S(x-z,\alpha(z),t) \leq \frac{t}{|x-z|^{d+\gamma_\infty(\alpha(z))}} \overset{|x-z|\leq 1}{\leq} \frac{t}{|x-z|^{d+\gamma_\infty^U}}.$$

This implies

$$J_2 \leq t \int_{t^{1/\gamma_\infty^U} < |x-z| \leq 1} \frac{1}{|x-z|^{d+\gamma_\infty^U}}\,dz = t d\omega_d \int_{t^{1/\gamma_\infty^U}}^1 \frac{1}{r^{1+\gamma_\infty^U}}\,dr \leq \frac{d\cdot\omega_d}{\gamma_\infty^U} =: C_2$$

for all $t \leq 1$. It remains to estimate J_3. For $x \in \mathbb{R}^d$ and $r > 0$ define

$$\begin{aligned}\gamma_\infty^L(x,r) &:= \inf\{\gamma_\infty(\alpha(z)); z \in B[x,r]\},\\ \gamma_\infty^U(x,r) &:= \sup\{\gamma_\infty(\alpha(z)); z \in B[x,r]\},\end{aligned} \tag{4.35}$$

and

$$D(x) := D(x,t) := \{z \in \mathbb{R}^d; t^{1/\gamma_\infty^U(x,r(t))} < |x-z| \leq r(t)\}.$$

where $r(t) := t^{1/\gamma_\infty^U}$. For any $z \in D(x) \subseteq B[x,r(t)]$, we have $\gamma_\infty^U(x,r(t)) \geq \gamma_\infty(\alpha(z))$; now

$$1 \geq |x-z| \geq t^{1/\gamma_\infty^U(x,r(t))} \overset{t\leq 1}{\geq} t^{1/\gamma_\infty(\alpha(z))}, \qquad z \in D(x),$$

implies, by (4.9),

$$S(x-z,\alpha(z),t) \leq \frac{t}{|x-z|^{d+\gamma_\infty(\alpha(z))}} \overset{|x-z|\leq 1}{\leq} \frac{t}{|x-z|^{d+\gamma_\infty^U(x,r(t))}}$$

for any $z \in D(x)$. Consequently,

$$\begin{aligned}\int_{D(x)} S(x-z,\alpha(z),t)\,dz &\leq t\int_{D(x)} \frac{1}{|x-z|^{d+\gamma_\infty^U(x,r(t))}}\,dz\\ &\leq d\omega_d \int_{t^{1/\gamma_\infty^U(x,r(t))}}^1 \frac{1}{r^{1+\gamma_\infty^U(x,r(t))}}\,dr\\ &\leq \frac{d\omega_d}{\gamma_\infty^U(x,r(t))} \leq \frac{d\omega_d}{\gamma_\infty^L}.\end{aligned}$$

On the other hand, Lemma 4.15(i) gives

$$\int_{|x-z|\leq t^{1/\gamma_\infty^U(x,r(t))}} S(x-z,\alpha(z),t)\,dz \leq d\omega_d t^{-d/\gamma_\infty^L(x,r(t))+d/\gamma_\infty^U(x,r(t))}.$$

Since $\gamma_\infty(\alpha(\cdot))$ is Hölder continuous with exponent $\varrho \in (0,1]$, we have

$$\begin{aligned}0 \leq \gamma_\infty^U(x,r(t)) - \gamma_\infty^L(x,r(t)) &\leq 2\sup_{y\in B[x,r(t)]} |\gamma_\infty(\alpha(y)) - \gamma_\infty(\alpha(x))|\\ &\leq 2C_3|r(t)|^\varrho = 2C_3 t^{\varrho/\gamma_\infty^U}.\end{aligned}$$

Hence,

$$\begin{aligned}t^{-d/\gamma_\infty^L(x,r(t))+d/\gamma_\infty^U(x,r(t))} &= \exp\left(-d\frac{\gamma_\infty^U(x,r(t)) - \gamma_\infty^L(x,r(t))}{\gamma_\infty^L(x,r(t))\gamma_\infty^U(x,r(t))}\log t\right)\\ &\leq \exp\left(-\frac{2dC_3}{(\gamma_\infty^L)^2} t^{\varrho/\gamma_\infty^U}\log t\right)\end{aligned}$$

for all $t \in (0,1]$. Combining the above estimates yields

$$\begin{aligned}J_3 &= \left(\int_{D(x)} + \int_{|x-z|\leq t^{1/\gamma_\infty^U(x,r(t))}}\right) S(x-z,\alpha(z),t)\,dz\\ &\leq \frac{d\omega_d}{\gamma_\infty^L} + d\omega_d \sup_{t\in(0,1]} \exp\left(-\frac{2dC_3}{(\gamma_\infty^L)^2} t^{\varrho/\gamma_\infty^U}\log t\right) < \infty.\end{aligned}$$ □

Lemma 4.16 allows us to prove the following important subconvolution property.

4.17 Lemma *Under the assumptions of Lemma 4.16 there exists a constant $C > 0$ such that*

$$\int S(x-z,\alpha(z),t-s)S(z-y,\alpha(y),s)\,dz \le CS(x-y,\alpha(y),t) + Ctg_{\gamma_0^L\wedge\gamma_\infty^L}(x-y),$$

for all $x, y \in \mathbb{R}^d$ and $0 < s < t \le 1$ where

$$g_\beta(y) := g_{\beta,m}(y) := \frac{1}{1+|y|^{d+\beta}}\exp\left(-\frac{m}{4}|y|\right), \qquad y \in \mathbb{R}^d, \beta > 0.$$

Proof. Up to some small modifications we follow [60, Lemma 7.4.2]. We will use the notation introduced in the proof of Lemma 4.16, in particular $\gamma_\infty^U(x,r)$, $\gamma_\infty^L(x,r)$, cf. (4.35), and $r(t) := t^{1/\gamma_\infty^U}$. For brevity of notation, we set

$$I(A) := \int_A S(x-z,\alpha(z),t-s)S(z-y,\alpha(y),s)\,dz, \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

Roughly, the strategy of this proof is as follows. We choose $A \in \mathcal{B}(\mathbb{R}^d)$ in such a way that we can bound one of the two factors, i. e. $S(x-z,\alpha(z),t-s)$ or $S(z-y,\alpha(y),s)$, from above by $S(x-y,\alpha(y),t)$ for $z \in A$ *and* $z \in A^c$. This leaves us with integral of the remaining factor over the corresponding domain which we can estimate using Lemma 4.16.

By definition, we have $S(x,\alpha,t) = S_m(x,\alpha,t) = \exp(-\frac{m}{4}|x|)S_0(x,\alpha,t)$. Suppose we already *knew* that the claim holds for $m = 0$. Then the elementary estimate

$$\begin{aligned}
&S(x-z,\alpha(z),t-s)S(z-y,\alpha(y),s)\\
&= \exp\left(-\frac{m}{4}(|x-z|+|z-y|)\right)S_0(x-z,\alpha(z),t-s)S_0(z-y,\alpha(y),s)\\
&\le \exp\left(-\frac{m}{4}|x-y|\right)S_0(x-z,\alpha(z),t-s)S_0(z-y,\alpha(y),s)
\end{aligned}$$

gives

$$I(\mathbb{R}^d) \le \exp\left(-\frac{m}{4}|x-y|\right)\int S_0(x-z,\alpha(z),t-s)S_0(z-y,\alpha(y),s)\,dz;$$

thus, the assertion also follows for $m > 0$. Consequently, it suffices to consider the case $m = 0$. For fixed $x, y \in \mathbb{R}^d$ we define

$$D := \left\{z \in \mathbb{R}^d;\, |z-y| \le \frac{|x-y|}{2}\right\}$$

and note that

$$D \subseteq \left\{z \in \mathbb{R}^d;\, \frac{3}{2}|x-y| \ge |z-x| \ge \frac{1}{2}|x-y|\right\}. \tag{4.36}$$

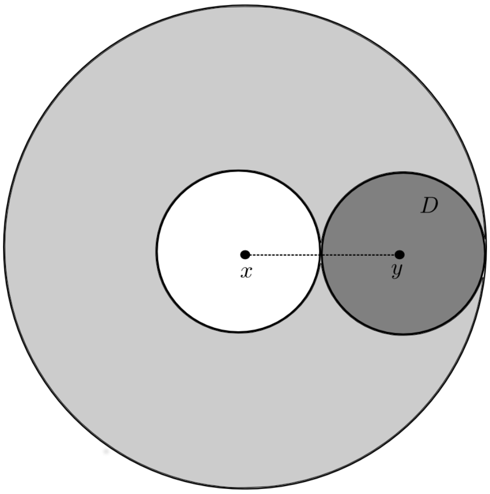

Figure 4.3: D is contained in the annulus $\{z \in \mathbb{R}^d; \frac{3}{2}|x-y| \geq |z-x| \geq \frac{1}{2}|x-y|\}$ coloured in light gray.

We have to distinguish several cases. Throughout the remaining part of the proof, the constants may vary from line to line.

(i) $|x-y| \geq 1$: For any $z \in D$ we have, by (4.36), $|x-z| \geq |x-y|/2 \geq 1/2$; hence, by (4.34) (with $A_1 = 1$, $A_2 = \frac{1}{2}$)

$$S(x-z,\alpha(z),t-s) \leq C\frac{t-s}{|x-z|^{d+\gamma_\infty(\alpha(z))\wedge\gamma_0(\alpha(z))}} \overset{z\in D}{\leq} C'\frac{t}{|x-y|^{d+\gamma_\infty(\alpha(z))\wedge\gamma_0(\alpha(z))}} \overset{|x-y|\geq 1}{\leq} C'\frac{t}{|x-y|^{d+\gamma_\infty^L\wedge\gamma_0^L}}$$

for all $z \in D$. Since

$$\int_D S(z-y,\alpha(y),s)\,dz \leq \int S(z-y,\alpha(y),s)\,dz \leq C$$

by Lemma 4.15, we get

$$I(D) \leq C'\frac{t}{|x-y|^{d+\gamma_\infty^L\wedge\gamma_0^L}}\int_D S(z-y,\alpha(y),s)\,dz \leq C'' t g_{\gamma_\infty^L\wedge\gamma_0^L}(x-y).$$

On the other hand, if $z \in D^c$ then $|z-y| \geq |x-y|/2 \geq 1/2$. Therefore, (4.34) gives

$$S(z-y,\alpha(y),s) \leq C\frac{s}{|z-y|^{d+\gamma_\infty(\alpha(y))\wedge\gamma_0(\alpha(y))}} \leq C'\frac{t}{|x-y|^{d+\gamma_\infty(\alpha(y))\wedge\gamma_0(\alpha(y))}}$$

for all $s \leq t$. This implies

$$\begin{aligned} I(D^c) &\leq C'\frac{t}{|x-y|^{d+\gamma_\infty(\alpha(y))\wedge\gamma_0(\alpha(y))}}\int_{D^c} S(x-z,\alpha(z),t-s)\,dz \\ &\overset{\text{L4.16}}{\leq} C''\frac{t}{|x-y|^{d+\gamma_\infty(\alpha(y))\wedge\gamma_0(\alpha(y))}} \\ &\overset{|x-y|\geq 1}{=} C'' S(x-y,\alpha(y),t). \end{aligned}$$

Combining the estimates for $I(D)$ and $I(D^c)$ proves the claim if $|x-y| \geq 1$.

(ii) $t^{1/\gamma_\infty^U} \leq |x-y| \leq 1$: As $t \leq 1$ and $\gamma_\infty^U \geq \gamma_\infty(\alpha(y))$, we have $1 \geq |x-y| \geq t^{1/\gamma_\infty^U} \geq t^{1/\gamma_\infty(\alpha(y))}$, and so

$$S(x-y,\alpha(y),t) \overset{(4.9)}{=} \frac{t}{|x-y|^{d+\gamma_\infty(\alpha(y))}}.$$

Therefore, the claim follows if we can show that

$$I(\mathbb{R}^d) = I(D) + I(D^c) \leq C\frac{t}{|x-y|^{d+\gamma_\infty(\alpha(y))}}. \tag{4.37}$$

For $z \in D$ we have

$$\frac{3}{2} \geq \frac{3}{2}|x-y| \overset{(4.36)}{\geq} |x-z| \overset{(4.36)}{\geq} \frac{1}{2}|x-y| \geq \frac{1}{2}t^{1/\gamma_\infty^U} \geq \frac{1}{2}t^{1/\gamma_\infty(\alpha(z))} \geq \frac{1}{2}(t-s)^{1/\gamma_\infty(\alpha(z))},$$

and therefore, by (4.34) (with $A_1 = 1/2$, $A_2 = 3/2$),

$$\begin{aligned} S(x-z,\alpha(z),t-s) \leq C\frac{t-s}{|x-z|^{d+\gamma_\infty(\alpha(z))}} &\leq C'\frac{t}{|x-y|^{d+\gamma_\infty(\alpha(z))}} \\ &\overset{|x-y|\leq 1}{\leq} C'\frac{t}{|x-y|^{d+\gamma_\infty^L(y,|x-y|)}} \end{aligned}$$

for any $z \in D \subset B[y,|x-y|]$ and $s \leq t$. Using this estimate and applying Lemma 4.16, we find

$$I(D) \leq C'\frac{t}{|x-y|^{d+\gamma_\infty^L(y,|x-y|)}} = C'\frac{t}{|x-y|^{d+\gamma_\infty(\alpha(y))}}|x-y|^{\gamma_\infty(\alpha(y))-\gamma_\infty^L(y,|x-y|)}.$$

This gives an estimate of the form (4.37) for $I(D)$ if

$$\sup_{|x-y|\leq 1} |x-y|^{\gamma_\infty(\alpha(y))-\gamma_\infty^L(y,|x-y|)} < \infty. \tag{4.38}$$

To see this, we note that by the Hölder continuity of $\gamma_\infty \circ \alpha$

$$0 \leq \gamma_\infty(\alpha(y)) - \gamma_\infty^L(y,r) \leq \sup_{y' \in B[y,r]} |\gamma_\infty(\alpha(y)) - \gamma_\infty(\alpha(y'))| \leq Cr^\varrho$$

for some constant C which does not depend on $r > 0$. Consequently,

$$|x-y|^{\gamma_\infty(\alpha(y))-\gamma_\infty^L(y,|x-y|)} \leq \exp\left(-C|x-y|^\varrho \log|x-y|\right),$$

and this proves (4.38). It remains to estimate $I(D^c)$. For $z \in D^c$ we have

$$|z-y| \geq \frac{|x-y|}{2} \geq \frac{1}{2}t^{1/\gamma_\infty^U} \geq \frac{1}{2}t^{1/\gamma_\infty(\alpha(y))}.$$

If $|z-y| \geq 1$, then

$$S(z-y,\alpha(y),s) \overset{(4.9)}{=} \frac{s}{|z-y|^{d+\gamma_\infty(\alpha(y))\wedge\gamma_0(\alpha(y))}} \leq s \overset{|x-y|\leq 1}{\leq} \frac{s}{|x-y|^{d+\gamma_\infty(\alpha(y))}};$$

on the other hand if $1 > |z-y| \geq \frac{1}{2}t^{1/\gamma_\infty(\alpha(y))}$, then by (4.34) (with $A_1 = \frac{1}{2}$, $A_2 = 1$)

$$S(z-y,\alpha(y),s) \leq C\frac{s}{|z-y|^{d+\gamma_\infty(\alpha(y))}} \overset{z\in D^c}{\leq} C'\frac{s}{|x-y|^{d+\gamma_\infty(\alpha(y))}}.$$

As $s \leq t$, this proves $S(z-y,\alpha(y),s) \leq C'\frac{s}{|x-y|^{d+\gamma_\infty(\alpha(y))}}$ for any $z \in D^c$. Thus, by Lemma 4.16,

$$I(D^c) \leq C'\frac{t}{|x-y|^{d+\gamma_\infty(\alpha(y))}} \int_{D^c} S(x-z,\alpha(z),t-s)\,dz \leq C''\frac{t}{|x-y|^{d+\gamma_\infty(\alpha(y))}}.$$

(iii) $t^{1/\gamma_\infty^U} \geq |x-y| \geq t^{1/\gamma_\infty^U(y,r(t))}$: As in the previous case, we are done if we succeed to show

$$I(\mathbb{R}^d) = I(D) + I(D^c) \leq C\frac{t}{|x-y|^{d+\gamma_\infty(\alpha(y))}}.$$

Again, we estimate $I(D)$ and $I(D^c)$ separately. The estimate for $I(D^c)$ is obtained in exactly the same way as in (ii). If $z \in D$, then

$$z \in D \subseteq B[y, |x-y|] \subseteq B[y, r(t)] \tag{$\star$}$$

as $r(t) = t^{1/\gamma_\infty^U}$. Thus,

$$\frac{3}{2} \geq \frac{3}{2}|x-y| \overset{(4.36)}{\geq} |x-z| \overset{(4.36)}{\geq} \frac{1}{2}|x-y| \geq \frac{1}{2}t^{1/\gamma_\infty^U(y,r(t))} \overset{(\star)}{\geq} \frac{1}{2}t^{1/\gamma_\infty(\alpha(z))} \geq \frac{1}{2}(t-s)^{1/\gamma_\infty(\alpha(z))}.$$

Applying (4.34) (with $A_1 = \frac{3}{2}$, $A_2 = \frac{1}{2}$), we get

$$\begin{aligned} S(x-z, \alpha(z), t-s) \leq C\frac{t-s}{|x-z|^{d+\gamma_\infty(\alpha(z))}} &\leq C'\frac{t}{|x-y|^{d+\gamma_\infty(\alpha(z))}} \\ &\overset{|x-y|\leq 1}{\leq} C'\frac{t}{|x-y|^{d+\gamma_\infty^U(y,r(t))}} \end{aligned}$$

for $z \in D \subseteq B[y, r(t)]$ and $s \leq t$. Now it follows from Lemma 4.16 that

$$I(D) \leq C'\frac{t}{|x-y|^{d+\gamma_\infty^U(y,r(t))}} \int_D S(z-y, \alpha(y), s)\, dz \leq C''\frac{t}{|x-y|^{d+\gamma_\infty^U(y,r(t))}}.$$

An argument very similar to the proof of (4.38), based on the Hölder continuity of $\gamma_\infty \circ \alpha$, shows

$$\sup_{t\in(0,1]} \sup_{|x-y|\leq r(t)} |x-y|^{\gamma_\infty(\alpha(y))-\gamma_\infty^U(y,r)(t)} < \infty.$$

Combining this with the estimate for $I(D)$, this proves $I(D) \leq C'''\frac{t}{|x-y|^{d+\gamma_\infty(\alpha(y))}}$.

(iv) $|x-y| \leq t^{1/\gamma_\infty^U(y,r(t))}$: We claim that

$$S(x-y, \alpha(y), t) \geq ct^{-d/\gamma_\infty(\alpha(y))} \tag{4.39}$$

for some constant $c > 0$ which does not depend on t, x, y. *Indeed:* If $|x-y| \leq t^{1/\gamma_\infty(\alpha(y))}$, then $S(x-y, \alpha(y), t) = t^{-d/\gamma_\infty(\alpha(y))}$ by the very definition of S, i. e. (4.39) holds with $c = 1$. If $t^{1/\gamma_\infty(\alpha(y))} < |x-y| \leq t^{1/\gamma_\infty^U(y,r(t))}$, then

$$\begin{aligned} S(x-y, \alpha(y), t) \overset{(4.9)}{=} \frac{t}{|x-y|^{d+\gamma_\infty(\alpha(y))}} &\geq \frac{t}{t^{(d+\gamma_\infty(\alpha(y)))/\gamma_\infty^U(y,r(t))}} \\ &= f(t)t^{-d/\gamma_\infty(\alpha(y))} \end{aligned}$$

where

$$\begin{aligned} f(t) := f(t,y) &:= t^{1+d/\gamma_\infty(\alpha(y))}t^{-(d+\gamma_\infty(\alpha(y)))/\gamma_\infty^U(y,r(t))} \\ &= \exp\left(-\left[-1 + \frac{\gamma_\infty(\alpha(y))}{\gamma_\infty^U(y,r(t))} - \frac{d}{\gamma_\infty(\alpha(y))} + \frac{d}{\gamma_\infty^U(y,r(t))}\right]\log t\right). \end{aligned}$$

The Hölder continuity of $\gamma_\infty \circ \alpha$ implies

$$0 < \inf_{y\in\mathbb{R}^d} \inf_{t\in(0,1]} f(t,y) \le \sup_{y\in\mathbb{R}^d} \sup_{t\in(0,1]} f(t,y) < \infty, \tag{4.40}$$

see Lemma A.5 for more details. This gives (4.39).

Because of (4.39) it suffices to show $I(\mathbb{R}^d) \le Ct^{-d/\gamma_\infty(\alpha(y))}$. We define

$$E := \{z \in \mathbb{R}^d; |z-y| \ge t^{1/\gamma_\infty^U(y,r(t))}\}$$

and estimate $I(E)$ and $I(E^c)$ separately. If $z \in E$ and $|z-y| \ge 1$, then

$$S(z-y,\alpha(y),s) \overset{(4.9)}{=} \frac{s}{|z-y|^{d+\gamma_\infty(\alpha(y))\wedge\gamma_0(\alpha(y))}} \le s \le t \le t^{-d/\gamma_\infty(\alpha(y))}$$

as $s \le t \le 1$. On the other hand, if $z \in E$ and $|z-y| < 1$, then

$$1 > |z-y| \ge t^{1/\gamma_\infty^U(y,r(t))} \ge t^{1/\gamma_\infty(\alpha(y))},$$

and therefore

$$\begin{aligned} S(x-y,\alpha(y),s) = \frac{s}{|z-y|^{d+\gamma_\infty(\alpha(y))}} &\le \frac{t}{t^{(d+\gamma_\infty(\alpha(y)))/\gamma_\infty^U(y,r(t))}} \\ &= f(t)t^{-d/\gamma_\infty(\alpha(y))} \overset{(4.40)}{\le} Ct^{-d/\gamma_\infty(\alpha(y))} \end{aligned}$$

for all $0 \le s \le t \le 1$. This shows $S(z-y,\alpha(y),s) \le Ct^{-d/\gamma_\infty(\alpha(y))}$ for any $z \in E$. Consequently, by Lemma 4.16,

$$I(E) \le Ct^{-d/\gamma_\infty(\alpha(y))} \int_E S(x-z,\alpha(z),t-s)\,dz \le C't^{-d/\gamma_\infty(\alpha(y))}.$$

It remains to estimate $I(E^c)$. If $s \ge t/2$, then we get from Lemma 4.15(i)

$$S(z-y,\alpha(y),s) \le Cs^{-d/\gamma_\infty(\alpha(y))} \overset{s\ge t/2}{\le} C't^{-d/\gamma_\infty(\alpha(y))}.$$

Applying Lemma 4.16 yields $I(E^c) \le C''t^{-d/\gamma_\infty(\alpha(y))}$. If $s < t/2$, then $t-s \ge t/2$ and, again by 4.15(i),

$$S(x-z,\alpha(z),t-s) \le C(t-s)^{-d/\gamma_\infty(\alpha(z))} \overset{t-s\ge t/2}{\le} C't^{-d/\gamma_\infty(\alpha(z))}.$$

Invoking the Hölder continuity of $\gamma_\infty \circ \alpha$ another time, it is not difficult to see that this implies $S(x-z,\alpha(z),t-s) \le C''t^{-d/\gamma_\infty(\alpha(y))}$ for any $z \in E^c$. Combining this estimate with Lemma 4.16 proves the desired estimate for $I(E^c)$. □

The two next results concern the convolution properties of g_β.

4.18 Lemma *For any $\beta > 0$ there exists a constant $C = C(\beta) > 0$ such that*

$$\int g_\beta(x-z)g_\beta(z)\,dz \le Cg_\beta(x) \qquad \text{for all } x \in \mathbb{R}^d.$$

The constant C depends continuously on $\beta \in (0,\infty)$.

Proof. As in the previous proof it suffices to prove the statement for $m = 0$. Obviously, the function $[0, \infty) \ni y \mapsto g_\beta(y) = (1 + |y|^{d+\beta})^{-1}$ is non-increasing and

$$g_\beta\left(\frac{y}{2}\right) = 2^{d+\beta}\frac{1}{2^{d+\beta} + |y|^{d+\beta}} \leq 2^{d+\beta}\frac{1}{1 + |y|^{d+\beta}} = 2^{d+\beta} g_\beta(y), \qquad y \in \mathbb{R}^d. \tag{4.41}$$

Fix $x \in \mathbb{R}^d$. If we set

$$D := \left\{z \in \mathbb{R}^d; |z - x| \geq \frac{|x|}{2}\right\},$$

then, by the monotonicity of g_β, $g_\beta(x - z) \leq g_\beta(x/2)$ for any $z \in D$; hence,

$$\int_D g_\beta(x - z) g_\beta(z)\,dz \leq g_\beta\left(\frac{x}{2}\right)\int_D g_\beta(z)\,dz \overset{(4.41)}{\leq} 2^{d+\beta} g_\beta(x)\int g_\beta(z)\,dz =: C(\beta) g_\beta(x).$$

Clearly, the constant $C(\beta)$ is finite for any $\beta > 0$ and depends continuously on $\beta \in (0, \infty)$. For $z \in D^c$ we find by the reverse triangle inequality

$$|z|^{d+\beta} \geq (|x| - |z - x|)^{d+\beta} \overset{z \in D^c}{\geq} \left(\frac{|x|}{2}\right)^{d+\beta}.$$

Thus, $g_\beta(z) \leq g_\beta(x/2)$ for $z \in D^c$. Using the same argument as before, we get

$$\int_{D^c} g_\beta(x - z) g_\beta(z)\,dz \leq C(\beta) g_\beta(x). \qquad \square$$

Lemma 4.19 is essentially taken from [60, Lemma 7.4.3].

4.19 Lemma *For any $\alpha \in I$ and $\beta > 0$ there exists a constant $C = C(\alpha, \beta)$ such that*

$$\int S(x - z, \alpha, t) g_\beta(z - y)\,dz \leq C g_{\gamma_0(\alpha) \wedge \gamma_\infty(\alpha) \wedge \beta}(x - y) \qquad \textit{for all } \; x, y \in \mathbb{R}^d, t \in (0, 1].$$

C depends continuously on $\gamma_0(\alpha), \gamma_\infty(\alpha) \in (0, 2]$ and β.

Proof. As in the proof of Lemma 4.17 it suffices to consider the case $m = 0$. Moreover, a simple change of variables shows that it is enough to prove the statement for $y = 0$. Clearly, the claim follows if

$$\int S(x - z, \alpha, t) g_\beta(z)\,dz \leq C g_\beta(x) + C g_{\gamma_\infty(\alpha)}(x) + C g_{\gamma_0(\alpha) \wedge \gamma_\infty(\alpha)}(x).$$

A straightforward calculation shows that

$$C_1 := C_1(\alpha, \beta) := \sup_{x \in \mathbb{R}^d} \sup_{t \in (0,1]} \left(\int S(x - z, \alpha, t)\,dz + \int g_\beta(z)\,dz\right) < \infty,$$

and that this constant admits an upper bound which depends continuously on β and $\gamma_\infty(\alpha), \gamma_0(\alpha) \in (0, 2]$, cf. Lemma 4.15. If $|z| \geq |x|/2$, then $g_\beta(z) \leq g_\beta(x/2) \leq 2^{d+\beta} g_\beta(x)$ by (4.41). Consequently,

$$\int_{|z| \geq |x|/2} S(x - z, \alpha, t) g_\beta(z)\,dz \leq 2^{d+\beta} g_\beta(x)\int S(x - z, \alpha, t)\,dz \leq 2^{d+\beta} C_1 g_\beta(x).$$

It remains to estimate the integral

$$I := \int_{|z| \leq |x|/2} S(x - z, \alpha, t) g_\beta(z)\,dz.$$

We consider three cases separately:

(i) $|x| \leq t^{1/\gamma_\infty(\alpha)}$: If $|z| \leq |x|/2$, then $|x-z| \leq \frac{3}{2}|x| \leq \frac{3}{2}t^{1/\gamma_\infty(\alpha)}$. By (4.34) (with $A_1 = \frac{3}{2}$, $A_2 = 1$), we find $S(x-z,\alpha,t) \leq Ct^{-d/\gamma_\infty(\alpha)}$. As $g_\beta(z) \leq 1$, this implies

$$I \leq Ct^{-d/\gamma_\infty(\alpha)} \int_{|z|\leq t^{1/\gamma_\infty(\alpha)}/2} g_\beta(z)\,dz = Cd \cdot \omega_d t^{-d/\gamma_\infty(\alpha)} t^{d/\gamma_\infty(\alpha)} \overset{|x|\leq 1}{\leq} C' g_{\gamma_0(\alpha)\wedge\gamma_\infty(\alpha)\wedge\beta}(x).$$

(ii) $t^{1/\gamma_\infty(\alpha)} < |x| \leq 1$: For $z \in \mathbb{R}^d$ such that $|z| \leq |x|/2$, we have by the (reverse) triangle inequality

$$\frac{3}{2} \geq \frac{3}{2}|x| \geq |x-z| \geq \frac{1}{2}|x| \geq \frac{1}{2}t^{1/\gamma_\infty(\alpha)}.$$

It follows from (4.34) (with $A_1 = \frac{1}{2}$, $A_2 = \frac{3}{2}$) that

$$S(x-z,\alpha,t) \leq C\frac{t}{|x-z|^{d+\gamma_\infty(\alpha)}} \leq C'\frac{t}{|x|^{d+\gamma_\infty(\alpha)}}.$$

Thus,

$$I \leq C'\frac{t}{|x|^{d+\gamma_\infty(\alpha)}} \int_{|z|\leq|x|/2} g_\beta(z)\,dz \leq C'C_1 g_{\gamma_\infty(\alpha)}(x).$$

(iii) $|x| > 1$: By the reverse triangle inequality, $|x-z| \geq |x|/2 \geq \frac{1}{2}$ for any $z \in \mathbb{R}^d$ with $|z| \leq |x|/2$. Hence, by (4.34),

$$S(x-z,\alpha,t) \leq C\frac{t}{|x-z|^{d+\gamma_\infty(\alpha)\wedge\gamma_0(\alpha)}} \leq C'\frac{t}{|x|^{d+\gamma_\infty(\alpha)\wedge\gamma_0(\alpha)}}.$$

Consequently, we find

$$I \leq C'\frac{t}{|x|^{d+\gamma_\infty(\alpha)\wedge\gamma_0(\alpha)}} \int_{|z|\leq|x|/2} g_\beta(z)\,dz \leq C'C_1 g_{\gamma_\infty(\alpha)\wedge\gamma_0(\alpha)}(x). \qquad \square$$

The following corollary is a simple consequence of the convolution estimates we have just obtained. It is stated in such a way that we can directly apply it in the next section.

4.20 Corollary *Let $\sigma_1, \sigma_2 > -1$ and choose a constant $C = C(T) > 0$ such that all the results from this section hold true for $t \leq T$ with this fixed constant C. Then*

$$\int_0^t \int (t-s)^{\sigma_1} S(x-z,\alpha(z),t-s) s^{\sigma_2} S(z-y,\alpha(y),s)\,dz\,ds$$
$$\leq Ct^{1+\sigma_1+\sigma_2} B(1+\sigma_1,1+\sigma_2)\big[S(x-y,\alpha(y),t) + tg_{\gamma_\infty^L\wedge\gamma_0^L}(x-y)\big]$$

for all $t \leq T$ and $x, y \in \mathbb{R}^d$. Here B denotes the Beta function. Moreover,

$$\int_0^t \int (t-s)^{\sigma_1} \varphi(t-s,x,z) s^{\sigma_2} \chi(s,z,y)\,dz\,ds \leq Ct^{1+\sigma_1+\sigma_2} B(1+\sigma_1,1+\sigma_2) g_{\gamma_\infty^L\wedge\gamma_0^L}(x-y).$$

holds in any of the following cases:

(i) $\varphi(t,x,y) = S(x-y,\alpha(y),t)$ and $\chi(t,x,y) = g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)$,

(ii) $\varphi(t,x,y) = \chi(t,x,y) = g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)$.

Proof. A simple change of variables shows that

$$\int_0^t (t-s)^{\sigma_1} s^{\sigma_2}\,ds = t^{1+\sigma_1+\sigma_2} \int_0^1 (1-r)^{\sigma_1} r^{\sigma_2}\,dr = t^{1+\sigma_1+\sigma_2} B(1+\sigma_1, 1+\sigma_2). \qquad (\star)$$

Combining this with Lemma 4.17, we find for all $t \leq T$ and $x, y \in \mathbb{R}^d$

$$\begin{aligned}
&\int_0^t \int (t-s)^{\sigma_1} S(x-z, \alpha(z), t-s) s^{\sigma_2} S(z-y, \alpha(y), s)\,dz\,ds \\
&\overset{\text{L4.17}}{\leq} C(S(x-y, \alpha(y), t) + t g_{\gamma_0^L \wedge \gamma_\infty^L}(x-y)) \int_0^t (t-s)^{\sigma_1} s^{\sigma_2}\,ds \\
&\overset{(\star)}{=} C t^{1+\sigma_1+\sigma_2} B(1+\sigma_1, 1+\sigma_2)(S(x-y, \alpha(y), t) + t g_{\gamma_0^L \wedge \gamma_\infty^L}(x-y)).
\end{aligned}$$

This proves the first claim. The other estimates follow analogously from Lemma 4.18 and Lemma 4.19, respectively. □

4.3 A candidate for the transition density

Our aim in this section is to show that our candidate for the transition density

$$p(t,x,y) := p_0(t,x,y) + \sum_{i \geq 1} (p_0 \circledast F^{\circledast i})(t,x,y), \qquad t > 0, x, y \in \mathbb{R}^d,$$

is well-defined and that the mapping $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \ni (t,x,y) \mapsto p(t,x,y)$ is continuous. The proof consists of the following steps: First, we derive on- and off-diagonal estimates for the auxiliary function F,

$$F(t,x,y) = \frac{1}{(2\pi)^d} \int (\psi_{\alpha(y)}(\xi) - \psi_{\alpha(x)}(\xi)) e^{-i\xi\cdot(x-y)} e^{-t\psi_{\alpha(y)}(\xi)}\,d\xi,$$

cf. Lemma 4.21. Using these estimates and the results from the previous section, we obtain, by induction, upper bounds for the iterated time-space convolutions $p_0 \circledast F^{\circledast i}$. This, in turn, allows us to deduce the local uniform convergence of the sum $\sum_{i\geq 1}(p_0 \circledast F^{\circledast i})$, see Theorem 4.25.

Throughout this section, we assume that the assumptions of Theorem 3.2 are satisfied; that is, $I \subseteq \mathbb{R}^n$ is an open convex set, $(\psi_\alpha)_{\alpha \in I} \in A(m)$ is an admissible family of continuous negative definite functions and $\alpha : I \to \mathbb{R}^d$ a Hölder continuous mapping with Hölder exponent $\varrho(\alpha) \in (0,1]$.

4.21 Lemma *For any $\gamma \leq \frac{1}{\gamma_\infty^U}$ and $T > 0$ there exists a constant $C = C(T) > 0$ such that*

$$|F(t,x,y)| \leq C t^{-1+\varrho(\alpha)\gamma}(1 + \ell(ct^{-1/\gamma_\infty^L})) S(x-y, \alpha(y), t) + C t^{(-d+\gamma_\infty^U)\gamma} g_{\gamma_0^L \wedge \gamma_\infty^L}(x-y)$$

for all $x, y \in \mathbb{R}^d$ and $t \in (0,T]$.

For the proof of Lemma 4.21 we need the following auxiliary result. We remind the reader that A^β denotes the pseudo-differential operator with symbol ψ_β, cf. (4.2).

4.22 Lemma *(i)* $|F(t,x,y)| \leq 2\sup_{\beta\in I}|A^\beta p_t^{\alpha(y)}(x-y)|$ *for all* $t>0$ *and* $x,y\in\mathbb{R}^d$.

(ii) There exists a constant $C>0$ *such that*

$$|F(t,x,y)| \leq C|x-y|^{\varrho(\alpha)} \sup_{j=1,\ldots,n}\sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} |\partial_{\beta_j} A^\beta p_t^{\alpha(y)}(x-y)|$$

for all $t>0$ *and* $x,y\in\mathbb{R}^d$*; here* $\sigma_{\alpha(x),\alpha(y)} := \{\lambda\alpha(y)+(1-\lambda)\alpha(x); \lambda\in[0,1]\}\subseteq\mathbb{R}^n$ *denotes the line segment between* $\alpha(x)$ *and* $\alpha(y)$.

Proof. (i) By the triangle inequality, we have

$$\begin{aligned}
&|F(t,x,y)| \\
&\leq \frac{1}{(2\pi)^d}\left|\int \psi_{\alpha(x)}(\xi)e^{-i\xi\cdot(x-y)}e^{-t\psi_{\alpha(y)}(\xi)}\,d\xi\right| + \frac{1}{(2\pi)^d}\left|\int \psi_{\alpha(y)}(\xi)e^{-i\xi\cdot(x-y)}e^{-t\psi_{\alpha(y)}(\xi)}\,d\xi\right| \\
&= |A^{\alpha(x)}p_t^{\alpha(y)}(x-y)| + |A^{\alpha(y)}p_t^{\alpha(y)}(x-y)| \leq 2\sup_{\beta\in I}|A^\beta p_t^{\alpha(y)}(x-y)|.
\end{aligned}$$

(ii) If we set $\sigma_{\alpha(x),\alpha(y)}(\lambda) := (1-\lambda)\alpha(x)+\lambda\alpha(y)$, $\lambda\in[0,1]$, for fixed $x,y\in\mathbb{R}^d$, then it follows from the gradient theorem that

$$\begin{aligned}
\Psi_{\alpha(y)}(r)-\Psi_{\alpha(x)}(r) &= \int_{\sigma_{\alpha(x),\alpha(y)}} \nabla_\beta\Psi_\beta(r)\,d\beta \\
&= \int_0^1 \nabla_\beta\Psi_\beta(r)\Big|_{\beta=\sigma_{\alpha(x),\alpha(y)}(\lambda)}\cdot(\alpha(y)-\alpha(x))\,d\lambda \\
&= \sum_{j=1}^n(\alpha_j(y)-\alpha_j(x))\int_0^1 \partial_{\beta_j}\Psi_\beta(r)\Big|_{\beta=\sigma_{\alpha(x),\alpha(y)}(\lambda)}\,d\lambda
\end{aligned}$$

for all $r\in\mathbb{R}$. Hence,

$$\psi_{\alpha(y)}(\xi)-\psi_{\alpha(x)}(\xi) = \sum_{j=1}^n(\alpha_j(y)-\alpha_j(x))\int_0^1 \partial_{\beta_j}\psi_\beta(\xi)\Big|_{\beta=\sigma_{\alpha(x),\alpha(y)}(\lambda)}\,d\lambda$$

for all $\xi\in\mathbb{R}^d$. Applying Fubini's theorem and the differentiation lemma for parameter-dependent integrals[3], we get

$$\begin{aligned}
&F(t,x,y) \\
&= \frac{1}{(2\pi)^d}\sum_{j=1}^n(\alpha_j(y)-\alpha_j(x))\int_{\mathbb{R}^d}\int_0^1 \partial_{\beta_j}\psi_\beta(\xi)\Big|_{\beta=\sigma_{\alpha(x),\alpha(y)}(\lambda)} e^{-i\xi\cdot(x-y)}e^{-t\psi_{\alpha(y)}(\xi)}\,d\lambda\,d\xi \\
&= \frac{1}{(2\pi)^d}\sum_{j=1}^n(\alpha_j(y)-\alpha_j(x))\int_0^1 \partial_{\beta_j}\int_{\mathbb{R}^d}\psi_\beta(\xi)\Big|_{\beta=\sigma_{\alpha(x),\alpha(y)}(\lambda)} e^{-i\xi\cdot(x-y)}e^{-t\psi_{\alpha(y)}(\xi)}\,d\xi\,d\lambda \\
&= \frac{1}{(2\pi)^d}\sum_{j=1}^n(\alpha_j(y)-\alpha_j(x))\int_0^1 \partial_{\beta_j}A^\beta p_t^{\alpha(y)}(x-y)\Big|_{\beta=\sigma_{\alpha(x),\alpha(y)}(\lambda)}\,d\lambda. \qquad (4.42)
\end{aligned}$$

As $\alpha(\cdot)$, hence $\alpha_j(\cdot)$, is Hölder continuous with exponent $\varrho(\alpha)$, this proves the claim. □

[3]Both theorems are applicable because of the growth assumption (C3) on ψ_α.

Proof of Lemma 4.21. We restrict ourselves to the case $m = 0$ and $T \leq 1$. The proof for $m > 0$ is exactly the same, but the calculations are more lengthy because of the additional term $\exp(-\frac{m}{4}|x-y|)$ in (almost) every line. In this proof we will frequently use the elementary estimate $t^\mu \leq t^\nu$ for $t \leq T \leq 1$ and $\mu \geq \nu$; if $T > 1$ then we have to use the estimate $t^\mu \leq T^{\mu-\nu} t^\nu$. We consider several cases separately.

(i) $|x-y| \leq t^{1/\gamma_\infty(\alpha(y))}$: Applying Lemma 4.22(ii) and Corollary 4.11, we find

$$\begin{aligned} &|F(t,x,y)| \\ &\leq C|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty(\alpha(y))})) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} t^{-(d+\gamma_\infty(\beta))/\gamma_\infty(\alpha(y))} \\ &\leq \frac{C}{t}|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty^L}))t^{-d/\gamma_\infty(\alpha(y))} \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} t^{-(\gamma_\infty(\beta)-\gamma_\infty(\alpha(y))/\gamma_\infty(\alpha(y))} \\ &\leq Ct^{-1+\varrho(\alpha)/\gamma_\infty(\alpha(y))}(1+\ell(ct^{-1/\gamma_\infty^L}))S(x-y,\alpha(y),t) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} t^{-(\gamma_\infty(\beta)-\gamma_\infty(\alpha(y))/\gamma_\infty(\alpha(y))} \end{aligned}$$

where we have used in the last step that, by (4.9), $S(x-y,\alpha(y),t) = t^{-d/\gamma_\infty(\alpha(y))}$. Since $t^{-1+\varrho(\alpha)/\gamma_\infty(\alpha(y))} \leq t^{-1+\varrho(\alpha)\gamma}$ for any $\gamma \leq \frac{1}{\gamma_\infty^U} \leq \frac{1}{\gamma_\infty(\alpha(y))}$ and $t \leq 1$, this proves the claim provided that

$$C' := \sup_{t\in(0,1]} \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} t^{-(\gamma_\infty(\beta)-\gamma_\infty(\alpha(y))/\gamma_\infty(\alpha(y))} < \infty.$$

To see this, we note that by the Hölder continuity of γ_∞ and $\alpha(\cdot)$

$$\begin{aligned} \left|\frac{\gamma_\infty(\beta)-\gamma_\infty(\alpha(y))}{\gamma_\infty(\alpha(y))}\right| &\leq \frac{C''}{\gamma_\infty^L}|\beta-\alpha(y)|^{\varrho(\gamma_\infty)} \leq \frac{C''}{\gamma_\infty^L}|\alpha(x)-\alpha(y)|^{\varrho(\gamma_\infty)} \\ &\leq \frac{C'''}{\gamma_\infty^L} t^{\varrho(\gamma_\infty)\varrho(\alpha)/\gamma_\infty(\alpha(y))} \leq \frac{C'''}{\gamma_\infty^L} t^{\varrho(\gamma_\infty)\varrho(\alpha)/\gamma_\infty^U}, \end{aligned}$$

for any $\beta \in \sigma_{\alpha(x),\alpha(y)}$, $t \leq 1$ and $|x-y| \leq t^{1/\gamma_\infty(\alpha(y))}$. Thus,

$$C' \leq \sup_{t\in(0,1]} \exp\left(-\frac{C'''}{\gamma_\infty^L} t^{\varrho(\gamma_\infty)\varrho(\alpha)/\gamma_\infty^U} \log t\right) < \infty.$$

(ii) $t^{1/\gamma_\infty(\alpha(y))} < |x-y| \leq t^\gamma$: It follows from Lemma 4.22(ii) and Corollary 4.11 that there exists a constant $C > 0$ such that

$$|F(t,x,y)| \leq C|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty^L})) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} \frac{1}{|x-y|^{d+\gamma_\infty(\beta)}}$$

for all $t \leq 1$ and $x,y \in \mathbb{R}^d$. As $t \leq 1$ we have $t^{1/\gamma_\infty(\alpha(y))} \leq |x-y| \leq 1$, and therefore $S(x-y,\alpha(y),t) = \frac{t}{|x-y|^{d+\gamma_\infty(\alpha(y))}}$ by the very definition of S. Thus,

$$\begin{aligned} &|F(t,x,y)| \\ &\leq \frac{C}{t}|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty^L}))S(x-y,\alpha(y),t) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} |x-y|^{\gamma_\infty(\alpha(y))-\gamma_\infty(\beta)} \end{aligned}$$

$$\leq Ct^{-1+\varrho(\alpha)\gamma}(1+\ell(t^{-1/\gamma_\infty^L}))S(x-y,\alpha(y),t) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} |x-y|^{\gamma_\infty(\alpha(y))-\gamma_\infty(\beta)}.$$

Using a very similar argument as in (i), it is not difficult to see that

$$C' := \sup_{t\in(0,1]} \sup_{|x-y|\leq t^\gamma} \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} |x-y|^{\gamma_\infty(\alpha(y))-\gamma_\infty(\beta)} < \infty.$$

This gives the desired estimate for $|F(t,x,y)|$.

(iii) $t^\gamma < |x-y| \leq 1$: Since

$$1 \geq |x-y| > t^\gamma \geq t^{1/\gamma_\infty^U} \geq t^{1/\gamma_\infty(\alpha(y))}$$

for all $t \leq 1$, Corollary 4.11 shows

$$|A^\beta p_t^{\alpha(y)}(x-y)| \leq C\frac{1}{|x-y|^{d+\gamma_\infty(\beta)}} \qquad \text{for all} \quad \beta\in I.$$

Combining this with Lemma 4.22(i), we get

$$|F(t,x,y)| \leq 2C\sup_{\beta\in I}\frac{1}{|x-y|^{d+\gamma_\infty(\beta)}} = 2C\sup_{\beta\in I}\left(\frac{1}{t^{\gamma(d+\gamma_\infty(\beta))}}\frac{1}{\left|\frac{|x-y|}{t^\gamma}\right|^{d+\gamma_\infty(\beta)}}\right).$$

As $t\leq 1$ and $\frac{|x-y|}{t^\gamma}\geq 1$, this implies

$$\begin{aligned}|F(t,x,y)| \leq 4Ct^{-\gamma(d+\gamma_\infty^U)}\frac{1}{1+\left|\frac{|x-y|}{t^\gamma}\right|^{d+\gamma_\infty^L}} &\leq 4Ct^{-\gamma(d+\gamma_\infty^U)}\frac{1}{1+|x-y|^{d+\gamma_\infty^L}} \\ &= 4Ct^{-\gamma(d+\gamma_\infty^U)}g_{\gamma_\infty^L}(x-y).\end{aligned}$$

(iv) $|x-y|>1$: By Corollary 4.11, we have

$$|A^\beta p_t^{\alpha(y)}(x-y)| \leq C\frac{1}{|x-y|^{d+\gamma_\infty(\beta)\wedge\gamma_0(\beta)}} \qquad \text{for all} \quad \beta\in I.$$

Hence,

$$\begin{aligned}|F(t,x,y)| \leq 2C\sup_{\beta\in I}\frac{1}{|x-y|^{d+\gamma_\infty(\beta)\wedge\gamma_0(\beta)}} &\overset{|x-y|\geq 1}{\leq} 2C\frac{1}{|x-y|^{d+\gamma_\infty^L\wedge\gamma_0^L}} \\ &= 4Cg_{\gamma_\infty^L\wedge\gamma_0^L}(x-y).\end{aligned}$$ □

Since ℓ is slowly varying, we have $\lim_{r\to\infty} r^{-\varepsilon}\ell(r) = 0$ for any $\varepsilon>0$ (cf. Lemma A.2). Therefore the lemma we have just proved yields

$$|F(t,x,y)| \leq Ct^{-1+\kappa_1}S(x-y,\alpha(y),t) + Ct^{-1+\kappa_2}g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)$$

with $\kappa_1 := \varrho(\alpha)\gamma - \varepsilon/\gamma_\infty^L$ and $\kappa_2 := \gamma(-d+\gamma_\infty^U)+1$. Choosing $\varepsilon>0$ and $\gamma>0$ sufficiently small, we can achieve $\kappa := \min\{\kappa_1,\kappa_2\}>0$. [4] Then

$$|F(t,x,y)| \leq Ct^{-1+\kappa}(S(x-y,\alpha(y),t)+g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)) =: G(t,x,y); \tag{4.43}$$

here and below the constant $C\geq\max\{1,\kappa^{-1},T\}$ is chosen sufficiently large such that the estimates of Lemma 4.17, Lemma 4.18, Lemma 4.19 and Lemma 4.21 hold true for $t\leq T$ with this fixed constant $C=C(T)$. For later reference we make the following observation.

[4] We choose κ this way to keep calculations as simple as possible.

4.23 Remark If $U:(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is such that

$$|U(t,x,y)|\le C'\begin{cases} t^{-1}|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty^L}))S(x-y,\alpha(y),t), & |x-y|\le t^\gamma,\\ \sup_{\beta\in I}\frac{1}{|x-y|^{d+\gamma_\infty(\beta)}}\exp\left(-\frac{m}{4}|x-y|\right), & t^\gamma<|x-y|\le 1,\\ \sup_{\beta\in I}\frac{1}{|x-y|^{d+\gamma_\infty(\beta)\wedge\gamma_0(\beta)}}\exp\left(-\frac{m}{4}|x-y|\right), & |x-y|>1,\end{cases}\tag{4.44}$$

for all $t\in(0,T]$, $x,y\in\mathbb{R}^d$, $\gamma\in(0,\frac{1}{\gamma_\infty^U}]$ for some constant $C'=C'(T)$, then

$$|U(t,x,y)|\le C''G(t,x,y).$$

This is a direct consequence of the definition of G and the proof of Lemma 4.21. We note that (4.44) is, in particular, satisfied if $|U(t,x,y)|$ is bounded from above by a multiple of $\exp\left(-\frac{m}{4}|x-y|\right)$ times

$$\begin{cases} |x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty(\alpha(y))}))\sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}}t^{-(d+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}, & |x-y|\le t^{1/\gamma_\infty(\alpha(y))},\\ |x-y|^{\varrho(\alpha)}(1+\ell(c|x-y|^{-1}))\sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}}\frac{1}{|x-y|^{d+\gamma_\infty(\beta)}}, & t^{1/\gamma_\infty(\alpha(y))}\le|x-y|\le t^\gamma,\\ \sup_{\beta\in I}\frac{1}{|x-y|^{d+\gamma_\infty(\beta)}}, & t^\gamma<|x-y|\le 1,\\ \sup_{\beta\in I}\frac{1}{|x-y|^{d+\gamma_\infty(\beta)\wedge\gamma_0(\beta)}}, & |x-y|>1.\end{cases}\tag{4.45}$$

Using the previous lemma and the results from Section 4.2, it is not difficult to derive estimates for the time-space convolutions $(p_0\circledast F^{\circledast i})$, $i\ge 1$. We remind the reader that we have defined iteratively

$$G^{\circledast i}(t,x,y):=(G^{\circledast i-1}\circledast G)(t,x,y)\qquad\qquad G^{\circledast 1}(t,x,y):=G(t,x,y)$$

for $i\ge 2$ and that, by the associativity of the time-space convolution,

$$(H\circledast G^{\circledast i})(t,x,y)=((H\circledast G^{\circledast i-1})\circledast G)(t,x,y)$$

for any two functions G,H such that the above expressions are well-defined (cf. Chapter 1). From now on we will frequently encounter time-space convolutions of the form $(S\circledast f)$ for some function f; since the function S depends on the triplet (x,α,t) it might be unclear to the reader what we mean by this expression. We denote, in abuse of notation, the expression

$$\int_0^t\int S(x-z,\alpha(z),t-s)f(s,z,y)\,dz\,ds$$

by $(S\circledast f)(t,x,y)$. If we set $\tilde S(t,x,y):=S(x-y,\alpha(y),t)$, then this is equivalent to

$$(S\circledast f)(t,x,y)\overset{\text{abuse}}{:=}(\tilde S\circledast f)(t,x,y)\overset{\text{def}}{=}\int_0^t\int S(x-z,\alpha(z),t-s)f(s,z,y)\,dz\,ds.$$

4.24 Lemma *For $i\in\mathbb{N}$ define*

$$H^i(t,x,y):=4^iC^{3i+1}\frac{\Gamma(\kappa)^i}{\Gamma(i\kappa)}t^{i\kappa}\big(S(x-y,\alpha(y),t)+g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)\big).$$

Then:

(i) $(H^i \circledast G)(t,x,y) \le H^{i+1}(t,x,y)$ *for all* $t>0$, $x,y \in \mathbb{R}^d$.

(ii) $|(p_0 \circledast F^{\circledast i})(t,x,y)| \le C(S \circledast G^{\circledast i})(t,x,y) \le H^i(t,x,y)$ *for all* $t>0$, $x,y\in\mathbb{R}^d$.

(iii) $G^{\circledast i}(t,x,y) \le H^{i-1}(t,x,y)$ *for all* $t>0$, $x,y\in\mathbb{R}^d$ *and* $i\ge 2$.

Proof. We start to prove (i). For brevity of notation, we write γ instead of $\gamma_\infty^L \wedge \gamma_0^L$. Fix $t>0$ and $x,y\in\mathbb{R}^d$. By the very definition of the time-space convolution and H^i and G, we have $(H^i\circledast G)(t,x,y) = I_1+I_2+I_3+I_4$ where

$$I_1 := 4^i \frac{\Gamma(\kappa)^i}{\Gamma(i\kappa)} C^{3i+1} \int_0^t \int (t-s)^{i\kappa} S(x-z,\alpha(z),t-s) s^{-1+\kappa} S(z-y,\alpha(y),s)\, dz\, ds,$$
$$I_2 := 4^i \frac{\Gamma(\kappa)^i}{\Gamma(i\kappa)} C^{3i+1} \int_0^t \int (t-s)^{i\kappa} S(x-z,\alpha(z),t-s) s^{-1+\kappa} g_\gamma(z-y)\, dz\, ds,$$
$$I_3 := 4^i \frac{\Gamma(\kappa)^i}{\Gamma(i\kappa)} C^{3i+1} \int_0^t \int (t-s)^{i\kappa} g_\gamma(x-z) s^{-1+\kappa} S(z-y,\alpha(y),s)\, dz\, ds,$$
$$I_4 := 4^i \frac{\Gamma(\kappa)^i}{\Gamma(i\kappa)} C^{3i+1} \int_0^t \int (t-s)^{i\kappa} g_\gamma(x-z) s^{-1+\kappa} g_\gamma(z-y)\, dz\, ds.$$

Applying Corollary 4.20 we can easily estimate the terms:

$$\begin{aligned} I_1 &\overset{\text{C4.20}}{\le} 4^i \frac{\Gamma(\kappa)^i}{\Gamma(i\kappa)} C^{3i+2} (S(x-y,\alpha(y),t) + t g_\gamma(x-y)) t^{(i+1)\kappa} B(\kappa,1+i\kappa) \\ &\le 4^i \frac{\Gamma(\kappa)^i}{\Gamma(i\kappa)} C^{3i+3} (S(x-y,\alpha(y),t) + g_\gamma(x-y)) t^{(i+1)\kappa} B(\kappa,1+i\kappa). \end{aligned}$$

Using the monotonicity of the Beta function, i. e. $B(\kappa,1+i\kappa)\le B(\kappa,i\kappa)$, and the well-known identity

$$B(u,v) = \tfrac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \qquad u,v>0, \tag{$\star$}$$

we get $I_1 \le H^{i+1}(t,x,y)/4$. The estimates for I_2, I_3, I_4 are obtained in a very similar way.

$$I_2 \overset{\text{C4.20}}{\le} 4^i \frac{\Gamma(\kappa)^i}{\Gamma(i\kappa)} C^{3i+2} t^{(i+1)\kappa} B(\kappa,i\kappa) g_\gamma(x-y)$$

Now $(\star)$ gives $I_2 \le H^{i+1}(t,x,y)/4$. Similarly, we get from Corollary 4.20

$$\begin{aligned} I_3 &\le 4^i \frac{\Gamma(\kappa)^i}{\Gamma(i\kappa)} C^{3i+2} t^{(i+1)\kappa} B(\kappa,(i-1)\kappa) g_\gamma(x-y) \\ I_4 &\le 4^i \frac{\Gamma(\kappa)^i}{\Gamma(i\kappa)} C^{3i+2} t^{(i+1)\kappa} B(\kappa,i\kappa) g_\gamma(x-y). \end{aligned}$$

Invoking again the monotonicity of $B(\cdot,\cdot)$ and $(\star)$, we conclude $I_3+I_4 \le H^{i+1}(t,x,y)/2$. Combining the estimates gives (i). We prove (ii) by induction.

- basis ($i=1$): By the very definition of p_0, we have $p_0(t,x,y) = p_t^{\alpha(y)}(x-y)$. Consequently, $|p_0(t,x,y)| \le CS(x-y,\alpha(y),t)$ by (4.10). Moreover, the definition of G, cf. (4.43), gives $|F|\le G$. Consequently, $|(p_0\circledast F)(t,x,y)| \le C(S\circledast G)(t,x,y)$.[5] It remains to show that $C(S\circledast G)(t,x,y)\le H^1(t,x,y)$. To this end, we split up the integral:

$$(S\circledast G)(t,x,y) = I_1 + I_2$$

[5] p_0 as well as G, H^i are non-negative; therefore we may drop the modulus.

where by Corollary 4.20

$$\begin{aligned}I_1 &:= C\int_0^t\int S(x-z,\alpha(z),t-s)s^{-1+\kappa}S(z-y,\alpha(y),s)\,dz\,ds\\&\le \frac{C^2}{\kappa}t^\kappa(S(x-y,\alpha(y),t)+tg_{\gamma_\infty}(x-y))\le\frac{1}{4C}H^1(t,x,y)\end{aligned}$$

and

$$\begin{aligned}I_2 &:= C\int_0^t\int S(x-z,\alpha(z),t-s)s^{-1+\kappa}g_\gamma(z-y)\,dz\,ds\\&\le\frac{C^2}{\kappa}t^\kappa g_\gamma(x-y)\le\frac{1}{4C}H^1(t,x,y).\end{aligned}$$

This shows $C(S\circledast G)(t,x,y)\le\frac12 H^1(t,x,y)$.

- inductive step ($i\rightsquigarrow i+1$): Since the time-space convolution is associative and monotone, we have

$$|(p_0\circledast F^{\circledast i+1})(t,x,y)|=|((p_0\circledast F^{\circledast i})\circledast F)(t,x,y)|\le(|p_0\circledast F^{\circledast i}|\circledast|F|)(t,x,y)$$

Using $|F|\le G$ and the induction hypothesis (IH), we find

$$\begin{aligned}|(p_0\circledast F^{\circledast(i+1)})(t,x,y)| &\overset{\text{IH}}{\le} C((S\circledast G^{\circledast i})\circledast|F|)(t,x,y)\overset{\text{IH}}{\le}(H^i\circledast G)(t,x,y)\\&\overset{\text{(i)}}{\le} H^{i+1}(t,x,y).\end{aligned}$$

We also prove (iii) by induction. From Corollary 4.20 it is not difficult to see that $G^{\circledast 2}(t,x,y)=(G\circledast G)(t,x,y)$ satisfies $G^{\circledast 2}(t,x,y)\le H^1(t,x,y)$. If the claim holds for some $i\ge 2$, then

$$G^{\circledast(i+1)}(t,x,y)=(G^{\circledast i}\circledast G)(t,x,y)\le(H^{i-1}\circledast G)(t,x,y)\overset{(i)}{\le}H^i(t,x,y),$$

i. e. the claimed estimate also holds for $i+1$. □

Now we are ready to show that the candidate for the transition density is well-defined and to prove the first important properties.

4.25 Theorem *The series*

$$p(t,x,y):=p_0(t,x,y)+\sum_{i\in\mathbb{N}}(p_0\circledast F^{\circledast i})(t,x,y),\qquad x,y\in\mathbb{R}^d,t>0,$$

converges locally uniformly in $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. It has the following properties:

(i) $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\ni(t,x,y)\to p(t,x,y)$ is continuous.

(ii) The function $\Phi(t,x,y):=\sum_{i\ge1}F^{\circledast i}(t,x,y)$ is continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ and

$$p(t,x,y)=p_0(t,x,y)+(p_0\circledast\Phi)(t,x,y)\qquad\textit{for all }\ t>0,x,y\in\mathbb{R}^d.\tag{4.46}$$

(iii) For any $T > 0$ there exists a constant $C = C(T) > 0$ such that

$$|p(t,x,y) - p_0(t,x,y)| \le CtG(t,x,y) \tag{4.47}$$

$$|p(t,x,y)| \le C(S(x-y,\alpha(y),t) + tG(t,x,y)) \tag{4.48}$$

$$|\Phi(t,x,y)| \le CG(t,x,y) \tag{4.49}$$

for all $x, y \in \mathbb{R}^d$ and $t \in (0,T]$.

Proof. By the definition of H^i, we have

$$\begin{aligned}\sum_{i\in\mathbb{N}} H^i(t,x,y) &\le t^\kappa(S(x-y,\alpha(y),t) + g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)) \sum_{i\in\mathbb{N}} 4^i C^{3i} \frac{\Gamma(\kappa)^i}{\Gamma(i\kappa)} T^{(i-1)\kappa} \\ &= C' tG(t,x,y) \sum_{i\in\mathbb{N}} 4^i C^{3i} \frac{\Gamma(\kappa)^i}{\Gamma(i\kappa)} T^{(i-1)\kappa} \end{aligned} \tag{$\star$}$$

for all $t \le T$ and $x, y \in \mathbb{R}^d$. Note that for any constant $c > 0$ we have $\Gamma(i\kappa) \ge c^i$ for i sufficiently large (cf. Lemma A.6). Choosing c sufficiently large, it follows easily that the series in ($\star$) converges and this proves the local uniform convergence of $\sum_{i\in\mathbb{N}} H^i(t,x,y)$. Since Lemma 4.24(ii) gives

$$\left|\sum_{i\in\mathbb{N}} (p_0 \circledast F^{\circledast i})(t,x,y)\right| \le \sum_{i\in\mathbb{N}} H^i(t,x,y),$$

we find that p is well-defined and that $(t,x,y) \mapsto p(t,x,y)$ is continuous as a local uniform limit of continuous functions. Moreover, the above estimate obviously shows (4.47). (4.48) is a direct consequence of (4.46) and (4.10). We note that by Lemma 4.24(iii) and ($\star$)

$$|\Phi(t,x,y)| \le G(t,x,y) + \sum_{i\ge 2} G^{\circledast i}(t,x,y) \le G(t,x,y) + \sum_{i\ge 1} H^i(t,x,y) \le C'' G(t,x,y) < \infty$$

for $t \le T$, i. e. Φ is well-defined and (4.49) holds. The estimate also gives that Φ is continuous as a local uniform limit of continuous functions. An application of the monotone convergence theorem shows

$$(p_0 \circledast |\Phi|)(t,x,y) \le \sum_{i\ge 1} (p_0 \circledast G^{\circledast i})(t,x,y) \overset{\text{L4.24}}{\le} \sum_{i\ge 1} H^i(t,x,y) < \infty;$$

therefore we may apply the dominated convergence theorem to get (4.46). □

4.4 Time derivative

In this section we establish that $(0,\infty) \ni t \mapsto p(t,x,y)$ is differentiable for each $x, y \in \mathbb{R}^d$. The reasoning is similar as in Section 4.3: After deriving estimates for $\partial_t F(t,x,y)$, we perform a proof by induction to obtain estimates for $\partial_t(p_0 \circledast F^{\circledast i})(t,x,y)$ in terms of H^i, see Lemma 4.24 for the definition – and these estimates give us the differentiability of $p(\cdot,x,y)$.

Let us briefly explain what causes the main difficulty. Since p_0 solves for each fixed $y \in \mathbb{R}^d$ the equation

$$\partial_t p_0(t,x,y) = A^{\alpha(y)} p_0(t,x,y), \tag{4.50}$$

cf. Section 1.7, it follows from Corollary 4.9 that

$$|\partial_t p_0(t,x,y,)| \le C t^{-d/\gamma_\infty(\alpha(y))} S(x-y,\alpha(y),t).$$

This shows that taking the derivative with respect to t increases the order of the singularity at $t=0$. This means in particular that we cannot expect to simply interchange differentiation and integration to prove the differentiability of $(p_0 \circledast F)(\cdot,x,y)$; for example the identity

$$\frac{\partial}{\partial t}\int_0^\tau \int p_0(t-s,x,z)F(s,z,y)\,dz\,ds = \int_0^\tau \int \partial_t p_0(t-s,x,z)F(s,z,y)\,dz\,ds$$

does, in general, not hold true since the integrand on the right-hand side might not even be integrable. To avoid this problem, we will first rewrite the time-space convolution $p_0 \circledast F$,

$$(p_0 \circledast F)(t,x,y) = t\int_0^1 \int p_0(t(1-s),x,z)F(ts,z,y)\,dzds$$

and then prove the differentiability of this expression (with respect to t) using the differentiation lemma for parametrized integrals. The idea is from Kolokoltsov [60, p. 332]; let us, however, remark that Kolokoltsov provides the reader barely a sketch of the proof.

As in the previous section, we assume throughout that the assumptions of Theorem 3.2 are satisfied.

4.26 Lemma *For any $0 < \gamma \le \frac{1}{\gamma_\infty^U}$ and $T > 0$ there exists a constant $C = C(T) > 0$ such that*

$$|\partial_t F(t,x,y)| \le C t^{-2+\gamma\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty^L}))S(x-y,\alpha(y),t) + Ct^{(-d+2\gamma_\infty^U)\gamma} g_{\gamma_0^L \wedge \gamma_\infty^L}(x-y)$$

for all $x,y \in \mathbb{R}^d$ and $t \in (0,T]$. In particular, there exists a constant $C' = C'(T) > 0$ such that

$$|\partial_t F(t,x,y)| \le C' t^{-1} G(t,x,y)$$

for all $x,y \in \mathbb{R}^d$ and $t \in (0,T]$ with G defined in (4.43).

Proof. It follows from the differentiation lemma for parametrized integrals – which is applicable because of the growth assumption (C3) – that the partial derivative $\partial_t F(t,x,y)$ exists and

$$\partial_t F(t,x,y) = -\int \psi_{\alpha(y)}(\xi)(\psi_{\alpha(y)}(\xi) - \psi_{\alpha(x)}(\xi))e^{-i(x-y)\cdot\xi}e^{-t\psi_{\alpha(y)}}(\xi)\,d\xi.$$

The next step is to establish the counterpart of Lemma 4.22, i. e. show that

$$|\partial_t F(t,x,y)| \le 2\sup_{\beta \in I} |\partial_t A^\beta p_t^{\alpha(y)}(x-y)| \tag{$\star$}$$

and

$$|\partial_t F(t,x,y)| \leq C|x-y|^{\varrho(\alpha)} \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} \sup_{j\in\{1,\dots,n\}} |\partial_t \partial_{\beta_j} A^\beta p_t^{\alpha(y)}(x-y)|; \qquad (\star\star)$$

$(\star)$ is a simple consequence of the triangle inequality and $(\star\star)$ follows by differentiating the identity (4.42) with respect to t using the differentiation lemma for parametrized integrals. Since the remaining part of the proof is very close to the proof of Lemma 4.21, we only point out the differences between these two proofs.

(i) $|x-y| \leq t^{1/\gamma_\infty(\alpha(y))}$: By $(\star\star)$ and Corollary 4.11,

$$|\partial_t F(t,x,y)| \leq Ct^{-1}|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty(\alpha(y))})) \sup_{\beta\in\sigma_{\alpha(y),\alpha(x)}} t^{-(d+\gamma_\infty(\beta))/\gamma_\infty(\alpha(y))}.$$

Except of the additional factor t^{-1} this is exactly the same estimate as in the proof of Lemma 4.21.

(ii) $t^{1/\gamma_\infty(\alpha(y))} < |x-y| \leq t^\gamma$: Compared with the proof of Lemma 4.21 we get an additional factor $\frac{1}{|x-y|^{\gamma_\infty(\alpha(y))}}$. Since $|x-y| \geq t^{1/\gamma_\infty(\alpha(y))}$, we can bound this term from above by t^{-1}.

(iii) $t^\gamma < |x-y| < 1$: Corollary 4.11 shows

$$|\partial_t A^\beta p_t^{\alpha(y)}(x-y)| \leq C\frac{1}{|x-y|^{d+\gamma_0(\alpha(y))+\gamma_0(\beta)}}.$$

Proceeding as in the proof of Lemma 4.21, case (iii), we find

$$|\partial_t F(t,x,y)| \leq C' t^{\gamma(-d+2\gamma_\infty^U)} g_{2\gamma_\infty^L}(x-y) \leq C' t^{\gamma(-d+2\gamma_\infty^U)} g_{\gamma_0(\infty)\wedge\gamma_\infty^L}(x-y).$$

(iv) $|x-y| > 1$: By Corollary 4.11,

$$|\partial_t A^\beta p_t^{\alpha(y)}(x-y)| \leq C\frac{1}{|x-y|^{d+(\gamma_0(\beta)+\gamma_0(\alpha(y)))\wedge(\gamma_\infty(\beta)\wedge\gamma_\infty(\alpha(y)))}}.$$

As $|x-y| \geq 1$ it follows now easily from $(\star)$ that

$$|\partial_t F(t,x,y)| \leq C'\frac{1}{|x-y|^{d+2\gamma_0^L\wedge\gamma_\infty^L}} \leq 2C' g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y).$$

This proves the first estimate. As $\gamma/\gamma_\infty^U \leq 1$, it follows that

$$|\partial_t F(t,x,y)| \leq Ct^{-1}\left(t^{-1+\gamma\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty^L}))S(x-y,\alpha(y),t) + t^{\gamma(-d+\gamma_\infty^U)} g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)\right).$$

By virtue of our choice of κ, this gives the claimed estimate, cf. (4.43). □

The next theorem is the key result to prove the differentiability of $p(\cdot,x,y)$.

4.27 Theorem *The function Φ, defined in Theorem 4.25, is differentiable with respect to t and $\partial_t \Phi(t,x,y)$ is continuous in (t,x,y). For any $T > 0$ there exists a constant $C = C(T) > 0$ such that*

$$\left|\frac{\partial}{\partial t}(\Phi(t,x,y) - F(t,x,y))\right| \le CG(t,x,y)$$

and

$$\left|\frac{\partial}{\partial t}\Phi(t,x,y)\right| \le Ct^{-1}G(t,x,y)$$

for all $x, y \in \mathbb{R}^d$ and $t \in (0,T]$.

Proof. First of all, we note that for any two non-negative measurable functions f, g

$$\begin{aligned}(f \circledast g)(t,x,y) &= \int_0^t \int f(t-s,x,z) g(s,z,y)\, dz\, ds \\ &\overset{s \rightsquigarrow st}{=} t \int_0^1 \int f(t(1-s),x,z) g(st,z,y)\, dz\, ds. \qquad (4.51)\end{aligned}$$

We now claim that $\partial_t F^{\circledast k}$ exists for all $k \ge 2$, $\partial_t F^{\circledast k}(t,x,y)$ is continuous in (t,x,y) and

$$\left|\frac{\partial}{\partial t} F^{\circledast k}(t,x,y)\right| \le \frac{ck}{t} H^{k-1}(t,x,y), \qquad t \in (0,T], x, y \in \mathbb{R}^d \qquad (4.52)$$

for a suitable constant $c = c(T) > 0$. If we have shown this, then the result follows using the same reasoning as in the proof of Theorem 4.25. We prove (4.52) by induction.

- basis ($k = 2$): We want to use (4.51) with $f = g = F$ to prove the differentiability. First we show that the differentiation lemma for parametrized integrals is applicable. By Lemma 4.26 and (4.43), we have

  $$\begin{aligned}&\left|\frac{\partial}{\partial t}\big(F(t(1-s),x,z)F(ts,z,y)\big)\right| \\ &\quad \le (1-s)\left|\frac{\partial}{\partial \tau}F(\tau,x,z)F(ts,z,y)\right|\Big|_{\tau=t(1-s)} + s\left|F(t(1-s),x,z)\frac{\partial}{\partial \tau}F(\tau,z,y)\right|\Big|_{\tau=ts} \\ &\quad \le \frac{2C}{t}G(t(1-s),x,z)G(ts,x,z) =: \frac{2C}{t}u_1(s,t,x,z,y) \qquad (4.53)\end{aligned}$$

  It follows from some elementary estimates and Lemma 4.24 that the mapping

  $$(s,z) \mapsto \sup_{t \in [T_0,T]} u_1(s,t,x,z,y)$$

  is integrable for any $0 < T_0 < T < \infty$, cf. Lemma A.7. Consequently, we may apply the differentiation lemma for parametrized integrals:

  $$\begin{aligned}\frac{\partial}{\partial t}F^{\circledast 2}(t,x,y) &= \int_0^1 \int F(t(1-s),x,z)F(st,z,y)\, dz\, ds \\ &\quad + t\int_0^1 \int \partial_t(F(t(1-s),x,z)F(ts,z,y))\, dz\, ds =: I_1 + I_2.\end{aligned}$$

  As $0 < T_0 < T < \infty$ is arbitrary, this proves the differentiability on $(0,\infty)$. Moreover, since F and its derivatives with respect to t are continuous in $(t,x,y) \in (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$, this also shows that $\partial_t F^{\circledast 2}$ is continuous in (t,x,y). From (4.51) and Lemma 4.24 we get

$$|I_1| \overset{(4.51)}{=} t^{-1}|F^{\circledast 2}(t,x,y)| \leq t^{-1}G^{\circledast 2}(t,x,y) \overset{\text{L4.24}}{\leq} t^{-1}H^1(t,x,y).$$

Similarly, by (4.51) and (4.53),

$$\begin{aligned} |I_2| &\overset{(4.53)}{\leq} 2C\int_0^1\int G(t(1-s),x,z)G(ts,z,y)\,dz\,ds \\ &\overset{(4.51)}{=} 2Ct^{-1}G^{\circledast 2}(t,x,y) \overset{\text{L4.24}}{\leq} 2Ct^{-1}H^1(t,x,y). \end{aligned}$$

This gives (4.52) for $k=2$ with $c := 1+2C$.

- induction hypothesis: (4.52) holds for $k-1$.
- inductive step ($k-1 \to k$): We have to check that the assumptions of the differentiation lemma for parametrized integrals are satisfied. Similar to (4.53) we find

$$\begin{aligned} &\left|\frac{\partial}{\partial t}(F^{\circledast k-1}(t(1-s),x,z)F(st,z,y))\right| \\ &= (1-s)|\partial_\tau F^{\circledast k-1}(\tau,x,z)F(ts,z,y)|\Big|_{s=t(1-s)} + s|F^{\circledast k-1}(t(1-s),x,z)\partial_\tau F(\tau,z,y)|\Big|_{s=ts} \\ &\leq \frac{c(k-1)+C}{t}H^{k-2}(t(1-s),x,z)G(ts,z,y) =: \frac{C'}{t}u_k(s,t,x,z,y) \qquad (4.54) \end{aligned}$$

where we have used the induction hypothesis, (4.43), Lemma 4.24 and Lemma 4.26. As $\sup_{t\in[T_0,T]} u_k(s,t,x,z,y)$ is integrable for each $0<T_0<T<\infty$, cf. Lemma A.7, we conclude that the differentiation lemma is indeed applicable. Using (4.51), we get

$$\begin{aligned} \frac{\partial}{\partial t}F^{\circledast k}(t,x,y) &= \int_0^1\int F^{\circledast k-1}(t(1-s),x,z)F(ts,z,y)\,dz\,ds \\ &\quad + t\int_0^1\int \partial_t\big(F^{\circledast k-1}(t(1-s),x,z)F(st,z,y)\big)\,dz\,ds \\ &=: J_1 + J_2. \end{aligned}$$

Obviously, this shows that $\partial_t F^{\circledast k}(t,x,y)$ is continuous in $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. Moreover,

$$|J_1| \overset{(4.51)}{=} t^{-1}|F^{\circledast k}(t,x,y)| \leq t^{-1}G^{\circledast k}(t,x,y) \overset{L4.24}{\leq} t^{-1}H^{k-1}(t,x,y)$$

and

$$\begin{aligned} |J_2| &\overset{(4.54)}{\leq} (c^{k-1}+C)\int_0^1\int H^{k-2}(t(1-s),x,z)G(ts,z,y)\,dz\,ds \\ &\overset{(4.51)}{=} (c^{k-1}+C)t^{-1}(H^{k-2}\circledast G)(t,x,y) \\ &\overset{\text{L4.24}}{\leq} (c^{k-1}+C)t^{-1}H^{k-1}(t,x,y). \end{aligned}$$

Adding all up, we find

$$|F^{\circledast k}(t,x,y)| \leq (c(k-1)+C+1)t^{-1}H^{k-1}(t,x,y) \leq ckt^{-1}H^{k-1}(t,x,y). \qquad \square$$

4.28 Corollary *p is differentiable with respect to t and the derivative $\partial_t p(t,x,y)$ is continuous in $(t,x,y) \in (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. For any $T>0$ there exists a constant $C=C(T)>0$ such that*

$$\left|\frac{\partial}{\partial t}p(t,x,y)\right| \leq Ct^{-1}\left(S(x-y,\alpha(y),t)+g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)\right) \tag{4.55}$$

for all $x,y\in\mathbb{R}^d$ and $t\in(0,T]$.

Proof. Since $A_x p_0(t,x,y)=\partial_t p_0(t,x,y)$ we get from Corollary 4.11

$$\left|\frac{\partial}{\partial t}p_0(t,x,y)\right| = |A_x^{\alpha(y)}p_0(t,x,y)| \leq Ct^{-1}S(x-y,\alpha(y),t) \tag{4.56}$$

for all $x,y\in\mathbb{R}^d$ and $t\in(0,T]$. Combining this estimate with Theorem 4.27, it is not difficult to see that we may apply the differentiation lemma for parametrized integrals in (4.46) to conclude that p is differentiable with respect to t and that (4.55) holds true. □

4.5 Strong continuity of the prospective semigroup

For each $t>0$ define a linear operator P_t by

$$P_t f(x) := \int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy, \qquad x\in\mathbb{R}^d, f\in\mathcal{B}_b(\mathbb{R}^d).$$

It follows directly from (4.48) that P_t is well-defined (see Theorem 4.32 below). For $t=0$ we set $P_0 f := f$. Our ultimate aim is to show that $(P_t)_{t\geq 0}$ defines a strong Feller semigroup, i. e. to establish that

(i) $(P_t)_{t\geq0}$ is a (conservative) Markov semigroup.

(ii) $(P_t)_{t\geq0}$ is strongly continuous on $C_\infty(\mathbb{R}^d)$.

(iii) $P_t(C_\infty(\mathbb{R}^d))\subseteq P_t(C_\infty(\mathbb{R}^d))$ for each $t>0$ (Feller property).

(iv) $P_t(\mathcal{B}_b(\mathbb{R}^d))\subseteq C_b(\mathbb{R}^d)$ for all $t>0$ (strong Feller property).

In this section we concentrate on the properties (ii)-(iv). Both the Feller property and the strong Feller property are straightforward consequences of Theorem 4.25. Proving the strong continuity of $(P_t)_{t\geq0}$ requires more effort. Recall that

$$p(t,x,y)=p_0(t,x,y)+(p_0\circledast\Phi)(t,x,y), \qquad t>0, x,y\in\mathbb{R}^d,$$

cf. (4.46). The idea is to show first the strong continuity for the first order approximation p_0, that is

$$\sup_{x\in\mathbb{R}^d}\left|\int p_0(t,x,y)f(y)\,dy - f(x)\right| \xrightarrow{t\to0} 0 \qquad \text{for all} \quad f\in C_\infty(\mathbb{R}^d),$$

and then prove that the residue term $p_0\circledast\Phi$ is negligible, i. e.

$$\sup_{x\in\mathbb{R}^d}\left|\int (p_0\circledast\Phi)(t,x,y)f(y)\,dy\right| \xrightarrow{t\to0} 0 \qquad \text{for all} \quad f\in C_\infty(\mathbb{R}^d).$$

Some of the proofs in this section have been inspired by Knopova & Kulik [57]. As usual, we assume throughout this section that the assumptions of Theorem 3.2 are satisfied.

First, we collect some integrability properties of G (see (4.43) for the definition) which we will use very frequently.

4.29 Lemma *(i)* $\sup_{x\in\mathbb{R}^d}\sup_{t\leq T} t\int G(t,x,y)\,dy < \infty$.

(ii) $t\sup_{x\in\mathbb{R}^d}\int G(t,x,y)\,dy \xrightarrow{t\to 0} 0$.

(iii) $\sup_{x\in\mathbb{R}^d}\sup_{t\leq T}\int_0^t\int G(s,x,y)\,dy\,ds < \infty$.

(iv) $\int \sup_{|x|\leq R} G(t,x,y)\,dy < \infty$ *for all* $R>0$ *and* $t>0$.

(v) $\sup_{t\in[T_0,T]}\sup_{x\in\mathbb{R}^d}\int_{|y-x|\geq R} G(t,x,y)\,dy \xrightarrow{R\to\infty} 0$ *for any* $0<T_0<T<\infty$.

(vi) $\sup_{x\in\mathbb{R}^d}\int G(t,x,y)\,dy < \infty$ *for each fixed* $t>0$.

(vii) $\sup_{x\in\mathbb{R}^d}\sup_{r\in(0,1)}\int_0^t\int G(s+r,x,y)\,dy\,ds \xrightarrow{t\to 0} 0$.

We refer the reader to the appendix (Lemma A.8) for a detailed proof. One of the key tools to prove the strong continuity of $(P_t)_{t\geq 0}$ is the next lemma; it states some basic properties of the first order approximation p_0.

4.30 Lemma *(i)* $\sup_{x\in\mathbb{R}^d}|\int p_0(t,x,y)\,dy - 1| \xrightarrow{t\to 0} 0$.

(ii) $\sup_{x\in\mathbb{R}^d}\sup_{t\leq T}\int |p_0(t,x,y)|\,dy \leq C$ *for some constant* $C=C(T)$.

(iii) *For all* $\delta>0$ *we have* $\lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\int_{|y-x|\geq\delta}|p_0(t,x,y)|\,dy = 0$.

Since p_t^α is the density of a Lévy process, we know that p_t^α satisfies (i)-(iii) for each *fixed* $\alpha\in I$. Lemma 4.30 states that these properties are preserved if we allow $\alpha=\alpha(y)$ to depend on the spatial variable y. To prove Lemma 4.30 we need the following auxiliary statement. Its proof is very similar to the proof of Lemma 4.21.

4.31 Lemma *(i)* *There exists an absolute constant* $C>0$ *such that*

$$|p_0(t,x,y)-p_t^{\alpha(x)}(x,y)| \leq Ct|x-y|^{\varrho(\alpha)}\sup_{j\in\{1,\dots,n\}}\sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}}|\partial_{\beta_j}A^\beta p_t^\alpha(x-y)|\Big|_{\alpha=\beta}; \tag{4.57}$$

here $\sigma_{\alpha(x),\alpha(y)}\subseteq\mathbb{R}^n$ *denotes the line segment between* $\alpha(x)$ *and* $\alpha(y)$. *Moreover, there exists for any* $T>0$ *a constant* $C=C(T)>0$ *such that*

$$|p_0(t,x,y)-p_t^{\alpha(x)}(x,y)| \leq 2\sup_{\alpha\in I}S(x-y,\alpha,t) \qquad \text{for all } t\in(0,T], x,y\in\mathbb{R}^d. \tag{4.58}$$

(ii) *For any* $T>0$ *there exists a constant* $C=C(T)>0$ *such that*

$$|p_0(t,x,y)-p_t^{\alpha(x)}(x,y)| \leq CtG(t,x,y) \qquad \text{for all } t\in(0,T], x,y\in\mathbb{R}^d.$$

Proof. For brevity of notation, we put $\Delta := \Delta_t(x,y) := p_0(t,x,y) - p_t^{\alpha(x)}(x-y)$. If we set $\sigma_{\alpha(x),\alpha(y)}(\lambda) := (1-\lambda)\alpha(x) + \lambda\alpha(y)$, it follows from the gradient theorem that

$$\begin{aligned} e^{-t\Psi_{\alpha(y)}(r)} - e^{-t\Psi_{\alpha(x)}(r)} &= \int_{\sigma_{\alpha(x),\alpha(y)}} \partial_\beta (e^{-t\Psi_\beta(r)})\, d\beta \\ &= -t \sum_{j=1}^n (\alpha_j(y) - \alpha_j(x)) \int_0^1 \partial_{\beta_j} \Psi_\beta(r) e^{-t\Psi_\alpha(r)} \Big|_{\alpha=\beta=\sigma_{\alpha(x),\alpha(y)}(\lambda)} d\lambda \end{aligned}$$

for all $r \in \mathbb{R}$. Applying Fubini's theorem and the differentiation lemma for parametrized integrals, we find

$$\begin{aligned} \Delta &= \frac{1}{(2\pi)^d} \int e^{-i(x-y)\cdot\xi} (e^{-t\psi_{\alpha(y)}(\xi)} - e^{-t\psi_{\alpha(x)}(\xi)})\, d\xi \\ &= -\frac{t}{(2\pi)^d} \sum_{j=1}^n (\alpha_j(y) - \alpha_j(x)) \int_0^1 \int e^{-i(x-y)\cdot\xi} \partial_{\beta_j} \psi_\beta(\xi) e^{-t\psi_\alpha(\xi)} \Big|_{\alpha=\beta=\sigma_{\alpha(x),\alpha(y)}(\lambda)} d\xi\, d\lambda \\ &= -\frac{t}{(2\pi)^d} \sum_{j=1}^n (\alpha_j(y) - \alpha_j(x)) \int_0^1 \partial_{\beta_j} A^\beta p_t^\alpha(x-y) \Big|_{\alpha=\beta=\sigma_{\alpha(x),\alpha(y)}(\lambda)} d\lambda. \end{aligned}$$

Because of the Hölder continuity of $\alpha(\cdot)$, this implies (4.57). On the other hand, by (4.10),

$$|\Delta_t(x,y)| \leq 2 \sup_{\alpha\in I} p_t^\alpha(x-y) \overset{(4.10)}{\leq} 2C \sup_{\alpha\in I} S(x-y,\alpha,t).$$

This proves the first part of the lemma. The proof of (ii) is similar to the proof of Lemma 4.21, but for the readers' convenience we include the details. For brevity of notation, we will restrict ourselves to the case $m = 0$ and $T \leq 1$. Our aim is to show that $U(t,x,y) := t^{-1}\Delta_t(x,y)$ satisfies (4.44), i. e. that for any $\gamma \in (0, \frac{1}{\gamma_\infty^U}]$ there exists $C > 0$ such that

$$|\Delta_t(x,y)| \leq Ct \begin{cases} t^{-1}|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty^L}))S(x-y,\alpha(y),t), & |x-y| \leq t^\gamma, \\ \sup_{\beta\in I} \frac{1}{|x-y|^{d+\gamma_\infty(\beta)}}, & t^\gamma < |x-y| \leq 1, \\ \sup_{\beta\in I} \frac{1}{|x-y|^{d+\gamma_\infty(\beta)\wedge\gamma_0(\beta)}}, & |x-y| > 1, \end{cases}$$

for $t \in (0,1]$, $x,y \in \mathbb{R}^d$, because then the claim is a direct consequence of Remark 4.23. For fixed $t \in (0,T]$, $x,y \in \mathbb{R}^d$ and $\gamma \in (0, \frac{1}{\gamma_\infty^U}]$ we consider several cases separately.

(a) $|x-y| \leq t^{1/\gamma_\infty(\alpha(y))}$: For any $|x-y| \leq t^{1/\gamma_\infty(\alpha(y))}$ we have $S(x-y,\alpha(y),t) = t^{-d/\gamma_\infty(\alpha(y))}$. Therefore, it follows from (4.57) and Corollary 4.11 that

$$\begin{aligned} |\Delta| &\leq Ct|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty(\alpha(y))})) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} t^{-(d+\gamma_\infty(\beta))/\gamma_\infty(\beta)} \\ &\leq C|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty^L}))S(x-y,\alpha(y),t) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} t^{-d/\gamma_\infty(\beta)+d/\gamma_\infty(\alpha(y))}. \end{aligned}$$

By the Hölder continuity of γ_∞ and $\alpha(\cdot)$, we have

$$\left|\frac{1}{\gamma_\infty(\beta)} - \frac{1}{\gamma_\infty(\alpha(y))}\right| \leq \frac{|\gamma_\infty(\alpha(y)) - \gamma_\infty(\beta)|}{(\gamma_\infty^L)^2} \leq \frac{C'}{(\gamma_\infty^L)^2}|\alpha(x)-\alpha(y)|^{\varrho(\gamma_\infty)}$$

$$\leq \frac{C''}{(\gamma_\infty^L)^2}|x-y|^{\varrho(\gamma_\infty)\varrho(\alpha)} \leq \frac{C''}{(\gamma_\infty^L)^2} t^{\varrho(\gamma_\infty)\varrho(\alpha)/\gamma_\infty^U}$$

for all $t \leq 1$, $|x-y| \leq t^{1/\gamma_\infty(\alpha(y))} \leq t^{1/\gamma_\infty^U}$ and $\beta \in \sigma_{\alpha(x),\alpha(y)}$. Thus,

$$\sup_{t\in(0,1]} \sup_{|x-y|\leq t^{1/\gamma_\infty(\alpha(y))}} \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} t^{-d(1/\gamma_\infty(\beta)-1/\gamma_\infty(\alpha(y))}$$
$$\leq \sup_{t\in(0,1]} \exp\left(-\frac{C''}{(\gamma_\infty^L)^2} t^{\varrho(\gamma_\infty)\varrho(\alpha)/\gamma_\infty^U} \log t\right) < \infty$$

implying

$$|\Delta_t(x,y)| \leq C'''|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty^L}))S(x-y,\alpha(y),t).$$

(b) $t^{1/\gamma_\infty(\alpha(y))} < |x-y| \leq t^\gamma$: Combining (4.57) with Corollary 4.11, we find

$$|\Delta| \leq Ct|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty^L})) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} \frac{1}{|x-y|^{d+\gamma_\infty(\beta)}}.$$

Since

$$S(x-y,\alpha(y),t) = \frac{1}{|x-y|^{d+\gamma_\infty(\alpha(y))}} \qquad \text{for all} \quad t^{1/\gamma_\infty(\alpha(y))} < |x-y| \leq t^\gamma \leq 1$$

we get

$$|\Delta| \leq C|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty^L}))S(x-y,\alpha(y),t) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} |x-y|^{\gamma_\infty(\alpha(y))-\gamma_\infty(\beta)}.$$

We have already seen in the proof of Lemma 4.21, case (ii), that

$$C' := \sup_{t\in(0,1]} \sup_{|x-y|\leq t^\gamma} \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} |x-y|^{\gamma_\infty(\alpha(y))-\gamma_\infty(\beta)} < \infty.$$

Consequently,

$$|\Delta_t(x,y)| \leq C\,C'|x-y|^{\varrho(\alpha)}(1+\ell(ct^{-1/\gamma_\infty^L}))S(x-y,\alpha(y),t).$$

(c) $t^\gamma < |x-y| \leq 1$: Since $\gamma \leq \frac{1}{\gamma_\infty^U} \leq \frac{1}{\gamma_\infty(\alpha(y))}$, we have $t^{1/\gamma_\infty(\alpha(y))} \leq |x-y| \leq 1$. Therefore, it follows from (4.58) and the very definition of S, cf. (4.9), that

$$|\Delta_t(x,y)| \leq 2\sup_{\beta\in I} S(x-y,\beta,t) \leq 2t\sup_{\beta\in I} \frac{1}{|x-y|^{d+\gamma_\infty(\beta)}}.$$

(d) $|x-y| > 1$: It is a direct consequence of (4.58) and (4.9) that

$$|\Delta_t(x,y)| \leq 2\sup_{\beta\in I} S(x-y,\beta,t) \leq 2t\sup_{\beta\in I} \frac{1}{|x-y|^{d+\gamma_0(\beta)\wedge\gamma_\infty(\beta)}}. \qquad \square$$

Proof of Lemma 4.30. (i) For each $\alpha \in I$, p_t^α is a density; in particular

$$\int p_t^{\alpha(x)}(x,y)\,dy = \int p_t^{\alpha(x)}(x-y)\,dy = 1 \qquad \text{for all} \quad x \in \mathbb{R}^d. \qquad (\star)$$

By the previous lemma,

$$\left|\int p_0(t,x,y)\,dy - 1\right| \overset{(\star)}{=} \left|\int (p_0(t,x,y) - p_t^{\alpha(x)}(x-y))\,dy\right| \overset{\text{L4.31}}{\leq} Ct\int G(t,x,y)\,dy.$$

Consequently,

$$\sup_{x\in\mathbb{R}^d}\left|\int p_0(t,x,y)\,dy - 1\right| \leq Ct\sup_{x\in\mathbb{R}^d}\int G(t,x,y)\,dy \xrightarrow[L4.29(ii)]{t\to 0} 0.$$

(ii) This is a direct consequence of Lemma 4.31(ii), Lemma 4.29(i) and ($\star$):

$$\int |p_0(t,x,y)|\,dy \overset{\text{L4.31}}{\leq} Ct\int G(t,x,y)\,dy + \int p_t^{\alpha(x)}(x-y)\,dy \leq C' + 1.$$

(iii) For fixed $\delta > 0$ choose $t \leq 1$ sufficiently small such that $\delta \geq t^{1/\gamma_\infty^U}$. If $x, y \in \mathbb{R}^d$ are such that $|x-y| \geq \delta$, then

$$|x-y| \geq \delta \geq t^{1/\gamma_\infty^U} \overset{t\leq 1}{\geq} t^{1/\gamma_\infty(\alpha(x))}.$$

Therefore, we find from Corollary 4.11

$$\begin{aligned}
&\int_{|x-y|\geq\delta} p_t^{\alpha(x)}(x-y)\,dy \\
&\quad\leq Ct\left(\int_{1>|x-y|\geq\delta}\frac{1}{|x-y|^{d+\gamma_\infty(\alpha(x))}}\,dy + \int_{|x-y|>1}\frac{1}{|x-y|^{d+\gamma_0(\alpha(x))\wedge\gamma_\infty(\alpha(x))}}\,dy\right) \\
&\quad\leq Ct\left(\int_{1>|z|\geq\delta}\frac{1}{|z|^{d+\gamma_\infty^U}}\,dz + \int_{|z|>1}\frac{1}{|z|^{d+\gamma_0^L\wedge\gamma_\infty^L}}\,dz\right) = C't.
\end{aligned}$$

Applying Lemma 4.31(ii) we conclude

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d}\left|\int_{|y-x|\geq\delta} p_0(t,x,y)\,dy\right| &\leq C''t\sup_{x\in\mathbb{R}^d}\int G(t,x,y)\,dy + \sup_{x\in\mathbb{R}^d}\int_{|y-x|\geq\delta} p_t^{\alpha(x)}(x-y)\,dy \\
&\leq C''t\sup_{x\in\mathbb{R}^d}\int G(t,x,y)\,dy + C't \xrightarrow[L4.29(ii)]{t\to 0} 0.
\end{aligned}$$ □

Now we are ready to prove that $(P_t)_{t\geq 0}$ is strongly continuous and that it has the (strong) Feller property.

4.32 Theorem *(i) P_t has the strong Feller property for all $t > 0$: $P_t : \mathcal{B}_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$. Moreover, P_t is continuous:*

$$\|P_t\| := \sup\{\|P_t f\|_\infty;\, f \in \mathcal{B}_b(\mathbb{R}^d), \|f\|_\infty \leq 1\} < \infty.$$

(ii) P_t has the Feller property for all $t > 0$, i. e. $P_t f \in C_\infty(\mathbb{R}^d)$ for all $f \in C_\infty(\mathbb{R}^d)$.

(iii) $\|P_t f - f\|_\infty \xrightarrow{t\to 0} 0$ for any bounded uniformly continuous function f. In particular, $(P_t)_{t\geq 0}$ is strongly continuous at $t = 0$ on $C_\infty(\mathbb{R}^d)$.

Proof. (i) Fix $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $t > 0$. As $t > 0$ is fixed, there exists some constant $C' = C'(t) > 0$ such that $S(x-y, \alpha(y), t) \le C'G(t,x,y)$ for all $x, y \in \mathbb{R}^d$. Because of (4.48), we get

$$|p(t,x,y)| \le C(S(x-y,\alpha(y),t) + tG(t,x,y)) \le C(C'+t)G(t,x,y) \tag{$\star$}$$

for all $x, y \in \mathbb{R}^d$. Consequently, we get from Lemma 4.29(iv)

$$\int \sup_{|x|\le R} |p(t,x,y)|\,dy \le C(C'+t)\int \sup_{|x|\le R} G(t,x,y)\,dy < \infty \qquad \text{for all} \quad R > 0.$$

Since $x \mapsto p(t,x,y)$ is continuous, cf. Theorem 4.25(i), it follows easily from the dominated convergence theorem that

$$x \mapsto P_t f(x) = \int p(t,x,y) f(y)\,dy$$

is continuous. Moreover,

$$\|P_t f\|_\infty \le \|f\|_\infty C(C'+t) \sup_{x\in\mathbb{R}^d} \int G(t,x,y)\,dy \le C''\|f\|_\infty$$

by Lemma 4.29(vi).

(ii) Let $f \in C_\infty(\mathbb{R}^d)$. As $P_t f$ is continuous by (i), it remains to show that $|P_t f(x)| \to 0$ as $|x| \to \infty$. Fix $\varepsilon > 0$. Since $f \in C_\infty(\mathbb{R}^d)$ there exists $R > 0$ such that $|f(y)| \le \varepsilon$ for all $|y| \ge R$. Thus,

$$\left|\int_{|y|\ge R} p(t,x,y)f(y)\,dy\right| \le \varepsilon \int |p(t,x,y)|\,dy \overset{(\star)}{\le} C(C'+t)\varepsilon \int G(t,x,y)\,dy \overset{4.29(\text{vi})}{\le} C''\varepsilon$$

for some constant $C'' = C''(t)$. On the other hand, we have for $|x| \ge 2R$

$$\left|\int_{|y|<R} p(t,x,y)f(y)\,dy\right| \le \|f\|_\infty \int_{|y-x|\ge R} |p(t,x,y)|\,dy \le C\|f\|_\infty \int_{|y-x|\ge R} G(t,x,y)\,dy.$$

By Lemma 4.29(v) we can choose $R > 0$ sufficiently large such that the right-hand side is at most ε. Consequently,

$$|P_t f(x)| \le \left|\int_{|y|\ge R} p(t,x,y)f(y)\,dy\right| + \left|\int_{|y|<R} p(t,x,y)f(y)\,dy\right| \le (C''+1)\varepsilon$$

for $|x| \ge 2R$.

(iii) Let f be bounded and uniformly continuous. First we show that

$$\sup_{x\in\mathbb{R}^d}\left|\int p_0(t,x,y)f(y)\,dy - f(x)\right| \xrightarrow{t\to 0} 0. \tag{4.59}$$

Obviously,

$$\left|\int p_0(t,x,y)f(y)\,dy - f(x)\right|$$

$$\leq \int |f(y) - f(x)| |p_0(t,x,y)| \, dy + \|f\|_\infty \left| \int p_0(t,x,y) \, dy - 1 \right|.$$

Since we already know from Lemma 4.30(i) that the second term on the right-hand side converges uniformly (in x) to 0, (4.59) follows if

$$J(t) := \sup_{x \in \mathbb{R}^d} \int |f(y) - f(x)| |p_0(t,x,y)| \, dy \xrightarrow{t \to 0} 0.$$

To this end, we note that

$$\begin{aligned} J(t) &\leq \sup_{|x-y|<\delta} |f(x) - f(y)| \sup_{x \in \mathbb{R}^d} \int |p_0(t,x,y)| \, dy + 2\|f\|_\infty \sup_{x \in \mathbb{R}^d} \int_{|x-y| \geq \delta} |p_0(t,x,y)| \, dy \\ &\overset{\text{L4.30}}{\leq} C \sup_{|x-y|<\delta} |f(x) - f(y)| + 2\|f\|_\infty \sup_{x \in \mathbb{R}^d} \int_{|x-y| \geq \delta} |p_0(t,x,y)| \, dy \\ &\xrightarrow[L4.30(iii)]{t \to 0} C \sup_{|x-y|<\delta} |f(x) - f(y)| \xrightarrow{\delta \to 0} 0 \end{aligned}$$

where we have used in the last step that f is uniformly continuous. Now we are ready to prove that $\|P_t f - f\|_\infty \xrightarrow{t \to 0} 0$. Because of (4.59) it suffices to check that

$$\sup_{x \in \mathbb{R}^d} \left| \int (p(t,x,y) - p_0(t,x,y)) f(y) \, dy \right| \xrightarrow{t \to 0} 0.$$

This is a direct consequence of (4.47) and Lemma 4.29(ii):

$$\sup_{x \in \mathbb{R}^d} \left| \int (p(t,x,y) - p_0(t,x,y)) f(y) \, dy \right| \leq Ct\|f\|_\infty \sup_{x \in \mathbb{R}^d} \int G(t,x,y) \, dy \xrightarrow{t \to 0} 0.$$

Finally, if $f \in C_\infty(\mathbb{R}^d)$, then f is bounded and uniformly continuous. Therefore we get in particular $\|P_t f - f\|_\infty \xrightarrow{t \to 0} 0$ for all $f \in C_\infty(\mathbb{R}^d)$. □

For later reference we make the following observation.

4.33 Remark The proof of Theorem 4.32(iii) shows that

$$\sup_{f \in A} \sup_{x \in \mathbb{R}^d} \left| \int p_0(t,x,y) f(y) \, dy - f(x) \right| \xrightarrow{t \to 0} 0$$

and

$$\sup_{f \in A} \sup_{x \in \mathbb{R}^d} |P_t f(x) - f(x)| \xrightarrow{t \to 0} 0$$

for any family $A \subseteq C_b(\mathbb{R}^d)$ which is uniformly bounded and uniformly equicontinuous.

4.6 The approximating fundamental solution

So far, we have seen that the candidate $p(t,x,y)$ for the transition density is well-defined and that the associated family of operators $(P_t)_{t \geq 0}$ has certain analytical properties. It still remains to relate a Markov process with p (i. e. show that $(P_t)_{t \geq 0}$ is a Markov semigroup) and to check that the generator of the process coincides on $C_c^\infty(\mathbb{R}^d)$ with the pseudo-differential operator A with symbol $q(x,\xi) = \psi_{\alpha(x)}(\xi)$. It turns out that this is the hardest part of the parametrix construction.

If we *knew* that p is a fundamental solution to the Cauchy problem $\partial_t - A$, that is

$$(\partial_t - A_x)p(t,x,y) = 0, \qquad p(t,x,y) \xrightarrow[v]{t\to 0} \delta_x(dy), \tag{4.60}$$

then we would get

$$(\partial_t - A_x)P_t f(x) = 0$$

for "nice" functions f. It is well-known that for solutions u of the evolutionary equation

$$(\partial_t - A_x)u(t,x) = 0$$

the implication

$$u(0,\cdot) \geq 0 \implies u(t,\cdot) \geq 0 \text{ for all } t \geq 0$$

holds whenever $A : \mathcal{D}(A) \to C_b(\mathbb{R}^d)$ is an operator which satisfies the positive maximum principle and has a sufficiently rich domain (see e. g. [60, Theorem 4.1.1]). Since the positive maximum principle holds for the pseudo-differential operator A, it is not difficult to deduce that $(P_t)_{t\geq 0}$ is a Markov semigroup using this result.

Unfortunately, this reasoning doesn't work in our setting. Since we only know that the pseudo-differential operator A extends to $C_b^2(\mathbb{R}^d)$, we would have to show $p(t,\cdot,y) \in C_b^2(\mathbb{R}^d)$ in order to verify $p(t,\cdot,y) \in \mathcal{D}(A)$ (which is obviously necessary to make sense of (4.60)). Unless we add further assumptions on the smoothness of $\alpha(\cdot)$, we cannot expect this kind of smoothness, see (the proof of) Theorem 3.8.

One possibility to resolve this problem is to approximate p by a family of functions $p_\varepsilon, \varepsilon > 0$, which is sufficiently smooth and satisfies (4.60) in the limit as $\varepsilon \to 0$. This idea is due to Knopova & Kulik [57] and presents – compared to other approaches in the literature – a transparent way to establish that $(P_t)_{t\geq 0}$ is a Markov semigroup. Following [57] we define for each $\varepsilon > 0$

$$p_\varepsilon(t,x,y) := p_0(t+\varepsilon,x,y) + \int_0^t \int p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds, \qquad t>0, x,y \in \mathbb{R}^d, \tag{4.61}$$

and call p_ε *approximate fundamental solution.* The associated family of operators is denoted by

$$P_{t,\varepsilon}f(x) := \int p_\varepsilon(t,x,y)f(y)\,dy, \qquad x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d).$$

This section is structured as follows. First we show that p_ε converges (in a uniform sense) to p (Lemma 4.35). Next we establish some properties of $(P_{t,\varepsilon})_{t\geq 0}$ and obtain convergence results for $P_{t,\varepsilon}f$ as $\varepsilon \to 0$ (Lemma 4.36). The remaining part of this section is devoted to convergence results for

$$q_{t,\varepsilon} := (\partial_t - A_x)p_{t,\varepsilon}(t,x,y),$$

the time derivative $\partial_t p_{t,eps}$ and the associated families of operators as $\varepsilon \to 0$ (Lemma 4.39, Lemma 4.40 and Lemma 4.41); in particular we show that $q_{t,\varepsilon}$ is well-defined, i. e. that

$p_\varepsilon(t,\cdot,y)$ is sufficiently smooth (see Lemma 4.37 and Lemma 4.38). Throughout this section, we assume that the conditions of Theorem 3.2 are satisfied.

The first lemma in this section states elementary estimates and relations between the functions S, G and H^1, see (4.43) and Lemma 4.24 for the definition of G and H^k, respectively. We defer the proof to the appendix (Lemma A.9).

4.34 Lemma *For any fixed $0 < T_0 < T < \infty$ there exists a constant $C > 0$ such that the following statements hold true.*

(i) $\sup_{t\in[T_0,T]} G(t,x,y) \le CG(T,x,y)$ *for all* $x,y \in \mathbb{R}^d$.

(ii) $\sup_{t\in[0,T]} \sup_{x,y\in\mathbb{R}^d} G(t+\varepsilon,x,y) < \infty$ *for all* $\varepsilon > 0$.

(iii) $C^{-1}G(t,x,y) \le H^1(t,x,y) \le CG(t,x,y)$ *for all* $t \in [T_0,T]$ *and* $x,y \in \mathbb{R}^d$.

(iv) $S(x-y,\alpha(y),t) \le CG(t,x,y)$ *for all* $x,y \in \mathbb{R}^d$, $t \in [T_0,T]$.

(v) $\sup_{x\in\mathbb{R}^d} \int H^1(t+\varepsilon,x,y)\,dy \le C(t+\varepsilon)^\kappa$ *for all* $\varepsilon > 0$ *and* $t \in [0,T]$.[6]

We remind the reader of the following two estimates which we have derived in Section 4.1 and Section 4.3, respectively.

$$\begin{aligned} |p_0(t,x,y)| = |p_t^{\alpha(y)}(x-y)| &\overset{(4.10)}{\le} CS(x-y,\alpha(y),t) \\ |\Phi(t,x,y)| &\overset{(4.49)}{\le} CG(t,x,y) \end{aligned} \tag{4.62}$$

Now we turn to the question whether p_ε is a good approximation for p. Because of (4.46) it is not surprising that p_ε converges pointwise to p, but we will need a stronger type of convergence later on. The next lemma shows that the convergence is uniform on sets of the form $[T_0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ where $0 < T_0 < T < \infty$.

4.35 Lemma *Let $0 < T_0 < T < \infty$. Then:*

(i) $p_\varepsilon(t,x,y) \to p(t,x,y)$ *uniformly on* $[T_0,T]\times\mathbb{R}^d\times\mathbb{R}^d$.

(ii) There exists a constant $C = C(T_0,T) > 0$ *such that* $|p_\varepsilon(t,x,y)| \le CG(t+\varepsilon,x,y)$ *for all* $t \in [T_0,T]$, $x,y \in \mathbb{R}^d$ *and* $\varepsilon > 0$.

Proof. We start to prove (i). Fix $0 < T_0 < T < \infty$. By the definition of p_ε and (4.46),

$$\begin{aligned} |p_\varepsilon(t,x,y) - p(t,x,y)| &\le |p_0(t+\varepsilon,x,y) - p_0(t,x,y)| \\ &\quad + \left|\int_0^t \int (p_0(t-s+\varepsilon,x,z) - p_0(t-s,x,z))\Phi(s,z,y)\,dz\,ds\right|. \end{aligned} \tag{$\star$}$$

We are going to show that each of the terms on the right-hand side converges uniformly to 0 as $\varepsilon \to 0$. Since $p_0(t,x,y) = p_t^{\alpha(y)}(x-y)$, an application of Lemma 4.5 yields

$$|p_0(t+\varepsilon,x,y) - p_0(t,x,y)| = (2\pi)^{-d}\left|\int e^{-i(x-y)\cdot\xi} e^{-t\psi_\alpha(\xi)}(e^{-\varepsilon\psi_{\alpha(y)}(\xi)} - 1)\,d\xi\right|$$

[6] See page 94 for the definition of κ.

$$\overset{\text{L4.5}}{\leq} (2\pi)^{-d}\varepsilon \int e^{-t\operatorname{Re}\psi_{\alpha(y)}(\xi)}|\psi_{\alpha(y)}(\xi)|\max\{1, e^{-\varepsilon\operatorname{Re}\psi_{\alpha(y)}(\xi)}\}\,d\xi.$$

It follows from (3.3) and (C3) that there exists a constant $C > 0$ such that

$$|p_0(t+\varepsilon, x, y) - p_0(t, x, y)| \leq C\varepsilon e^{c_4 T}\left(1 + \int_{|\xi|\geq 1} |\xi|^2 e^{-T_0 c_1 |\xi|^{\gamma_\infty^L}}\,d\xi\right)$$

for all $x, y \in \mathbb{R}^d$ and $t \in [T_0, T]$. Consequently,

$$\sup_{x,y\in\mathbb{R}^d}\sup_{t\in[T_0,T]} |p_0(t+\varepsilon, x, y) - p_0(t, x, y)| \xrightarrow{\varepsilon\to 0} 0. \tag{4.63}$$

It remains to show that the second term in $(\star)$ converges to 0, i. e.

$$\sup_{x,y\in\mathbb{R}^d}\sup_{t\in[T_0,T]}\left|\int_0^t\int (p_0(t-s+\varepsilon, x, z) - p_0(t-s, x, z))\Phi(s, z, y)\,dz\,ds\right| \xrightarrow{\varepsilon\to 0} 0. \tag{4.64}$$

To this end, we split up the domain of integration into $[0, t-\delta]$ and $[t-\delta, t]$ for some fixed $\delta \in (0, t)$. Using (4.49) and Lemma 4.29(iv), we find

$$\begin{aligned}
&\int_0^{t-\delta}\int |p_0(t-s+\varepsilon, x, z) - p_0(t-s, x, z)|\,|\Phi(s, z, y)|\,dz\,ds \\
&\qquad \overset{(4.49)}{\leq} C \sup_{x,y\in\mathbb{R}^d}\sup_{\delta\leq r\leq T} |p_0(r+\varepsilon, x, z) - p_0(r, x, z)| \int_0^t\int G(s, z, y)\,dz\,ds \\
&\qquad \overset{\text{L4.29}}{\leq} C' \sup_{x,y\in\mathbb{R}^d}\sup_{\delta\leq r\leq T} |p_0(r+\varepsilon, x, z) - p_0(r, x, z)|.
\end{aligned}$$

The first part of this proof shows that this term converges uniformly to 0 as $\varepsilon \to 0$ for fixed $\delta > 0$, cf. (4.63). On the other hand, using a very similar reasoning as in the proof of Corollary 4.20, it is not difficult to see that there exists a constant $C = C(T, \kappa) > 0$ such that

$$\int_{t-\delta}^t\int S(t-s+r, x, z)G(s, z, y)\,dz\,ds \leq C\delta^{\kappa\wedge 1}(S(x-y, \alpha(y), t+r) + (t+r)g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y))$$

for all $r \geq 0$, $\delta \in (0, 1)$, $t \leq T$ and $x, y \in \mathbb{R}^d$; we refer the reader to Lemma A.10 for a detailed proof. If we combine this with the bounds for p_0 and Φ, cf. (4.62), we obtain

$$\begin{aligned}
&\int_{t-\delta}^t\int |p_0(t-s+r, x, z)|\,|\Phi(s, z, y)|\,dz\,ds \\
&\qquad\qquad \leq C' \int_{t-\delta}^t\int S(x-z, \alpha(z), t-s+r)G(s, z, y)\,dz\,ds \\
&\qquad\qquad \leq C\,C'\delta^{\kappa\wedge 1}(S(x-y, \alpha(y), t+r) + (t+r)g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y))
\end{aligned}$$

for all $r \geq 0$. Hence, by the triangle inequality,

$$\begin{aligned}
&\sup_{t\in[T_0,T]}\sup_{x,y\in\mathbb{R}^d}\int_{t-\delta}^t\int |p_0(t-s+\varepsilon, x, z) - p_0(t-s, x, z)|\,|\Phi(s, z, y)|\,dz\,ds \\
&\qquad \leq 2C\,C'\delta^{\kappa\wedge 1}\left[\sup_{t\in[T_0,T+1]}\sup_{x,y\in\mathbb{R}^d}(S(x-y, \alpha(y), t) + t g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y))\right]
\end{aligned}$$

for all $\varepsilon \in (0,1)$. Note that the expression in the bracket is finite because of the global estimate (4.11). Letting first $\varepsilon \to 0$ and then $\delta \to 0$ gives (4.64).
To prove (ii) we note that, by (4.61) and (4.62), there exists a constant $C = C(T, \varepsilon) > 0$ such that

$$\begin{aligned}|p_\varepsilon(t,x,y)| &\leq CS(x-y,\alpha(y),t+\varepsilon) + C\int_0^t \int S(x-z,\alpha(z),t+\varepsilon-s)G(s,z,y)\,dz\,ds \\ &\leq CS(x-y,\alpha(y),t+\varepsilon) + C\int_0^{t+\varepsilon} \int S(x-z,\alpha(z),t+\varepsilon-s)G(s,z,y)\,dz\,ds \\ &= CS(x-y,\alpha(y),t+\varepsilon) + C(S \circledast G)(t+\varepsilon,x,y)\end{aligned}$$

for all $t \in [0,T]$, $x, y \in \mathbb{R}^d$. Applying Lemma 4.24(ii) we find

$$|p_\varepsilon(t,x,y)| \leq CS(x-y,\alpha(y),t+\varepsilon) + CH^1(t+\varepsilon,x,y).$$

Now the claim follows from Lemma 4.34. □

Our next aim is to establish properties of the family of operators $(P_{t,\varepsilon})_{t\geq 0}$ associated with the approximate fundamental solution p_ε. Since we know from the previous lemma that $p_\varepsilon \xrightarrow{\varepsilon\to 0} p$ uniformly on $[T_0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ for any $0 < T_0 < T < \infty$, it is not too surprising that $P_{t,\varepsilon} f$ converges uniformly to $P_t f$ as $\varepsilon \to 0$. Moreover, because of the results from Section 4.5, we also expect that $(P_{t,\varepsilon})_{t\geq 0}$ is strongly continuous in some approximate sense.

4.36 Lemma *(i)* $\sup_{\varepsilon\in(0,1)} \sup_{t\in[T_0,T]} |P_{t,\varepsilon} f(x)| \xrightarrow{|x|\to\infty} 0$ *for any* $f \in C_\infty(\mathbb{R}^d)$ *and all* $0 < T_0 < T < \infty$.

(ii) $\|P_{t,\varepsilon} f - f\|_\infty \to 0$ *as* $\varepsilon, t \to 0$ *for any* $f \in C_\infty(\mathbb{R}^d)$.

(iii) $\|P_{t+s,\varepsilon} f - P_{s,\varepsilon} f\|_\infty \to 0$ *as* $\varepsilon, t \to 0$ *for any* $f \in C_c(\mathbb{R}^d)$ *and* $s > 0$.

(iv) $\sup_{t\in[0,T]} \|P_{t,\varepsilon} f - P_t f\|_\infty \xrightarrow{\varepsilon\to 0} 0$ *for all* $T > 0$ *and* $f \in C_\infty(\mathbb{R}^d)$.

Proof. The proof of statement (i) is similar to the proof of the Feller property of $(P_t)_{t\geq 0}$, cf. Lemma 4.32(ii). Fix $f \in C_\infty(\mathbb{R}^d)$ and $\delta > 0$. By Lemma 4.35(ii) there exists a constant $C > 0$ such that $|p_\varepsilon(t,x,y)| \leq CG(t+\varepsilon,x,y)$ for all $t \in [T_0, T+1]$, $\varepsilon \in (0,1)$ and $x, y \in \mathbb{R}^d$. Choose $R > 0$ sufficiently large such that $|f(y)| \leq \delta$ for all $|y| \geq R$. Then

$$\begin{aligned}|P_{t,\varepsilon} f(x)| &\leq \delta \int_{|y|\geq R} |p_\varepsilon(t,x,y)|\,dy + \|f\|_\infty \int_{|y|\leq R} |p_\varepsilon(t,x,y)|\,dy \\ &\leq C\delta \int G(t+\varepsilon,x,y)\,dy + C\|f\|_\infty \int_{|y-x|\geq R} G(t+\varepsilon,x,y)\,dy\end{aligned}$$

for all $|x| \geq 2R$. Now (i) follows from Lemma 4.29(i),(v). To prove (ii) we note that

$$|P_{t,\varepsilon} f(x) - f(x)| \leq \left| \int p_0(t+\varepsilon,x,y) f(y)\,dy - f(x) \right| + \left| \int (p_0(\cdot+\varepsilon,\cdot,\cdot) \circledast \Phi)(t,x,y) f(y)\,dy \right|.$$

We have already seen in (4.59) that the first term on the right-hand side converges uniformly to 0 as $\varepsilon, t \to 0$. Since

$$|(p_0(\cdot+\varepsilon,\cdot,\cdot) \circledast \Phi)(t,x,y)| \overset{(4.10)}{\underset{(4.49)}{\leq}} C \int_0^t \int S(x-z,\alpha(z),t-s+\varepsilon) G(s,z,y)\,dz\,ds$$

$$\begin{aligned} &\leq C\int_0^{t+\varepsilon}\int S(x-z,\alpha(z),t+\varepsilon-s)G(s,z,y)\,dz\,ds \\ &= C(S\circledast G)(t+\varepsilon,x,y)\overset{\text{L4.24}}{\leq} CH^1(t+\varepsilon,x,y), \end{aligned}$$

it follows that also the second term converges uniformly to 0 as $\varepsilon,t\to 0$:

$$\begin{aligned} \left|\int (p_0(\cdot+\varepsilon,\cdot,\cdot)\circledast\Phi)(t,x,y)f(y)\,dy\right| &\leq C\|f\|_\infty\int H^1(t+\varepsilon,x,y)\,dy \\ &\overset{\text{L4.34(v)}}{\leq} C'(t+\varepsilon)^\kappa \xrightarrow{t,\varepsilon\to 0} 0. \end{aligned}$$

This shows (ii). We proceed to prove (iii). Since f is bounded and has compact support, the assertion follows from the dominated convergence theorem if we can show that

$$\sup_{x,y\in\mathbb{R}^d}|p_\varepsilon(t+s,x,y)-p_\varepsilon(s,x,y)|\xrightarrow{\varepsilon,t\to 0} 0 \qquad \text{for all } \ s>0.$$

In Lemma 4.37 below we will show that $p_\varepsilon(\cdot,x,y)$ is differentiable and that there exists a constant $C=C(s)>0$ such that $|\partial_\tau p_\varepsilon(\tau,x,y)|\leq CG(\tau+\varepsilon,x,y)$ for all $\tau\in[s,s+1]$, $\varepsilon\in(0,1)$ and $x,y\in\mathbb{R}^d$, cf. (4.68). By the mean value theorem, we get

$$\sup_{x,y\in\mathbb{R}^d}|p_\varepsilon(t+s,x,y)-p_\varepsilon(s,x,y)|\leq Ct\sup_{x,y\in\mathbb{R}^d}\sup_{r\leq 2}G(s+r,x,y)$$

for all $\varepsilon\in(0,1)$, $t\in(0,1)$. Because of the boundedness of G, cf. Lemma 4.34(ii), it follows that

$$\sup_{x,y\in\mathbb{R}^d}|p_\varepsilon(t+s,x,y)-p_\varepsilon(s,x,y)|\leq C't\xrightarrow{\varepsilon,t\to 0} 0.$$

It remains to prove (iv). Fix $T>0$, $f\in C_\infty(\mathbb{R}^d)$ and $\delta>0$. Choose $R>0$ sufficiently large such that $|f(y)|\leq\delta$ for all $|y|\geq R$. By (ii) and Lemma 4.32(iii), there exist $T_0\in(0,T)$, $\varepsilon_0\in(0,1)$ such that

$$\|P_{t,\varepsilon}f-P_tf\|_\infty\leq\|P_{t,\varepsilon}f-f\|_\infty+\|P_tf-f\|_\infty\leq\delta \qquad \text{for all } \ t\in[0,T_0],\varepsilon\leq\varepsilon_0. \tag{$\star$}$$

On the other hand, we have $\|P_{t,\varepsilon}f-P_tf\|_\infty\leq J_1+J_2$ where

$$\begin{aligned} J_1 &:= \delta\sup_{x\in\mathbb{R}^d}\int(|p_\varepsilon(t,x,y)|+|p(t,x,y)|)\,dy \\ J_2 &:= \|f\|_\infty\lambda^d(B(0,R))\sup_{\substack{t\in[T_0,T]\\ x,y\in\mathbb{R}^d}}|p_\varepsilon(t,x,y)-p(t,x,y)|. \end{aligned}$$

It follows from Lemma 4.35(ii) and (4.48) that

$$J_1\leq C\delta\sup_{x\in\mathbb{R}^d}\int(G(t+\varepsilon,x,y)+S(x-y,\alpha(y),t)+tG(t,x,y))\,dy.$$

Hence, $J_1\leq C'\delta$ where

$$C'=C\sup_{t\in[T_0,T]}\sup_{x\in\mathbb{R}^d}\sup_{\varepsilon\in(0,1)}\int(G(t+\varepsilon,x,y)+S(x-y,\alpha(y),t)+tG(t,x,y))\,dy<\infty$$

by Lemma 4.15 and Lemma 4.29. Combining the estimates and applying Lemma 4.35(i) gives

$$\|P_{t,\varepsilon}f - P_t f\|_\infty \le C'\delta + \|f\|_\infty \lambda^d(B(0,R)) \sup_{\substack{t\in[T_0,T]\\ x,y\in\mathbb{R}^d}} |p_\varepsilon(t,x,y) - p(t,x,y)| \le (C'+1)\delta$$

for all $t \in [T_0, T]$ and $\varepsilon \le \varepsilon_1$ sufficiently small. Hence, by $(\star)$, $\|P_{t,\varepsilon}f - P_t f\|_\infty \le (C'+1)\delta$ for all $t \in [0,T]$ and $\varepsilon \le \min\{\varepsilon_0, \varepsilon_1\}$. □

In the remaining part of this section we discuss convergence results for $\partial_t p_\varepsilon$,

$$q_\varepsilon(t,x,y) := (\partial_t - A_x)p_\varepsilon(t,x,y)$$

and the families of operators $\partial_t P_{t,\varepsilon}$ and $Q_{t,\varepsilon}f := (\partial_t - A_x)P_{t,\varepsilon}f$ as $\varepsilon \to 0$. The first step is to verify that these expressions are well-defined. The next lemma shows in particular that $p_\varepsilon(\cdot,x,y)$ is differentiable and that $p_\varepsilon(t,\cdot,y) \in \mathcal{D}(A)$.

4.37 Lemma *For any $\varepsilon > 0$, $t > 0$ and $x, y \in \mathbb{R}^d$ it holds that*

$$\begin{aligned}\partial_t p_\varepsilon(t,x,y) = \partial_t p_0(t+\varepsilon,x,y) &+ \int p_0(\varepsilon,x,z)\Phi(t,z,y)\,dz \\ &+ \int_0^t \int \partial_t p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds\end{aligned} \tag{4.65}$$

and

$$A_x p_\varepsilon(t,x,y) = A_x p_0(t+\varepsilon,x,y) + \int_0^t \int A_x p(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds. \tag{4.66}$$

Proof. First we prove (4.65). Recall that

$$p_\varepsilon(t,x,y) = p_0(t+\varepsilon,x,y) + \int_0^t \int p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds.$$

Fix $T > 0$. Since p_0 is (for fixed y) a solution to (4.50), we have

$$\left|\frac{\partial}{\partial t} p_0(t,x,y)\right| = |A_x^{\alpha(y)} p_0(t,x,y)| \overset{C4.11}{\le} Ct^{-1}S(x-y,\alpha(y),t)$$

for $t \in (0,T]$. Therefore, we get for all $t \in [0,T]$, $s \in [0,t]$

$$\begin{aligned}|\partial_t p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)| &\le C(t-s+\varepsilon)^{-1}S(x-z,\alpha(z),t-s+\varepsilon)|\Phi(s,z,y)| \\ &\underset{(4.11)}{\overset{(4.49)}{\le}} C'G(s,z,y)\end{aligned}$$

for some constant $C' = C'(\varepsilon,T)$. Since $\int_0^T \int G(s,z,y)\,dz\,ds < \infty$, cf. Lemma 4.15, we may apply the differentiation lemma for parametrized integrals to conclude that

$$\frac{\partial}{\partial t}\int_0^\tau \int p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds = \int_0^\tau \int \partial_t p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds$$

for all $t, \tau \in [0,T]$. Now (4.65) follows from the chain rule and the very definition of p_ε. It remains to prove (4.66). Since

$$\mathcal{F}(p_0(t+\varepsilon,\cdot,y))(\xi) = (2\pi)^{-d} e^{iy\cdot\xi} e^{-(t+\varepsilon)\Psi_{\alpha(y)}(|\xi|)}$$

the growth condition on $\operatorname{Re}\Psi_{\alpha(y)}$ entails that $|\psi_{\alpha(x)}(\xi)\mathcal{F}(p_0(t+\varepsilon,\cdot,y))(\xi)| \in L^1(d\xi)$. This implies $p_0(t+\varepsilon,\cdot,y) \in \mathcal{D}(A)$ for all $t \geq 0$. Consequently, we are done if we can show that

$$A_x\left(\int_0^t \int p_0(t-s+\varepsilon,\cdot,z)\Phi(s,z,y)\,dz\,ds\right)(x) = \int_0^t \int A_x p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds.$$

To prove this identity we apply Fubini's theorem twice to interchange the pseudo-differential operator A with the double integral. Lemma 4.24(i) shows that there exists a constant $C = C(\varepsilon,T)$ such that

$$\begin{aligned}&\int_0^t \int S(x-z,\alpha(z),t-s+\varepsilon)G(s,z,y)\,dz\,ds \\ &\quad\leq \int_0^{t+\varepsilon}\int S(x-z,\alpha(z),t+\varepsilon-s)G(s,z,y)\,dz\,ds \overset{\text{L4.24}}{\leq} H^1(t+\varepsilon,x,y)\end{aligned} \tag{$\star$}$$

for all $t \in (0,T]$ and $x,y \in \mathbb{R}^d$. By (4.62), we get

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} |p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)|\,dz\,ds\,dx \\ &\qquad\leq C'\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} S(x-z,\alpha(z),t-s+\varepsilon)G(s,z,y)\,dz\,ds\,dx \\ &\qquad\overset{(\star)}{\leq} C'\int_{\mathbb{R}^d} H^1(t+\varepsilon,x,y)\,dx < \infty.\end{aligned}$$

An application of Fubini's theorem now gives

$$\begin{aligned}&\mathcal{F}\left(\int_0^t\int p_0(t-s+\varepsilon,\cdot,z)\Phi(s,z,y)\,dz\,ds\right)(\xi) \\ &\quad= (2\pi)^{-d}\int_{\mathbb{R}^d}\left(\int_0^t\int p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds\right)e^{-ix\cdot\xi}\,dx \\ &\quad\overset{\text{Fub}}{=} (2\pi)^{-d}\int_0^t\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} e^{-ix\cdot\xi}p_0(t-s+\varepsilon,x,z)\,dx\right)\Phi(s,z,y)\,dz\,ds \\ &\quad= \int_0^t\int_{\mathbb{R}^d}\mathcal{F}(p_0(t-s+\varepsilon,\cdot,z))(\xi)\Phi(s,z,y)\,dz\,ds\end{aligned} \tag{$\star\star$}$$

for all $\xi \in \mathbb{R}^d$. Since $p_0(r,x,z) = p^{\alpha(z)}(r,x-z)$ a simple change of variables shows

$$\begin{aligned}|\mathcal{F}(p_0(t-s+\varepsilon,\cdot,z))(\xi)| &= (2\pi)^{-d}\left|\int p^{\alpha(z)}(t-s+\varepsilon,x-z)e^{-ix\cdot\xi}\,dx\right| \\ &\leq (2\pi)^{-d}\sup_{\alpha\in I}\left|\int p^{\alpha}(t-s+\varepsilon,x-z)e^{-ix\cdot\xi}\,dx\right| \\ &= (2\pi)^{-d}\sup_{\alpha\in I}\left|\int p^{\alpha}(t-s+\varepsilon,u)e^{-iu\cdot\xi}\,du\right| \\ &= (2\pi)^{-d}\sup_{\alpha\in I}\left|e^{-(t-s+\varepsilon)\Psi_\alpha(\xi)}\right|\end{aligned}$$

for all $z,\xi \in \mathbb{R}^d$. Because of the growth conditions on $\operatorname{Re}\Psi_\alpha$, cf. (3.3) and (C3), we get

$$|\mathcal{F}(p_0(t-s+\varepsilon,\cdot,z))(\xi)| \leq (2\pi)^{-d}e^{c_4(t+\varepsilon)}\mathbb{1}_{\{|\xi|\leq 1\}} + (2\pi)^{-d}e^{-c_1\varepsilon|\xi|^{\gamma_\infty^L}}\mathbb{1}_{\{|\xi|>1\}}$$

for all $s \in [0,t]$, $z,\xi \in \mathbb{R}^d$. Combining this estimate with (4.49) and Lemma 4.29(iv) it follows easily that

$$\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\psi_{\alpha(x)}(\xi)\mathcal{F}(p_0(t-s+\varepsilon,\cdot,y))(\xi)\Phi(s,z,y)|\,d\xi\,dz\,ds < \infty.$$

for each $x, y \in \mathbb{R}^d$. Another application of Fubini's theorem shows

$$\begin{aligned} &A_x\bigg(\int_0^t\int p_0(t-s+\varepsilon,\cdot,z)\Phi(s,z,y)\,dz\,ds\bigg)(x) \\ &\quad= -\int e^{ix\cdot\xi}\psi_{\alpha(x)}(\xi)\mathcal{F}\bigg(\int_0^t\int p_0(t-s+\varepsilon,\cdot,z)\Phi(s,z,y)\,dz\,ds\bigg)(\xi)\,d\xi \\ &\quad\overset{(\star\star)}{=} -\int e^{ix\cdot\xi}\psi_{\alpha(x)}(\xi)\int_0^t\int \mathcal{F}(p_0(t-s+\varepsilon,\cdot,z))(\xi)\Phi(s,z,y)\,dz\,ds\,d\xi \\ &\quad\overset{\text{Fub}}{=} -\int_0^t\int\bigg(\int \psi_{\alpha(x)}(\xi)e^{ix\cdot\xi}\mathcal{F}(p_0(t-s+\varepsilon,\cdot,z))(\xi)\,d\xi\bigg)\Phi(s,z,y)\,dz\,ds \\ &\quad= \int_0^t\int A_x p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds. \end{aligned}$$

□

The next lemma is concerned with the regularity of $P_{t,\varepsilon}f(x)$ with respect to t and x.

4.38 Lemma *(i)* $P_{t,\varepsilon}f \in C_\infty^2(\mathbb{R}^d)$ *for any* $f \in C_\infty(\mathbb{R}^d)$, $t > 0$.

(ii) $(0,\infty) \ni t \mapsto P_{t,\varepsilon}f(x)$ *is continuously differentiable for each* $f \in \mathcal{B}_b(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$.

(iii) $(0,\infty) \ni t \mapsto P_t f(x)$ *is continuously differentiable for all* $f \in \mathcal{B}_b(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$,

$$\frac{\partial}{\partial t}P_t f(x) = \int \frac{\partial}{\partial t}p(t,x,y)f(y)\,dy.$$

Proof. (i) By Theorem 4.12, $x \mapsto p_0(t,x,y) = p^{\alpha(y)}(t,x,y)$ is twice continuously differentiable and there exists a constant $C = C(T) > 0$ such that

$$\left|\frac{\partial}{\partial x_j}p_0(t,x,y)\right| \le Ct^{-1/\gamma_\infty^L}S(x-y,\alpha(y),t)$$

for all $t \in (0,T]$, $x, y \in \mathbb{R}^d$ and $j \in \{1,\dots,d\}$. Consequently,

$$\begin{aligned} \left|\frac{\partial}{\partial x_j}(p_0(t-s+\varepsilon,x,z)\Phi(s,z,y))\right| &\le C\varepsilon^{-1/\gamma_\infty^L}S(x-z,\alpha(z),t-s+\varepsilon)|\Phi(s,z,y)| \quad (\star) \\ &\le C'G(s,z,y) \end{aligned}$$

for some constant $C' = C'(\varepsilon,t)$ where we have used the boundedness of S, cf. (4.11), and (4.49). This shows that G is a dominating function. Moreover, by Lemma 4.29,

$$\int_0^t\int \sup_{|x|\le R} G(s,x,y)\,dy\,ds < \infty \qquad \text{for all} \quad R > 0.$$

Therefore, the differentiation lemma for parametrized integrals yields

$$\partial_{x_j}p_\varepsilon(t,x,y) = \partial_{x_j}p_0(t+\varepsilon,x,y) + \int_0^t\int \partial_{x_j}p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds.$$

In particular,

$$\begin{aligned} |\partial_{x_j}p_\varepsilon(t,x,y)| &\underset{T4.12}{\overset{(\star)}{\le}} Ct^{-1/\gamma_\infty^L}S(x-y,\alpha(y),t+\varepsilon) \\ &\quad + C\varepsilon^{-1/\gamma_\infty^L}\int_0^{t+\varepsilon}\int S(x-z,\alpha(z),t-s+\varepsilon)G(s,z,y)\,dz\,ds \end{aligned}$$

$$\overset{\text{L4.24}}{\leq} Ct^{-1/\gamma_\infty^L} S(x-y,\alpha(y),t+\varepsilon) + C\varepsilon^{-1/\gamma_\infty^L} H^1(t+\varepsilon,x,y)$$
$$\overset{\text{L4.34}}{\leq} C'' G(t+\varepsilon,x,y) \tag{4.67}$$

for some constant $C'' = C''(\varepsilon,t)$. Now it follows from Lemma 4.29(iv) that we may apply the differentiation lemma for parametrized integrals:

$$\frac{\partial}{\partial x_j} P_{t,\varepsilon} f(x) = \int \frac{\partial}{\partial x_j} p_\varepsilon(t,x,y) f(y)\,dy.$$

Since we have already shown that $|\partial_{x_j} p_\varepsilon(t,x,y)| \leq C''' G(t+\varepsilon,x,y)$, we can use the same reasoning as in Lemma 4.36(i) to conclude that $\partial_{x_j} P_{t,\varepsilon} f \in C_\infty(\mathbb{R}^d)$. Using the corresponding estimate for the derivatives of order 2, cf. Theorem 4.12, we find by another application of the differentiation lemma for parametrized integrals that $P_{t,\varepsilon} f \in C_\infty^2(\mathbb{R}^d)$.

(ii) Let $f \in \mathcal{B}_b(\mathbb{R}^d)$. We are going to show that $t \mapsto P_{t,\varepsilon} f(x)$ is differentiable on $[0,T]$ for fixed $T < \infty$, $\varepsilon > 0$ and $x \in \mathbb{R}^d$. We have already seen in Lemma 4.37 that

$$\partial_t p_\varepsilon(t,x,y) = \partial_t p_0(t+\varepsilon,x,y) + \int p_0(\varepsilon,x,z)\Phi(t,z,y)\,dz$$
$$+ \int_0^t \int \partial_t p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds.$$

Since $\partial_t p_0(t,x,y) = A_x p_0(t,x,y)$, cf. (4.50), it follows from Corollary 4.11 that there exists a constant $C = C(T)$ such that

$$|\partial_\tau p_0(\tau,x,y)| = |A_x^{\alpha(y)} p_0(\tau,x,y)| = |A_x^{\alpha(y)} p_\tau^{\alpha(y)}(x-y)| \overset{\text{C4.11}}{\leq} C\tau^{-1} S(x-y,\alpha(y),\tau)$$
$$\leq C\varepsilon^{-1} S(x-y,\alpha(y),\tau)$$

for all $\tau \in (\varepsilon, T+\varepsilon]$ and $x,y \in \mathbb{R}^d$. Combining this with (4.49), we get

$$|\partial_t p_\varepsilon(t,x,y)| \leq C' S(x-y,\alpha(y),t+\varepsilon) + C' \int S(x-z,\alpha(z),\varepsilon) G(t,z,y)\,dz$$
$$+ C' \int_0^t \int S(x-z,\alpha(z),t-s+\varepsilon) G(s,z,y)\,dz\,ds$$

for a suitable constant $C' = C'(\varepsilon,T)$. Using Lemma 4.34(iv), Corollary 4.20 and Lemma 4.24, we find

$$|\partial_t p_\varepsilon(t,x,y)| \leq C'' G(t+\varepsilon,x,y) \qquad \text{for all } \ x,y \in \mathbb{R}^d, t \in [0,T]. \tag{4.68}$$

Hence, by Lemma 4.34,

$$\int \sup_{t\in[0,T]} |\partial_t p_\varepsilon(t,x,y)|\,dy \leq C'' \int \sup_{t\in[0,T]} G(t+\varepsilon,x,y)\,dy \leq C''' \int G(T+\varepsilon,x,y)\,dy < \infty.$$

This shows that the assumptions of the differentiation lemma for parametrized integrals are satisfied. Therefore, we obtain

$$\frac{\partial}{\partial t} P_{t,\varepsilon} f(x) = \int \frac{\partial}{\partial t} p_\varepsilon(t,x,y) f(y)\,dy \tag{4.69}$$

for all $t \in (0,T]$ and $x \in \mathbb{R}^d$.

(iii) Fix $0 < T_0 < T < \infty$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$. Corollary 4.28 combined with Lemma 4.34 shows that there exists a constant $C = C(T_0,T) > 0$ such that

$$\left|\frac{\partial}{\partial t} p(t,x,y)\right| \leq CG(t,x,y) \qquad \text{for all } \; x,y \in \mathbb{R}^d, t \in [T_0,T].$$

Consequently,

$$\begin{aligned}\int \sup_{t\in[T_0,T]} \left|\frac{\partial}{\partial t} p(t,x,y) f(y)\right| dy &\leq C\|f\|_\infty \int \sup_{t\in[T_0,T]} G(t,x,y)\, dy \\ &\overset{\text{L4.34}}{\leq} C'\|f\|_\infty \int G(T,x,y)\, dy.\end{aligned}$$

We conclude from Lemma 4.29(vi) that

$$\int \sup_{t\in[T_0,T]} \left|\frac{\partial}{\partial t} p(t,x,y) f(y)\right| dy < \infty.$$

Applying the differentiation lemma for parametrized integrals proves the differentiability on (T_0,T). □

Now we are ready to prove that $q_\varepsilon = (\partial_t - A_x)p_\varepsilon \xrightarrow{\varepsilon\to 0} 0$ uniformly on compact sets; this result justifies the name "approximate fundamental solution".

4.39 Lemma $q_\varepsilon \to 0$ *uniformly on compact subsets of* $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.

Proof. Fix $0 < T_0 < T < \infty$ and a compact set $K \subseteq \mathbb{R}^d$. As $\partial_t p_0(t,x,y) = A_x^{\alpha(y)} p_0(t,x,y)$ for each fixed $y \in \mathbb{R}^d$ we obtain directly from Lemma 4.37 and the definition of F, cf. (4.4), that

$$\begin{aligned} q_\varepsilon(t,x,y) &= (A_x^{\alpha(y)} - A_x)p_0(t+\varepsilon,x,y) + \int_0^t \int (A_x^{\alpha(y)} - A_x)p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds \\ &\quad + \int p_0(\varepsilon,x,z)\Phi(t,z,y)\,dz \\ &= -F(t+\varepsilon,x,y) - \int_0^t \int F(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds \\ &\quad + \int p_0(\varepsilon,x,z)\Phi(t,z,y)\,dz. \end{aligned}$$

As

$$\begin{aligned} F(t+\varepsilon,x,y) &= \sum_{k\geq 1} F^{\circledast k}(t+\varepsilon,x,y) - \sum_{k\geq 2} F^{\circledast k}(t+\varepsilon,x,y) \\ &= \sum_{k\geq 1} F^{\circledast k}(t+\varepsilon,x,y) - \sum_{k\geq 1} (F \circledast F^{\circledast k})(t+\varepsilon,x,y) \\ &= \Phi(t+\varepsilon,x,y) - (F\circledast\Phi)(t+\varepsilon,x,y) \\ &= \Phi(t+\varepsilon,x,y) - \int_0^{t+\varepsilon}\int F(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds \end{aligned}$$

we get $q_\varepsilon = J_1 + J_2$ where

$$J_1 := \int_t^{t+\varepsilon}\int F(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds,$$

$$J_2 := \int p_0(\varepsilon,x,z)\Phi(t,z,y)\,dz - \Phi(t+\varepsilon,x,y).$$

We estimate J_1 and J_2 separately. As $|F| \leq G$ and $|\Phi| \leq CG$ for some constant $C = C(T)$, cf. (4.43) and (4.49), we have for all $t \leq T$ and $x,y \in \mathbb{R}^d$

$$|J_1| \leq C \int_t^{t+\varepsilon} \int G(t-s+\varepsilon,x,z)G(s,z,y)\,dz\,ds.$$

Using the convolution estimates from Section 4.2, it is not difficult to see that this implies

$$|J_1| \leq C'\varepsilon^{\kappa\wedge 1}G(t+\varepsilon,x,y) \quad \text{for all}\, \varepsilon \in (0,1), x,y \in \mathbb{R}^d, t \in [T_0,T]$$

for some constant $C' = C'(T_0,T)$; since the proof is similar to the proof of Lemma A.10 we omit the details. Because of the boundedness of G, cf. Lemma 4.34(ii), the expression on the right-hand side converges uniformly to 0 on $[T_0,T] \times \mathbb{R}^d \times \mathbb{R}^d$. It remains to estimate J_2. Obviously,

$$|J_2| \leq |\Phi(t+\varepsilon,x,y) - \Phi(t,x,y)| + \left| \int p_0(\varepsilon,x,z)\Phi(t,z,y)\,dz - \Phi(t,x,y)\right|.$$

The first term on the right-hand side converges uniformly to 0 on $[T_0,T] \times K \times K$ since Φ is uniformly continuous on the compact set $[T_0,T] \times K \times K$ (cf. Theorem 4.25). We want to deduce the uniform convergence of the second term using Remark 4.33; to this end we have to show that $A := \{\Phi(t,\cdot,y); t \in [T_0,T], y \in K\}$ is uniformly equicontinuous and uniformly bounded. Fix $\delta > 0$. Since

$$|\Phi(t,x,y)| \overset{(4.49)}{\leq} CG(t,x,y) \leq \frac{Ct}{|x-y|^{d+\gamma_\infty^L \wedge \gamma_0^L}} \qquad \text{for all} \;\; |x-y| \gg 1$$

and K is compact, we can choose $R \geq 1$ such that $|\Phi(t,x,y)| \leq \delta/2$ for all $|x| \geq R$, $y \in K$, $t \in [T_0,T]$. This implies

$$|\Phi(t,x,y) - \Phi(t,z,y)| \leq \delta \qquad \text{for all} \;\; x,z \in \mathbb{R}^d \backslash B[0,R], y \in K, t \in [T_0,T].$$

On the other hand, the mapping $[T_0,T] \times B[0,2R] \times K \ni (t,x,y) \mapsto \Phi(t,x,y)$ is uniformly continuous, i. e. there exists $\varrho > 0$ such that

$$|\Phi(t,x,y) - \Phi(t,z,y)| \leq \delta \qquad \text{for all} \;\; |z-x| \leq \varrho, x,z \in B[0,2R], y \in K, t \in [T_0,T].$$

Combining both estimates, we get the uniform equicontinuity of A. The uniform boundedness is a direct consequence of (4.49) and Lemma 4.34(ii). □

Next we show that the convergence carries over to the family of operators

$$Q_{t,\varepsilon}f(x) := (\partial_t - A_x)P_{t,\varepsilon}f(x).$$

4.40 Lemma *For any $f \in C_\infty(\mathbb{R}^d)$ it holds that*

(i) $Q_{t,\varepsilon}f \xrightarrow{\varepsilon\to 0} 0$ *uniformly on compact subsets of* $(0,\infty) \times \mathbb{R}^d$,

(ii) $\int_0^t Q_{s,\varepsilon} f\,ds \xrightarrow{\varepsilon\to 0} 0$ *uniformly on* $[0,T]\times K$ *for any compact set* $K\subseteq\mathbb{R}^d$ *and* $T>0$.

Proof. Let $0<T_0<T<\infty$ and $K\subseteq\mathbb{R}^d$ be compact. By Lemma 4.38, $Q_{t,\varepsilon}f$ is well-defined for $f\in C_\infty(\mathbb{R}^d)$. Moreover,

$$Q_{t,\varepsilon}f(x) = (\partial_t - A_x)P_{t,\varepsilon}f(x) = \int (\partial_t - A_x)p_\varepsilon(t,x,y)f(y)\,dy = \int q_\varepsilon(t,x,y)f(y)\,dy$$

for all $x\in\mathbb{R}^d$; we defer the proof of this identity to Lemma A.11. Moreover, it is not difficult to see from Lemma 4.37 and the estimates from Section 4.1 that

$$|q_\varepsilon(t,x,y)| \leq CG(t+\varepsilon,x,y). \tag{4.70}$$

Now let $f\in C_\infty(\mathbb{R}^d)$. For fixed $\delta>0$ choose $R>0$ such that $|f(y)|\leq\delta$ for all $|y|\geq R$. Then

$$\begin{aligned}|Q_{t,\varepsilon}f(x)| &\leq \delta\int_{|y|\geq R}|q_\varepsilon(t,x,y)|\,dy + \|f\|_\infty\int_{|y|\leq R}|q_\varepsilon(t,x,y)|\,dy\\ &\overset{(4.70)}{\leq} C\delta\int G(t+\varepsilon,x,y)\,dy + \|f\|_\infty\int_{|y|\leq R}|q_\varepsilon(t,x,y)|\,dy. \end{aligned}\tag{4.71}$$

(i) Obviously (4.71) implies

$$|Q_{t,\varepsilon}f(x)| \leq \left(C\sup_{z\in\mathbb{R}^d}\sup_{t\in[T_0,T+1]}\int G(t,z,y)\,dy\right)\delta + \|f\|_\infty\int_{|y|\leq R}|q_\varepsilon(t,x,y)|\,dy$$

for all $\varepsilon\in(0,1)$. Note that the expression in the bracket is finite by Lemma 4.29(i). It follows from the previous lemma that there exists $\varepsilon_0>0$ such that

$$\sup_{x\in K}\sup_{t\in[T_0,T]}\int_{|y|\leq R}|q_\varepsilon(t,x,y)|\,dy \leq \lambda^d(B[0,R])\sup_{x,y\in K\cup B[0,R]}\sup_{t\in[T_0,T]}|q_\varepsilon(t,x,y)|\leq\delta$$

for all $\varepsilon\leq\varepsilon_0$. Thus,

$$\sup_{x\in K}|Q_{t,\varepsilon}f(x)|\leq C'\delta \qquad \text{for all}\quad \varepsilon\leq\varepsilon_0.$$

(ii) We obtain from (4.71)

$$\begin{aligned}&\int_0^t |Q_{s,\varepsilon}f(x)|\,ds\\ &\quad\leq\left(C\sup_{z\in\mathbb{R}^d}\sup_{t\in[0,T+1]}\int_0^t\int G(s,z,y)\,dy\,ds\right)\delta + \|f\|_\infty\int_0^t\int_{|y|\leq R}|q_\varepsilon(s,x,y)|\,dy\,ds\end{aligned}$$

for all $x\in\mathbb{R}^d$ and $\varepsilon\in(0,1)$. By Lemma A.8(vii), we can choose $\varrho>0$ sufficiently small such that

$$\sup_{z\in\mathbb{R}^d}\sup_{\varepsilon\leq 1}\int_0^\varrho\int G(s+\varepsilon,z,y)\,dy\,ds\leq\delta.$$

Moreover, because of Lemma 4.37 there exists $\varepsilon_0\in(0,1)$ such that

$$\sup_{x,y\in K\cup B[0,R]}\sup_{t\in[\varrho,T]}|q_\varepsilon(t,x,y)|\leq\delta$$

for all $\varepsilon \le \varepsilon_0$. Hence,

$$\int_0^t \int_{|y|\le R} |q_\varepsilon(s,x,y)|\,dy\,ds \le C \sup_{z\in\mathbb{R}^d} \sup_{\varepsilon\le 1} \int_0^\varrho \int G(t+\varepsilon,z,y)\,dy \\ + T\lambda^d(B[0,R]) \sup_{t\in[\varrho,T]} \sup_{x,y\in K\cup B[0,R]} |q_\varepsilon(t,x,y)|$$

for all $x \in K$ and $\varepsilon \le \varepsilon_0$. Together with the above estimate for $\int_0^t |Q_{s,\varepsilon} f(x)|\,ds$ this yields (ii). □

We close this section with a convergence result for the time derivatives $\partial_t p_\varepsilon$ and $\partial_t P_{t,\varepsilon}$.

4.41 Lemma *Let $0 < T_0 < T < \infty$. Then:*

(i) $\partial_t p_\varepsilon(t,x,y) \to \partial_t p(t,x,y)$ uniformly on $[T_0,T]\times\mathbb{R}^d\times\mathbb{R}^d$.

(ii) $\partial_t P_{t,\varepsilon} f(x) \to \partial_t P_t f(x)$ uniformly on $[T_0,T]\times\mathbb{R}^d$ for any $f \in C_\infty(\mathbb{R}^d)$.

Proof. Throughout this proof, the value of the constants may vary from line to line. Fix $0 < T_0 < T < \infty$. By the differentiation lemma for parametrized integrals, we have

$$|\partial_t p_0(t+\varepsilon,x,y) - \partial_t p_0(t,x,y)| = (2\pi)^{-d} \left| \int \psi_{\alpha(y)}(\xi) e^{-i(x-y)\cdot\xi} e^{-t\psi_{\alpha(y)}(\xi)} (e^{-\varepsilon\psi_{\alpha(y)}(\xi)} - 1)\,d\xi \right|.$$

Using the same reasoning as in the first part of the proof of Lemma 4.35(i), we get

$$\partial_t p_0(t+\varepsilon,x,y) \xrightarrow{\varepsilon\to 0} \partial_t p_0(t,x,y) \quad \text{uniformly on } [T_0,T]\times\mathbb{R}^d\times\mathbb{R}^d. \tag{4.72}$$

Recall that

$$p_\varepsilon(t,x,y) = p_0(t+\varepsilon,x,y) + \int_0^t \int p_0(t-s+\varepsilon,x,z)\Phi(s,z,y)\,dz\,ds \\ = p_0(t+\varepsilon,x,y) + t\int_0^1 \int p_0(t(1-r)+\varepsilon,x,z)\Phi(rt,z,y)\,dz\,dr.$$

If we formally differentiate this identity with respect to t, we find

$$\partial_t p_\varepsilon(t,x,y) = \partial_t p_0(t+\varepsilon,x,y) + \int_0^1 \int p_0(t(1-r)+\varepsilon,x,z)\Phi(rt,z,y)\,dz\,dr \\ + t\int_0^1 \int (1-r)\partial_\tau p_0(\tau,x,z)\big|_{\tau=(1-r)t+\varepsilon} \Phi(rt,z,y)\,dz\,dr \\ + t\int_0^1 \int r p_0(r(1-t)+\varepsilon,x,z)\partial_\tau \Phi(\tau,z,y)\big|_{\tau=rt}\,dz\,dr.$$

We obtain an analogous expression for $\partial_t p$ by formally differentiating (4.46) with respect to t. To make these calculations rigorous we have to justify that the differentiation lemma for parametrized integrals is applicable. Since the reasoning is very similar to the proof of Theorem 4.27 and the proof of (4.65), we omit the details. Consequently, we obtain

$$|\partial_t p_\varepsilon(t,x,y) - \partial_t p(t,x,y)| \le J_1 + J_2 + J_3 + J_4$$

where

$$J_1 := |\partial_t p_0(t+\varepsilon,x,y) - \partial_t p_0(t,x,y)|$$

$$J_2 := \left| \int_0^1 \int (p_0(t(1-r)+\varepsilon,x,z) - p_0(t(1-r),x,z))\Phi(rt,z,y)\,dz\,dr \right|$$
$$J_3 := t \left| \int_0^1 \int (1-r) \left(\partial_\tau p_0(\tau,x,z)\big|_{\tau=(1-r)t+\varepsilon} - \partial_\tau p_0(\tau,x,z)\big|_{\tau=t(1-r)} \right) \Phi(rt,z,y)\,dz\,dr \right|$$
$$J_4 := t \left| \int_0^1 \int r(p_0(t(1-r)+\varepsilon,x,z) - p_0(t(1-r),x,z))\partial_\tau \Phi(\tau,z,y)\big|_{\tau=rt}\,dz\,dr \right|.$$

We estimate the terms separately. We have already seen that J_1 converges uniformly to 0 on $[T_0,T]\times\mathbb{R}^d\times\mathbb{R}^d$, cf. (4.72). Moreover,

$$\begin{aligned} J_2 &\overset{(4.51)}{=} \frac{1}{t}\left|(p_0(\cdot+\varepsilon,\cdot,\cdot)\circledast\Phi)(t,x,y) - (p_0\circledast\Phi)(t,x,y)\right| \\ &\leq \frac{1}{T_0}\left|(p_0(\cdot+\varepsilon,\cdot,\cdot)\circledast\Phi)(t,x,y) - (p_0\circledast\Phi)(t,x,y)\right|. \end{aligned}$$

Therefore, it follows from (4.64) that J_2 converges uniformly to 0. In order to estimate J_3 and J_4 we use a similar argument as in the proof of Lemma 4.35(i). If we set

$$J_{31} := t \left| \int_0^{1-\delta} \int (1-r) \left(\partial_\tau p_0(\tau,x,z)\big|_{\tau=(1-r)t+\varepsilon} - \partial_\tau p_0(\tau,x,z)\big|_{\tau=t(1-r)} \right) \Phi(rt,z,y)\,dz\,dr \right|$$
$$J_{32} := t \left| \int_{1-\delta}^1 \int (1-r) \left(\partial_\tau p_0(\tau,x,z)\big|_{\tau=(1-r)t+\varepsilon} - \partial_\tau p_0(\tau,x,z)\big|_{\tau=t(1-r)} \right) \Phi(rt,z,y)\,dz\,dr \right|$$

for fixed $\delta > 0$, then $J_3 \leq J_{31} + J_{32}$. By (4.49) and a change of variables,

$$\begin{aligned} J_{31} &\overset{(4.49)}{\leq} Ct \sup_{s\in[\delta T_0,T+1]} |\partial_t p_0(s+\varepsilon,x,y) - \partial_t p_0(s,x,y)| \int_0^1 \int G(rt,z,y)\,dz\,dr \\ &= C \sup_{s\in[\delta T_0,T+1]} |\partial_t p_0(s+\varepsilon,x,y) - \partial_t p_0(s,x,y)| \int_0^t \int G(s,z,y)\,dz\,ds \\ &\overset{\text{L4.29}}{\leq} C' \sup_{s\in[\delta T_0,T+1]} |\partial_t p_0(s+\varepsilon,x,y) - \partial_t p_0(s,x,y)| \end{aligned}$$

for all $\varepsilon \in (0,1)$ for some constant $C' = C'(T) > 0$. The right-hand side converges uniformly to 0 on $[T_0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ for fixed $\delta > 0$, cf. (4.72). On the other hand,

$$J_{32} \leq 2t \sup_{u\in[0,1]} \left| \int_{1-\delta}^1 \int (1-r)\partial_\tau p_0(\tau,x,z)\big|_{\tau=t(1-r)+u} \Phi(rt,z,y)\,dz\,dr \right|$$

for all $\varepsilon \in (0,1)$. Since

$$\begin{aligned} t&\left| \int_{1-\delta}^1 \int (1-r)\partial_\tau p_0(\tau,x,z)\big|_{\tau=t(1-r)+u} \Phi(rt,z,y)\,dz\,dr \right| \\ &\underset{(4.49)}{\overset{(4.56)}{\leq}} C \int_{1-\delta}^1 \int S(x-z,\alpha(z),t(1-r)+u)G(rt,z,y)\,dz\,dr \\ &= \frac{C}{t}\int_{(1-\delta)t}^t \int S(x-z,\alpha(z),t-s+u)G(s,z,y)\,dz\,ds \\ &\overset{\text{LA.10}}{\leq} \frac{C'}{T_0}\delta^{\kappa\wedge 1}(S(x-y,\alpha(y),t+u) + (t+u)g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)) \end{aligned}$$

for some absolute constant $C' = C'(T) > 0$, we obtain

$$J_{32} \leq \frac{2C}{T_0}\delta^{\kappa\wedge 1} \sup_{s\in[T_0,T+1]} (S(x-y,\alpha(y),s) + s g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)).$$

It follows from the boundedness of S, cf. (4.11), that $J_{32} \to 0$ uniformly on $[T_0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ as $\delta\to 0$. Letting first $\varepsilon\to 0$ and then $\delta\to 0$ yields $J_3\to 0$ uniformly on $[T_0,T]\times\mathbb{R}^d\times\mathbb{R}^d$. Splitting up the integral J_4 in the same way, we get from Theorem 4.27

$$\begin{aligned} J_{41} &\overset{\text{T4.27}}{\leq} \frac{C}{t} \sup_{s\in[\delta T_0,T]} |p_0(s+\varepsilon,x,y)-p_0(s,x,y)| \int_0^t \int G(s,z,y)\,dz\,ds \\ &\overset{\text{L4.29}}{\leq} C' \sup_{s\in[\delta T_0,T]} |p_0(s+\varepsilon,x,y)-p_0(s,x,y)|, \end{aligned}$$

and by (4.63) the right-hand side converges uniformly to 0 as $\varepsilon\to 0$. Moreover,

$$\begin{aligned} J_{42} &\leq 2t \sup_{u\in[0,1]} \left| \int_{1-\delta}^1 \int r p_0(t(1-r)+u,x,z)\partial_\tau\Phi(\tau,z,y)\Big|_{\tau=rt}\,dz\,dr \right| \\ &\underset{T4.27}{\overset{(4.10)}{\leq}} 2\frac{C}{T_0} \sup_{u\in[0,1]} \int_{1-\delta}^1 \int S(x-z,\alpha(z),t(1-r)+u)G(rt,z,y)\,dz\,dr. \end{aligned}$$

Applying Lemma A.10 another time we find – as for J_{32} – that $J_{42} \xrightarrow{\varepsilon\to 0} 0$ uniformly on $[T_0,T]\times\mathbb{R}^d\times\mathbb{R}^d$. If we let $\varepsilon\to 0$ and $\delta\to 0$, we obtain $J_4 \xrightarrow{\varepsilon\to 0} 0$ uniformly to 0 on $[T_0,T]\times\mathbb{R}^d\times\mathbb{R}^d$. Combining the above estimates yields (i). It remains to prove (ii), i. e.

$$\sup_{t\in[T_0,T]} \sup_{x\in\mathbb{R}^d} |\partial_t P_{t,\varepsilon} f(x) - \partial_t P_t f(x)| \xrightarrow{\varepsilon\to 0} 0 \qquad \text{for all} \quad f\in C_\infty(\mathbb{R}^d).$$

The argument is very similar to the proof of Lemma 4.36(iv). For fixed $\delta>0$ choose $R>0$ such that $|f(y)|\leq\delta$ for all $|y|\geq R$. Since (the proof of) Lemma 4.40 shows

$$\partial_t P_{t,\varepsilon} f(x) - \partial_t P_t f(x) = \int \partial_t(p_\varepsilon(t,x,y)-p(t,x,y))f(y)\,dy,$$

we get

$$\begin{aligned} |\partial_t P_{t,\varepsilon} f(x) - \partial_t P_t f(x)| \leq{}& \|f\|_\infty \lambda^d(B(0,R)) \sup_{y\in\mathbb{R}^d} |\partial_t p_\varepsilon(t,x,y)-\partial_t p(t,x,y)| \\ &+ \delta \int_{|y|\geq R} (|\partial_t p_\varepsilon(t,x,y)| + |\partial_t p(t,x,y)|)\,dy \end{aligned}$$

It follows from (4.68) and Corollary 4.28 that there exists a constant $C=C(T_0,T)>0$ such that

$$|\partial_t p_\varepsilon(t,x,y)| + |\partial_t p(t,x,y)| \leq C(G(t+\varepsilon,x,y)+G(t,x,y))$$

for all $t\in[T_0,T]$ and $x,y\in\mathbb{R}^d$. Consequently, we obtain from Lemma 4.29(i) that there exists a constant $C'=C'(T_0,T)>0$ such that

$$\begin{aligned} &|\partial_t P_{t,\varepsilon} f(x) - \partial_t P_t f(x)| \\ &\leq \|f\|_\infty \lambda^d(B(0,R)) \sup_{y\in\mathbb{R}^d} |\partial_t p_\varepsilon(t,x,y)-\partial_t p(t,x,y)| + C\delta \sup_{x\in\mathbb{R}^d} \sup_{s\in[T_0,T+1]} \int G(s,x,y)\,dy \\ &\overset{\text{L4.29}}{\leq} \|f\|_\infty \lambda^d(B(0,R)) \sup_{y\in\mathbb{R}^d} |\partial_t p_\varepsilon(t,x,y)-\partial_t p(t,x,y)| + C'\delta \end{aligned}$$

for all $\varepsilon\in(0,1)$, $t\in[T_0,T]$ and $x\in\mathbb{R}^d$. Since we already know from the first part of this lemma that the first term on the right-hand side converges uniformly to 0 on $[T_0,T]\times\mathbb{R}^d$, we conclude

$$\sup_{t\in[T_0,T]} \sup_{x\in\mathbb{R}^d} |\partial_t P_{t,\varepsilon} f(x) - \partial_t P_t f(x)| \leq (C'+1)\delta \qquad \text{for all} \;\; \varepsilon \text{ sufficiently small.}$$

□

4.7 $(P_t)_{t\geq 0}$ as a Markov semigroup

In this section we show that $(P_t)_{t\geq 0}$ is a conservative Markov semigroup and determine the generator of the semigroup on $C_c^\infty(\mathbb{R}^d)$. The key tools are the convergence results from the previous section and the positive maximum principle. Let us remind the reader that any pseudo-differential operator A with negative definite symbol satisfies the positive maximum principle on $C_b^2(\mathbb{R}^d)$, i. e.

$$u(x) = \sup_{y\in\mathbb{R}^d} u(y) \implies Au(x) \leq 0 \tag{PMP}$$

for any $u \in C_b^2(\mathbb{R}^d)$. The proofs we present in this section are very close to the proofs in [57, Section 4.2], but for the readers' convenience we include all the details. As in the previous sections, we assume that the conditions of Theorem 3.2 are satisfied.

4.42 Theorem *$(P_t)_{t\geq 0}$ is positivity preserving on $\mathcal{B}_b(\mathbb{R}^d)$, i. e. $P_t f \geq 0$ for all $f \in \mathcal{B}_b(\mathbb{R}^d)$, $f \geq 0$. Moreover, $p_t \geq 0$ for all $t > 0$.*

The statement is taken from [57, Lemma 4.3], but we have to modify the proof slightly; see the remark below the proof.

Proof. Suppose that $(P_t)_{t\geq 0}$ is not positivity preserving on $C_\infty(\mathbb{R}^d)$. Then there exist $T > 0$ and $f \in C_\infty(\mathbb{R}^d)$, $f \geq 0$, such that

$$\inf_{t\leq T} \inf_{x\in\mathbb{R}^d} P_t f(x) \leq \inf_{x\in\mathbb{R}^d} P_T f(x) < 0.$$

Since $P_{t,\varepsilon} f$ converges uniformly to $P_t f$, cf. Lemma 4.36(iv), there exist $\varepsilon_0 \in (0,1)$, $\delta > 0$, $\theta > 0$ such that

$$\inf_{t\leq T} \inf_{x\in\mathbb{R}^d} (P_{t,\varepsilon} f(x) + \theta t) \leq -\delta \qquad \text{for all } \ \varepsilon \leq \varepsilon_0.$$

As $f \geq 0$, we find from Lemma 4.36(ii) that there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and $T_0 > 0$ such that

$$P_{t,\varepsilon} f(x) + \theta t \geq P_{t,\varepsilon} f(x) \geq -\frac{\delta}{2} \qquad \text{for all } \ t \leq T_0, \varepsilon < \varepsilon_1, x \in \mathbb{R}^d. \tag{4.73}$$

On the other hand, Lemma 4.36(i) shows

$$\sup_{\varepsilon\in(0,\varepsilon_1)} \sup_{t\in[T_0,T]} |P_{t,\varepsilon} f(x)| \xrightarrow{|x|\to\infty} 0.$$

This implies that we can choose $R > 0$ such that

$$P_{t,\varepsilon} f(x) + \theta t \geq -\frac{\delta}{2} \qquad \text{for all } \ |x| > R, t \in [T_0, T], \varepsilon < \varepsilon_1.$$

Together with (4.73), we get for $\varepsilon \leq \varepsilon_1$

$$-\delta \geq \inf_{t\leq T} \inf_{x\in\mathbb{R}^d} (P_{t,\varepsilon} f(x) + \theta t) = \inf_{t\in(T_0,T]} \inf_{|x|\leq R} (P_{t,\varepsilon} f(x) + \theta t).$$

Because of the continuity of the mapping $(t,x) \mapsto P_{t,\varepsilon} f(x)$, this shows that for each $\varepsilon \in (0, \varepsilon_1)$ there exists $(t_\varepsilon, x_\varepsilon) \in (T_0, T] \times B[0,R]$ such that the above infimum is attained.

For brevity of notation, we set $g_\varepsilon(t,x) := P_{t,\varepsilon} f(x) + \theta t$. Since $g_\varepsilon(t_\varepsilon, x_\varepsilon) = \inf_{x\in\mathbb{R}^d} g_\varepsilon(t_\varepsilon, x)$ and $g_\varepsilon(t_\varepsilon, \cdot) \in C^2_\infty(\mathbb{R}^d)$ by Lemma 4.38 it follows from (PMP) that

$$A_x g_\varepsilon(t,x)\big|_{(t,x)=(t_\varepsilon,x_\varepsilon)} \geq 0.$$

for all $\varepsilon \leq \varepsilon_1$. Moreover, we have

$$\partial_t g_\varepsilon(t,x)\big|_{(t,x)=(t_\varepsilon,x_\varepsilon)} \leq 0 \qquad \text{for all } \; \varepsilon \leq \varepsilon_1.$$

Indeed: Suppose this was not true, then Taylor's formula would yield the existence of $t'_\varepsilon \in (T_0, t_\varepsilon)$ such that $g_\varepsilon(t'_\varepsilon, x_\varepsilon) < g_\varepsilon(t_\varepsilon, x_\varepsilon)$. Obviously, this contradicts the fact that g_ε attains its infimum in $(t_\varepsilon, x_\varepsilon)$. Note that it is crucial that we have excluded the case that the infimum is attained at the left boundary, i. e. $t_\varepsilon = T_0$.[7]
Consequently, we find

$$(\partial_t - A_x) g_\varepsilon(t,x)\big|_{(t,x)=(t_\varepsilon,x_\varepsilon)} \leq 0 \qquad \text{for all } \; \varepsilon \leq \varepsilon_1.$$

On the other hand, it follows from Lemma 4.40(i) that

$$(\partial_t - A_x) g_\varepsilon(t,x)\big|_{(t,x)=(t_\varepsilon,x_\varepsilon)} = Q_{t_\varepsilon,\varepsilon} f(x_\varepsilon) + \theta \xrightarrow{\varepsilon\to 0} \theta > 0.$$

This contradicts the previous inequality. Hence, $(P_t)_{t\geq 0}$ is positivity preserving on $C_\infty(\mathbb{R}^d)$. An application of Riesz' representation theorem, cf. e. g. [87, Theorem 2.14], yields that $(P_t)_{t\geq 0}$ is also positivity preserving on $\mathcal{B}_b(\mathbb{R}^d)$ and that $p_t \geq 0$ for all $t > 0$. □

Remark Knopova & Kulik [57] claim that $Q_{t,\varepsilon} f$ converges uniformly to 0 on the set $[T_0, T] \times \mathbb{R}^d$ for $f \in C_\infty(\mathbb{R}^d)$ (cf. [57, Lemma 4.2]). Since we have only established uniform convergence on compact subsets of $(0,\infty) \times \mathbb{R}^d$, we have, in contrast to [57], to ensure that the family $(x_\varepsilon)_{\varepsilon>0}$ is contained in a compact set.

4.43 Theorem *(i) $(P_t)_{t\geq 0}$ has the semigroup property, i. e. for all $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $s,t \geq 0$ it holds that $P_t P_s f = P_{t+s} f$.*

(ii) $P_t f - f = \int_0^t P_s A f \, ds$ for all $f \in C_c^\infty(\mathbb{R}^d)$ and $t \geq 0$.

Using Theorem 4.43(ii), it is not difficult to see that $(P_t)_{t\geq 0}$ is conservative (see Corollary 4.44 below) and that $C_c^\infty(\mathbb{R}^d)$ is contained in the generator of the semigroup $(P_t)_{t\geq 0}$.

Proof of Theorem 4.43. Since the reasoning is similar to the proof of Theorem 4.42, we do not give all the details.

(i) Suppose there exist $f \in C_c(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $s, T > 0$ such that

$$P_T P_s f(x) - P_{T+s} f(x) < 0. \tag{$\star$}$$

[7] Just consider e. g. $f(x) := x$ on $[0,1]$, then the minimum is attained in $x = 0$ but $f'(0) = 1 > 0$.

By Lemma 4.36(iv) and Theorem 4.32(ii), there exist $\varepsilon_0 \in (0,1)$, $\delta > 0$, $\theta > 0$ such that $g_\varepsilon(t,x) := P_{t,\varepsilon} P_s f(x) - P_{t+s,\varepsilon} f(x) + \theta t$ satisfies

$$\inf_{t \leq T} \inf_{x \in \mathbb{R}^d} g_\varepsilon(t,x) \leq -\delta \qquad \text{for all } \varepsilon \leq \varepsilon_0.$$

Combining Lemma 4.36(iii),(iv) with Theorem 4.32(ii), we find

$$\|g_\varepsilon(t,\cdot)\|_\infty \leq \theta t + \|P_{t+s,\varepsilon} f - P_{s,\varepsilon} f\|_\infty + \|P_{s,\varepsilon} f - P_s f\|_\infty + \|P_{t,\varepsilon} P_s f - P_s f\|_\infty \xrightarrow{t,\varepsilon \to 0} 0.$$

Consequently, there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and $T_0 > 0$ such that $g_\varepsilon(t,x) \geq -\delta/2$ for all $x \in \mathbb{R}^d$, $t \leq T_0$ and $\varepsilon \leq \varepsilon_1$. On the other hand, it follows again from Lemma 4.36(i) that there exists $R > 0$ such that $g_{t,\varepsilon}(x) \geq -\delta/2$ for all $t \in [T_0, T]$, $|x| \geq R$ and $\varepsilon \leq \varepsilon_1$. As in the proof of the previous lemma, we can therefore choose $(t_\varepsilon, x_\varepsilon) \in (T_0, T] \times B[0,R]$ such that

$$g_\varepsilon(t_\varepsilon, x_\varepsilon) = \inf_{t \leq T} \inf_{x \in \mathbb{R}^d} g_\varepsilon(t,x)$$

and

$$(\partial_t - A_x) g_\varepsilon(t,x)\big|_{(t,x)=(t_\varepsilon,x_\varepsilon)} \geq 0 \qquad \text{for all } \varepsilon \leq \varepsilon_1.$$

On the contrary, we have by Lemma 4.40(i) and Theorem 4.32(ii)

$$\begin{aligned}(\partial_t - A_x) g_\varepsilon(t,x)\big|_{(t,x)=(t_\varepsilon,x_\varepsilon)} &= Q_{t,\varepsilon} P_s f(x)\big|_{(t,x)=(t_\varepsilon,x_\varepsilon)} - Q_{t+s,\varepsilon} f(x)\big|_{(t,x)=(t_\varepsilon,x_\varepsilon)} + \theta \\ &\xrightarrow{\varepsilon \to 0} \theta > 0.\end{aligned}$$

If we replace f by $-f$ in $(\star)$, we get $P_{T+s} f - P_T P_s f = 0$ for all $f \in C_c(\mathbb{R}^d)$. Applying Riesz' representation theorem to $f \mapsto P_{T+s} f - P_T P_s f$ gives $P_{T+s} f - P_T P_s f = 0$ for all $f \in \mathcal{B}_b(\mathbb{R}^d)$.

(ii) Suppose that

$$P_T f(x) - f(x) < \int_0^T P_s A f(x)\, ds$$

for some $f \in C_c^\infty(\mathbb{R}^d)$, $T > 0$ and $x \in \mathbb{R}^d$. Since $Af \in C_\infty(\mathbb{R}^d)$, cf. Theorem 1.27, it follows from Lemma 4.36(ii),(iv) that there exist $\theta > 0$, $\delta > 0$, $\varepsilon_0 \in (0,1)$, $T_0 > 0$ such that

$$g_\varepsilon(t,x) := P_{t,\varepsilon} f(x) - f(x) - \int_0^t P_{s,\varepsilon} A f(x)\, ds + \theta t,$$

satisfies $\inf_{t \leq T} \inf_{x \in \mathbb{R}^d} g_\varepsilon(t,x) \leq -\delta$ and

$$g_\varepsilon(t,x) \geq -\frac{\delta}{2} \qquad \text{for all } x \in \mathbb{R}^d, t \leq T_0, \varepsilon \leq \varepsilon_0.$$

Using a very similar argument as in the proof of Lemma 4.42 we find $R > 0$, $\varepsilon_1 \in (0, \varepsilon_0)$ and $(t_\varepsilon, x_\varepsilon) \in (T_0, T] \times B[0,R]$ such that g_ε attains its minimum on $[0,T] \times \mathbb{R}^d$ in $(t_\varepsilon, x_\varepsilon)$ for each $\varepsilon \leq \varepsilon_1$. Since $g_\varepsilon(t,\cdot) \in C_\infty^2(\mathbb{R}^d)$ by Lemma 4.38, we still have

$$(\partial_t - A_x) g_\varepsilon(t,x)\big|_{(t,x)=(t_\varepsilon,x_\varepsilon)} \leq 0 \qquad \text{for all } \varepsilon \leq \varepsilon_1$$

by the positive maximum principle. On the other hand, it is not difficult to see that Fubini's theorem implies

$$A\left(\int_0^t P_{s,\varepsilon}f(\cdot)\,ds\right)(x) = \int_0^t AP_{s,\varepsilon}f(x)\,ds,$$

cf. Lemma A.12 for a proof. Combining this with the fundamental theorem of calculus and Lemma 4.38 we get

$$\begin{aligned}
&(\partial_t - A_x)g_\varepsilon(t,x)\\
&\quad= Q_{t,\varepsilon}f(x) + Af(x) - P_{t,\varepsilon}Af(x) + \int_0^t AP_{s,\varepsilon}Af(x)\,ds + \theta\\
&\quad= Q_{t,\varepsilon}f(x) + Af(x) - P_{t,\varepsilon}Af(x) + \int_0^t (\partial_s P_{s,\varepsilon}Af(x) - Q_{s,\varepsilon}Af(x))\,ds + \theta\\
&\quad= Q_{t,\varepsilon}f(x) - \int_0^t Q_{s,\varepsilon}Af(x)\,ds + \theta.
\end{aligned}$$

Hence, by Lemma 4.40,

$$(\partial_t - A_x)g_\varepsilon(t,x)\big|_{(t,x)=(t_\varepsilon,x_\varepsilon)} \xrightarrow{\varepsilon\to 0} \theta > 0$$

which is again a contradiction. Replacing f by $-f$ yields $P_tf - f = \int_0^t P_sAf\,ds$ for all $f \in C_c^\infty(\mathbb{R}^d)$ and $t \geq 0$. □

4.44 Corollary $(P_t)_{t\geq 0}$ *is conservative, i. e.* $P_t1 = 1$ *for all* $t \geq 0$.

Proof. Obviously, the claim is trivially satisfied for $t = 0$; therefore we may assume that $t > 0$. Pick $f \in C_c^\infty(\mathbb{R}^d)$ such that $f(x) = 1$ for all $|x| \leq 1$ and $f(x) = 0$ for $|x| \geq 2$. Then $f_k(x) := f(x/k)$, $x \in \mathbb{R}^d$, satisfies by Theorem 4.43(ii)

$$P_tf_k(x) - f_k(x) = \int_0^t P_sAf_k(x)\,ds \qquad \text{for all } \; x \in \mathbb{R}^d, k \in \mathbb{N}. \tag{$\star$}$$

As

$$|P_tf_k(x) - P_t1(x)| \leq P_t(\mathbb{1}_{B(0,k)^c})(x) \overset{(4.48)}{\leq} C\int_{|y|\geq k}(S(x-y,\alpha(y),t) + G(x-y,\alpha(y),t))\,dy$$

an application of the monotone convergence theorem yields

$$P_tf_k(x) \xrightarrow{k\to\infty} P_t1(x) \qquad \text{for all } \; x \in \mathbb{R}^d.$$

Since we also know that $f_k(x) \xrightarrow{k\to\infty} 1$ for all $x \in \mathbb{R}^d$, the claim follows by letting $k \to \infty$ in $(\star)$ if we can show that

$$\lim_{k\to\infty}\int_0^t P_sAf_k(x)\,ds = 0 \qquad \text{for all } \; x \in \mathbb{R}^d.$$

To this end, we note that

$$\begin{aligned}
Af_k(x) = -\int e^{ix\cdot\xi}\psi_{\alpha(x)}(\xi)\hat{f}_k(\xi)\,d\xi &= -k^d\int e^{ix\cdot\xi}\psi_{\alpha(x)}(\xi)\hat{f}(k\xi)\,d\xi\\
&= -\int e^{ix\cdot\eta/k}\psi_{\alpha(x)}\left(\frac{\eta}{k}\right)\hat{f}(\eta)\,d\eta.
\end{aligned}$$

This shows, on the one hand, that $Af_k(x) \xrightarrow{k\to\infty} 0$ for all $x \in \mathbb{R}^d$ and on the other hand that

$$|Af_k(x)| \le C := c \int (1+|\eta|^2)\hat{f}(\eta)\,d\eta$$

for all $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$; here c is chosen such that $|\psi_{\alpha(x)}(\xi)| \le c(1+|\xi|^2)$ for all $\xi \in \mathbb{R}^d$. Using (4.48) and Lemma 4.29, we find

$$\begin{aligned}\int_0^t \int p(s,x,y) \sup_{k\in\mathbb{N}} |Af_k(y)|\,dy\,ds &\le C \int_0^t \int |p(s,x,y)|\,dy\,ds \\ &\le C'C \int_0^t \int G(s,x,y)\,dy\,ds < \infty.\end{aligned}$$

Therefore we may apply the dominated convergence theorem to conclude

$$\int_0^t P_s Af_k(x)\,ds = \int_0^t \int p(s,x,y) Af_k(y)\,dy\,ds \xrightarrow{k\to\infty} 0 \qquad \text{for all } x \in \mathbb{R}^d. \qquad \square$$

4.8 Proof of the main results

Finally we are ready to prove the main results. For the proof of Theorem 3.7 and Theorem 3.8 see Section 4.9.

Proof of Theorem 3.2. Theorem 4.32, Theorem 4.42, Theorem 4.43(i) and Corollary 4.44 show that $(P_t)_{t\ge0}$ is a strong Feller semigroup. Denote by $(L, \mathcal{D}(L))$ the generator of the semigroup. Moreover, it follows from Theorem 1.27 and (C3) that $Af \in C_\infty(\mathbb{R}^d)$ for any $f \in C_c^\infty(\mathbb{R}^d)$. Combining Lemma 4.43(ii) and the fundamental theorem of calculus, we get

$$\lim_{t\to0} \frac{P_t f(x) - f(x)}{t} = Af(x) \in C_\infty(\mathbb{R}^d) \qquad \text{for all } x \in \mathbb{R}^d.$$

Hence, by Proposition 1.20, $f \in \mathcal{D}(L)$ and $Lf = Af$. This proves $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(L)$ and $A|_{C_c^\infty(\mathbb{R}^d)} = L|_{C_c^\infty(\mathbb{R}^d)}$. By Theorem 1.18, there exists a Feller process $(X_t)_{t\ge0}$ whose semigroup equals $(P_t)_{t\ge0}$. Finally, we have seen in Theorem 4.25 that the mapping $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d \ni (t,x,y) \mapsto p(t,x,y)$ is continuous, and this finishes the proof. $\square$

Proof of Proposition 3.3. By Theorem 3.2, we have $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(L)$ and $Af = Lf$ for any $f \in C_c^\infty(\mathbb{R}^d)$. Since $q(x,\xi) = \psi_{\alpha(x)}(\xi)$ has bounded coefficients, this already implies $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(L)$ and the stated identity for Lf (see [19, Theorem 2.37] for details). As $(L, \mathcal{D}(L))$ is a closed operator, we get

$$\overline{(L, C_\infty^2(\mathbb{R}^d))}^{\|\cdot\|_\infty} \subseteq (L, \mathcal{D}(L)).$$

To prove "⊇" let us first establish that

$$(P_t f, \partial_t P_t f) \in \overline{(L, C_\infty^2(\mathbb{R}^d))}^{\|\cdot\|_\infty} \qquad \text{for all } t>0, f \in C_\infty(\mathbb{R}^d). \qquad (\star)$$

We remind the reader that we have shown in Lemma 4.38 that $P_{t,\varepsilon} f \in C_\infty^2(\mathbb{R}^d)$ and that $P_t f$ is differentiable with respect to t for any $f \in C_\infty(\mathbb{R}^d)$. Fix $f \in C_\infty(\mathbb{R}^d)$ and $t > 0$.

Lemma 4.36 shows that $P_{t,\varepsilon}f$ converges uniformly to $P_t f$ as $\varepsilon \to 0$. On the other hand, as $P_{t,\varepsilon}f \in C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(L)$, we have $LP_{t,\varepsilon}f = \partial_t P_{t,\varepsilon}f$, and therefore

$$\|LP_{t,\varepsilon}f - \partial_t P_t f\|_\infty = \|\partial_t P_{t,\varepsilon}f - \partial_t P_t f\|_\infty \xrightarrow[L4.41]{\varepsilon\to 0} 0.$$

Consequently, $(P_t f, \partial_t P_t f) \in \overline{(L, C_\infty^2(\mathbb{R}^d))}^{\|\cdot\|_\infty}$. This proves $(\star)$. Now take $f \in \mathcal{D}(L)$. Then $P_t f \in \mathcal{D}(L)$ for any $t > 0$ and, therefore $\partial_t P_t f = LP_t f$. It follows from $(\star)$ that

$$(P_t f, LP_t f) = (P_t f, \partial_t P_t f) \in \overline{(L, C_\infty^2(\mathbb{R}^d))}^{\|\cdot\|_\infty}.$$

The strong continuity of $(P_t)_{t\geq 0}$ yields $P_t f \xrightarrow{t\to 0} f$ uniformly. As $Lf \in C_\infty(\mathbb{R}^d)$ the strong continuity also entails

$$\|LP_t f - Lf\|_\infty = \|P_t Lf - Lf\|_\infty \xrightarrow{t\to 0} 0.$$

Hence, $\overline{(L, C_\infty^2(\mathbb{R}^d))}^{\|\cdot\|_\infty} \ni (P_t f, LP_t f) \xrightarrow{t\to 0} (f, Lf)$ uniformly, i. e. $(f, Lf) \in \overline{(L, C_\infty^2(\mathbb{R}^d))}^{\|\cdot\|_\infty}$. This proves $\overline{(L, C_\infty^2(\mathbb{R}^d))} = (L, \mathcal{D}(L))$, i. e. that $C_\infty^2(\mathbb{R}^d)$ is a core of $(L, \mathcal{D}(L))$. Since we already know that $(X_t)_{t\geq 0}$ is a rich Feller process and $(C_c^\infty(\mathbb{R}^d), \|\cdot\|_{(2)})$ is dense in $(C_\infty^2(\mathbb{R}^d), \|\cdot\|_{(2)})$, it follows from Remark 1.39 that $C_c^\infty(\mathbb{R}^d)$ is a core of $(L, \mathcal{D}(L))$. □

Proof of Corollary 3.4. We have shown in Theorem 4.32 that

$$\lim_{t\to 0} \int p(t,x,y) f(y)\, dy = f(x), \qquad x \in \mathbb{R}^d,$$

for any bounded and uniformly continuous function $f : \mathbb{R}^d \to \mathbb{R}$. As $P_t 1 = \int p(t,x,y)\, dy = 1$ for all $t > 0$, this is, by the portmanteau theorem [10, Theorem 1.2.1], equivalent to $p(t,x,\cdot) \to \delta_x$ weakly. Moreover, we know from Corollary 4.28 that $p(\cdot,x,y)$ is differentiable on $(0,\infty)$ for all $x, y \in \mathbb{R}^d$. It remains to check that $p(t,\cdot,y) \in \mathcal{D}(L)$ for all $t > 0$, $y \in \mathbb{R}^d$ and that $(\partial_t - L_x)p(t,x,y) = 0$. The argument is similar to the proof of Proposition 3.3(ii). In the proof of Lemma 4.38(i) we have seen that $p_\varepsilon(t,\cdot,y) \in C_\infty^2(\mathbb{R}^d)$ for all $\varepsilon > 0$, $y \in \mathbb{R}^d$ and $t > 0$. As $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(L)$ by Proposition 3.3, we have

$$\partial_t p_\varepsilon(t,x,y) = L_x p_\varepsilon(t,x,y). \tag{$\star$}$$

Since

$$\|p_\varepsilon(t,\cdot,y) - p(t,\cdot,y)\|_\infty \xrightarrow[L4.35]{\varepsilon\to 0} 0$$

and

$$\|L_x p_\varepsilon(t,\cdot,y) - \partial_t p(t,\cdot,y)\| \overset{(\star)}{=} \|\partial_t p_\varepsilon(t,\cdot,y) - \partial_t p(t,\cdot,y)\| \xrightarrow[L4.41]{\varepsilon\to 0} 0,$$

we get $(p(t,\cdot,y), \partial_t p(t,\cdot,y)) \in \overline{(L, C_\infty^2(\mathbb{R}^d))}^{\|\cdot\|_\infty}$. By Proposition 3.3, this implies $p(t,\cdot,y) \in \mathcal{D}(L)$ and $L_x p(t,x,y) = \partial_t p(t,x,y)$. □

Proof of Corollary 3.5. (i) This is a direct consequence of Example 1.32.

(ii) This follows from Proposition 3.3 and Theorem 1.37. □

Proof of Theorem 3.6. This is obvious from the estimates we have derived in Theorem 4.25 and Corollary 4.28. □

4.9 Proof of Theorem 3.7, Theorem 3.8 and Corollary 3.9

Proof of Theorem 3.7. We have used condition (C1), i. e. the rotational invariance of ψ_α, only in Section 4.1. Consequently, it suffices to check that the results from Section 4.1 still hold in dimension $d = 1$ if ψ_α is not necessarily symmetric. With two exceptions (Theorem 4.12 and Corollary 4.14) the modifications in the proofs are similar and therefore we just explain, as an example, how to prove Theorem 4.1 for $m > 0$. Before we do so, some words on the proof of Theorem 4.12 and Corollary 4.14 in dimension $d = 1$:

- Theorem 4.12: In Section 4.1 this theorem was proved using results from Fourier analysis, and for this method of proof the assumption of symmetry is crucial. However, in dimension $d = 1$, Theorem 4.12 is a direct consequence of (the one-dimensional version of) Theorem 4.7 and the differentiation lemma for parametrized integrals.

- Corollary 4.14: Corollary 4.14 is not relevant for this part of the proof; we need it only for the proof of Corollary 3.9(ii),(iii) (and there symmetry of ψ_α is assumed).

Let $(\psi_\alpha)_{\alpha\in I}$ be a family of continuous negative definite functions such that $\Psi_\alpha := \psi_\alpha$ satisfies (C2)-(C4). Since we haven't used the symmetry in the proof of the on-diagonal estimate, it is enough to prove the off-diagonal estimate

$$p_t^\alpha(x) \le C\frac{t}{|x|^{1+\gamma_\infty(\alpha)}}\exp\left(-\frac{m}{4}|x|\right), \qquad t \in (0,T], x \in \mathbb{R}.$$

For fixed $x > 0$ we choose $z_0 \in \mathbb{C}$, $|z_0| = 1$, with $\operatorname{Re} z_0 > 0$, $\operatorname{Im} z_0 > 0$ such that $-i\frac{m}{2} + rz_0 \in \Omega$ and $-i\frac{m}{2} - r\bar{z}_0 \in \Omega$ for all $r \ge 0$. A straightforward calculation shows that there exist a sequence $(R_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ such that $R_n \to \infty$ and

$$\operatorname{Re}\left(z_0\int_0^{R_n} e^{-i(R_n-r)x_0z_0}\,dr\right) = 0 = \operatorname{Re}\left(\bar{z}_0\int_{-R_n}^0 e^{irx\bar{z}_0}\,dr\right) \qquad \text{for all } n \in \mathbb{N}. \tag{$\star$}$$

Now we define curves as follows (cf. Figure 4.4):

$$\begin{aligned}
\gamma_1(\theta) &:= R_n e^{-i\theta}, && \theta \in [0, f_n]\\
\gamma_2(r) &:= -i\frac{m}{2} + (R_n - r)z_0, && r \in [0, R_n]\\
\gamma_3(r) &:= -i\frac{m}{2} - r\bar{z}_0, && r \in [0, R_n]\\
\gamma_4(\theta) &:= R_n e^{-i\theta}, && \theta \in [g_n, \pi],
\end{aligned}$$

where $f_n \in (0,\pi/2)$ and $g_n \in (\frac{\pi}{2},\pi)$ are chosen in such a way that $\gamma_1(f_n) = \gamma_2(0)$ and $\gamma_3(R_n) = \gamma_4(g_n)$. Because of our choice of R_n, we have

$$\begin{aligned}
\operatorname{Re}\left(\int_{\gamma_2} e^{-ix\xi}e^{-t\psi_\alpha(\xi)}\,d\xi\right) &= \operatorname{Re}\left(z_0\exp\left(-\frac{m}{2}x\right)\int_0^{R_n} e^{-ix(R_n-r)z_0}e^{-t\psi_\alpha(\xi)}\,d\xi\right)\\
&\overset{(\star)}{=} \operatorname{Re}\left(z_0\exp\left(-\frac{m}{2}x\right)\int_0^{R_n} e^{-ix(R_n-r)z_0}(e^{-t\psi_\alpha(\xi)}-1)\,d\xi\right).
\end{aligned}$$

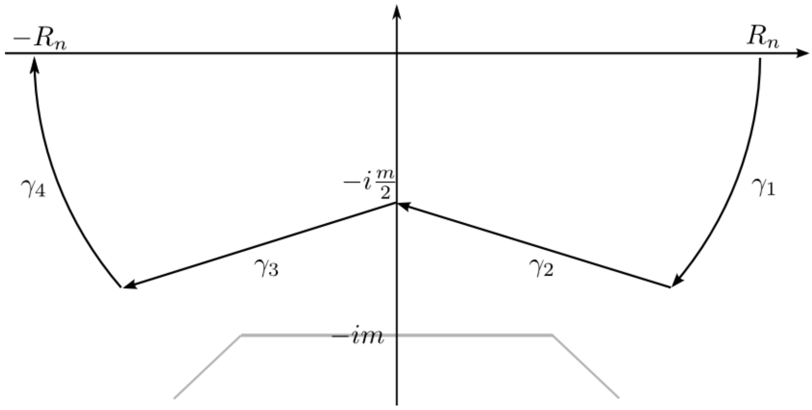

Figure 4.4: The contour of integration to obtain the heat kernel estimate in dimension $d = 1$.

Applying Lemma 4.5 and using the growth assumptions (3.3),(C4), we find constants $C = C(T)$, $C' = C'(T, z_0)$ such that

$$\begin{aligned}\left|\mathrm{Re}\left(\int_{\gamma_2} e^{-ix\xi}e^{-t\psi_\alpha(\xi)}\,d\xi\right)\right| &\leq Ct\exp\left(-\frac{m}{2}x\right)\int_0^{R_n} e^{-xr\,\mathrm{Im}\,z_0}\max\{1, r^{\gamma_\infty^\alpha}\}\,dr\\ &\leq C't\frac{1}{|x|^{1+\gamma_\infty(\alpha)}}\exp\left(-\frac{m}{4}x\right)\end{aligned}$$

for all $t \in (0, T]$. Since the paths are defined in a symmetric way and also the assumptions on $\Psi_\alpha(\xi) = \psi_\alpha(\xi)$ are symmetric, we get the same estimate if we replace γ_2 by γ_3. Moreover, using exactly the same argument as in the proof of Theorem 4.1, we also get

$$\lim_{n\to\infty}\left|\int_{\gamma_1} e^{-ix\xi}e^{-t\psi_\alpha(\xi)}\,d\xi\right| = \lim_{n\to\infty}\left|\int_{\gamma_4} e^{-ix\xi}e^{-t\psi_\alpha(\xi)}\,d\xi\right| = 0.$$

Adding all up, we conclude from Cauchy's theorem that

$$\begin{aligned}p_t^\alpha(x) &= \frac{1}{2\pi}\lim_{n\to\infty}\mathrm{Re}\left(\int_{-R_n}^{R_n} e^{-ix\xi}e^{-t\psi_\alpha(\xi)}\,d\xi\right)\\ &= \frac{1}{2\pi}\lim_{n\to\infty}\mathrm{Re}\left(\sum_{i=1}^4\int_{\gamma_i} e^{-ix\xi}e^{-t\psi_\alpha(\xi)}\,d\xi\right)\\ &\leq \frac{C'}{\pi}\frac{t}{|x|^{1+\gamma_\infty(\alpha)}}\exp\left(-\frac{m}{4}x\right)\end{aligned}$$

for all $x > 0$. For $x < 0$ we define, in an analogous way, the paths in the upper half plane. □

The remaining part of this section is concerned with the proof of Theorem 3.8. First we state some estimates which we will need later on.

4.45 Lemma *Define*

$$\begin{aligned}A^{\beta,k}p_t^\alpha(x) &:= -\frac{1}{2\pi}\int_{\mathbb{R}} \xi^k\psi_\beta(\xi)e^{-ix\xi}e^{-t\psi_\alpha(\xi)}\,d\xi\\ B^\beta p_t^\alpha(x) &:= -\frac{1}{2\pi}\int_{\mathbb{R}} \psi_\beta(\xi)\cdot\partial_\alpha\psi_\alpha(\xi)e^{-ix\xi}e^{-t\psi_\alpha(\xi)}\,d\xi\end{aligned}$$

for $t > 0$, $\alpha, \beta \in I$, $k \in \mathbb{N}_0$ and $x \in \mathbb{R}$. Then, under the assumptions of Theorem 3.8, there exist $c = c(T) > 0$ and $C = C(T) > 0$ such that

$$|A^{\beta,k} p_t^\alpha(x)| \le C \exp\left(-\frac{m}{4}|x|\right) \min\left\{ t^{-(1+k+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}, \frac{|x|}{1+|x|^{2+k+\gamma_0(\beta)}} + \frac{1}{|x|^{1+k+\gamma_\infty(\beta)}} \right\} \tag{4.74}$$

$$|\partial_{\beta_j} A^{\beta,k} p_t^\alpha(x)| \le C \exp\left(-\frac{m}{4}|x|\right) \min\left\{ (1+\ell(ct^{-1/\gamma_\infty(\alpha)}))t^{-(1+k+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}, \frac{|x|}{1+|x|^{2+k+\gamma_0(\beta)}} + \frac{1+\ell(c|x|^{-1})}{|x|^{1+k+\gamma_\infty(\beta)}} \right\} \tag{4.75}$$

$$|B^{\beta} p_t^\alpha(x)| \le C \exp\left(-\frac{m}{4}|x|\right) \min\left\{ (1+\ell(ct^{-1/\gamma_\infty(\alpha)}))t^{-(1+\gamma_\infty(\alpha)+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}, \frac{|x|}{1+|x|^{2+\gamma_0(\alpha)+\gamma_0(\beta)}} + \frac{1+\ell(c|x|^{-1})}{|x|^{1+\gamma_\infty(\alpha)+\gamma_\infty(\beta)}} \right\} \tag{4.76}$$

$$|\partial_{\beta_j} B^{\beta} p_t^\alpha(x)| \le C \exp\left(-\frac{m}{4}|x|\right) \min\left\{ (1+\ell(ct^{-1/\gamma_\infty(\alpha)}))t^{-(1+\gamma_\infty(\alpha)+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}, \frac{|x|}{1+|x|^{2+\gamma_0(\alpha)+\gamma_0(\beta)}} + \frac{1+\ell(c|x|^{-1})}{|x|^{1+\gamma_\infty(\alpha)+\gamma_\infty(\beta)}} \right\} \tag{4.77}$$

$$|\partial^2_{\beta_j} A^{\beta} p_t^\alpha(x)| \le C \exp\left(-\frac{m}{4}|x|\right) \min\left\{ (1+\ell(ct^{-1/\gamma_\infty(\alpha)}))t^{-(1+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}, \frac{|x|}{1+|x|^{2+\gamma_0(\beta)}} + \frac{1+\ell(c|x|^{-1})}{|x|^{1+\gamma_\infty(\beta)}} \right\} \tag{4.78}$$

for all $t \in (0,T]$, $x, y \in \mathbb{R}$, $j \in \{1, \dots, n\}$ and $\alpha, \beta \in I$.

Proof. Using (C3), (C4) and condition (C5), this follows directly from Theorem 4.7. Note that Theorem 4.7 holds without the assumption of symmetry of ψ_α and H_j, see the first part of the proof of Theorem 3.7. □

Roughly, the strategy to prove the differentiability of $x \mapsto p(t,x,y)$ is similar to the proof of the differentiability with respect to t (cf. Section 4.4). The first step is to derive off- and on-diagonal estimates for $\partial_x F(t,x,y)$ and then deduce estimates for $\partial_x(p_0 \circledast F^{\circledast k})$ for $k \in \mathbb{N}$. We follow the idea presented by Kolokoltsov [60, pp. 332]. Let us start with the estimates for $\partial_x F$.

4.46 Lemma *Under the assumptions of Theorem 3.8, $x \mapsto F(t,x,y)$ is differentiable for all $t > 0$, $y \in \mathbb{R}$ and there exists a constant $C = C(T) > 0$ such that*

$$\left|\frac{\partial}{\partial x} F(t,x,y)\right| \le C t^{-1/\gamma_\infty^L} G(t,x,y) \qquad \text{for all } t \in (0,T], x, y \in \mathbb{R}.$$

Proof. Because of the growth assumptions (3.3) and (C4), the differentiation lemma for parametrized integrals shows

$$\frac{\partial}{\partial x} F(t,x,y) = -i\frac{1}{2\pi} \int \xi(\psi_{\alpha(x)}(\xi) - \psi_{\alpha(y)}(\xi)) e^{-i\xi(x-y)} e^{-t\psi_{\alpha(y)}(\xi)}\, d\xi$$

$$+\frac{\alpha'(x)}{2\pi}\int \partial_\alpha \psi_\alpha(\xi)\big|_{\alpha=\alpha(x)} e^{-i(x-y)\xi} e^{-t\psi_{\alpha(y)}(\xi)}.$$

We denote the integral expressions on the right-hand side by J_1 and J_2, respectively. To estimate J_1 we proceed as in the proof of Lemma 4.22. Writing

$$\psi_{\alpha(x)}(\xi) - \psi_{\alpha(y)}(\xi) = \int_{\alpha(y)}^{\alpha(x)} \partial_\alpha \psi_\alpha(\xi)\, d\alpha$$

and applying Fubini's theorem, we obtain

$$|J_1| \le |\alpha(x)-\alpha(y)| \sup_{\beta \in \sigma_{\alpha(x),\alpha(y)}} |\partial_\beta A^{\beta,1} p_t^{\alpha(y)}(x-y)|; \tag{4.79}$$

here $\sigma_{\alpha(x),\alpha(y)} := [\alpha(x)\wedge\alpha(y), \alpha(x)\vee\alpha(y)]$ denotes the smallest closed interval containing $\alpha(x)$ and $\alpha(y)$. On the other hand, the triangle inequality gives

$$|J_1| \le 2\sup_{\beta\in I} |A^{\beta,1} p_t^{\alpha(y)}(x-y)|. \tag{4.80}$$

Moreover,

$$|J_2| = |\partial_\beta A^\beta p_t^{\alpha(y)}(x-y)|\big|_{\beta=\alpha(x)}. \tag{4.81}$$

If we can show that $U_1(t,x,y) := t^{1/\gamma_\infty^L}|J_1(t,x,y)|$ and $U_2(t,x,y) := t^{1/\gamma_\infty^L}|J_2(t,x,y)|$ both satisfy an estimate of the form (4.45), then the claim follows from Remark 4.23. To keep notation simple, we only consider the non-exponential case, i. e. $m=0$. Fix $\gamma \in (0, \frac{1}{\gamma_\infty^U}]$, $T>0$ and $t\in(0,T]$.

(i) $|x-y| \le t^{1/\gamma_\infty(\alpha(y))}$: It follows from (4.79), (4.75) and the Lipschitz continuity of α that

$$\begin{aligned}|U_1(t,x,y)| &\le C_1|\alpha(x)-\alpha(y)| t^{1/\gamma_\infty^L}(1+\ell(ct^{-1/\gamma_\infty(\alpha(y))})) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} t^{-(2+\gamma_\infty(\beta))/\gamma_\infty(\alpha)} \\ &\le C_1'|x-y|(1+\ell(ct^{-1/\gamma_\infty(\alpha(y))})) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} t^{-(1+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}\end{aligned}$$

Similarly, (4.81) combined with Corollary 4.11 gives

$$\begin{aligned}|U_2(t,x,y)| &\le C_2|\alpha(x)-\alpha(y)| t^{1/\gamma_\infty^L}(1+\ell(ct^{-1/\gamma_\infty(\alpha(y))})) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} t^{-(1+\gamma_\infty(\beta))/\gamma_\infty(\alpha)} \\ &\le C_2'|x-y|(1+\ell(ct^{-1/\gamma_\infty(\alpha(y))})) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} t^{-(1+\gamma_\infty(\beta))/\gamma_\infty(\alpha)}.\end{aligned}$$

(ii) $t^{1/\gamma_\infty(\alpha(y))} \le |x-y| \le t^\gamma$: By (4.79) and (4.75),

$$|U_1(t,x,y)| \le C_1|\alpha(x)-\alpha(y)| t^{1/\gamma_\infty^L}(1+\ell(c|x-y|^{-1})) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} \frac{1}{|x-y|^{2+\gamma_\infty(\beta)}}.$$

As

$$t^{1/\gamma_\infty^L}\frac{1}{|x-y|} \le t^{1/\gamma_\infty^L} t^{-1/\gamma_\infty(\alpha(y))} \qquad \text{for all } |x-y| \ge t^{1/\gamma_\infty(\alpha(y))} \tag{$\star$}$$

and the right-hand side is bounded in $t \in (0,T]$, we get

$$|U_1(t,x,y)| \leq C_1'|x-y|(1+\ell(c|x-y|^{-1})) \sup_{\beta\in\sigma_{\alpha(x),\alpha(y)}} \frac{1}{|x-y|^{1+\gamma_\infty(\beta)}}.$$

The corresponding estimate for U_2 is, as in the first case, a direct consequence of (4.81) and Corollary 4.11.

(iii) $t^\gamma \leq |x-y| \leq 1$: Using (4.80) and (4.74), we find

$$|U_1(t,x,y)| \leq C_1 t^{1/\gamma_\infty^L} \sup_{\beta\in I} \frac{1}{|x-y|^{2+\gamma_\infty(\beta)}} \overset{(\star)}{\leq} C_1' \sup_{\beta\in I} \frac{1}{|x-y|^{1+\gamma_\infty(\beta)}}.$$

From (4.81) and Corollary 4.11 we get

$$|U_2(t,x,y)| \leq C_2 t^{1/\gamma_\infty^L} \sup_{\beta\in I} \frac{1+\ell(c|x-y|^{-1})}{|x-y|^{1+\gamma_\infty(\beta)}} \leq C_2' \sup_{\beta\in I} \frac{1}{|x-y|^{1+\gamma_\infty(\beta)}};$$

here we have used that the monotonicity of ℓ and Lemma A.2 imply

$$\sup_{t\in(0,T]} \sup_{|x-y|\geq t^\gamma} t^{1/\gamma_\infty^L}(1+\ell(c|x-y|^{-1})) \leq \sup_{t\in(0,T]} t^{1/\gamma_\infty^L}(1+\ell(ct^\gamma)) < \infty.$$

(iv) $|x-y| \geq 1$: The reasoning is completely analogous to the previous case. □

Recall that

$$G(t,x,y) := Ct^{-1+\kappa}(S(x-y,\alpha(y),t) + g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)), \qquad t>0, x,y\in\mathbb{R},$$

cf. (4.43). In order to formulate the next auxiliary result, we define a new function, which we call in abuse of notation also G, by

$$G(t,x,y,z) := Ct^{-1+\kappa}(S(x-y,\alpha(z),t) + g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)), \qquad t>0, x,y,z\in\mathbb{R}.$$

Note that both definitions are consistent in the sense that $G(t,x,y) = G(t,x,y,y)$.

4.47 Lemma *Under the assumptions of Theorem 3.8, there exists a constant $C = C(T)$ such that*

$$\left|\frac{\partial}{\partial x} F(t,x-y,x-z)\right| \leq CG(t,y,z,x-z) \qquad \textit{for all } \; t\in(0,T], x,y,z\in\mathbb{R}.$$

Proof. By the definition of F, see (4.4), we have

$$F(t,x-y,x-z) = \frac{1}{2\pi}\int (\psi_{\alpha(x-y)}(\xi) - \psi_{\alpha(x-z)}(\xi))e^{-i\xi(y-z)}e^{-t\psi_{\alpha(x-z)}(\xi)}\,d\xi.$$

As in the proof of the previous lemma, we may apply the differentiation lemma for parametrized integrals:

$$\begin{aligned}&\frac{\partial}{\partial x}F(t,x-y,x-z)\\ &\quad= \frac{1}{2\pi}\sum_{j=1}^n \int (\partial_{\alpha_j}\psi_{\alpha(x-y)}(\xi)\alpha_j'(x-y) - \partial_{\alpha_j}\psi_{\alpha(x-z)}(\xi)\alpha_j'(x-z))e^{-i\xi(y-z)}e^{-t\psi_{\alpha(x-z)}(\xi)}\,d\xi\end{aligned}$$

$$-\frac{t}{2\pi}\sum_{j=1}^n \alpha_j'(x-z)\int \partial_{\alpha_j}\psi_{\alpha(x-z)}(\xi)(\psi_{\alpha(x-y)}(\xi)-\psi_{\alpha(x-z)}(\xi))e^{-i\xi(y-z)}e^{-t\psi_{\alpha(x-z)}(\xi)}\,d\xi$$
$$=\frac{1}{2\pi}\sum_{j=1}^n (J_{j,1}+J_{j,2}+J_{j,3})$$

where

$$\begin{aligned}
J_{j,1} &:= (\alpha_j'(x-y)-\alpha_j'(x-z))\int \partial_{\beta_j}\psi_\beta(\xi)\big|_{\beta=\alpha(x-y)}e^{-i\xi(y-z)}e^{-t\psi_{\alpha(x-z)}(\xi)}\,d\xi \\
&= -(\alpha_j'(x-y)-\alpha_j'(x-z))\partial_{\beta_j}A^\beta p_t^{\alpha(x-z)}(y-z)\big|_{\beta=\alpha(x-y)}, \\
J_{j,2} &:= \alpha_j'(x-z)\int \left(\partial_{\beta_j}\psi_\beta(\xi)\big|_{\beta=\alpha(x-y)}-\partial_{\beta_j}\psi_\beta(\xi)\big|_{\beta=\alpha(x-z)}\right)e^{-i\xi(y-z)}e^{-t\psi_{\alpha(x-z)}(\xi)}\,d\xi \\
&= -\alpha_j'(x-z)(\partial_{\beta_j}A^\beta p_t^{\alpha(x-z)}(y-z)\big|_{\beta=\alpha(x-y)}-\partial_{\beta_j}A^\beta p_t^{\alpha(x-z)}(y-z)\big|_{\beta=\alpha(x-z)}), \\
J_{j,3} &:= -t\alpha_j'(x-z)\int \partial_{\beta_j}\psi_\beta(\xi)\big|_{\beta=\alpha(x-z)}(\psi_{\alpha(x-y)}(\xi)-\psi_{\alpha(x-z)}(\xi))e^{-i\xi(y-z)}e^{-t\psi_{\alpha(x-z)}(\xi)}\,d\xi.
\end{aligned}$$

for $j\in\{1,\dots,n\}$. Fix $j\in\{1,\dots,n\}$. As $\alpha\in C_b^2(\mathbb{R})$ we know that α_j' is Lipschitz continuous. Consequently,

$$|J_{j,1}|\le C|y-z|\,|\partial_{\beta_j}A^\beta p_t^{\alpha(x-z)}(y-z)\big|_{\beta=\alpha(x-y)}|.$$

On the other hand, it follows from the mean value theorem that

$$|J_{j,2}|\le \|\alpha'\|_\infty|y-z|\sup_{\beta\in I}|\partial_\beta^2 A^\beta p_t^{\alpha(x-z)}(y-z)|.$$

Moreover, using the reasoning from the proof of Lemma 4.22, we find from the triangle inequality (for the first term) and the gradient theorem and Fubini's theorem (for the second term) that

$$|J_{j,3}|\le t\|\alpha'\|_\infty\min\left\{2\sup_{\beta\in I}|B^\beta p_t^{\alpha(x-z)}(y-z)|,\ \sup_{\beta\in\sigma_{\alpha(x-y),\alpha(x-z)}}|\partial_\beta B^\beta p_t^{\alpha(x-z)}(y-z)|\right\},$$

see Lemma 4.45 for the definition of $B^\beta p_t^\alpha$. Combining these estimates with Lemma 4.45, we find that

$$U(t,x,y,z):=|J_{j,1}(t,x,y,z)|+|J_{j,2}(t,x,y,z)|+|J_{j,3}(t,x,y,z)|$$

satisfies an estimate of the form (4.45). Since the calculations are lengthy, but straightforward, and similar to the proof of the previous lemma, we omit the details. Finally, the assertion follows from Remark 4.23. □

The final step is to derive an upper bound for $\partial_x(p_0\circledast F^{\circledast k})(t,x,y)$. Note that by Theorem 4.12

$$|\partial_x p_0(t,x,y)|=|\partial_x p^{\alpha(y)}(t,x-y)|\le Ct^{-1/\gamma_\infty(\alpha(y))}S(x-y,\alpha(y),t).$$

Because of the strong singularity at $t=0$, we cannot expect to justify the application of the dominated convergence theorem to conclude

$$\frac{\partial}{\partial x}(p_0\circledast F)(t,x,y)=\frac{\partial}{\partial x}\int_0^t\int p_0(t-s,x,z)F(s,z,y)\,dz\,ds$$

$$= \int_0^t \int \frac{\partial}{\partial x} p_0(t-s,x,z) F(s,z,y)\,dz\,ds,$$

see also the discussion at the beginning of Section 4.4. As in Section 4.4 the key point is to rewrite the time-space convolution in a clever way before taking the derivative, see (4.84) below.

4.48 Lemma *Under the assumptions of Theorem 3.8, there exists for any $T > 0$ a constant $C = C(T) > 0$ such that*

$$\left| \frac{\partial}{\partial x}(p_0 \circledast F^{\circledast k})(t,x,y) \right| \le C \left(\frac{k}{t}\right)^{1/\gamma_\infty^U} H^k(t,x,y)$$

for all $t \in (0,T]$, $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$; see Lemma 4.24 for the definition of H^k.

Proof. First, we make two observations. By the definition of p^α and the differentiation lemma for parametrized integrals, we have

$$\begin{aligned} \frac{\partial}{\partial z} p_t^{\alpha(z)}(x,y) &= -\frac{1}{2\pi} t\alpha'(z) \int \partial_\beta \psi_\beta(\xi)\big|_{\beta=\alpha(z)} e^{-i(x-y)\xi} e^{-t\psi_{\alpha(z)}(\xi)}\,d\xi \\ &= t\alpha'(z) \partial_\beta A^\beta p_t^{\alpha(z)}(x-y)\big|_{\beta=\alpha(z)}. \end{aligned}$$

Therefore, it follows from Corollary 4.11 and the boundedness of α' that there exists a constant $C_1 = C_1(T) > 0$ such that

$$\left| \frac{\partial}{\partial z} p_t^{\alpha(z)}(x,y) \right| \le C_1 S(x-y,\alpha(z),t) \qquad \text{for all} \quad t \in (0,T], x, y \in \mathbb{R}. \tag{4.82}$$

Moreover, we have

$$(f \circledast g^{\circledast k})(t,x,y) = \int_0^t \int (f \circledast g^{\circledast(k-1)})(t-s_1,x,z_1) g(s_1,z_1,y)\,dz_1\,ds_1.$$

for any two suitable integrable functions f, g. If we denote by

$$\Omega(t) := \left\{ s \in \mathbb{R}^k; s_i \ge 0, \sum_{i=1}^k s_i \le t \right\}$$

a k-simplex, then we obtain by induction that

$$(f \circledast g^{\circledast k})(t,x,y) = \int_{\Omega(t)} \int_{\mathbb{R}^k} f\left(t - \sum_{i=1}^k s_i, x, z_1\right) \prod_{i=1}^{k-1} g(s_i,z_i,z_{i+1}) g(s_k,z_k,y)\,dz\,ds. \tag{4.83}$$

Define iteratively

$$\Omega_0(t) := \Omega(t/2) \qquad \Omega_j(t) := \{s \in \Omega(t); s_j = \max_{i=1,\ldots,k} s_i\} \setminus \bigcup_{i=0}^{j-1} \Omega_i(t), \quad j = 1,\ldots,k.$$

As $\Omega(t) = \dot\bigcup_{j=0}^k \Omega_j(t)$, we have

$$(f \circledast g^{\circledast k})(t,x,y) = \sum_{i=0}^k \int_{\Omega_i(t)} \int_{\mathbb{R}^k} f\left(t - \sum_{i=1}^k s_i, x, z_1\right) \prod_{i=1}^{k-1} g(s_i,z_i,z_{i+1}) g(s_k,z_k,y)\,dz\,ds. \tag{4.84}$$

Moreover, if $s \in \Omega_j(t)$ for some $j \in \{1,\dots,k\}$, then $s \in \Omega(t)\backslash\Omega_0(t)$, and so

$$\frac{t}{2} \overset{s\in\Omega\backslash\Omega_0}{\leq} \sum_{i=1}^k s_i \leq k \max_{i=1,\dots,k} s_i \overset{s\in\Omega_j}{\leq} ks_j,$$

i. e.

$$s_j \geq \frac{t}{2k} \qquad \text{for all} \quad s \in \Omega_j(t). \tag{4.85}$$

Fix $T > 0$, $t \in (0,T]$ and $y \in \mathbb{R}$. Obviously, (4.84) implies that $\partial_x(p_0 \circledast F^{\circledast k})(t,x,y)$ exists if

$$I_j := \frac{\partial}{\partial x} \int_{\Omega_j(t)} \int p_0\left(t - \sum_{i=1}^k s_i, x, z_1\right) \prod_{i=1}^{k-1} F(s_i,z_i,z_{i+1}) F(s_k,z_k,y)\,dz\,ds$$

exists for all $j \in \{0,\dots,k\}$. We consider the terms I_j, $j = 0,\dots,k$, separately. Throughout the remaining part of the proof, we will formally apply the differentiation lemma for parametrized integrals to obtain the existence of the derivatives and derive estimates. We show in Lemma A.13 that the differentiation lemma is indeed applicable.

It follows from the differentiation lemma for parametrized integrals, Theorem 4.12 and (4.43) that there exists a constant $C_0 = C_0(T)$ such that

$$\begin{aligned} |I_0| &= \left| \int_{\Omega(t/2)} \int \partial_x p_0\left(t - \sum_{i=1}^k s_i, x, z_1\right) \prod_{i=1}^{k-1} F(s_i,z_i,z_{i+1}) F(s_k,z_k,y)\,dz\,ds \right| \\ &\leq C_0 \int_{\Omega(t/2)} \int \left(t - \sum_{i=1}^k s_i\right)^{-1/\gamma_\infty^L} S\left(t - \sum_{i=1}^k s_i, x, z_1\right) \prod_{i=1}^{k-1} G(s_i,z_i,z_{i+1}) G(s_k,z_k,y)\,dz\,ds \\ &\leq C_0 \left(\frac{t}{2}\right)^{-1/\gamma_\infty^L} \int_{\Omega(t)} \int S\left(t - \sum_{i=1}^k s_i, x, z_1\right) \prod_{i=1}^{k-1} G(s_i,z_i,z_{i+1}) G(s_k,z_k,y)\,dz\,ds. \end{aligned} \tag{4.86}$$

Using (4.83) and Lemma 4.24 another time, we get

$$|I_0| \leq C_0 \left(\frac{t}{2}\right)^{-1/\gamma_\infty^L} (S \circledast G^{\circledast k})(t,x,y) \leq C_0 \left(\frac{t}{2}\right)^{-1/\gamma_\infty^L} H^k(t,x,y).$$

It remains to estimate I_j for $j \in \{1,\dots,k\}$. For brevity of notation, we define an auxiliary function $P_{j,K} : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ by

$$P_{j,K}(s,x,z,y) := \left(\prod_{\substack{i=1\\ i\notin K}}^{j-1} F(s_i, x-z_i, x-z_{i+1})\right) F(s_j, x-z_j, z_{j+1}) \left(\prod_{\substack{i=j+1\\ i\notin K}}^{k-1} F(s_k, z_i, z_{i+1})\right) F(s_k,z_k,y)$$

for $j \in \{1,\dots,k\}$ and $K \subseteq \{1,\dots,k\}$. A change of variables ($z_i \rightsquigarrow x - z_i$ for $i = 1,\dots,j$) shows

$$I_j = \frac{\partial}{\partial x} \int_{\Omega_j(t)} \int p_0\left(t - \sum_{i=1}^k s_i, x, x - z_1\right) P_{j,\varnothing}(s,x,z,y)\,dz\,ds.$$

Applying, formally, the differentiation lemma for parametrized integrals, we get

$$\begin{aligned} I_j &= \int_{\Omega_j(t)} \int \left[\partial_x p_0 \left(t - \sum_{i=1}^k s_i, x, x - z_1 \right) \right] P_{j,\varnothing}(s,x,z,y)\, dz\, ds \\ &+ \sum_{\ell=1}^{j-1} \int_{\Omega_j(t)} \int p_0 \left(t - \sum_{i=1}^k s_i, x, x - z_1 \right) P_{j,\{\ell\}}(s,x,z,y) \partial_x F(s_\ell, x - z_\ell, x - z_{\ell+1})\, dz\, ds \\ &+ \int_{\Omega_j(t)} \int p_0 \left(t - \sum_{i=1}^k s_i, x, x - z_1 \right) P_{j,\varnothing}(s,x,z,y) \frac{\partial_x F(s_j, x - z_j, z_{j+1})}{F(s_j, x - z_j, z_{j+1})}\, dz\, ds \\ &=: I_{j,1} + I_{j,2} + I_{j,3}. \end{aligned} \tag{4.87}$$

(For $j = k$ we set $z_{k+1} = z_{j+1} := y$.) Since, by (4.3) and (4.83),

$$\left| \partial_x p_0 \left(t - \sum_{i=1}^k s_i, x, x - z_1 \right) \right| = \left| \partial_x p^{\alpha(x - z_1)} \left(t - \sum_{i=1}^k s_i, z_1 \right) \right| \le C_1 S \left(z_1, \alpha(x - z_1), t - \sum_{i=1}^k s_i \right) \tag{4.88}$$

it follows that

$$|I_{j,1}| \le C_1 \int_{\Omega_j(t)} \int S \left(z_1, \alpha(x - z_1), t - \sum_{i=1}^k s_i \right) |P_{j,\varnothing}(s,x,z,y)|\, dz\, ds.$$

Lemma 4.47 shows that there exists a constant $C_2 = C_2(T)$ such that

$$|I_{j,2}| \le C_2 \sum_{\ell=1}^{j-1} \int_{\Omega_j(t)} \int p_0 \left(t - \sum_{i=1}^k s_i, x, x - z_1 \right) |P_{j,\{\ell\}}(s,x,z,y)| G(s_\ell, z_\ell, z_{\ell+1}, x - z_\ell)\, dz\, ds$$

for all $t \in (0,T]$ and $x, y \in \mathbb{R}$. Finally, (4.85) and Lemma 4.46 give

$$\begin{aligned} |I_{j,3}| &\le C_3 \int_{\Omega_j(t)} \int p_0 \left(t - \sum_{i=1}^k s_i, x, x - z_1 \right) |P_{j,\varnothing}(s,x,z,y)| s_j^{-1/\gamma_\infty^L} \frac{G(s_k, x - z_j, z_{j+1})}{|F(s_j, x - z_j, z_{j+1})|}\, dz\, ds \\ &\le C_3 \left(\frac{t}{2k} \right)^{-1/\gamma_\infty^L} \int_{\Omega_j(t)} \int p_0 \left(t - \sum_{i=1}^k s_i, x, x - z_1 \right) |P_{j,\varnothing}(s,x,z,y)| \frac{G(s_k, x - z_j, z_{j+1})}{|F(s_j, x - z_j, z_{j+1})|}\, dz\, ds. \end{aligned}$$

(Note that, on the right-hand side, the term $F(s_j, x - z_j, z_{j+1})$ in the denominator cancels with the one in $P_{j,\varnothing}$.) Reversing the above change of variables ($z_i \rightsquigarrow z_i + x$ for $i = 1, \dots, j$) and using (4.43) and (4.83), we get

$$\begin{aligned} |I_j| &\le C' \left((k+1) + k^{-1/\gamma_\infty^L} \right) t^{-1/\gamma_\infty^L} (S \circledast G^{\circledast k})(t,x,y) \\ &\overset{\text{L4.24}}{\le} C' \left((k+1) + k^{-1/\gamma_\infty^L} \right) t^{-1/\gamma_\infty^L} H^k(t,x,y) \end{aligned}$$

for an absolute constant $C' = C'(T)$ (not depending on k). □

Proof of Theorem 3.8. Lemma 4.48 gives

$$\sum_{i \ge 1} \left| \frac{\partial}{\partial x} (p_0 \circledast F^{\circledast i})(t,x,y) \right| \le C t^{-1/\gamma_\infty^L} \sum_{i \ge 1} i^{1/\gamma_\infty^L} H^i(t,x,y) \le C' t^{-1/\gamma_\infty^L} t G(t,x,y)$$

for all $t \in (0,T]$ and $x,y \in \mathbb{R}$ (see the proof of Theorem 4.25). Consequently, the differentiation lemma for parametrized integrals shows that

$$\sum_{i\geq 1}(p_0 \circledast F^{\circledast i})(t,x,y) = (p_0 \circledast \Phi)(t,x,y)$$

is differentiable with respect to x and

$$\left|\frac{\partial}{\partial x}(p_0 \circledast \Phi)(t,x,y)\right| \leq C' t^{-1/\gamma_\infty^L} tG(t,x,y).$$

Since $p_0(t,x,y) = p_t^{\alpha(y)}(x-y)$ is also differentiable with respect to x, we find that the transition probability $p = p_0 + (p_0 \circledast \Phi)$ is differentiable. Moreover, as

$$\left|\frac{\partial}{\partial x}p_0(t,x,y)\right| \leq ct^{-1/\gamma_\infty(\alpha(y))}S(x-y,\alpha(y),t) \leq c't^{-1/\gamma_\infty^L}S(x-y,\alpha(y),t), \quad t \in (0,T], x,y \in \mathbb{R},$$

cf. Theorem 4.11, we get the claimed estimate for $\partial_x p(t,x,y)$. □

Proof of Corollary 3.9. (i) Fix $T > 0$ and $f \in \mathcal{B}_b(\mathbb{R})$. In Lemma 4.38(iii) we have seen that

$$\frac{\partial}{\partial x}P_t f(x) = \int \frac{\partial}{\partial x}p(t,x,y)f(y)\,dy$$

for all $x \in \mathbb{R}$ and $t > 0$. Moreover, by Theorem 3.8, there exists a constant $C > 0$ such that

$$\left|\frac{\partial}{\partial x}p(t,x,y)\right| \leq C(S(x-y,\alpha(y),t) + tG(t,x,y))$$

for all $x,y \in \mathbb{R}$ and $t \in (0,T]$. Combining this with Lemma 4.16 and Lemma 4.29(i), we conclude

$$\sup_{x\in\mathbb{R}}\left|\frac{\partial}{\partial x}P_t f(x)\right| \leq C't^{-1/\gamma_\infty^L}\|f\|_\infty.$$

(ii) Fix $T > 0$. By Corollary 4.14, there exist constants $C_1, C_2 > 0$ such that

$$p_0(t,x,y) = p_t^{\alpha(y)}(x-y) \geq C_1 t^{-1/\gamma_\infty(\alpha(y))}\left(1 - C_2 t^{-1/\gamma_\infty(y)}|x-y|\right)$$

for all $x,y \in \mathbb{R}$ and $t \in (0,T]$. On the other hand, we have by (4.47), the definition of G and (4.11)

$$|p(t,x,y) - p_0(t,x,y)| \leq C_3 tG(t,x,y) \leq C_3' t^{\kappa - 1/\gamma_\infty(\alpha(y))}.$$

Hence, by the triangle inequality,

$$\begin{aligned}|p(t,x,y)| &\geq |p_0(t,x,y)| - |p(t,x,y) - p_0(t,x,y)| \\ &\geq C_1 t^{-1/\gamma_\infty(\alpha(y))}\left(1 - C_2 t^{-1/\gamma_\infty(y)}|x-y| - \frac{C_3'}{C_1}t^\kappa\right).\end{aligned}$$

Since the transition density p is non-negative, this finishes the proof.

(iii) First, we show that for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$ there exists $T = T(n,x)$ such that

$$p(t,x,y) > 0 \qquad \text{for all } \; y \in [-n,n], t \geq T. \tag{$\star$}$$

To this end, we fix $x \in \mathbb{R}$, $n \in \mathbb{N}$ and note that by (ii) there exist $t_0 > 0$ and $\delta > 0$ such that

$$p(t_0,y,z) > 0 \qquad \text{for all } \; |y-z| \leq \delta.$$

Now if $|x-y| \leq \frac{3}{2}\delta$, then we can choose $r > 0$ sufficiently small such that

$$|x-z| \leq \delta \quad \text{and} \quad |y-z| \leq \delta \qquad \text{for all } \; z \in B := B[(x+y)/2, r].$$

Hence, by the Markov property,

$$p(2t_0,x,y) = \int_B p(t_0,x,z)p(t_0,z,y)\,dz > 0$$

for all $y \in \mathbb{R}$ with $|x-y| \leq \frac{3}{2}\delta$. Iterating the procedure, we find

$$p(kt_0,x,y) > 0 \qquad \text{for all } \; y \in \mathbb{R}, |y-x| \leq \left(1 + \frac{k}{2}\right)\delta.$$

If we choose $k \in \mathbb{N}$ sufficiently large, this gives ($\star$). Now we are ready to prove the λ-irreducibility. Let $B \in \mathcal{B}(\mathbb{R})$ with $\lambda(B) > 0$ and fix $x \in \mathbb{R}$. Because of the continuity of the measure λ, there exists $n \in \mathbb{N}$ such that $\lambda(B \cap [-n,n]) > 0$. Choosing $T = T(n,x) > 0$ as in ($\star$), we obtain

$$\begin{aligned}\int_{(0,\infty)} \mathbb{P}^x(X_t \in B)\,dt &= \int_{(0,\infty)} \int_B p(t,x,y)\,dy\,dt \\ &\geq \int_{[T,\infty)} \int_{B \cap [-n,n]} p(t,x,y)\,dy\,dt > 0.\end{aligned}$$

□

5

Applications

In this chapter, we present applications of the parametrix results which we have established in the previous chapters. We discuss three different kinds of applications. In the first part of this chapter, Section 5.1 and Section 5.2, we investigate variable order subordination. Section 5.1 is concerned with symbols of the form $q(x,\xi) = f_{\alpha(x)}(|\xi|^2)$ where $(f_\alpha)_{\alpha\in I}$ is a family of Bernstein functions. In particular, we obtain existence results for normal tempered stable-like, relativistic stable-like and Lamperti stable-like processes. The existence of Feller processes with symbols of varying order is studied in Section 5.2. Section 5.3 is devoted to jump processes of mixed type; for example we will see that for any Hölder continuous function $\chi : \mathbb{R}^d \to [0,1]$ there exists a Feller process with symbol

$$q(x,\xi) := \chi(x)|\xi|^\alpha + (1-\chi(x))((|\xi|^2+m^2)^{\alpha/2} - m^\alpha), \qquad x,\xi \in \mathbb{R}^d$$

(mixed stable-relativistic stable process). Finally, in Section 5.4, we prove existence and uniqueness results for solutions of Lévy-driven SDEs with Hölder continuous coefficients. It would be interesting to study path properties of the Feller processes whose existence we derive in this chapter, such as transience and recurrence (cf. [89]), ergodicity (cf. [90]) or the asymptotic growth behaviour of the sample paths (cf. [58, 94]).

5.1 Variable order subordination

Let $\psi_\alpha : \mathbb{R}^d \to \mathbb{C}$, $\alpha \in I$, be a family of continuous negative definite functions and $\alpha : \mathbb{R}^d \to I$ a Hölder continuous function. Theorem 3.2 provides sufficient conditions for the existence of a Feller process with symbol $q(x,\xi) := \psi_{\alpha(x)}(\xi)$. In dimension $d \geq 2$ Theorem 3.2 requires that each ψ_α is rotationally invariant. The most prominent examples for rotationally invariant negative definite functions are of the form $f(|\xi|^2)$ for a Bernstein function f. Therefore, it is natural to consider families of continuous negative definite functions which can be written as

$$\psi_\alpha(\xi) = f_\alpha(|\xi|^2), \qquad \xi \in \mathbb{R}^d,$$

for a family of Bernstein functions $(f_\alpha)_{\alpha\in I}$. Using the mapping properties of the function $\mathbb{C} \ni z \mapsto z^2$, we can formulate (C1)-(C4) in terms of $(f_\alpha)_{\alpha\in I}$.

5.1 Lemma *Let $I \subseteq \mathbb{R}^n$ be an open convex set, $m \geq 0$ and let $f_\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha \in I$, be a family of Bernstein functions such that $f_\alpha(0) = 0$ for all $\alpha \in I$. Assume that for any $R > 0$ there exist constants c_1, c_2, c_3 and $\vartheta > 0$ such that the following conditions are satisfied.*

(B1) *For each $\alpha \in I$ the Bernstein function f_α admits an extension F_α which is holomorphic on*

$$\Lambda(m,R,\vartheta) := \Lambda(R,\vartheta) := C(2\vartheta) \cup \{z \in \mathbb{C}; -m^2 < \operatorname{Re} z < m R, |\operatorname{Im} z| < m R\}; \tag{5.1}$$

here

$$C(2\vartheta) := \{z \in \mathbb{C}\backslash\{0\}; |\arg z| < 2\vartheta\}$$

denotes a cone of opening angle 2ϑ, see Figure 5.1 below.

(B2) *There exist measurable mappings $\beta_0 : I \to (0,1]$ and $\beta_\infty : I \to (0,1]$ such that*

$$|F_\alpha(z)| \leq c_1(|z|^{\beta_0(\alpha)} \mathbb{1}_{\{|z|\leq 1\}} + |z|^{\beta_\infty(\alpha)} \mathbb{1}_{\{|z|>1\}}), \qquad z \in \Lambda(R,\vartheta),$$

and $\beta_0^L := \inf_{\alpha \in I} \beta_0(\alpha) > 0$, $\beta_\infty^L := \inf_{\alpha\in I} \beta_\infty(\alpha) > 0$. Moreover, $\alpha \mapsto \beta_\infty(\alpha)$ is Hölder continuous.

(B3) $\operatorname{Re} F_\alpha(z) \geq c_2 |z|^{\beta_\infty(\alpha)}$ *for all $z \in \Lambda(R,\theta)$, $|z| \geq 1$.*

(B4) *For all $j = 1, \dots, n$ the partial derivative $\partial_{\alpha_j} F_\alpha$ exists and is holomorphic on $\Lambda(R,\vartheta)$. There exists an increasing function $\ell : (0,\infty) \to (0,\infty)$ which is slowly varying (at ∞) such that*

$$|\partial_{\alpha_j} F_\alpha(z)| \leq c_3(1 + \ell(|z|))\big(|z|^{\beta_0(\alpha)} \mathbb{1}_{\{|z|\leq 1\}} + |z|^{\beta_\infty(\alpha)} \mathbb{1}_{\{|z|>1\}}\big)$$

for all $z \in \Lambda(R,\vartheta)$, $\alpha \in I$ and $j \in \{1, \dots, n\}$.

Then the family of continuous negative definite functions $\psi_\alpha(\xi) := f_\alpha(|\xi|^2)$, $\xi \in \mathbb{R}^d$, satisfies (C1)-(C4) *for $\gamma_\infty(\alpha) := 2\beta_\infty(\alpha)$ and $\gamma_0(\alpha) := 2\beta_0(\alpha)$. In particular, the results from Section 3.1 are applicable. Furthermore, if*

(B5) *For all $j = 1, \dots, n$ the partial derivative $\partial^2_{\alpha_j} F_\alpha$ exists and is holomorphic on $\Lambda(R,\vartheta)$. Moreover, there exists a constant $c_4 > 0$ such that*

$$|\partial^2_{\alpha_j} F_\alpha(z)| \leq c_4(1 + \ell(|z|))(|z|^{\beta_0(\alpha)} \mathbb{1}_{\{|z|\leq 1\}} + |z|^{\beta_\infty(\alpha)} \mathbb{1}_{\{|z|>1\}})$$

for any $z \in \Lambda(R,\vartheta)$, $\alpha \in I$ and $j \in \{1, \dots, n\}$; here ℓ is the slowly varying function from (B4).

holds true, then $(\psi_\alpha)_{\alpha \in I}$ satisfies (C5).

Note that $\Lambda(R,\vartheta) = C(2\vartheta)$ if $m = 0$ and that

$$z \in \Lambda(R,\vartheta), |z| \gg 1 \implies z \in C(2\vartheta). \tag{5.2}$$

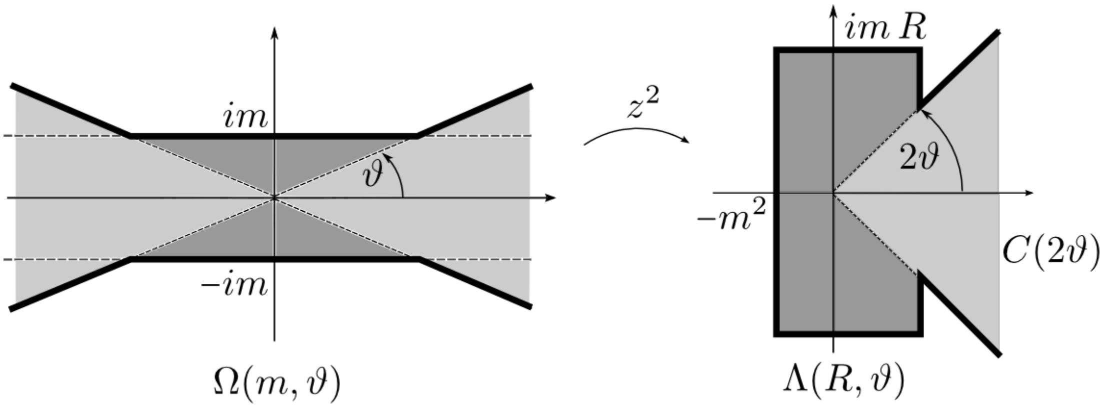

Figure 5.1: Since $\operatorname{Re}(z^2) = (\operatorname{Re} z)^2 - (\operatorname{Im} z)^2 > -m^2$ for any $z \in \{\zeta \in \mathbb{C}; |\operatorname{Im}\zeta| < m\}$ we can choose $R > 0$ such that $z \mapsto z^2$ maps $\Omega(m,\vartheta)$ to $\Lambda(R,\vartheta)$.

Proof of Lemma 5.1. We can choose $R \geq m^2$ sufficiently large such that

$$z^2 \in \Lambda(R,\vartheta) \qquad \text{for all} \quad z \in \Omega(m,\vartheta) \tag{5.3}$$

where $\Omega(m,\vartheta)$ denotes the domain defined in (C2), see Figure 5.1. Therefore, it is obvious that the family $\Psi_\alpha(z) := F_\alpha(z^2)$, $\alpha \in I$, satisfies (C1)-(C4) on $\Omega(m,\vartheta)$. □

The outline of this section is as follows. First we state an existence result for symmetric stable-like processes. Then, by smoothing the negative definite function $\psi(\xi) = |\xi|^\alpha$ and the Bernstein function $f(\lambda) = \lambda^{\alpha/2}$, we derive existence results for TLP-like, NTS-like and relativistic stable-like processes (see Figure 5.2), cf. Theorem 5.3, Theorem 5.4 and Theorem 5.7. The last part of this section is concerned with Lamperti stable-like processes. For a summary of the results and further examples we refer the reader to Table 5.1.

Most of the examples which we present in this section discuss the existence of Feller processes with symbols of the form $q(x,\xi) = f_{\alpha(x)}(|\xi|^2)$. Let us remark that the results from Section 3.1 apply, more generally, to symbols of the form

$$q(x,\xi) = f_{\alpha(x)}(\psi_{\beta(x)}(\xi))$$

where $(\psi_\beta)_{\beta\in J}$ is a suitable family of continuous negative definite functions. This leads to, so called, *variable order subordination* which has been studied by Evans & Jacob [33] in the last years. They provide sufficient conditions under which a pseudo-differential operator with symbol of the form

$$q(x,\xi) = f(x, p(x,\xi))$$

extends to the generator of a Feller semigroup; here, $f(x,\cdot)$ is a Bernstein function for each fixed x and $p(x,\xi)$ a symbol. Since their approach relies on symbolic calculus, it requires in particular a high regularity of f (with respect to x).

Throughout this section, $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space. We will frequently use that the principal value of the complex logarithm

$$\operatorname{Log} z := \log|z| + i \arg z, \qquad z \in \mathbb{C}, \arg z \in (-\pi,\pi]$$

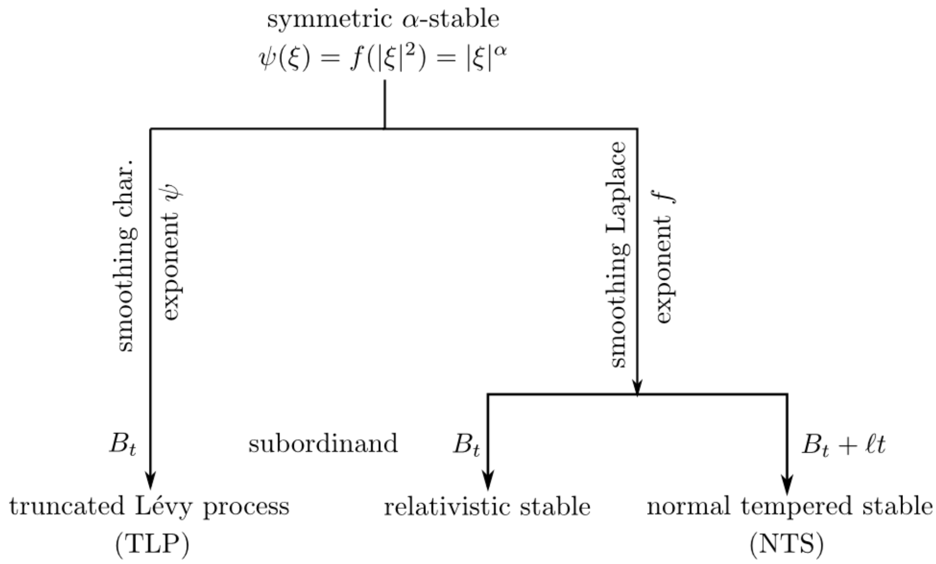

Figure 5.2: We derive the characteristic exponents of relativistic stable Lévy processes, truncated Lévy processes (TLP) and normal tempered stable (NTS) Lévy processes by smoothing the continuous negative definite function $|\xi|^\alpha$ and the Bernstein function $\lambda^{\alpha/2}$, respectively.

is holomorphic on $\mathbb{C}\backslash(-\infty,0]$.

5.2 Theorem (Stable-like process) *For any Hölder continuous mapping $\alpha:\mathbb{R}^d\to(0,2]$ such that*

$$\alpha^L := \inf_{x\in\mathbb{R}^d}\alpha(x) > 0$$

there exists a strong Feller process $(X_t)_{t\geq 0}$ with symbol $q(x,\xi) := |\xi|^{\alpha(x)}$. The process $(X_t)_{t\geq 0}$ has the following properties:

- *The transition probability $p:(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with $\gamma_0(\alpha(x)) = \gamma_\infty(\alpha(x)) = \alpha(x)$ and $m=0$.*
- *$C_c^\infty(\mathbb{R}^d)$ is a core for the generator $(L,\mathcal{D}(L))$ of $(X_t)_{t\geq 0}$ and $C_\infty^2(\mathbb{R}^d)\subseteq\mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.*
- *$(X_t)_{t\geq 0}$ is the unique solution to the $(L,C_c^\infty(\mathbb{R}^d))$-martingale problem.*
- *If $d=1$ and $\alpha\in C_b^2(\mathbb{R})$, then the transition density $p=p(t,x,y)$ admits a derivative with respect to x satisfying the heat kernel estimates from Theorem 3.8. Furthermore, the lower bound (3.5) holds true, and the Feller process $(X_t)_{t\geq 0}$ is λ-irreducible.*

Theorem 5.2 is slightly more general than the result by Kolokoltsov [60] (cf. Example 1.24) since we do not need to assume that α is bounded away from 2. Bass [4, Corollary 2.3] obtained the existence of a unique solution to the $(L,C_b^2(\mathbb{R}^d))$-martingale problem in dimension $d = 1$ under the weaker assumption that α is Dini continuous[1]. Let us remark that the existence of a Feller process with symbol $q(x,\xi) = |\xi|^{\alpha(x)}$ can also be established using results from symbolic calculus – but under much stronger assumptions on the regularity of α (typically, $\alpha \in C^{5d+3}(\mathbb{R}^d)$ or even $\alpha \in C^\infty(\mathbb{R}^d)$, cf. Hoh [43] and Potrykus [81]).

Proof of Theorem 5.2. We are going to show that the family of Bernstein functions

$$f_\alpha(\lambda) := \lambda^{\alpha/2}, \qquad \lambda > 0, \alpha \in I := [\alpha^L, 2],$$

satisfies the assumptions of Lemma 5.1 for $m = 0$; then the assertion is a direct consequence of the results from Section 3.1. Define

$$F_\alpha(z) := \exp\left(\frac{\alpha}{2}\operatorname{Log} z\right), \qquad z \in \mathbb{C}, \alpha \in I,$$

and fix $\vartheta \in (0, \frac{\pi}{4}]$.

(B1) Since Log is holomorphic on $\mathbb{C}\backslash(-\infty,0]$, it is obvious that F_α is holomorphic on $C(\vartheta) := \{z \in \mathbb{C}\backslash\{0\}; |\arg z| < \vartheta\}$. Moreover, since $\operatorname{Log}\lambda = \log\lambda$ for all $\lambda > 0$, we have $F_\alpha(\lambda) = f_\alpha(\lambda)$ for all $\lambda > 0$ and so F_α is a holomorphic extension to $C(\vartheta)$.

(B2) As

$$|F_\alpha(z)| = \left|\exp\left(\frac{\alpha}{2}(\log|z| + i\arg z)\right)\right| = |z|^{\alpha/2} \qquad \text{for all} \quad z \in \mathbb{C}, \tag{$\star$}$$

(B2) is satisfied with $\beta_\infty(\alpha) = \beta_0(\alpha) := \alpha/2$. By assumption,

$$\beta_\infty^L = \beta_0^L = \inf_{\alpha\in I}\frac{\alpha}{2} = \frac{\alpha^L}{2} > 0.$$

(B3) Since $\arg z \in (-\vartheta,\vartheta)$ for any $z \in C(\vartheta)$, we have

$$\operatorname{Re} F_\alpha(z) = |z|^{\alpha/2}\cos\left(\frac{\alpha}{2}\arg z\right) \geq |z|^{\alpha/2}\cos\left(\frac{\alpha}{2}\vartheta\right) \qquad \text{for all} \quad z \in C(\vartheta).$$

Hence, as $\vartheta < \pi/4$,

$$\operatorname{Re} F_\alpha(z) \geq |z|^{\alpha/2}\cos\left(\frac{\pi}{4}\right)$$

for any $\alpha \in I \subseteq (0,2]$ and $z \in C(\vartheta)$, i.e. (B3) holds with $\beta_\infty(\alpha) := \alpha/2$.

(B4)&(B5) Clearly,

$$\frac{\partial}{\partial\alpha}F_\alpha(z) = \frac{\operatorname{Log} z}{2}\exp\left(\frac{\alpha}{2}\operatorname{Log} z\right) = \frac{\operatorname{Log} z}{2}F_\alpha(z)$$

and

$$\frac{\partial^2}{\partial\alpha^2}F_\alpha(z) = \left(\frac{\operatorname{Log} z}{2}\right)^2\exp\left(\frac{\alpha}{2}\operatorname{Log} z\right) = \left(\frac{\operatorname{Log} z}{2}\right)^2 F_\alpha(z)$$

are holomorphic on $C(\vartheta)$. As $|\operatorname{Log} z|^2 \leq (\log|z|)^2 + \pi^2$ by the very definition of complex logarithm, it is obvious from ($\star$) that the growth conditions (B4) and (B5) hold with $\ell(r) := 1 \vee |\log(1+r)|^2$, $r > 0$. □

[1] A mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ is *Dini continuous* if $w(r) := \sup_{|x-y|\leq r}|f(y) - f(x)|$ satisfies $\int_0^1 w(r)/r\,dr < \infty$. In particular any Hölder continuous function is Dini continuous.

Remark Obviously, there are different ways to extend functions from the real line to the complex plane. We stress that, for our method, a careful choice of the extension is important. Often we will have to rewrite the function in a clever way before extending it. For example, if we consider the function

$$(0,\infty) \ni \xi \mapsto \xi^\alpha = \exp\left(\alpha \log \xi\right)$$

then the extension

$$\Psi_\alpha(z) := \exp\left(\alpha \operatorname{Log} z\right)$$

is not a good choice because it is not holomorphic on the negative real line. Instead, we note that

$$\xi^\alpha = (\xi^2)^{\alpha/2} = \exp\left(\frac{\alpha}{2}\log(\xi^2)\right), \qquad \xi > 0,$$

implies that

$$\Psi_\alpha(z) := \exp\left(\frac{\alpha}{2}\operatorname{Log}(z^2)\right),$$

is also an extension; that's exactly the extension we have used in the proof of Theorem 5.2.

It is well-known that symmetric α-stable Lévy processes have infinite variance. This implies, in particular, that α-stable Lévy processes don't have any exponential moments – the reason is, from an analytical point of view, that the characteristic exponent $\xi \mapsto |\xi|^\alpha$ is not sufficiently smooth at $\xi = 0$. Our next aim is to derive two existence results on tempered Lévy processes by smoothing the Bernstein function $f(\lambda) := \lambda^{\alpha/2}$. Let us briefly explain the idea. Let f be a Bernstein function of the form

$$f(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda r})\, \mu(dr), \qquad \lambda > 0.$$

Obviously, f differentiable on $(0,\infty)$, but, in general, not differentiable (from the right) at $\lambda = 0$. In order to avoid the point $\lambda = 0$, it is a natural idea to consider the shifted Bernstein function

$$g(\lambda) := f(\lambda + \varrho^2) - f(\varrho^2)$$

for fixed $\varrho > 0$. Clearly, g is differentiable[2] on $[0,\infty)$ and $g(\lambda) \approx f(\lambda)$ for $\lambda \gg 1$. Since

$$\begin{aligned} g(\lambda) &= \int_{(0,\infty)} (1 - e^{-(\lambda+\varrho^2)r})\, \mu(dr) - \int_{(0,\infty)} (1 - e^{-\varrho^2 r})\, \mu(dr) \\ &= \int_{(0,\infty)} e^{-\varrho^2 r}(1 - e^{-\lambda r})\, \mu(dr) \end{aligned}$$

we see that smoothing the Bernstein function is equivalent to tempering the Lévy measure of the Bernstein function. Rosiński [86] observed that there is also an interpretation on the level of stochastic processes.

Smoothing the Bernstein function $f(\lambda) = \lambda^{\alpha/2}$, $\alpha \in (0,2)$, we find that

$$\psi_{\alpha,\varrho}(\xi) := f(|\xi|^2 + \varrho^2) - f(\varrho^2) = (|\xi|^2 + \varrho^2)^{\alpha/2} - \varrho^\alpha, \qquad \xi \in \mathbb{R}^d,$$

[2] In fact, it has a holomorphic extension to $\{z \in \mathbb{C}; \operatorname{Re} z > \varrho^2\}$, see [100, Theorem 6.2].

is a continuous negative definite function for $\varrho > 0$ and $\alpha \in (0,2)$. Therefore, we can associate a family of Lévy processes with $\psi_{\alpha,\varrho}$ via the Lévy–Khintchine formula, the so-called *relativistic stable Lévy processes*. Relativistic stable Lévy processes are of interest in mathematical physics (more precisely, in relativistic quantum mechanics, see e. g. [21, 71, 42] and the references therein). We obtain the following existence result on relativistic stable-like processes.

5.3 Theorem (Relativistic stable-like process) *Let $\alpha : \mathbb{R}^d \to (0,2]$ and $\varrho : \mathbb{R}^d \to (0,\infty)$ be Hölder continuous mappings such that*

$$\alpha^L := \inf_{x \in \mathbb{R}^d} \alpha(x) > 0$$

and

$$0 < \varrho^L := \inf_{x \in \mathbb{R}^d} \varrho(x) \le \sup_{x \in \mathbb{R}^d} \varrho(x) =: \varrho^U < \infty.$$

Then there exists a strong Feller process $(X_t)_{t \ge 0}$ with symbol

$$q(x,\xi) := (|\xi|^2 + \varrho(x)^2)^{\alpha(x)/2} - \varrho(x)^{\alpha(x)}.$$

$(X_t)_{t\ge0}$ has the following additional properties:

- *The transition probability $p : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with $\gamma_0(\alpha(x),\varrho(x)) = 2$, $\gamma_\infty(\alpha(x),\varrho(x)) = \alpha(x)$ and any $m \in (0,\varrho^L)$.*
- *$C_c^\infty(\mathbb{R}^d)$ is a core for the generator $(L, \mathcal{D}(L))$ of $(X_t)_{t\ge0}$ and $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.*
- *$(X_t)_{t\ge0}$ is the unique solution to the $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem.*
- *If $d = 1$ and $\alpha \in C_b^2(\mathbb{R})$, then the transition density $p = p(t,x,y)$ admits a derivative with respect to x satisfying the heat kernel estimates from Theorem 3.8. Furthermore, the lower bound* (3.5) *holds true, and the Feller process $(X_t)_{t\ge0}$ is λ-irreducible.*

The heat kernel estimates for (killed) relativistic stable Lévy processes obtained by Chen, Kim & Song [24] show that the upper bound for the transition density p from Theorem 3.6 is sharp.

Proof of Theorem 5.3. Set $I := [\alpha^L, 2] \times [\varrho^L, \varrho^U]$ and fix $m \in (0,\varrho^L)$, $\vartheta \in (0,\pi/8)$. In the proof of the previous theorem we have seen that

$$F_\alpha(z) := \exp\left(\frac{\alpha}{2} \operatorname{Log} z\right)$$

extends the Bernstein function $\lambda^{\alpha/2}$ from $(0,\infty)$ to the complex plane. If we define

$$\tilde{F}_{\alpha,\varrho}(z) := F_\alpha(z + \varrho^2) - F_\alpha(\varrho^2),$$

then $\tilde{F}_{\alpha,\varrho}$ is an extension of the Bernstein function $f_{\alpha,\varrho}(\lambda) = (\lambda + \varrho^2)^{\alpha/2} - \varrho^\alpha$. As in the proof of the previous theorem, we are going to check that this family satisfies (B1)-(B5).

(B1) Because $\operatorname{Log} z$ is holomorphic on $\mathbb{C}\backslash(-\infty, 0]$, it follows that $\tilde{F}_{\alpha,\varrho}$ is holomorphic on

$$\{z \in \mathbb{C}; \operatorname{Re}(z + \varrho^2) > 0\} = \{z \in \mathbb{C}; \operatorname{Re} z > -\varrho^2\}.$$

As $m < \varrho^L$, this implies that $\tilde{F}_{\alpha,\varrho}$ is holomorphic on $\Lambda(m, R, \vartheta)$ for any $R > 0$ and $(\alpha, \varrho) \in I$ (see (5.1) for the definition of $\Lambda(m, R, \vartheta)$).

(B2)&(B3) Using (5.2) and the fact that F_α satisfies (B2),(B3) on $C(2\vartheta)$ (see the proof of Theorem 5.2), we find

$$\operatorname{Re} \tilde{F}_{\alpha,\varrho}(z) \geq c|\operatorname{Re} z + \varrho^2|^{\alpha/2} - \varrho^\alpha \geq c'|\operatorname{Re} z|^{\alpha/2}$$

and

$$|\tilde{F}_{\alpha,\varrho}(z)| \leq C|z + \varrho^2|^{\alpha/2} + (\varrho^U)^2 \leq C'|z|^{\alpha/2}$$

for any $(\alpha, \varrho) \in I$ and $z \in \Lambda(R, \vartheta)$, $|z| \gg 1$. On the other hand, Taylor's formula yields

$$|\tilde{F}_{\alpha,\varrho}(z)| = |F_\alpha(z + \varrho^2) - F_\alpha(\varrho^2)| \leq |z| \sup_{\zeta \in B[\varrho^2, (\varrho^L)^2/2]} \left| \frac{\partial}{\partial \zeta} F_\alpha(\zeta) \right| \leq C''|z|$$

for all $(\alpha, \varrho) \in I$ and $z \in \Lambda(R, \vartheta)$, $|z| \leq (\varrho^L)^2/2$; here C'' denotes an absolute constant (not depending on (α, ϱ)).[3] From this and the boundedness of $\tilde{F}_{\alpha,\varrho}$ on compact subsets of $\Lambda(R, \vartheta)$, we find that $\tilde{F}_{\alpha,\varrho}$ satisfies (B2) with $\beta_0(\alpha, \varrho) = 1$, $\beta_\infty(\alpha, \varrho) = \alpha/2$.

(B4)&(B5) This follows again from the fact that F_α satisfies (B4) and (B5) on $C(2\vartheta)$ and Taylor's formula. The computations are straightforward and similar to the proof of (B2). □

We have seen that a relativistic stable Lévy process $(X_t)_{t\geq 0}$ with characteristic exponent $\psi(\xi) = (|\xi|^2 + \varrho^2)^{\alpha/2} - \varrho^\alpha$ equals (in distribution) the subordinate process B_{S_t}, $t \geq 0$, where $(B_t)_{t\geq 0}$ is a Brownian motion and $(S_t)_{t\geq 0}$ an independent subordinator with Laplace exponent $f_{\alpha,\varrho}(\lambda) = (\lambda + \varrho^2)^{\alpha/2} - \varrho^\alpha$. We can generalize this family of Lévy processes a little further by adding a drift part, i. e. we consider the subordinate process

$$Y_t := B_{S_t} + 2bS_t$$

for $b \in \mathbb{R}^d$. It follows from Theorem 1.16 that $(Y_t)_{t\geq 0}$ is a Lévy process with characteristic exponent

$$\begin{aligned} f_{\alpha,\varrho}\left(|\xi|^2 - 2ib \cdot \xi\right) &= (|\xi|^2 - 2ib \cdot \xi + \varrho^2)^{\alpha/2} - \varrho^\alpha \\ &= (\varrho^2 + |b|^2 + (\xi - ib) \cdot (\xi - ib))^{\alpha/2} - \varrho^\alpha. \end{aligned} \tag{5.4}$$

Introducing another scaling parameter $\delta > 0$ and drift parameter $\ell \in \mathbb{R}^d$, we find that

$$-i\ell \cdot \xi + \delta(\varrho^2 + |b|^2 + (\xi - ib) \cdot (\xi - ib))^{\alpha/2} - \delta\varrho^\alpha$$

[3] For this estimate it is essential that $\varrho^L > 0$.

is a family of continuous negative definite functions. It is convenient to define $\kappa^2 := \varrho^2 + |b|^2$:

$$-i\ell\cdot\xi + \delta(\kappa^2 + (\xi - ib)\cdot(\xi - ib))^{\alpha/2} - \delta(\kappa^2 - |b|^2)^{\alpha/2}.$$

A Lévy process with a characteristic exponent of this form is called *normal tempered stable* Lévy process (NTS, for short). For $\alpha = 1$ this family of Lévy processes includes *normal inverse Gaussian* (NIG) Lévy processes.

Because of the possible applications in finance, it is of interest to prove the existence of NTS-like processes. Barndorff-Nielsen & Levendorskiĭ [3] obtained an existence result for NIG-like processes under restrictive assumptions; they require in particular smoothness of κ, b, δ, ℓ. In dimension $d = 1$ we can weaken these assumptions significantly using the parametrix construction. Here, we state the result for the particular case $\ell = 0$, $\delta = 1$; the general case will be discussed in Section 5.3, see Example 5.14.

5.4 Theorem (NTS-like process) *Let $b : \mathbb{R} \to \mathbb{R}$, $\alpha : \mathbb{R} \to (0,2)$ and $\kappa : \mathbb{R} \to (0,\infty)$ be Hölder continuous functions such that $\|b\|_\infty < \infty$ and*

$$\begin{aligned} &0 < \alpha^L := \inf_{x\in\mathbb{R}} \alpha(x) \le \sup_{x\in\mathbb{R}} \alpha(x) =: \alpha^U \\ &0 < \kappa^L := \inf_{x\in\mathbb{R}} \kappa(x) \le \sup_{x\in\mathbb{R}} \kappa(x) =: \kappa^U < \infty \\ &0 < \kappa^L - \|b\|_\infty. \end{aligned}$$

Then there exists a strong Feller process $(X_t)_{t\ge0}$ with symbol

$$q(x,\xi) := (\kappa(x)^2 + (\xi - ib(x))^2)^{\alpha(x)/2} - (\kappa(x)^2 - b(x)^2)^{\alpha(x)/2}, \qquad x,\xi \in \mathbb{R}.$$

$(X_t)_{t\ge0}$ has the following additional properties:

- *The transition probability $p : (0,\infty)\times\mathbb{R}\times\mathbb{R} \to [0,\infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with $\gamma_0(\alpha(x),\kappa(x),b(x)) = 2$, $\gamma_\infty(\alpha(x),\kappa(x),b(x)) = \alpha(x)$ and any $m \in (0, \kappa^L - \|b\|_\infty)$.*
- *$C_c^\infty(\mathbb{R})$ is a core for the generator $(L, \mathcal{D}(L))$ of $(X_t)_{t\ge0}$ and $C_\infty^2(\mathbb{R}) \subseteq \mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.*
- *$(X_t)_{t\ge0}$ is the unique solution to the $(L, C_c^\infty(\mathbb{R}))$-martingale problem.*

Note that we have to restrict ourselves to dimension $d = 1$ since $q(x,\cdot)$ is not symmetric.

Proof of Theorem 5.4. We pick up the notation from the proof of the previous theorem; in particular

$$\tilde{F}_{\alpha,\kappa}(z) := \exp\left(\frac{\alpha}{2}\operatorname{Log}(z+\kappa^2)\right) - \exp\left(\frac{\alpha}{2}\operatorname{Log}(\kappa^2)\right).$$

Fix $m \in (0, \kappa^L - \|b\|_\infty)$, $\vartheta \in (0, \pi/8]$ and set $I := [\alpha^L, 2) \times [\kappa^L, \kappa^U] \times [-\|b\|_\infty, \|b\|_\infty]$. Since $z \mapsto z - ib$ is just a shift, it follows from (5.3) that we can choose $R > 0$ large such that

$$\Omega(m,\vartheta) \xrightarrow{z-ib} \Omega(m + \|b\|_\infty, \vartheta) \underset{(5.3)}{\overset{z^2}{\hookrightarrow}} \Lambda(m + \|b\|_\infty, R, \vartheta) \tag{5.5}$$

for all $b \in [-\|b\|_\infty, \|b\|_\infty]$. (Here, $A \overset{f}{\hookrightarrow} B$ is short for $f(A) \subseteq B$.) Moreover, we note that

$$q(x,\xi) = \tilde{F}_{\alpha(x),\kappa(x)}((\xi - ib(x))^2) - \tilde{F}_{\alpha(x),\kappa(x)}((-ib(x))^2) \qquad \text{for all} \quad x, \xi \in \mathbb{R}.$$

Consequently, the claim follows from Theorem 3.7 if we can show that

$$z \mapsto \tilde{F}_{\alpha,\kappa}((z-ib)^2) - \tilde{F}_{\alpha,\kappa}((-ib)^2) =: F_{\alpha,\kappa,b}(z) \tag{5.6}$$

satisfies (C3),(C4) on $\Omega(m,\vartheta)$ for $(\alpha,\kappa,b) \in I$. Since $m + \|b\|_\infty < \kappa^L$, we know from the proof of Theorem 5.3 that $z \mapsto \tilde{F}_{\alpha,\kappa}(z)$ satisfies (B3) and (B4) on $\Lambda(m + \|b\|_\infty, R, \vartheta)$. Therefore, it follows readily from the boundedness of $b(\cdot)$ and (5.5) that

$$\operatorname{Re} \tilde{F}_{\alpha,\kappa}((z-ib)^2) \geq c\left|(z-ib)^2\right|^{\alpha/2} \geq c'|z|^\alpha$$

for $(\alpha,\kappa,b) \in I$ and $z \in \Omega(m,\vartheta)$, $|z| \gg 1$. As $\sup_{(\alpha,\kappa,b)\in I} \tilde{F}_{\alpha,\kappa}((-ib)^2) < \infty$, this implies

$$\operatorname{Re} F_{\alpha,\kappa,b}(z) \geq c''|z|^\alpha \qquad \text{for all} \quad (\alpha,\kappa,b) \in I, z \in \Omega(m,\vartheta), |z| \gg 1.$$

Similarly, we get

$$|F_{\alpha,\kappa,b}(z)| + \left|\frac{\partial}{\partial\alpha} F_{\alpha,\kappa,b}(z)\right| + \left|\frac{\partial}{\partial\kappa} F_{\alpha,\kappa,b}(z)\right| \leq C|z|^\alpha$$

for all $(\alpha,\kappa,b) \in I$ and $z \in \Omega(m,\vartheta)$, $|z| \gg 1$. Since

$$\frac{\partial}{\partial b} F_{\alpha,\kappa,b}(z) = -\frac{i(z-ib)\alpha}{(z-ib)^2+\kappa^2} \exp\left(\frac{\alpha}{2} \operatorname{Log}((z-ib)^2+\kappa^2)\right) + \frac{b^2\alpha}{\kappa^2-b^2} \exp\left(\frac{\alpha}{2} \log(\kappa^2-b^2)\right)$$

and, by assumption,

$$\inf_{(\alpha,\kappa,b)\in I} (\kappa^2 - b^2) \geq (\kappa^L)^2 - \|b\|_\infty^2 > 0,$$

we find, by the reverse triangle inequality,

$$\begin{aligned}\left|\frac{\partial}{\partial b} F_{\alpha,\kappa,b}(z)\right| &\leq 4\|b\|_\infty \frac{|z| + \|b\|_\infty}{(|z| - \|b\|_\infty)^2 - \kappa^2} (|z|^2 + \|b\|_\infty^2 + \kappa^2)^{\alpha/2} + C \\ &\leq C'|z|^\alpha\end{aligned}$$

for all $|z| \gg 1$, $z \in \Omega(m,\vartheta)$, and $(\alpha,\kappa,b) \in I$. It remains to check that there exists a constant $C > 0$ such that

$$|F_{\alpha,\kappa,b}(z)| + \left|\frac{\partial}{\partial\alpha} F_{\alpha,\kappa,b}(z)\right| + \left|\frac{\partial}{\partial\kappa} F_{\alpha,\kappa,b}(z)\right| + \left|\frac{\partial}{\partial b} F_{\alpha,\kappa,b}(z)\right| \leq C|z|^2$$

for all $z \in \Omega(m,\vartheta)$, $|z| \ll 1$ and $(\alpha,\kappa,b) \in I$. As in the proof of Theorem 5.3, this is a direct consequence of Taylor's formula and the definition of $F_{\alpha,\kappa,b}$, cf. (5.6). The computations are straightforward, but lengthy; we omit the details. □

We have seen that smoothing the Laplace exponent of the α-stable subordinator $\lambda^{\alpha/2}$, $\alpha \in (0,2)$, leads (via subordination) to relativistic stable Lévy processes and NTS Lévy processes. The next part of this section shows that smoothing the characteristic exponent of an α-stable Lévy process $\psi(\xi) = |\xi|^\alpha$, $\alpha \in (0,2)$, yields a different class of Lévy processes. We start with the following, more general, result which shows that smoothing a continuous negative definite function is equivalent to tempering the associated Lévy measure. For simplicity of notation we state it only in dimension $d = 1$.

5.5 Lemma *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a rotationally invariant negative definite function without Gaussian part, i.e. $\psi(\xi) = \Psi(|\xi|)$ with*

$$\Psi(r) = 2\int_{(0,\infty)} (1 - \cos(yr))\,\nu(dy) = 2\operatorname{Re}\left(\int_{(0,\infty)} (1 - e^{iry} + iry\mathbb{1}_{(0,1)}(y))\,\nu(dy)\right)$$

for a symmetric Lévy measure ν on $(\mathbb{R}\backslash\{0\}, \mathcal{B}(\mathbb{R}\backslash\{0\}))$.

(i) The function

$$\Psi(z) := 2\operatorname{Re}\left(\int_{(0,\infty)} (1 - e^{izy} + izy\mathbb{1}_{(0,1)}(y))\,\nu(dy)\right) \tag{5.7}$$

extends Ψ from $\mathbb{R}$ to the upper half plane $\{z \in \mathbb{C}; \operatorname{Im} z \geq 0\}$.

(ii) If we define

$$\Psi_\varrho(r) := 2\int_{(0,\infty)} (1 - \cos(ry))e^{-\varrho y}\,\nu(dy), \qquad r \geq 0,$$

for fixed $\varrho \geq 0$, then $\psi_\varrho(\xi) := \Psi_\varrho(|\xi|)$ is a continuous negative definite function on $\mathbb{R}$ and

$$\psi_\varrho(\xi) = \Psi(|\xi| + i\varrho) - \Psi(i\varrho), \qquad \xi \in \mathbb{R}. \tag{5.8}$$

Proof. (i) Fix $y \geq 0$ and $z \in \mathbb{C}$ with $\operatorname{Im} z \geq 0$. Since

$$|1 - e^{izy}| \leq 1 + e^{-y\operatorname{Im} z} \leq 2$$

and, by Taylor's formula,

$$|1 - e^{izy} - izy| \leq \frac{1}{2}|z|^2\,|y|^2,$$

the integral on the right-hand side of (5.7) is finite.

(ii) It is clear from the definition that

$$\psi_\varrho(\xi) = \Psi_\varrho(|\xi|) = 2\int_{(0,\infty)} (1 - \cos(|\xi|\,y))e^{-\varrho y}\,\nu(dy), \qquad \xi \in \mathbb{R},$$

is a continuous negative definite function. Therefore, it just remains to prove (5.8). By (5.7),

$$\begin{aligned}\Psi(i\varrho) &= 2\operatorname{Re}\left(\int_{(0,\infty)} (1 - e^{-\varrho y} - \varrho y\mathbb{1}_{(0,1)}(y))\,\nu(dy)\right)\\ &= 2\int_{(0,\infty)} (1 - e^{-\varrho y} - \varrho y\mathbb{1}_{(0,1)}(y))\,\nu(dy)\end{aligned}$$

and

$$\begin{aligned}\Psi(|\xi| + i\varrho) &= 2\operatorname{Re}\left(\int_{(0,\infty)} (1 - e^{iy|\xi| - \varrho y} + i(|\xi| + i\varrho)y\mathbb{1}_{(0,1)}(y))\,\nu(dy)\right)\\ &= 2\int_{(0,\infty)} (1 - e^{-\varrho y}\cos(y|\xi|) - \varrho y\mathbb{1}_{(0,1)}(y))\,\nu(dy).\end{aligned}$$

Subtracting both expressions and using the definition of ψ_ϱ yields (5.8). □

Next we apply Lemma 5.5 to $\psi(\xi) = |\xi|^\alpha$.

5.6 Example Let $\alpha \in (0,2)\backslash\{1\}$ and $\varrho > 0$. For the tempered Lévy measure

$$\nu_{\alpha,\varrho}(dy) := e^{-\varrho|y|}|y|^{-1-\alpha}\,dy$$

it holds that

$$\int_{\mathbb{R}\backslash\{0\}} (1-\cos(\xi y))\,\nu_{\alpha,\varrho}(dy) = 2\frac{\Gamma(2-\alpha)}{\alpha(1-\alpha)}\left((|\xi|^2+\varrho^2)^{\alpha/2}\cos\left(\alpha\arctan\left[\frac{|\xi|}{\varrho}\right]\right)-\varrho^\alpha\right) \tag{5.9}$$

for all $\xi \in \mathbb{R}$.

The identity was, to our knowledge, first stated by Koponen [61]. A proof can be, e. g., found in [65, Lemma 2.6]. We give an alternative proof using Lemma 5.5. Letting $\alpha \to 1$ using l'Hôpital's rule[4], it is possible to calculate the continuous negative definite function with Lévy measure $\nu(dy) := e^{-\varrho|y|}|y|^{-2}$ explicitly.

Proof of Example 5.6. In order to apply Lemma 5.5(ii), we have to calculate the integral

$$J(z) := \int_{(0,\infty)} (1-e^{izy}+izy\mathbb{1}_{(0,1)}(y))\frac{1}{y^{1+\alpha}}\,dy$$

for $z \in \mathbb{C}$, $\operatorname{Im} z > 0$. To this end, we will first calculate $J(ir)$ for $r > 0$ and then apply the identity theorem for holomorphic functions. By the definition of J, we have

$$J(ir) = \int_{(0,\infty)} (1-e^{-ry}-ry\mathbb{1}_{(0,1)}(y))\frac{1}{y^{1+\alpha}}\,dy$$

for any $r > 0$. Applying twice the integration by parts formula gives

$$\begin{aligned}
&\int_{(0,1)} (1-e^{-ry}-ry)\frac{1}{y^{1+\alpha}}\,dy \\
&= \left[\frac{1-e^{-ry}-ry}{-\alpha y^\alpha}\right]_{y=0}^1 + \frac{r}{\alpha}\int_0^1 (e^{-ry}-1)\frac{1}{y^\alpha}\,dy \\
&= \frac{-1+e^{-r}+r}{\alpha} + \frac{r}{\alpha}\left[\frac{e^{-ry}-1}{(1-\alpha)y^{\alpha-1}}\right]_{y=0}^1 + \frac{r^2}{\alpha(1-\alpha)}\int_0^1 e^{-ry}y^{1-\alpha}\,dy \\
&= \frac{-1+e^{-r}+r}{\alpha} + \frac{r}{\alpha}\frac{e^{-r}-1}{1-\alpha} + \frac{r^2}{\alpha(1-\alpha)}\int_0^1 e^{-ry}y^{1-\alpha}\,dy.
\end{aligned}$$

Similarly, we find

$$\int_{(1,\infty)} (1-e^{-ry})\frac{1}{y^{1+\alpha}}\,dy = \frac{1-e^{-r}}{\alpha} - \frac{r}{\alpha}\frac{e^{-r}}{1-\alpha} + \frac{r^2}{\alpha(1-\alpha)}\int_1^\infty e^{-ry}y^{1-\alpha}\,dy.$$

Consequently,

$$J(ir) = \frac{r}{\alpha} - \frac{r}{\alpha(1-\alpha)} + \frac{r^2}{\alpha(1-\alpha)}\int_0^\infty e^{-ry}y^{1-\alpha}\,dy$$

[4]Since $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$ for any $x \in \mathbb{R}$, the right-hand side of (5.9) is for $\alpha = 1$, formally, an expression of the form 0/0.

$$= \frac{r}{\alpha-1} + \frac{r^2}{\alpha(1-\alpha)} r^{\alpha-2} \int_0^\infty e^{-u} u^{1-\alpha}\, du$$
$$= \frac{r}{\alpha-1} + \frac{\Gamma(2-\alpha)}{\alpha(1-\alpha)} r^\alpha =: \frac{r}{\alpha-1} + c_\alpha r^\alpha.$$

This shows that J equals

$$\tilde{J}(z) := -\frac{iz}{\alpha-1} + c_\alpha \left(\frac{z}{i}\right)^\alpha$$

on the positive imaginary axis. Since both J and $\tilde{J}$ are holomorphic on the upper half plane $H^+ := \{z \in \mathbb{C}; \operatorname{Im} z > 0\}$, the identity theorem for holomorphic function yields $J = \tilde{J}$ on H^+. Consequently,

$$\Psi(z) := 2\operatorname{Re}\left(\int_{(0,\infty)} (1 - e^{izy} + izy\mathbb{1}_{(0,1)}(y)) \frac{1}{y^{1+\alpha}}\, dy\right) = 2\operatorname{Re} J(z)$$

equals

$$\Psi(z) = 2\operatorname{Re}\left(-\frac{iz}{\alpha-1} + c_\alpha \left[\frac{z}{i}\right]^\alpha\right) = \frac{2\operatorname{Im} z}{\alpha-1} + 2c_\alpha |z|^\alpha \cos(\alpha \arg z)$$

for all $z \in H^+$. Hence,

$$\Psi(r+i\varrho) - \Psi(i\varrho) = 2c_\alpha (r^2+\varrho^2)^{\alpha/2} \cos\left(\alpha \arctan\left[\frac{r}{\varrho}\right]\right) - 2c_\alpha \varrho^\alpha$$

for all $r \geq 0$ and $\varrho > 0$. Applying Lemma 5.5 finishes the proof. □

Example 5.6 shows in particular that

$$\xi \mapsto \psi_{\alpha,\varrho}(\xi) := \operatorname{sgn}(1-\alpha)\left[(|\xi|^2+\varrho^2)^{\alpha/2} \cos\left(\alpha \arctan \frac{|\xi|}{\varrho}\right) - \varrho^\alpha\right]$$

is a continuous negative definite function for any $\alpha \in (0,2)\backslash\{1\}$ and $\varrho > 0$. Using the elementary identity

$$\cos(2\arctan x) = \frac{1-x^2}{1+x^2}$$

we find that $\psi_{2,\varrho}(\xi) = |\xi|^2$, i. e. $\psi_{\alpha,\varrho}$ is also a (continuous) negative definite function for $\alpha = 2$. We follow Matacz [75] and call a Lévy process with characteristic exponent $\psi_{\alpha,\varrho}$ a (symmetric) *truncated Lévy process* (TLP, for short). Let us remark that different authors use different names for this process, e. g. KoBoL process (Boyarchenko & Levendorskiĭ [14]) and CGMY process (Carr, Geman, Madan & Yor [22]). Using the parametrix construction, we obtain the following existence result on TLP-like processes.

5.7 Theorem (TLP-like process) *Let $\alpha : \mathbb{R}^d \to (0,1)$ and $\varrho : \mathbb{R}^d \to (0,\infty)$ be Hölder continuous functions such that*

$$0 < \alpha^L := \inf_{x\in\mathbb{R}^d} \alpha(x) \leq \sup_{x\in\mathbb{R}^d} \alpha(x) =: \alpha^U < 1,$$
$$0 < \varrho^L := \inf_{x\in\mathbb{R}^d} \varrho(x) \leq \sup_{x\in\mathbb{R}^d} \varrho(x) =: \varrho^U < \infty.$$

Then there exists a strong Feller process $(X_t)_{t\geq 0}$ with symbol

$$q(x,\xi) := (|\xi|^2 + \varrho(x)^2)^{\alpha(x)/2} \cos\left(\alpha(x) \arctan \frac{|\xi|}{\varrho(x)}\right) - \varrho(x)^{\alpha(x)}, \qquad x,\xi \in \mathbb{R}^d.$$

$(X_t)_{t\geq 0}$ has the following properties:

- *The transition probability $p:(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d \to [0,\infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with $\gamma_0(\alpha(x),\varrho(x)) = 2$, $\gamma_\infty(\alpha(x),\varrho(x)) = \alpha(x)$ and any $m \in (0,\varrho^L)$.*
- *$C_c^\infty(\mathbb{R}^d)$ is a core for the generator $(L,\mathcal{D}(L))$ of $(X_t)_{t\geq 0}$ and $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.*
- *$(X_t)_{t\geq 0}$ is the unique solution to the $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem.*
- *If $d=1$ and $\alpha \in C_b^2(\mathbb{R})$, then the transition density $p = p(t,x,y)$ admits a derivative with respect to x satisfying the heat kernel estimates from Theorem 3.8. Furthermore, the lower bound (3.5) holds true, and the Feller process $(X_t)_{t\geq 0}$ is λ-irreducible.*

For the proof of Theorem 5.7 we need the following auxiliary result; we defer the proof to the appendix (see Lemma A.15).

5.8 Lemma *The mapping*

$$\arctan : \mathbb{C}\backslash\{z\in\mathbb{C}; |\operatorname{Im} z| \geq 1\} \to \mathbb{C},\, z \mapsto \int_0^z \frac{1}{1+\zeta^2}\,d\zeta$$

is holomorphic and satisfies

$$|\operatorname{Im}(\arctan z)| \leq 4|\sin(\arg z)|$$

for all $z \in \Omega(0,\pi/4) = \{z\in\mathbb{C}\backslash\{0\}; |\arg z \mod \pi| \leq \pi/4\}$ (see (3.1) for the definition). Moreover, for any $\varepsilon > 0$ there exist $R > 0$ and $\vartheta \in (0,\pi/2)$ such that

$$\frac{\pi}{2}(1-\varepsilon) \leq |\operatorname{Re}(\arctan z)| \leq \frac{\pi}{2}(1+\varepsilon) \qquad \text{for all } \; z\in\Omega(0,\vartheta), |z| \geq R.$$

Proof of Theorem 5.7. Fix $m \in (0,\varrho^L)$ and set $I := [\alpha^L,\alpha^U]\times[\varrho^L,\varrho^U]$,

$$\Psi_{\alpha,\varrho}(r) := (r^2+\varrho^2)^{\alpha/2}\cos\left(\alpha \arctan\left[\frac{r}{\varrho}\right]\right) - \varrho^\alpha.$$

We are going to show that $(\Psi_{\alpha,\varrho})_{(\alpha,\varrho)\in I}$ satisfies (C1)-(C4) on $\Omega(m,\vartheta)$ for $\vartheta \in (0,\pi/4)$ sufficiently small. Note that

$$z\in\Omega(m,\vartheta), |z| \gg 1 \implies z \in \Omega(0,\vartheta). \tag{$\star$}$$

(C1) Obvious.

(C2) Both $z \mapsto \arctan(z/\varrho)$ and $z \mapsto (z^2+\varrho^2)^{\alpha/2}$ are holomorphic on the domain $\mathbb{C}\backslash\{z \in \mathbb{C}; \operatorname{Re} z = 0, |\operatorname{Im} z| \geq \varrho^L\}$ for $(\alpha,\varrho) \in I$. As $m < \varrho^L$ this gives in particular analyticity on $\Omega(m,\pi/4)$. Therefore, it is obvious that

$$\Psi_{\alpha,\varrho}(z) =: (z^2+\varrho^2)^{\alpha/2} \cos\left(\alpha \arctan\left[\frac{z}{\varrho}\right]\right) - \varrho^\alpha$$

is holomorphic on $\Omega(m,\pi/4)$.

(C3) For any $\zeta \in \mathbb{C}$, $\operatorname{Re}\zeta \geq 0$, and $\varrho > 0$, we have $|\arg(\zeta+\varrho^2)| \leq |\arg(\zeta)|$. Since $z \mapsto z^2$ maps $\Omega(0,\vartheta)$ to $\{z \in \mathbb{C}; \operatorname{Re} z \geq 0\}$, it follows from $(\star)$ that

$$|\arg(z^2+\varrho^2)| \leq |\arg(z^2)| = 2|\arg z| \leq 2\vartheta \qquad \text{for all } \; z \in \Omega(m,\vartheta), |z| \gg 1.$$

Therefore, we find for $\vartheta \in (0,\pi/4)$ sufficiently small

$$\begin{aligned}\operatorname{Re}((z^2+\varrho^2)^{\alpha/2}) &= |z^2+\varrho^2|^{\alpha/2} \cos\left(\frac{\alpha}{2}\arg(z^2+\varrho^2)\right)\\ &\geq (|z|^2-\varrho^2)^{\alpha/2}\cos\left(\frac{\vartheta}{\alpha}\right) \geq \frac{1}{2}|z|^\alpha \cos\left(\frac{\vartheta}{\alpha^L}\right)\end{aligned}$$

and

$$|\operatorname{Im}((z^2+\varrho^2)^{\alpha/2})| = |z^2+\varrho^2|^{\alpha/2}\left|\sin\left(\frac{\alpha}{2}\arg(z^2+\varrho^2)\right)\right| \leq 4|z|^\alpha \sin\left(\frac{\vartheta}{\alpha^L}\right)$$

for all $z \in \Omega(m,\vartheta)$, $|z| \gg 1$ and $(\alpha,\varrho) \in I$. Using the elementary identity

$$\operatorname{Im}\cos z = -\sin(\operatorname{Im} z)\sinh(\operatorname{Im} z)$$

it follows from Lemma 5.8 and $(\star)$ that we can choose $\vartheta > 0$ sufficiently small such that

$$\begin{aligned}\left|\operatorname{Im}\cos\left(\alpha\arctan\frac{z}{\varrho}\right)\right| &\leq \sin\left(4\alpha\left|\sin\left(\arg\frac{z}{\varrho}\right)\right|\right)\sinh\left(4\alpha\left|\sin\left(\arg\frac{z}{\varrho}\right)\right|\right)\\ &\leq \sin\left(4\alpha\sin\frac{\vartheta}{\varrho}\right)\sinh\left(4\alpha\sin\frac{\vartheta}{\varrho}\right) \leq \frac{1}{4}\end{aligned}$$

for all $z \in \Omega(m,\vartheta)$, $|z| \gg 1$ and $(\alpha,\varrho) \in I$. Choose $\delta > 0$ such that $\alpha^U(1+\delta) < 1$. By Lemma 5.8 and $(\star)$, there exist $R > 0$ and $\vartheta > 0$ such that

$$(1-\delta)\frac{\pi}{2} \leq \left|\operatorname{Re}\left(\arctan\frac{z}{\varrho}\right)\right| \leq \frac{\pi}{2}(1+\delta) \qquad \text{for all } \; |z| \geq R, z \in \Omega(m,\vartheta), \varrho > \varrho^L.$$

Using the monotonicity of cosine on $(-\pi/2,\pi/2)$ and

$$\operatorname{Re}\cos z = \cos(\operatorname{Re} z)\cosh(\operatorname{Im} z), \qquad z \in \mathbb{C},$$

we get

$$\operatorname{Re}\cos\left(\alpha\arctan\frac{z}{\varrho}\right) = \cos\left(\alpha\operatorname{Re}\arctan\frac{z}{\varrho}\right)\underbrace{\cosh\left(\alpha\operatorname{Im}\arctan\frac{z}{\varrho}\right)}_{\geq 1}$$

$$\geq \cos\left(\alpha \operatorname{Re}\arctan\frac{z}{\varrho}\right) \geq \cos\left(\alpha^U(1+\delta)\frac{\pi}{2}\right) =: c_0 > 0$$

for all $z \in \Omega(m,\vartheta)$, $|z| \geq R$. Combining the above estimates shows

$$\begin{aligned}\operatorname{Re}\Psi_{\alpha,\varrho}(z) &= \operatorname{Re}((z^2+\varrho^2)^{\alpha/2}) \operatorname{Re}\cos\left(\alpha\arctan\frac{z}{\varrho}\right)\\ &\quad - \operatorname{Im}((z^2+\varrho^2)^{\alpha/2}) \operatorname{Im}\cos\left(\alpha\arctan\frac{z}{\varrho}\right)\\ &\geq |z|^\alpha\left[\frac{c_0}{2}\cos\left(\frac{\vartheta}{\alpha^L}\right) - \frac{4}{4}\sin\left(\frac{\vartheta}{\alpha^L}\right)\right]\end{aligned}$$

for all $z \in \Omega(m,\vartheta)$, $|z| \gg 1$ and $(\alpha,\varrho) \in I$. If we choose $\vartheta > 0$ sufficiently small, then the expression in the bracket is strictly larger than 0, and this means that (C3-1) holds with $\gamma_\infty(\alpha,\varrho) = \alpha$. On the other hand, it is not difficult to see that

$$|\Psi_{\alpha,\varrho}(z)| \leq C|z|^\alpha$$

for $z \in \Omega(m,\vartheta)$, $|z| \gg 1$ (see also the proof of Theorem 5.3). Finally, it follows again from Taylor's formula that

$$|\Psi_{\alpha,\varrho}(z)| \leq c|z|^2 \qquad \text{for all } \; z \in \Omega(m,\vartheta), |z| \ll 1.$$

Consequently, (C3-2) holds with $\gamma_\infty(\alpha,\varrho) = \alpha$ and $\gamma_0(\alpha,\varrho) = 2$.

(C4)&(C5) Using a similar reasoning as in the proof of (C3), it follows that (C4) and (C5) hold for a slowly varying function ℓ which grows for $r \to \infty$ at most as $|\log(r)|^2$; we omit the details of the proof. □

In Theorem 5.7 we assumed that $\alpha : \mathbb{R}^d \to (0,1)$ is bounded away from 1. The proof of Theorem 5.7 shows that the statement of Theorem 5.7 remains valid if α takes values in $(1,2]$ and is bounded away from 1; only the proof of the growth condition (C3-1) has to be modified slightly.

The last part of this section is devoted to Lamperti stable(-like) processes. We call a subordinator $(S_t)_{t\geq 0}$ *Lamperti stable* if its Laplace exponent is of the form

$$f(t) = \frac{\Gamma(t+\varrho+\alpha)}{\Gamma(t+\varrho)} - \frac{\Gamma(\alpha+\varrho)}{\Gamma(\varrho)}, \qquad t \geq 0$$

for some constants $\alpha \in (0,1)$,$\varrho > 0$. Using the Pochhammer symbol $(\lambda)_\alpha := \Gamma(\lambda+\alpha)/\Gamma(\lambda)$, this can be more compactly written as

$$f(t) = (t+\varrho)_\alpha - (\varrho)_\alpha, \qquad t \geq 0.$$

Caballero et. al [20, Proposition 3.1 & Theorem 3.1] have shown that f is a Bernstein function by deriving its Lévy-Khintchine representation:

$$f(t) = c_{\alpha,\varrho}t + \tilde{c}_{\alpha,\varrho}\int_{(0,\infty)} (1-e^{-ty})\frac{e^{(1-\varrho)y}}{(e^y-1)^{1+\alpha}}\,dy, \qquad t > 0. \tag{5.10}$$

Note that the Lévy measure behaves around the origin as the Lévy measure of an α-stable subordinator. For a further discussion of Lamperti stable subordinators and, more generally, Lamperti stable Lévy processes we refer the reader to Patie [79] and Caballero et. al [20].

5.9 Theorem (Lamperti stable-like process) *Let $\alpha : \mathbb{R}^d \to (0,1)$ and $\varrho : \mathbb{R}^d \to (0,\infty)$ be Hölder continuous functions such that*

$$0 < \alpha^L := \inf_{x\in\mathbb{R}^d} \alpha(x) \leq \sup_{x\in\mathbb{R}^d} \alpha(x) \leq 1,$$
$$0 < \varrho^L := \inf_{x\in\mathbb{R}^d} \varrho(x) \leq \sup_{x\in\mathbb{R}^d} \varrho(x) =: \varrho^U < \infty.$$

Then there exists a strong Feller process $(X_t)_{t\geq 0}$ with symbol

$$q(x,\xi) := \frac{\Gamma(|\xi|^2 + \varrho(x) + \alpha(x))}{\Gamma(|\xi|^2 + \varrho(x))} - \frac{\Gamma(\varrho(x) + \alpha(x))}{\Gamma(\varrho(x))}, \qquad x,\xi \in \mathbb{R}^d.$$

$(X_t)_{t\geq 0}$ has the following properties:

- *The transition probability $p : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with $\gamma_0(\alpha(x),\varrho(x)) = 2$, $\gamma_\infty(\alpha(x),\varrho(x)) = 2\alpha(x)$ and any $m \in (0,\sqrt{\varrho^L})$.*
- *$C_c^\infty(\mathbb{R}^d)$ is a core for the generator $(L,\mathcal{D}(L))$ of $(X_t)_{t\geq 0}$ and $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.*
- *$(X_t)_{t\geq 0}$ is the unique solution to the $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem.*

Proof. Set $I := [\alpha^L,\alpha^U] \times [\varrho^L,\varrho^U]$ and fix $m \in (0,\sqrt{\varrho^L})$. Since $z \mapsto \Gamma(z)$ is holomorphic on the right half-plane and $\Gamma(z) \neq 0$ for all $z \in \mathbb{C}\backslash(-\mathbb{N}_0)$ (see e. g. [78, Section 5.2]), it is obvious

$$F_{\alpha,\varrho}(z) := \frac{\Gamma(z+\varrho+\alpha)}{\Gamma(z+\varrho)} - \frac{\Gamma(\alpha+\varrho)}{\Gamma(\varrho)}$$

is holomorphic on $\{z \in \mathbb{C}; \operatorname{Re} z > -m^2\}$ for any $(\alpha,\varrho) \in I$. Moreover, $(0,\infty) \ni \lambda \mapsto F_{\alpha,\varrho}(\lambda)$ is a Bernstein function (cf. (5.10)). As $q(x,\xi) = F_{\alpha(x),\varrho(x)}(|\xi|^2)$, it follows from Lemma 5.1 and the results from Section 3.1 that it suffices to show that $F_{\alpha,\varrho}$ satisfies (B3)-(B4) on

$$\Lambda(m,R,\vartheta) := \Lambda(R,\vartheta) := C(2\vartheta) \cup \{z \in \mathbb{C}; -m^2 < \operatorname{Re} z < R, |\operatorname{Im} z| < R\}$$

for all $(\alpha,\varrho) \in I$ for any $\vartheta \in (0,\pi/4)$ sufficiently small and $R > 0$ (see Figure 5.1). For the proof we need the following properties of the Gamma function:

(a) $\Gamma(z+1) = \Gamma(z)z$ for all $z \in \mathbb{C}\backslash(-\mathbb{N}_0)$ (cf. [78, (5.5.1)]).

(b) For any compact set $K \subseteq (0,\infty)$ it holds that

$$\lim_{\substack{|z|\to\infty \\ |\arg z|<\pi/2}} \sup_{a,b\in K} \left| \frac{1}{z^{a-b}} \frac{\Gamma(z+a)}{\Gamma(z+b)} - 1 \right| = 0.$$

This is a consequence of Stirling's formula [78, (5.11.7)] (see also [78, (5.11.12)]). Note that this implies

$$\operatorname{Re}\left(\frac{\Gamma(z+b)}{\Gamma(z+a)}\right) \geq \frac{1}{2} \operatorname{Re} z^{a-b} = |z|^{a-b} \cos((a-b)\arg z) \qquad \text{for all } |z| \gg 1, |\arg z| < \frac{\pi}{2}.$$

(c) The Psi function (also: Digamma function) $\psi(z) := \Gamma'(z)/\Gamma(z)$ and its derivative ψ' are continuous on the right half plane (cf. [78, 5.15.1]). Moreover, there exists a constant $C > 0$ such that

$$|\psi(z)| \leq C \log|z| \qquad \text{for all } \ |z| \geq 1, |\arg z| < \frac{\pi}{2},$$

cf. [1, 6.3.18].

Clearly, (b) implies that $F_{\alpha,\varrho}$ satisfies $\operatorname{Re} F_{\alpha,\varrho}(z) \geq c|z|^\alpha$ and $|F_{\alpha,\varrho}(z)| \leq C|z|^\alpha$ for $|z| \gg 1$, $z \in \Lambda(R,\vartheta)$ if we choose $\vartheta \in (0,\pi/4)$ sufficiently small. On the other hand, Taylor's formula yields

$$|F_{\alpha,\varrho}(z)| = |F_{\alpha,\varrho}(z) - F_{\alpha,\varrho}(0)| \leq |z| \sup_{|\zeta|\leq|z|} \left| \frac{d}{d\zeta} F_{\alpha,\varrho}(\zeta) \right|.$$

As

$$\frac{d}{d\zeta} F_{\alpha,\varrho}(\zeta) = \frac{\Gamma'(\zeta+\varrho+\alpha)}{\Gamma(\zeta+\varrho)} - \frac{\Gamma(\zeta+\varrho+\alpha)}{\Gamma(\zeta+\varrho)^2}\Gamma'(\zeta+\varrho) = \frac{\Gamma(\varrho+\alpha+\zeta)}{\Gamma(\varrho+\zeta)}(\psi(\varrho+\alpha+\zeta) - \psi(\varrho+\zeta)),$$

it follows from the continuity of ψ and Γ and the fact that $\varrho^L > 0$, $\alpha^L > 0$ that

$$\sup_{|\zeta|\leq\varrho^L/2} \left| \frac{d}{d\zeta} F_{\alpha,\varrho}(\zeta) \right| < \infty;$$

hence,

$$|F_{\alpha,\varrho}(z)| \leq C'|z| \qquad \text{for all } \ |z| \ll 1, z \in \Lambda(R,\vartheta), (\alpha,\varrho) \in I.$$

It remains to check that the derivatives $\partial_\alpha F_{\alpha,\varrho}$ and $\partial_\varrho F_{\alpha,\varrho}$ satisfy (B4). Since the reasoning is very similar, we only verify the growth conditions for $\partial_\alpha F_{\alpha,\varrho}$. Obviously,

$$\frac{\partial}{\partial\alpha} F_{\alpha,\varrho}(z) = \frac{\Gamma'(z+\varrho+\alpha)}{\Gamma(z+\varrho)} - \frac{\Gamma'(\varrho+\alpha)}{\Gamma(\varrho)} = \psi(z+\varrho+\alpha)\frac{\Gamma(z+\varrho+\alpha)}{\Gamma(z+\varrho)} - \psi(\varrho+\alpha)\frac{\Gamma(\varrho+\alpha)}{\Gamma(\varrho)}.$$

It follows from (b) and (c) that $|\partial_\alpha F_{\alpha,\varrho}(z)| \leq C'|z|^\alpha \log|z|$ for all $|z| \gg 1$, $z \in \Lambda(R,\vartheta)$. Applying Taylor's formula and using the continuity of ψ' gives the necessary growth condition for small $|z|$. □

In Table 5.1 we summarize the results from this section and present further examples of of admissible continuous negative definite functions. For any symbol q listed in Table 5.1 there exists a Feller process $(X_t)_{t\geq 0}$ with symbol q which enjoys the properties listed in Section 3.1. If there are any results known (on existence, path properties,...), we collect them in the column "literature". With the exception of Example No. 3, the symbols can be written in the form $q(x,\xi) = f_{\alpha(x)}(\xi)$ for a family of Bernstein functions $(f_\alpha)_{\alpha\in I}$ (see [100] for an extensive list of Bernstein functions). Numbers marked with an asterisk require Extension 3.10 or Extension 3.11 for the proof. For an interval $I \subseteq \mathbb{R}$ we denote by $C^{>0}(I)$ the space of bounded functions $f : \mathbb{R}^d \to I$ which are Hölder continuous with Hölder exponent > 0 and satisfy

$$f^L(x) := \inf_{x\in\mathbb{R}^d} f(x) \in I \quad \text{and} \quad f^U(x) := \sup_{x\in\mathbb{R}^d} f(x) \in I.$$

	Name	Symbol
1	symmetric α-stable-like (Theorem 5.2)	$\lvert\xi\rvert^{\alpha(x)}$
2	relativistic stable-like (Theorem 5.3)	$(\lvert\xi\rvert^2 + \varrho^2(x))^{\alpha(x)/2} - \varrho(x)^\alpha$
3	NTS-like (Theorem 5.4, Example 5.14)	$(\kappa(x)^2 + (\xi - ib(x))^2)^{\alpha(x)/2} - (\kappa(x)^2 - b(x)^2)^{\alpha(x)/2}$
4		$\frac{\lvert\xi\rvert^2}{\sqrt{\lvert\xi\rvert^2+\varrho(x)}}$
5		$\frac{\lvert\xi\rvert^2}{(\lvert\xi\rvert^2+\varrho(x))^{\alpha(x)}}$
6		$\frac{\lvert\xi\rvert^{\beta(x)}-1}{\lvert\xi\rvert^{\alpha(x)}-1} - 1$ (extended by continuity at $\xi = 1$)
7		$-\frac{\lvert\xi\rvert^{\alpha(x)}-1}{\lvert\xi\rvert^{\alpha(x)-2}-1}$ (extended by continuity at $\xi = 1$)
8		$\frac{\lvert\xi\rvert^{\alpha(x)}-1}{\lvert\xi\rvert^{\alpha(x)-2}-1} - 1$ (extended by continuity at $\xi = 1$)
9		$\lvert\xi\rvert^2 \frac{\lvert\xi\rvert^{\alpha(x)}-\varrho(x)^{\alpha(x)}}{\lvert\xi\rvert^2-\varrho(x)^2}$ (extended by continuity at $\xi = \varrho(x)$)
10		$(\lvert\xi\rvert^{-\alpha(x)} + \lvert\xi\rvert^{-\beta(x)})^{-1}$ (extended by continuity at $\xi = 0$)
11		$\lvert\xi\rvert(1 - e^{-2\varrho(x)\lvert\xi\rvert})$
12		$\lvert\xi\rvert(1 + e^{-2\varrho(x)\lvert\xi\rvert})$
13		$\lvert\xi\rvert^2 \log(1 + \varrho(x)/\lvert\xi\rvert^2)$
14*		$\varrho(x)\lvert\xi\rvert^2(\lvert\xi\rvert^2 + 1) \log(1 + \lvert\xi\rvert^{-2})$

Table 5.1: Examples of admissible symbols

Assumptions	Dim.	Parameters (Thm. 3.6)	Literature
$\alpha \in C^{>0}((0,2])$	$d \geq 1$	$\gamma_\infty(\alpha(x)) = \gamma_0(\alpha(x)) = \alpha(x)$ $m = 0$	Lévy: e. g. [92] Lévy-type: [60, 4]
$\alpha \in C^{>0}((0,2])$ $\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\alpha(x), \varrho(x)) = 2$, $m \in (0, \varrho^L)$ $\gamma_\infty(\alpha(x), \varrho(x)) = \alpha(x)$	Lévy: e. g. [24, 21, 52] Lévy-type: -
$\alpha \in C^{>0}((0,2])$ $\kappa \in C^{>0}((0,\infty))$ $b \in C^{>0}(\mathbb{R})$ $\kappa^L - \|b\|_\infty > 0$	$d = 1$	$\gamma_0(\alpha(x), b(x), \kappa(x)) = 2$ $\gamma_\infty(\alpha(x), b(x), \kappa(x)) = \alpha(x)$ $m \in (0, \kappa^L - \|b\|_\infty)$	Lévy: e. g. [2, 88] (NIG) Lévy-type: [3] (NIG)
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = 2$, $\gamma_\infty(\varrho(x)) = 1$ $m \in (0, \sqrt{\varrho^L})$	Lévy: Lévy-type:
$\alpha \in C^{>0}((0,2))$ $\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\alpha(x), \varrho(x)) = 2$, $m \in (0, \sqrt{\varrho^L})$ $\gamma_\infty(\alpha(x), \varrho(x)) = 2 - \alpha(x)$	Lévy: Lévy-type:
$\alpha, \beta \in C^{>0}((0,1])$ $(\beta - \alpha)^L > 0$	$d \geq 1$	$\gamma_0(\alpha(x), \beta(x)) = \alpha(x)$, $m = 0$ $\gamma_\infty(\varrho(x)) = \beta(x) - \alpha(x)$	Lévy: Lévy-type:
$\alpha \in C^{>0}((0,2))$	$d \geq 1$	$\gamma_0(\alpha(x)) = 2 - \alpha(x)$, $\gamma_\infty(\alpha(x)) = 2$ $m = 0$	Lévy: Lévy-type:
$\alpha \in C^{>0}((2,4])$	$d \geq 1$	$\gamma_0(\alpha(x)) = \alpha(x) - 2$, $\gamma_\infty(\alpha(x)) = 2$ $m = 0$	Lévy: Lévy-type:
$\alpha \in C^{>0}((0,2))$ $\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\alpha(x), \varrho(x)) = 2$, $m = 0$ $\gamma_\infty(\alpha(x), \varrho(x)) = \alpha(x)$	Lévy: Lévy-type:
$\alpha, \beta \in C^{>0}((0,2])$	$d \geq 1$	$\gamma_0(\alpha(x), \beta(x)) = \alpha(x) \vee \beta(x)$, $m = 0$ $\gamma_\infty(\alpha(x), \beta(x)) = \alpha(x) \wedge \beta(x)$	Lévy: Lévy-type:
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = 2$, $\gamma_\infty(\varrho(x)) = 1$ $m = 0$	Lévy: Lévy-type:
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 1$ $m = 0$	Lévy: Lévy-type:
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 2$ $m = 0$	Lévy: Lévy-type:
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 2$ $m = 0$	Lévy: Lévy-type:

	Name	Symbol
15		$\varrho(x)\lvert\xi\rvert^2(1-\lvert\xi\rvert^2\log(1+\lvert\xi\rvert^{-2}))$
16*		$\varrho(x)\frac{\lvert\xi\rvert^2(\lvert\xi\rvert^2+1)}{(\lvert\xi\rvert^2+2)\log(\lvert\xi\rvert^2+2)}$
17		$\lvert\xi\rvert\arctan(\varrho(x)\lvert\xi\rvert)$
18	TLP-like (Theorem 5.7)	$(\lvert\xi\rvert^2+\varrho(x)^2)^{\alpha(x)/2}\cos\left(\alpha(x)\arctan\frac{\lvert\xi\rvert}{\varrho(x)}\right)-\varrho(x)^{\alpha(x)}$
19		$\varrho(x)\lvert\xi\rvert\frac{\cosh^2(\sqrt{2}\lvert\xi\rvert)}{\sinh(2\sqrt{2}\lvert\xi\rvert)}$
20		$\varrho(x)\lvert\xi\rvert\frac{\sinh^2(\sqrt{2}\lvert\xi\rvert)}{\sinh(2\sqrt{2}\lvert\xi\rvert)}$
21		$\lvert\xi\rvert\coth((2\lvert\xi\rvert)^{-1})-\lvert\xi\rvert^2$
22		$\varrho(x)\log(\sinh(\sqrt{2}\lvert\xi\rvert))-\log(\sqrt{2}\lvert\xi\rvert)$
23	symmetric Meixner-like	$\varrho(x)\log(\cosh(\sqrt{2}\lvert\xi\rvert))$
24		$\lvert\xi\rvert\log(1+\varrho(x)\tanh(b(x)\lvert\xi\rvert))$
25		$\frac{\Gamma(\varrho(x)/2(\lvert\xi\rvert^2+1))}{\Gamma(\varrho(x)\lvert\xi\rvert^2/2)}$
26		$\lvert\xi\rvert^2\frac{\Gamma(\alpha(x)\lvert\xi\rvert^2+1-\alpha(x))}{\Gamma(\alpha(x)\lvert\xi\rvert^2+1)}$
27		$\frac{\Gamma(\alpha(x)\lvert\xi\rvert^2+1)}{\Gamma(\alpha(x)\lvert\xi\rvert^2+1-\alpha(x))}-\frac{1}{\Gamma(1-\alpha(x))}$
28	Lamperti stable-like (Theorem 5.9)	$\frac{\Gamma(\lvert\xi\rvert^2+\alpha(x)+\varrho(x))}{\Gamma(\lvert\xi\rvert^2+\varrho(x))}-\frac{\Gamma(\alpha(x)+\varrho(x))}{\Gamma(\varrho(x))}$

Table 5.1: Examples of admissible symbols

Assumptions	Dim.	Parameters (Thm. 3.6)	Literature
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 2$ $m = 0$	Lévy: Lévy-type:
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 2$ $m \in (0, \sqrt{2})$	Lévy: Lévy-type:
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 1$ $m = 0$	Lévy: Lévy-type:
$\alpha \in C^{>0}((0,2))$ $\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\alpha(x), \varrho(x)) = 2,\ m \in (0, \varrho^L)$ $\gamma_\infty(\alpha(x), \varrho(x)) = \alpha(x)$	Lévy: e. g. [14, 39, 75] Lévy-type:
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 1$ $m = 0$	Lévy: Lévy-type:
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = \gamma_\infty(\varrho(x)) = 1$ $m = 0$	Lévy: Lévy-type:
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = 1,\ \gamma_\infty(\varrho(x)) = 2$ $m = 0$	Lévy: Lévy-type:
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = 2,\ \gamma_\infty(\varrho(x)) = 1$ $m = 0$	Lévy: Lévy-type:
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = 2,\ \gamma_\infty(\varrho(x)) = 1$ $m = 0$	Lévy: [80, 52, 103, 41] Lévy-type: [15, 18]
$b \in C^{>0}((0,\infty))$ $\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(b(x), \varrho(x)) = \gamma_\infty(b(x), \varrho(x)) = 1$ $m = 0$	Lévy: Lévy-type:
$\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\varrho(x)) = 2,\ \gamma_\infty(\varrho(x)) = 1$ $m \in (0, \sqrt{\varrho^L})$	Lévy: Lévy-type:
$\alpha \in C^{>0}((0,1))$	$d \geq 1$	$\gamma_0(\alpha(x)) = 2,\ \gamma_\infty(\alpha(x)) = 2 - \alpha(x)$ $m \in (0, \sqrt{1/\alpha^U - 1})$	Lévy: Lévy-type:
$\alpha \in C^{>0}((0,1))$	$d \geq 1$	$\gamma_0(\alpha(x)) = 2,\ \gamma_\infty(\alpha(x)) = \alpha(x)$ $m \in (0, \sqrt{1/\alpha^U - 1})$	Lévy: Lévy-type:
$\alpha \in C^{>0}((0,1))$ $\varrho \in C^{>0}((0,\infty))$	$d \geq 1$	$\gamma_0(\alpha(x), \varrho(x)) = 2, m \in (0, \sqrt{\varrho^L})$ $\gamma_\infty(\alpha(x), \varrho(x)) = 2\alpha(x)$	Lévy: [79, 20] Lévy-type:

5.2 Feller processes with symbols of varying order

Hoh [43] studied the existence of Feller processes with symbols of varying order, i. e. symbols q which can be written in the form

$$q(x,\xi) = p(x,\xi)^{\alpha(x)}, \qquad x,\xi \in \mathbb{R}^d.$$

Since his approach relies on symbolic calculus, Hoh typically requires smoothness of α, i. e. $\alpha \in C^\infty((0,1])$ (cf. [43, Theorem 7.10]). In this section we derive two existence results which show that smoothness is not a necessary assumption. We start with the following, less abstract, result.

5.10 Theorem *Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a rotationally invariant continuous negative definite function, $\psi(0) = 0$, which satisfies the following conditions.*

(i) There exists a mapping $\Psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(\xi) = \Psi(|\xi|)$, $\xi \in \mathbb{R}^d$, and Ψ has a holomorphic extension to

$$\Omega := \Omega(\vartheta) := \{z \in \mathbb{C}\backslash\{0\}; \arg z \in (-\vartheta,\vartheta) \cup (\pi-\vartheta,\pi+\vartheta)\}$$

for some $\vartheta \in (0,\pi/2)$.

(ii) There exist $c_1, c_2 > 0$ and $\beta_0, \beta_\infty \in (0,2]$ such that

$$|\Psi(z)| \leq c_1(|z|^{\beta_0}\mathbb{1}_{\{|z|\leq 1\}} + |z|^{\beta_\infty}\mathbb{1}_{\{|z|>1\}}), \qquad z \in \Omega,$$

and

$$|\operatorname{Re}\Psi(z)| \geq c_2|\operatorname{Re} z|^{\beta_\infty}, \qquad z \in \Omega, |z| \gg 1.$$

(iii) Ψ satisfies the sector condition, i. e. there exists a constant $c_3 > 0$ such that

$$|\operatorname{Im}\Psi(z)| \leq c_3|\operatorname{Re}\Psi(z)| \qquad \text{for all } \ z \in \Omega.$$

Let $\alpha : \mathbb{R}^d \to (0,1]$ be a Hölder continuous mapping such that $\alpha^L := \inf_{x\in\mathbb{R}^d} \alpha(x) > 0$. Then there exists a strong Feller process $(X_t)_{t\geq 0}$ with symbol $q(x,\xi) := \psi(\xi)^{\alpha(x)}$. The process $(X_t)_{t\geq 0}$ has the following properties:

- *The transition probability $p : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with $\gamma_0(\alpha(x)) = \beta_0\alpha(x)$, $\gamma_\infty(\alpha(x)) = \beta_\infty\alpha(x)$ and $m = 0$.*
- *$C_c^\infty(\mathbb{R}^d)$ is a core for the generator $(L,\mathcal{D}(L))$ of $(X_t)_{t\geq 0}$ and $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.*
- *$(X_t)_{t\geq 0}$ is the unique solution to the $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem.*

In dimension $d = 1$, we can drop the assumption of rotational invariance of ψ. Moreover, we note that for $z \in \Omega$, $|z| \geq 1$, the sector condition (iii) is a direct consequence of (ii).

Proof of Theorem 5.10. It follows from Example 1.15 and the remark following the proof of Theorem 1.16 that $\psi_\alpha(\xi) = \psi(\xi)^\alpha$ is a continuous negative definite function for all $\alpha \in I := [\alpha^L, 1]$. Moreover, because of the sector condition, there exists $\tilde{\vartheta} \in (0, \pi/2)$ such that $\Omega(\vartheta) \overset{\Psi}{\hookrightarrow} \Omega(\tilde{\vartheta})$. (Here, $A \overset{f}{\hookrightarrow} B$ is short for $f(A) \subseteq B$.) If we set[5]

$$z^{1/2} := \exp\left(\frac{1}{2} \operatorname{Log} z\right).$$

for $z \in \mathbb{C}$, then

$$\Omega(\vartheta) \overset{\Psi}{\hookrightarrow} \Omega(\tilde{\vartheta}) \overset{z^2}{\hookrightarrow} C(2\tilde{\vartheta}) := \{z \in \mathbb{C}\backslash\{0\}; |\arg z| < 2\tilde{\vartheta}\} \overset{z^{1/2}}{\longhookrightarrow} C(\tilde{\vartheta}). \qquad (\star)$$

Since $\operatorname{Log} z$ is holomorphic on $\mathbb{C}\backslash(-\infty, 0]$, we find that

$$\Psi_\alpha(z) := \exp\Big(\alpha \operatorname{Log}\big((\Psi(z)^2)^{1/2}\big)\Big)$$

extends ψ_α holomorphically from the real line to $\Omega(\vartheta)$. It remains to check the growth conditions (C3) and (C4). As $|e^{\operatorname{Log} z}| = |z|$, we have $|\Psi_\alpha(z)| = |\Psi(z)|^\alpha$ and, because of (ii), this implies that (C3-2) is satisfied with $\gamma_0(\alpha) := \alpha\beta_0$ and $\gamma_\infty(\alpha) := \alpha\beta_\infty$. On the other hand,

$$\operatorname{Re} \Psi_\alpha(z) = |\Psi(z)|^\alpha \cos\Big[\alpha \arg\Big((\Psi(z)^2)^{1/2}\Big)\Big] \overset{(\star)}{\geq} |\Psi(z)|^\alpha \cos(\tilde{\vartheta}) \geq c_2 \cos(\tilde{\vartheta}) |z|^{\alpha\beta_\infty}$$

for all $z \in \Omega(\vartheta)$, $|z| \gg 1$, and $\alpha \in I$, and so (C3-1) holds true. Finally, we note that

$$\frac{\partial}{\partial \alpha} \Psi_\alpha(z) = \operatorname{Log}\big((\Psi(z)^2)^{1/2}\big) \Psi_\alpha(z).$$

Using (C3) and the elementary estimate $|\operatorname{Log} z|^2 \leq (\log |z|)^2 + \pi^2$, it follows that (C4) is satisfied with

$$\ell(r) := 1_{(0,2)}(r) + \log r \mathbb{1}_{[2,\infty)}(r), \qquad r > 0.$$

Now the claim follows from the results presented in Section 3.1. □

Remark Let $\psi : \mathbb{R} \to \mathbb{R}$ and $\alpha : \mathbb{R} \to (0, 1]$ be as in Theorem 5.10, and suppose additionally that $\alpha \in C_b^2(\mathbb{R})$. Then it follows easily from Theorem 3.8, Corollary 3.9 and the proof of Theorem 5.10 that the Feller process $(X_t)_{t \geq 0}$ with symbol $q(x, \xi) := (\psi(\xi))^{\alpha(x)}$ is irreducible with respect to Lebesgue measure and that the partial derivative $\partial_x p(t, x, y)$ (exists and) satisfies the heat kernel estimates from Theorem 3.8.

Using a similar reasoning as in the proof of the previous theorem, we can easily obtain the following more abstract result. Let us mention that Theorem 5.11 applies to many of the symbols listed in Table 5.1.

5.11 Theorem *Let $I \subseteq \mathbb{R}^n$ an open convex set and $\psi_\beta : \mathbb{R}^d \to \mathbb{C}$, $\beta \in I$, be a family of continuous negative definite functions satisfying* (C1)-(C3) *on*

$$\Omega(\vartheta) := \{z \in \mathbb{C}\backslash\{0\}; \arg z \in (-\vartheta, \vartheta) \cup (\pi - \vartheta, \pi + \vartheta)\}$$

for some $\vartheta \in (0, \pi/2)$. Assume additionally that

[5] Mind that $(z^2)^{1/2} \neq z$ for $z \in \mathbb{C}$ with $|\arg z| > \pi/2$.

(C4") *The partial derivative* $\frac{\partial}{\partial \beta_j}\Psi_\beta(r)$ *exists for all* $r \in \mathbb{R}$ *and extends holomorphically to* $\Omega(\vartheta)$ *for all* $j \in \{1,\dots,n\}$ *and* $\alpha \in I$*. There exist an increasing slowly varying (at* ∞*) function* $\ell : (0,\infty) \to (0,\infty)$ *and a constant* $c_4 > 0$ *such that*

$$\left|\frac{\partial_{\beta_j}\Psi_\beta(z)}{\Psi_\beta(z)}\right| \leq c_4(1+\ell(|z|)) \qquad \text{for all} \;\; z \in \Omega(\vartheta), j = 1,\dots,n.$$

and

(S) $(\psi_\beta)_{\beta\in I}$ *satisfies the sector condition, i. e. there exists a constant* $c > 0$ *such that*

$$|\operatorname{Im}\Psi_\beta(z)| \leq c|\operatorname{Re}\Psi_\beta(z)| \qquad \text{for all} \;\; z \in \Omega(\vartheta), \beta \in I.$$

Then for any two Hölder continuous mappings $\alpha : \mathbb{R}^d \to (0,1]$ *and* $\beta : \mathbb{R}^d \to I$ *such that* $\alpha^L := \inf_{x\in\mathbb{R}^d}\alpha(x) > 0$*, there exists a strong Feller process* $(X_t)_{t\geq 0}$ *with symbol*

$$q(x,\xi) := (\psi_{\beta(x)}(\xi))^{\alpha(x)}, \qquad x,\xi \in \mathbb{R}^d.$$

The process $(X_t)_{t\geq0}$ *has the following properties:*

- *The transition probability* $p : (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d \to [0,\infty)$ *is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with* $\tilde\gamma_0(\alpha(x),\beta(x)) := \alpha(x)\gamma_0(\beta(x))$, $\tilde\gamma_\infty(\alpha(x),\beta(x)) := \alpha(x)\gamma_\infty(\beta(x))$ *and* $m = 0$*; here* $\gamma_0(\beta(x))$ *and* $\gamma_\infty(\beta(x))$ *are the mappings associated with* $(\psi_\beta)_{\beta\in I}$ *by the growth condition* (C3)*.*
- $C_c^\infty(\mathbb{R}^d)$ *is a core for the generator* $(L,\mathcal{D}(L))$ *of* $(X_t)_{t\geq0}$ *and* $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(L)$*. Moreover, p is a fundamental solution to the Cauchy problem for the operator* $\partial_t - L$*.*
- $(X_t)_{t\geq0}$ *is the unique solution to the* $(L, C_c^\infty(\mathbb{R}^d))$*-martingale problem.*

Remarks (i) In dimension $d = 1$ Theorem 5.11 remains valid if we just assume that $(\psi_\beta)_{\beta\in I}$ satisfies (C2) and (C3), i.e. we can drop the assumption of rotational invariance.

(ii) Clearly, (C4") implies (C4).

Proof of Theorem 5.11. It follows exactly as in the proof of the previous theorem that

$$\tilde\Psi_{\alpha,\beta}(z) := \exp\Big(\alpha \operatorname{Log}\big((\Psi_\beta(z)^2)^{1/2}\big)\Big)$$

satisfies (C1)-(C3) on $\Omega(\vartheta)$ for any $(\alpha,\beta) \in \tilde I := [\alpha^L,1]\times I$ and that the partial derivative $\partial_\alpha\tilde\Psi_{\alpha,\beta}$ has a holomorphic extension which satisfies (C4). Since

$$\frac{\partial}{\partial\beta_j}\tilde\Psi_{\alpha,\beta}(z) = \alpha\frac{\partial_{\beta_j}\Psi_\beta(z)}{\Psi_\beta(z)}\tilde\Psi_{\alpha,\beta}(z)$$

it is a direct consequence of (C4") that also the partial derivatives $\partial_{\beta_j}\tilde\Psi_{\alpha,\beta}$ satisfy the growth condition (C4). This proves that the family $(\tilde\Psi_{\alpha,\beta})_{(\alpha,\beta)\in\tilde I}$ is admissible, and therefore the claim follows from the results presented in Section 3.1. $\square$

In dimension $d = 1$ we get the following corollary.

5.12 Corollary *Let $I \subseteq \mathbb{R}^n$ be an open convex set and let $\psi_\beta : \mathbb{R} \to \mathbb{R}$, $\beta \in I$, be as in Theorem 5.11, i. e. a family of continuous negative definite functions satisfying* (C1)-(C3), (C4") *and* (S). *Suppose, additionally, that the following condition holds.*

(C5") *The partial derivative $\frac{\partial^2}{\partial \beta_j^2}\Psi_\beta(r)$ exists for all $r \in \mathbb{R}$ and has a holomorphic extension to $\Omega(\vartheta)$ for all $j \in \{1,\dots,n\}$ and $\alpha \in I$. There exist an increasing slowly varying (at ∞) function $\ell : (0,\infty) \to (0,\infty)$ and a constant $c_5 > 0$ such that*

$$\left| \frac{\partial^2_{\beta_j}\Psi_\beta(z)}{\Psi_\beta(z)} \right| \le c_5(1+\ell(|z|)) \qquad \text{for all} \;\; z \in \Omega(\vartheta), j = 1,\dots,n.$$

Let $\alpha : \mathbb{R} \to (0,1]$ and $\beta : \mathbb{R} \to I$ be such that $\alpha, \beta \in C_b^2(\mathbb{R})$ and $\inf_{x\in\mathbb{R}} \alpha(x) > 0$. By Theorem 5.11, there exists a Feller process $(X_t)_{t\ge0}$ with symbol $q(x,\xi) := (\psi_{\beta(x)}(\xi))^{\alpha(x)}$ and transition density p. Then:

(i) $p = p(t,x,y)$ is differentiable with respect to x.

(ii) $\partial_x p(t,x,y)$ satisfies the heat kernel estimates from Theorem 3.8 and the lower bound (3.5) *holds true.*

(iii) $(X_t)_{t\ge0}$ is λ-irreducible.

Proof. We pick up the notation from the proof of Theorem 5.11, in particular

$$\tilde{\Psi}_{\alpha,\beta}(z) := \exp\Big(\alpha \operatorname{Log}\big((\Psi_\beta(z)^2)^{1/2}\big)\Big).$$

Then

$$\frac{\partial^2}{\partial \alpha^2}\tilde{\Psi}_{\alpha,\beta}(z) = \big[\operatorname{Log}\big((\Psi_\beta(z)^2)^{1/2}\big)\big]^2 \tilde{\Psi}_{\alpha,\beta}(z)$$

and

$$\frac{\partial^2}{\partial \beta_j^2}\tilde{\Psi}_{\alpha,\beta}(z) = \left(\alpha \frac{\partial_{\beta_j^2}\Psi_\beta(z)}{\Psi_\beta(z)} + (\alpha^2-\alpha)\left[\frac{\partial_{\beta_j}\Psi_\beta(z)}{\Psi_\beta(z)}\right]^2\right)\tilde{\Psi}_{\alpha,\beta}(z).$$

It follows directly from the growth conditions (C4") and (C5") that the assumptions of Theorem 3.8 are satisfied. On the other hand, $(\psi_\beta(\cdot))^\alpha$ is symmetric, and so Corollary 3.9 is applicable. Applying both results, finishes the proof. □

5.3 Mixing

Recall that a family $(\psi_\alpha)_{\alpha\in I}$ of continuous negative definite functions is admissible, and write $(\psi_\alpha)_{\alpha\in I} \in A(m)$ satisfies (C1)-(C4) for given $m \ge 0$. In this case, we write $(\psi_\alpha)_{\alpha\in I} \in A(m)$.

5.13 Proposition *Let $I, \tilde{I} \subseteq \mathbb{R}^n$ be convex open sets and $m, \tilde{m} \ge 0$. Let $K_1 \subseteq (0,\infty)$ and $K_2 \subseteq \mathbb{R}$ be compact convex sets.*

(i) *($d = 1$) Let $\psi_\alpha : \mathbb{R} \to \mathbb{C}$ be such that $(\psi_\alpha)_{\alpha\in I} \in A(m)$ and $\gamma_\infty^L = \inf_{\alpha\in I} \gamma_\infty(\alpha) \geq 1$. Then $(-i\beta\xi + \delta\psi_\alpha(\xi))_{(\alpha,\beta,\delta)\in J} \in A(m)$ for $J := I \times K_2 \times K_1$.*

(ii) *Let $(\psi_\alpha)_{\alpha\in I} \in A(m)$, $(\tilde{\psi}_\kappa)_{\kappa\in\tilde{I}} \in A(\tilde{m})$. Then $(\beta\psi_\alpha(\cdot) + \delta\tilde{\psi}_\kappa(\cdot))_{(\alpha,\kappa,\beta,\delta)\in J} \in A(m \wedge \tilde{m})$ for $J := I \times \tilde{I} \times K_1 \times K_1$.*

Since the proof is straightforward, we omit it. Combining Proposition 5.13 and Theorem 5.4, we obtain a general existence result on NTS-like processes.

5.14 Example Let $b : \mathbb{R} \to \mathbb{R}$, $\alpha : \mathbb{R} \to [1, 2]$, $\ell : \mathbb{R} \to \mathbb{R}$, $\delta : \mathbb{R} \to (0,\infty)$ and $\kappa : \mathbb{R} \to (0, \infty)$ be Hölder continuous functions such that $\|b\|_\infty + \|\ell\|_\infty < \infty$ and

$$\begin{aligned} &0 < \kappa^L := \inf_{x\in\mathbb{R}} \kappa(x) \leq \sup_{x\in\mathbb{R}} \kappa(x) =: \kappa^U < \infty \\ &0 < \kappa^L - \|b\|_\infty \\ &0 < \delta^L := \inf_{x\in\mathbb{R}} \delta(x) \leq \sup_{x\in\mathbb{R}} \delta(x) =: \delta^U. \end{aligned}$$

Then there exists a strong Feller process $(X_t)_{t\geq 0}$ with symbol

$$q(x,\xi) := i\ell(x)\xi + \delta(x)[\kappa(x)^2 + (\xi - ib(x))^2]^{\alpha(x)/2} - \delta(x)[\kappa(x)^2 - b(x)^2]^{\alpha(x)/2}, \qquad x, \xi \in \mathbb{R}.$$

$(X_t)_{t\geq 0}$ has the following additional properties:

- The transition probability $p : (0, \infty) \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with $\gamma_0(\alpha(x)) = 2$, $\gamma_\infty(\alpha(x)) = \alpha(x)$ and any $m \in (0, \kappa^L - \|b\|_\infty)$.
- $C_c^\infty(\mathbb{R})$ is a core for the generator $(L, \mathcal{D}(L))$ of $(X_t)_{t\geq 0}$ and $C_\infty^2(\mathbb{R}) \subseteq \mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.
- $(X_t)_{t\geq 0}$ is the unique solution to the $(L, C_c^\infty(\mathbb{R}))$-martingale problem.

The following example is also a direct consequence of Proposition 5.13.

5.15 Example (Mixed stable-relativistic stable process) Let $\chi : \mathbb{R} \to [0, 1]$, $\alpha : \mathbb{R} \to (0, 2]$ and $\varrho : \mathbb{R} \to (0, \infty)$ be Hölder continuous functions such that

$$\begin{aligned} &0 < \inf_{x\in\mathbb{R}} |\alpha(x)| \\ &0 < \inf_{x\in\mathbb{R}} |\varrho(x)| \leq \sup_{x\in\mathbb{R}} |\varrho(x)| < \infty. \end{aligned}$$

Define

$$q(x,\xi) := (1 - \chi(x))|\xi|^{\alpha(x)} + \chi(x)[(|\xi|^2 + \varrho(x)^2)^{\alpha/2} - \varrho(x)^{\alpha(x)}], \qquad x, \xi \in \mathbb{R}.$$

Then, by the previous proposition and the results established in Section 3.1, there exists a strong Feller process $(X_t)_{t\geq 0}$ with symbol $q(x, \xi)$ and $(X_t)_{t\geq 0}$ has the following properties:

- The transition probability $p : (0, \infty) \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with $\gamma_0(\chi(x)) = \gamma_\infty(\chi(x)) = \alpha(x)$ and $m = 0$.
- $C_c^\infty(\mathbb{R})$ is a core for the generator $(L, \mathcal{D}(L))$ of $(X_t)_{t\geq 0}$ and $C_\infty^2(\mathbb{R}) \subseteq \mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.
- $(X_t)_{t\geq 0}$ is the unique solution to the $(L, C_c^\infty(\mathbb{R}))$-martingale problem.

If $\alpha \in C_b^2(\mathbb{R})$, then it follows from Theorem 3.8 that the transition density $p = p(t, x, y)$ admits a derivative with respect to x satisfying the heat kernel estimates from Theorem 3.8, the lower bound (3.5) holds true and $(X_t)_{t\geq 0}$ is λ-irreducible by Corollary 3.9.
Interesting choices are e. g.

$$\chi_1(x) := \begin{cases} 0, & x \leq -k^{-1}, \\ \frac{kx+1}{2}, & x \in (-k^{-1}, k^{-1}), \\ 1, & x \geq k^{-1} \end{cases}$$

or

$$\chi_2(x) := \sum_{n\in\mathbb{Z}} \kappa(x + 2n)$$

for a piecewise linear function κ such that $\kappa(x) = 0$ for $x \in (-\infty, -k^{-1}) \cup (1 + k^{-1}, \infty)$ and $\kappa(x) = 1$ for $x \in (k^{-1}, 1 - k^{-1})$; $k \in \mathbb{N}$, $k \geq 2$, is a fixed number.

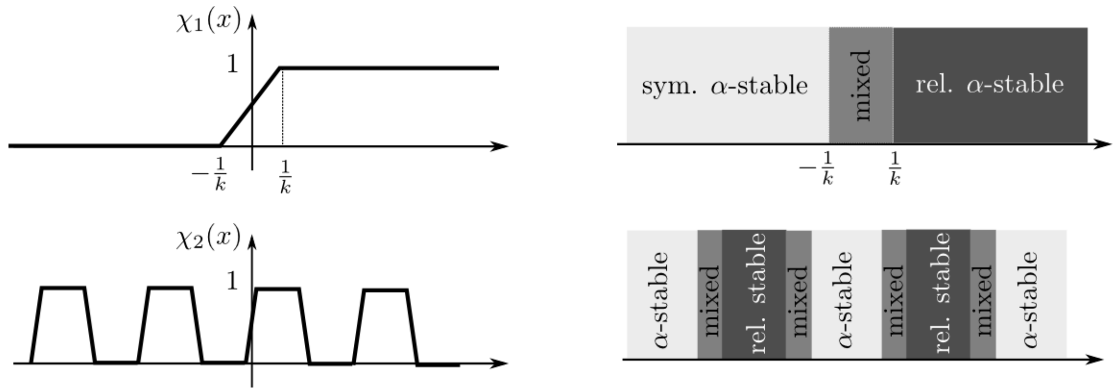

Figure 5.3: The functions χ_1 and χ_2 and the behaviour of the corresponding symbol for the particular case that $\alpha(x) = \alpha \in (0, 2)$ is constant.

More generally, we can consider symbols of the form

$$q(x, \xi) = (1 - \chi(x))\psi_{\alpha(x)}(\xi) + \chi(x)\tilde{\psi}_{\alpha(x)}(\xi), \qquad x, \xi \in \mathbb{R}^d$$

where α is Hölder continuous mapping and $(\psi_\alpha)_\alpha$ and $(\tilde{\psi}_\alpha)_\alpha$ are families of admissible continuous negative definite functions (see e. g. Table 5.1 for examples) such that either

(i) $\inf_{x\in\mathbb{R}^d} \chi(x) > 0$ and $\tilde{\gamma}_\infty(\alpha) \geq \gamma_\infty(\alpha)$

or

(ii) $\tilde{\gamma}_\infty(\alpha) = \gamma_\infty(\alpha)$.

Here $\gamma_\infty(\cdot)$ and $\tilde{\gamma}_\infty(\cdot)$ denote the mappings associated with $(\psi_\alpha)_\alpha$ and $(\tilde{\psi}_\alpha)_\alpha$, respectively, by the growth condition (C3). These conditions are needed to ensure that the symbol q satisfies (C3). It would be of interest to study the sample path behaviour of such mixed processes; see Böttcher [17] and Sandric [89, 90] for some results on transience & recurrence. We close this example with the following remark: Imkeller & Willrich [47] obtained results on the existence of solutions to the martingale problem for discontinuous χ; note, however, that we cannot expect the solution to be a (rich) Feller process because $q(\cdot,\xi)$ is not continuous (cf. Theorem 1.26).

Next we consider a different kind of mixed processes.

5.16 Proposition *Let $I := [\alpha^L, \alpha^U] \subseteq (0,2]$, $I \neq \emptyset$, and $J \subseteq \mathbb{R}^n$ an open set. Let $f : I \times J \to (0,\infty)$ be a bounded function such that*

(a) $\beta \mapsto f(\alpha,\beta)$ is partially differentiable for each $\alpha \in I$ and $\sup_{(\alpha,\beta)\in I\times J} |\partial_{\beta_j} f(\alpha,\beta)| < \infty$ for all $j \in \{1,\dots,n\}$,

(b) $f^L := \inf_{(\alpha,\beta)\in I\times J} f(\alpha,\beta) > 0$.

Define

$$\tilde{\psi}_\beta(\xi) := \int_I |\xi|^\alpha f(\alpha,\beta)\,d\alpha \qquad \text{for all } \ \xi \in \mathbb{R}^d, \beta \in J.$$

If $\beta : \mathbb{R}^d \to J$ is a Hölder continuous function, then there exists a strong Feller process $(X_t)_{t\geq 0}$ with symbol

$$q(x,\xi) := \tilde{\psi}_{\beta(x)}(\xi) = \int_I |\xi|^\alpha f(\alpha,\beta(x))\,d\alpha, \qquad x,\xi \in \mathbb{R}^d.$$

The process $(X_t)_{t\geq 0}$ has the following properties:

- *The transition probability $p : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with $\gamma_0(\beta(x)) = \alpha^L$, $\gamma_\infty(\beta(x)) = \alpha^U$ and $m = 0$.*
- *$C_c^\infty(\mathbb{R}^d)$ is a core for the generator $(L, \mathcal{D}(L))$ of $(X_t)_{t\geq 0}$ and $C_\infty^2(\mathbb{R}^d) \subseteq \mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.*
- *$(X_t)_{t\geq 0}$ is the unique solution to the $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem.*

It follows from the well-known identity

$$|\xi|^\alpha = c_{\alpha,d} \int_{\mathbb{R}^d} (1-\cos(y\cdot\xi)) \frac{1}{|y|^{d+\alpha}}\,dy, \qquad \xi \in \mathbb{R}^d, \alpha \in (0,2),$$

that

$$\tilde{\psi}_\beta(\xi) = \int_{\mathbb{R}^d} (1-\cos(y\cdot\xi)) \left(\int_I c_{\alpha,d} f(\alpha,\beta) \frac{1}{|y|^{d+\alpha}}\,d\alpha \right) dy,$$

for a certain normalizing constant $c_{\alpha,d}$. This shows that the Lévy measure associated with the continuous negative definite function $\tilde{\psi}_\beta$ is given by

$$\tilde{\nu}_\beta(dy) = \left(\int_I c_{\alpha,d} f(\alpha,\beta) \frac{1}{|y|^{d+\alpha}}\, d\alpha\right) dy.$$

Jump processes of such a mixed type have been studied in a more general framework by Chen, Kim & Kumagai [23]. Let us remark that Proposition 5.16 is not a particular case of [23]; Chen, Kim & Kumagai prove the existence of certain Markov processes of mixed type, but they do not investigate whether the associated semigroup is Feller.

Proof of Proposition 5.16. Throughout this proof, we denote by $\Omega := \Omega(0,\pi/8)$ the domain defined in (3.1). We have already seen in the proof of Theorem 5.2 that $\psi_\alpha(\xi) := |\xi|^\alpha$, $\alpha \in I$, satisfies (C1)-(C4) on Ω. By (C1) (for $(\psi_\alpha)_{\alpha\in I}$), there exists for each $\alpha \in I$ a holomorphic mapping $\Psi_\alpha : \Omega \to \mathbb{C}$ such that $\psi_\alpha(\xi) = \Psi_\alpha(|\xi|)$. The assertion follows from Extension 3.10 if we can show that $(\tilde{\psi}_\beta)_{\beta\in J}$ satisfies (C1),(C2),(C3') and (C4).

(C1)&(C2) Define $\tilde{\Psi}_\beta(z) := \int_I \Psi_\alpha(z) f(\alpha,\beta)\, d\alpha$ for $z \in \Omega$. Clearly, $\tilde{\Psi}_\beta$ is holomorphic on Ω and

$$\tilde{\psi}_\beta(\xi) = \int_I \Psi_\alpha(|\xi|) f(\alpha,\beta)\, d\alpha = \tilde{\Psi}_\beta(|\xi|), \qquad \xi \in \mathbb{R}^d.$$

(C3') Using (C2) (for $(\psi_\alpha)_{\alpha\in I}$), we find

$$\begin{aligned}\operatorname{Re} \tilde{\Psi}_\beta(z) = \int_I \operatorname{Re} \Psi_\alpha(z) f(\alpha,\beta)\, d\alpha &\geq c \int_{[\alpha^L,\alpha^U]} |\operatorname{Re} z|^\alpha f(\alpha,\beta)\, d\alpha \\ &\geq c f^L \int_{[\alpha^L,\alpha^U]} |\operatorname{Re} z|^\alpha\, d\alpha \\ &= c f^L \frac{|\operatorname{Re} z|^{\alpha^U} - |\operatorname{Re} z|^{\alpha^L}}{\log|\operatorname{Re} z|} \\ &\geq c' \frac{|\operatorname{Re} z|^{\alpha^U}}{\log|\operatorname{Re} z|}\end{aligned}$$

for all $z \in \Omega$, $|z| \gg 1$. Similarly, the boundedness of f implies

$$|\tilde{\Psi}_\beta(z)| \leq \int_I |\Psi_\alpha(z)|\, f(\alpha,\beta)\, d\alpha \leq C\|f\|_\infty \int_{[\alpha^L,\alpha^U]} |z|^\alpha\, d\alpha$$

for all $z \in \Omega$. Hence,

$$|\tilde{\Psi}_\beta(z)| \leq C\|f\|_\infty (\alpha^U - \alpha^L)\big(\mathbb{1}_{\{|z|\leq 1\}} |z|^{\alpha^L} + \mathbb{1}_{\{|z|>1\}} |z|^{\alpha^U}\big), \qquad z \in \Omega.$$

(C4) Since f and $\partial_\beta f$ are bounded, we may apply the differentiation lemma for parameter-dependent integrals to interchange differentiation and integration. Then, it follows exactly as in the previous step of the proof that

$$|\partial_\beta \tilde{\Psi}_\beta(z)| \leq C'' \mathbb{1}_{\{|z|\leq 1\}} |z|^{\alpha^L} + C'' \mathbb{1}_{\{|z|>1\}} |z|^{\alpha^U}, \qquad z \in \Omega. \qquad \square$$

Remarks (i) Under the additional assumptions that $d = 1$, $\beta(\cdot) \in C_b^2(\mathbb{R})$ and $f(\alpha, \cdot)$ is twice partially differentiable,

$$\sup_{j \in \{1,\dots,n\}} \sup_{(\alpha,\beta) \in I \times J} |\partial^2_{\beta_j} f(\alpha, \beta)| < \infty,$$

it follows easily from Theorem 3.8 and Corollary 3.9 that the transition density p has a derivative with respect to x which satisfies the heat kernel estimates from Theorem 3.8 and that $(X_t)_{t \geq 0}$ is irreducible with respect to Lebesgue measure.

(ii) The proof of Proposition 5.16 actually shows the following more general statement: Let I, f, $\beta(\cdot)$ be as in Proposition 5.16 and $(\psi_\alpha)_{\alpha \in I} \in A(m)$ for some $m \geq 0$ such that $\gamma_\infty(\alpha) = \alpha$ for all $\alpha \in I$. If we define

$$q(x, \xi) = \int_I \psi_\alpha(\xi) f(\alpha, \beta(x)) \, d\alpha, \qquad x, \xi \in \mathbb{R}^d,$$

then there exists a strong Feller process with symbol q and the process has the properties listed in Proposition 5.16.

5.4 Solutions of Lévy-driven SDEs with Hölder continuous coefficients

For a Lévy process $(L_t)_{t \geq 0}$ we call the SDE

$$dX_t = b(X_{t-}) \, dt + \sigma(X_{t-}) \, dL_t, \qquad X_0 = x$$

a *Lévy-driven SDE*. If b and σ are bounded Lipschitz continuous functions, then it is well-known that a unique (strong) solution to the SDE exists and that the solution is a Feller process, cf. Schilling & Schnurr [99]. It is natural to ask whether we can weaken the assumption on the regularity of b and σ, i.e. to ask whether the solution (if it exists) is still a Feller process if we replace "Lipschitz continuity" by "Hölder continuity". There are some results on existence and uniqueness of (weak) solutions of Lévy-driven SDEs with Hölder continuous coefficients (see e.g. Priola [82], Huang [45] or the recent publication [26] by Chen & Zhang), but only few authors address the question whether the solution is a Feller process. One of those publications is the work by Knopova & Kulik [57] where the particular case that $(L_t)_{t \geq 0}$ is a symmetric α-stable Lévy process is discussed in full detail. Our aim is to present sufficient conditions on the Lévy process $(L_t)_{t \geq 0}$ which ensure that the solution to a Lévy-driven SDE with Hölder continuous coefficients (exists and) is a Feller process.
Throughout this section, we have to restrict ourselves to one-dimensional Lévy processes $(L_t)_{t \geq 0}$ because the symbols which we will encounter are not rotationally invariant. Let us remark that the results presented in this section can be extended to dimension $d > 1$ if we succeed in proving Conjecture 3.12 (see Section 3.2).

We start with the following abstract result.

5.17 Proposition *Let $I \subseteq \mathbb{R}^n$ be a convex open set and $\psi_\alpha : \mathbb{R} \to \mathbb{C}$, $\alpha \in I$, a family of continuous negative definite functions satisfying* (C2)-(C4) *for some $m \geq 0$. Suppose additionally that*

$$\gamma_\infty^L := \inf_{\alpha \in I} \gamma_\infty(\alpha) > 1$$

and there exists a constant $c > 0$ such that

$$\left| \frac{\partial}{\partial z} \psi_\alpha(z) \right| \leq c \mathbb{1}_{\{|z| \leq 1\}} + c|z|^{\gamma_\infty(\alpha)-1} \mathbb{1}_{\{|z|>1\}} \qquad \text{for all } z \in \Omega(m), \alpha \in I, \tag{5.11}$$

cf. (C2),(C3) *for the definition of $\gamma_\infty(\alpha)$ and $\Omega(m)$. Let $\alpha : \mathbb{R} \to I$, $b : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}$ be Hölder continuous functions such that $\|b\|_\infty < \infty$ and*

$$0 < \sigma^L := \inf_{x \in \mathbb{R}} |\sigma(x)| \leq \sup_{x \in \mathbb{R}} |\sigma(x)| =: \sigma^U < \infty.$$

Then there exists a strong Feller process $(X_t)_{t \geq 0}$ with symbol

$$q(x, \xi) = -ib(x)\xi + \psi_{\alpha(x)}(\sigma(x)\xi), \qquad x, \xi \in \mathbb{R}.$$

The process $(X_t)_{t \geq 0}$ has the following additional properties:

- *The transition probability $p : (0, \infty) \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with $\tilde{\gamma}_0(b(x), \sigma(x), \alpha(x)) = \min\{\gamma_0(\alpha(x)), 1\}$, $\tilde{\gamma}_\infty(b(x), \sigma(x), \alpha(x)) = \gamma_\infty(\alpha(x))$ and any $\tilde{m} \in (0, m/\sigma^U)$.*
- *$C_c^\infty(\mathbb{R})$ is a core for the generator $(L, \mathcal{D}(L))$ of $(X_t)_{t \geq 0}$ and $C_\infty^2(\mathbb{R}) \subseteq \mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.*
- *$(X_t)_{t \geq 0}$ is the unique solution to the $(L, C_c^\infty(\mathbb{R}))$-martingale problem.*

Proof. This is an obvious consequence of Theorem 3.7. □

In Table 5.1 we have listed symbols which satisfy (C2)-(C4). Consequently, we may apply Theorem 5.17 to any of those symbols provided that the additional assumptions $\gamma_\infty^L > 1$ and (5.11) hold true.

5.18 Remarks (i) If $b = 0$ we may drop the assumption $\gamma_\infty^L > 1$. Moreover, if there is no drift part, we can easily extend Proposition 5.17 to dimension $d > 1$ under the additional assumption that ψ_α is rotationally invariant for each $\alpha \in I$ and $\sigma(x)$ is a rotation matrix for all $x \in \mathbb{R}^d$.

(ii) (5.11) holds for $z \in \Omega(m)$, $|z| \gg 1$ if

$$\left| \frac{\partial}{\partial z} \psi_\alpha(z) \right| \leq c \frac{|\psi_\alpha(z)|}{|z|}, \qquad |z| \gg 1, z \in \Omega(m)$$

which is a quite natural assumption. Since

$$f'(\lambda) \leq \frac{f(\lambda)}{\lambda}, \qquad \lambda > 0,$$

holds for any Bernstein function f (this is a direct consequence of the Lévy-Khintchine representation (1.8) and the elementary estimate $\lambda^{-1}(1-e^{-\lambda r}) \geq e^{-\lambda r} r$), we find that (5.11) is in particular satisfied if $\psi_\alpha(\xi) = f_\alpha(|\xi|^2)$ for a Bernstein function f_α.

(iii) Naively, one might think that $(X_t)_{t\geq 0}$ is a solution to a "Feller-driven" SDE

$$dX_t = b(X_{t-})\,dt + \sigma(X_{t-})\,dL_t, \qquad X_0 = x,$$

where $(L_t)_{t\geq 0}$ is a Feller process with symbol $p(x,\xi) := \psi_{\alpha(x)}(\xi)$. Note, however, we cannot expect to make this rigorous since it is well-known that the solution of such an SDE is, in general, not Markovian unless $(L_t)_{t\geq 0}$ is a Lévy process (see [83, Section V.6]).

For the particular case that I consists of a single element, Proposition 5.17 yields the following existence and uniqueness result for Lévy-driven SDEs.

5.19 Corollary *Let $(L_t)_{t\geq 0}$ be a one-dimensional Lévy process such that its characteristic exponent $\psi : \mathbb{R} \to \mathbb{C}$ satisfies the following conditions:*

(C2) *ψ has a holomorphic extension to*

$$\Omega := \Omega(m^L, \vartheta) := \{z \in \mathbb{C}; |\operatorname{Im} z| < m^L\} \cup \{z \in \mathbb{C}\backslash\{0\}; \arg z \in (-\vartheta, \vartheta) \cup (\pi - \vartheta, \pi + \vartheta)\}$$

for some $m^L \geq 0$ and $\vartheta \in (0, \pi/2)$ (see Figure 3.1).

(C3) *There exist $\alpha \in (0,2]$, $\beta \in (1,2)$ and constants $c_1, c_2 > 0$ such that*

$$\operatorname{Re}\psi(z) \geq c_1 |\operatorname{Re} z|^\beta \qquad \text{for all } |z| \gg 1, z \in \Omega$$

and

$$|\psi(z)| \leq c_2 \begin{cases} |z|^\alpha, & |z| \ll 1, z \in \Omega, \\ |z|^\beta, & |z| \gg 1, z \in \Omega. \end{cases}$$

(C6) *There exists a constant $c_3 > 0$ such that $|\psi'(z)| \leq c_3 |z|^{\beta-1}$ for all $z \in \Omega$, $|z| \gg 1$.*

Let $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ be Hölder continuous functions such that $\|b\|_\infty < \infty$ and

$$0 < \sigma^L := \inf_{x \in \mathbb{R}} |\sigma(x)| \leq \sup_{x \in \mathbb{R}} |\sigma(x)| =: \sigma^U < \infty.$$

Then there exists a unique weak solution to the SDE

$$dX_t = b(X_{t-})\,dt + \sigma(X_{t-})\,dL_t, \qquad X_0 = x. \tag{5.12}$$

The solution is a strong Feller process with symbol $q(x,\xi) = -ib(x)\xi + \psi(\sigma(x)\xi)$ and it has following additional properties:

(i) The transition probability $p : (0,\infty) \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with $\gamma_0(b(x), \sigma(x)) = \min\{\alpha, 1\}$, $\gamma_\infty(b(x), \sigma(x)) = \beta$ and any $m \in (0, m^L/\sigma^U)$.

(ii) $C_c^\infty(\mathbb{R})$ *is a core for the generator* $(L, \mathcal{D}(L))$ *of* $(X_t)_{t\geq 0}$ *and* $C_\infty^2(\mathbb{R}) \subseteq \mathcal{D}(L)$*. Moreover,* p *is a fundamental solution to the Cauchy problem for the operator* $\partial_t - L$*.*

(iii) $(X_t)_{t\geq 0}$ *is the unique solution to the* $(L, C_c^\infty(\mathbb{R}^d))$*-martingale problem.*

Corollary 5.19 covers, in particular, Feller processes with a generator of the form $Lf(x) = b(x)\nabla f(x) + Af(x)$ where A is a "good" Lévy generator. Starting with the work by Tanaka & Tsuchiya [107] such gradient perturbations of Lévy generators have received a lot of attention, see e. g. Biler et. al [9], Bogdan & Jakubowski [13] and Jakubowski & Szczypkowski [55] ($A = \Delta^{\alpha/2}$), Veretennikov [109], Priola [82] and Nourdin & Simon [77] (pathwise uniqueness of the SDE (5.12) and regularizing effects of the Lévy noise).

Proof of Corollary 5.19. Applying Proposition 5.17 we find that there exists a strong Feller process $(X_t)_{t\geq 0}$ with symbol $q(x,\xi) = -ib(x)\xi + \psi(\sigma(x)\xi)$ which satisfies (i)-(iii). It just remains to prove that it is a weak solution to (5.12) and that it is the unique weak solution (in law). Since $(X_t)_{t\geq 0}$ is, by (iii), a solution to the $(L, C_\infty^2(\mathbb{R}))$-martingale problem and $C_c^2(\mathbb{R}) \subseteq \mathcal{D}(L)$, it follows from [63, Theorem 2.3] that $(X_t)_{t\geq 0}$ is a weak solution to the SDE (5.12). On the other hand, it is not difficult to see from Itô's formula that any weak solution to (5.12) is a solution to the $(L, C_\infty^2(\mathbb{R}))$-martingale problem. Since we already proved that the martingale problem has a unique solution, this shows the uniqueness of the weak solution. □

Finally we apply Corollary 5.19 to some of the symbols listed in Table 5.1.

5.20 Example Let $(L_t)_{t\geq 0}$ be one of the following Lévy processes:

(a) symmetric α-stable Lévy process with characteristic exponent $\psi(\xi) = |\xi|^\alpha$, $\alpha > 1$.

(b) relativistic stable Lévy process with characteristic exponent $\psi(\xi) = (|\xi|^2 + \varrho^2)^{\alpha/2} - \varrho^\alpha$, $\alpha > 1$, $\varrho > 0$.

(c) truncated Lévy process with char. exp. $\psi(\xi) = (|\xi|^2 + \varrho^2)^{\alpha/2} \cos(\alpha \arctan \frac{|\xi|}{\varrho}) - \varrho^\alpha$, $\alpha > 1$, $\varrho > 0$.

(d) Lamperti stable Lévy process with characteristic exponent $\psi(\xi) = (|\xi|^2 + \varrho)_\alpha - (\varrho)_\alpha$, $\alpha > 1$, $\varrho > 0$; here $(\lambda)_\alpha := \Gamma(\lambda + \alpha)/\Gamma(\lambda)$ denotes the Pochhammer symbol.

(e) normal tempered stable Lévy process with characteristic exponent $\psi(\xi) = (\kappa^2 + (\xi - i\beta)^2)^{\alpha/2} - (\kappa^2 - \beta^2)^{\alpha/2}$, $\alpha > 1$, $\kappa > 0$, $\beta \in [-\kappa, \kappa]$.

(f) Brownian motion.

Let $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ be Hölder continuous functions such that $\|b\|_\infty < \infty$ and

$$0 < \sigma^L := \inf_{x\in\mathbb{R}} |\sigma(x)| \leq \sup_{x\in\mathbb{R}} |\sigma(x)| =: \sigma^U < \infty.$$

Then there exists a unique weak solution to the SDE

$$dX_t = b(X_{t-})\,dt + \sigma(X_{t-})\,dL_t, \qquad X_0 = x.$$

The solution is a strong Feller process with symbol $q(x,\xi) = -ib(x)\xi + \psi(\sigma(x)\xi)$ and it has following additional properties:

(i) The transition probability $p : (0,\infty) \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ is continuous, differentiable with respect to t and satisfies the heat kernel estimates from Theorem 3.6 with

  (a) $\gamma_0(b(x),\sigma(x)) = 1$, $\gamma_\infty(b(x),\sigma(x)) = \alpha$, $m = 0$,

  (b),(c) $\gamma_0(b(x),\sigma(x)) = 1$, $\gamma_\infty(b(x),\sigma(x)) = \alpha$, $m \in (0, \varrho/\sigma^U)$,

  (d) $\gamma_0(b(x),\sigma(x)) = 1$, $\gamma_\infty(b(x),\sigma(x)) = \alpha$, $m \in (0, \sqrt{\varrho}/\sigma^U)$,

  (e) $\gamma_0(b(x),\sigma(x)) = 1$, $\gamma_\infty(b(x),\sigma(x) = \alpha$, $m \in (0, (\kappa-\beta)/\sigma^U)$.

  (f) $\gamma_0(b(x),\sigma(x)) = 1$, $\gamma_\infty(b(x),\sigma(x)) = 2$, $m > 0$.

(ii) $C_c^\infty(\mathbb{R}) \subseteq \mathcal{D}(L)$ is a core for the generator $(L,\mathcal{D}(L))$ of $(X_t)_{t\geq 0}$ and $C_\infty^2(\mathbb{R}) \subseteq \mathcal{D}(L)$. Moreover, p is a fundamental solution to the Cauchy problem for the operator $\partial_t - L$.

(iii) $(X_t)_{t\geq 0}$ is the unique solution to the $(L, C_\infty^2(\mathbb{R}))$-martingale problem.

Proof. We have already seen in Section 5.1 that the characteristic exponents in (a)-(e) satisfy the conditions (C2),(C3) from Corollary 5.19; for the Brownian motion this is obvious. Moreover, by calculating the derivative ψ' explicitly (or applying Remark 5.18(ii)), it follows easily that the growth condition (C6) is also satisfied. Therefore, the assertion follows from Corollary 5.19. □

5.21 Remarks (i) If Conjecture 3.12 holds true, then we can extend Corollary 5.19 and Example 5.20 to dimension $d \geq 1$ and, moreover, we can weaken the assumptions on the growth of ψ (e. g. in the above examples we can replace the assumption $\alpha > 1$ by the balance condition $\alpha(1+\varrho(b)) > 1$; here $\varrho(b)$ denotes the Hölder coefficient of the drift coefficient b).

(ii) To our knowledge only the symmetric α-stable case has been discussed in the literature before (see e. g. Knopova & Kulik [57]); even for the case that $(L_t)_{t\geq 0}$ is a Brownian motion we could not find a reference. [6]

(iii) Huang [45] considers SDEs driven by a class of tempered stable (dominated) Lévy processes. Using a parametrix construction, he has recently established the existence of a unique weak solution and derived heat kernel estimates for the solution under the additional assumption that b is Lipschitz continuous. His proof is based on heat kernel estimates for tempered stable Lévy processes derived by Sztonyk [105]. It

[6] For a Brownian motion $(L_t)_{t\geq 0}$ it *is* known that the SDE has a unique weak solution (this follows e. g. by combining [104, Theorem 5.6] and [56, Corollary 4.8]), but not that the solution is a Feller process.

would be of interest to generalize this approach and prove a more general statement along the following lines:

Let $(L_t)_{t\geq 0}$ be a Lévy process with characteristic exponent ψ such that the density

$$p_t(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t\psi(\xi)} e^{-ix\cdot\xi}\, d\xi, \qquad x \in \mathbb{R}^d$$

exists for all $t > 0$ and that p_t satisfies "good" heat kernel estimates. Moreover, let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ be Hölder continuous functions with Hölder exponent $\varrho(\alpha)$ and $\varrho(b)$, respectively, such that $\|b\|_\infty < \infty$ and

$$0 < \sigma^L := \inf_{x\in\mathbb{R}} |\sigma(x)| \leq \sup_{x\in\mathbb{R}} |\sigma(x)| =: \sigma^U < \infty.$$

If $\varrho(b)$ is "sufficiently large", then there exists a unique weak solution to the SDE

$$dX_t = b(X_{t-})\, dt + \sigma(X_{t-})\, dL_t, \qquad X_0 = x.$$

The solution is a Feller process with symbol $q(x,\xi) = -ib(x)\xi + \psi(\sigma(x)\xi)$.

Note that, for our parametrix construction, we do not only need heat kernel estimates for p_t, but all the bounds listed in Corollary 4.11 – and these are, in general, hard to verify.

A
Appendix

A.1 Slowly varying functions

In this section we collect some results on slowly varying functions. In particular, we state the Karamata–Tauberian theorem the way we use it in Section 4.1. Our standard reference is the monograph by Bingham, Goldie & Teugels [11].

A.1 Definition A function $\ell : (0,\infty) \to (0,\infty)$ is *slowly varying (at infinity)* if

$$\lim_{x\to\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1 \qquad \text{for all} \quad \lambda > 0.$$

$L : (0,\infty) \to (0,\infty)$ is called *regularly varying (at infinity) of index* $\alpha > 0$ if

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = \lambda^\alpha \qquad \text{for all} \quad \lambda > 0.$$

Any regularly varying function L of index $\alpha > 0$ admits a representation of the form

$$L(x) = x^\alpha \ell(x)$$

for a slowly varying function ℓ, see [11, Theorem 1.4.1]. There is the following statement on the growth of slowly varying functions.

A.2 Lemma *Let* $\ell : \mathbb{R} \to [0,\infty)$ *be a locally bounded slowly varying function. Then*

$$x^{-\alpha}\ell(x) \xrightarrow{x\to\infty} 0 \qquad \textit{for all} \quad \alpha > 0.$$

Proof. For fixed $\alpha > 0$ choose $\varepsilon > 0$ sufficiently small such that $1 + \varepsilon < 2^\alpha$. Since

$$\left| \frac{\ell(2x)}{\ell(x)} - 1 \right| \le \varepsilon \qquad \text{for all} \quad x \ge R = R(\varepsilon)$$

we find $\ell(2x) \le (1+\varepsilon)\ell(x)$ for all $x \ge R$. By iteration, this yields $\ell(2^n x) \le (1+\varepsilon)^n \ell(x)$ for all $x \ge R$ and $n \in \mathbb{N}$. For $y \in [2^n R, 2^{n+1}R]$ we have $x := 2^{-n}y \in [R, 2R]$, and therefore

$$\sup_{y\in[2^nR,2^{n+1}R]} \ell(y) \le (1+\varepsilon)^n \sup_{x\in[R,2R]} \ell(x).$$

Consequently,

$$y^{-\alpha}\ell(y) \le (2^nR)^{-\alpha} \sup_{y\in[2^nR,2^{n+1}R]} \ell(y) \le \left(\frac{1+\varepsilon}{2^\alpha}\right)^n \sup_{x\in[R,2R]} \ell(x)$$

for all $y \in [2^n R, 2^{n+1} R]$. Since the right-hand side converges to 0 as $n \to \infty$, this finishes the proof. □

Lemma A.2 shows that slowly varying functions are, compared to power functions, negligible. This also motivates the following theorem.

A.3 Theorem (Karamata–Tauberian theorem) *Let ℓ be increasing and slowly varying (at infinity). Then the limit*

$$\lim_{t\to 0} \frac{\int_{(0,\infty)} r^\beta \ell(r)^k e^{-tr^\alpha}\, dr}{\ell(t^{-1/\alpha})^k t^{-(\beta+1)/\alpha}}$$

exists for each $\alpha, \beta > 0$ and $k \in \mathbb{N}_0$.

Since

$$\int_{(0,\infty)} r^\beta e^{-tr^\alpha}\, dr = t^{-(\beta+1)/\alpha}$$

the Karamata–Tauberian theorem gives

$$\int_{(0,\infty)} r^\beta \ell(r)^k e^{-tr^\alpha}\, dr \approx \ell(t^{-1/\alpha})^k \int_{(0,\infty)} r^\beta e^{-tr^\alpha}\, dr$$

for small $t > 0$. This means, roughly, that we can pull the slowly varying function ℓ outside the integral and then compute the integral as usual.

Proof of Theorem A.3. For a function g denote by

$$(Lg)(s) := s \int_{(0,\infty)} e^{-sr} g(r)\, dr$$

the Laplace–Stieltjes transform of g (cf. [11, p. 37]). By the "classical" Karamata–Tauberian Theorem, cf. [11, Theorem 1.7.1], the following statements are equivalent for any increasing function g, slowly varying function $\tilde{\ell}$ and $\varrho \in \mathbb{R}$:

(i) The limit

$$\lim_{r\to\infty} \frac{g(r)}{r^\varrho \tilde{l}(r)}$$

exists.

(ii) The limit

$$\lim_{t\to 0} \frac{(Lg)(t)}{t^{-\varrho} \tilde{\ell}(1/t)}$$

exists.

If we choose

$$\varrho := \frac{\beta+1}{\alpha} - 1 \quad \tilde{\ell}(x) := \ell(r^{1/\alpha})^k \quad g(x) := x^\varrho \tilde{\ell}(x)$$

then (i) is trivially satisfied. Since

$$\begin{aligned} (Lg)(t) &= t \int_{(0,\infty)} e^{-tr} x^{\frac{\beta+1}{\alpha}-1} \ell(r^{1/\alpha})^k\, dr \\ &\overset{u^\alpha = r}{=} \alpha t \int_{(0,\infty)} e^{-tu^\alpha} u^\beta \ell(u)^k\, du \end{aligned}$$

the assertion follows directly from the implication "$(i) \implies (ii)$". □

A.2 Auxiliary results

A.4 Lemma *(i) For $m > 0$ and $z_0 \in \mathbb{C}$, $|z_0| = 1$, with $\operatorname{Re} z_0 > 0$, $\operatorname{Im} z_0 > 0$ we set $\gamma_R(r) := -i\frac{m}{2} - rz_0$, $r \in [0, R]$. Then $\operatorname{Re}\left(\int_{\gamma_R}(ir)^{(d-1)/2+k}W_{0,d/2-1}(2ir|x|)\,dr\right) \xrightarrow{R\to\infty} 0$ for any $k \geq 0$ and $x \neq 0$.*

(ii) Set $\gamma_R(r) := re^{-i\Theta}$,$r \in [R^{-1}, R]$, for fixed $\Theta \in (0, \frac{\pi}{2})$. Then

$$\operatorname{Re}\left(\int_{\gamma_R}(ir)^{(d-1)/2+k}W_{0,d/2-1}(2ir)\,dr\right) \xrightarrow{R\to\infty} 0$$

for any $k \geq 0$ and $x \neq 0$.

Proof. We only prove (i); the proof of the second statement is very similar. For fixed $R > 0$ define $\mu_1(r) := -ir$, $r \in [\frac{m}{2}, R]$ and $\mu_2(r) = -iR + r$, $r \in [0, h(R)]$ where $h(R)$ is chosen such that $\gamma_R(R) = \mu_2(h(R))$. Note that $h(R) \asymp R$ for $R \gg 1$. Obviously,

$$\operatorname{Re}\left(\int_{\mu_1}(ir)^{(d-1)/2+k}W_{0,d/2-1}(2ir|x|)\,dr\right) = \operatorname{Re}\left(-i\int_{m/2}^{R} r^{(d-1)/2+k}W_{0,d/2-1}(2r|x|)\,dr\right) = 0.$$

On the other hand, applying (4.27) and using $|\mu_2(r)| \leq 2R$ we get

$$\begin{aligned}\left|\int_{\mu_2}(ir)^{(d-1)/2+k}W_{0,d/2-1}(2ir|x|)\,dr\right| &\leq \int_0^{h(R)} |\mu_2(r)|^{(d-1)/2+k}|W_{0,d/2-1}(2i|x|\mu_2(r))|\,dr\\ &\leq c(2R)^{(d-1)/2+k}e^{-R|x|}R \xrightarrow{R\to\infty} 0.\end{aligned}$$

By Cauchy's theorem,

$$\begin{aligned}\operatorname{Re}\left(\int_{\gamma_R}(ir)^{(d-1)/2}W_{0,d/2-1}(2ir|x|)\,dr\right) &= \operatorname{Re}\left(\int_{\mu_1}+\int_{\mu_2}(ir)^{(d-1)/2}W_{0,d/2-1}(2ir|x|)\,dr\right)\\ &\xrightarrow{R\to\infty} 0.\end{aligned}$$

□

A.5 Lemma *The function*

$$f(t) := f(t,y) := t^{1+d/\gamma_\infty(\alpha(y))}t^{-(d+\gamma_\infty(\alpha(y)))/\gamma_\infty^U(y,r(t))}$$

satisfies (4.40), *i. e.*

$$0 < \inf_{y\in\mathbb{R}^d}\inf_{t\in(0,1]} f(t,y) \leq \sup_{y\in\mathbb{R}^d}\sup_{t\in(0,1]} f(t,y) < \infty.$$

Proof. Recall that $r(t) := t^{1/\gamma_\infty^U}$,

$$\gamma_\infty^U(y,r) := \sup\{\gamma_\infty(\alpha(y)); y \in B[y,r]\}$$

and that $\gamma_\infty \circ \alpha$ is Hölder continuous with exponent $\varrho \in (0,1]$. Obviously,

$$f(t) = \exp\left(-\left[-1 + \frac{\gamma_\infty(\alpha(y))}{\gamma_\infty^U(y,r(t))} - \frac{d}{\gamma_\infty(\alpha(y))} + \frac{d}{\gamma_\infty^U(y,r(t))}\right]\log t\right).$$

Because of the Hölder continuity of $\gamma_\infty(\alpha(\cdot))$, we have

$$\begin{aligned} 0 \le \gamma_\infty^U(y,r(t)) - \gamma_\infty(\alpha(y)) &= \sup_{y\in B[y,r(t)]} |\gamma_\infty(\alpha(y)) - \gamma_\infty(\alpha(y))| \\ &\le Cr(t)^\varrho = Ct^{\varrho/\gamma_\infty^U}. \end{aligned} \tag{$\star$}$$

Consequently, we find

$$\begin{aligned} 0 \le \frac{d}{\gamma_\infty(\alpha(y))} - \frac{d}{\gamma_\infty^U(y,r(t))} &= d\frac{\gamma_\infty^U(y,r(t)) - \gamma_\infty(\alpha(y))}{\gamma_\infty^U(y,r(t))\gamma_\infty(\alpha(y))} \\ &\overset{(\star)}{\le} d\frac{Ct^{\varrho/\gamma_\infty^U}}{(\gamma_\infty^L)^2} \end{aligned}$$

and

$$0 \le 1 - \frac{\gamma_\infty(\alpha(y))}{\gamma_\infty^U(y,r(t))} = \frac{\gamma_\infty^U(y,r(t)) - \gamma_\infty(\alpha(y))}{\gamma_\infty^U(y,r(t))} \overset{(\star)}{\le} \frac{Ct^{\varrho/\gamma_\infty^U}}{\gamma_\infty^L}.$$

Hence,

$$\exp\left(-\left[d\frac{C}{(\gamma_\infty^L)^2} + \frac{C}{m_\infty}\right]t^{\varrho/\gamma_\infty^U}\log t\right) \ge f(t) \ge \exp\left(\left[d\frac{C}{(\gamma_\infty^L)^2} + \frac{C}{m_\infty}\right]t^{\varrho/\gamma_\infty^U}\log t\right).$$

Since the lower and the upper bound are continuous and strictly positive on $(0,\infty)$ and converge to 1 as $t \to 0$, this finishes the proof. □

A.6 Lemma *For any $c > 0$ and $\kappa > 0$ there exists $N \in \mathbb{N}$ such that $\Gamma(n\kappa) \ge c^n$ for all $n \ge N$.*

Proof. If $c > 1$, then the elementary estimate

$$\Gamma(n\kappa) \ge \int_{c^{2/\kappa}}^\infty e^{-x}x^{n\kappa-1}\,dx \ge (c^{2/\kappa})^{n\kappa-1}\int_{c^{2/\kappa}}^\infty e^{-x}\,dx = c^{2n-2/\kappa}\int_{c^{2/\kappa}}^\infty e^{-x}\,dx$$

shows that $\Gamma(n\kappa) \ge c^n$ for n sufficiently large. On the other hand if $c \in (0,1]$, then

$$\Gamma(n\kappa) \ge \int_0^\infty e^{-x}\,dx = 1 \ge c^n \qquad \text{for all } \ n \in \mathbb{N}.$$ □

A.7 Lemma *The functions*

$$u_1(s,t,x,z,y) := G(t(1-s),x,y)G(ts,z,y)$$

and

$$u_k(s,t,x,z,y) := H^{k-1}(t(1-s),x,z)G(ts,z,y), \qquad k \ge 2,$$

defined in the proof of Theorem 4.28 satisfy

$$I_k := \int_0^1 \int \sup_{t\in[T_0,T]} u_k(s,t,x,z,y)\,dz\,ds < \infty$$

for all $k \in \mathbb{N}$, $0 < T_0 < T < \infty$ and $x,y \in \mathbb{R}^d$.

Proof. By the definition of S, cf. (4.9), and (4.7) we have

$$
\begin{aligned}
S(z,\alpha,tr) &= \exp\left(-\frac{m}{4}|z|\right)\begin{cases}\min\left\{(tr)^{-d/\gamma_\infty(\alpha)}, \frac{tr}{|z|^{d+\gamma_\infty(\alpha)}}\right\}, & |z|\le 1,\\ \frac{tr}{|z|^{d+\gamma_\infty(\alpha)\wedge\gamma_0(\alpha)}}, & |z|>1\end{cases}\\
&\le \max\{T, T_0^{-d/\gamma_\infty^L}\}\exp\left(-\frac{m}{4}|z|\right)\begin{cases}\min\left\{r^{-d/\gamma_\infty(\alpha)}, \frac{r}{|z|^{d+\gamma_\infty(\alpha)}}\right\}, & |z|\le 1,\\ \frac{r}{|z|^{d+\gamma_\infty(\alpha)\wedge\gamma_0(\alpha)}}, & |z|>1\end{cases}\\
&= \max\{T, T_0^{-d/\gamma_\infty^L}\}S(z,\alpha,r) =: C_S(T_0,T)S(z,\alpha,r)
\end{aligned}
$$

for all $t\in[T_0,T]$, $r\ge 0$ and $z\in\mathbb{R}^d$. Consequently,

$$
\begin{aligned}
G(tr,z,y) &= C(tr)^{-1+\kappa}S(z-y,\alpha(y),tr) + C(tr)^{-1+\kappa}g_{\gamma_0^L\wedge\gamma_\infty^L}(z-y)\\
&\le C_S(T_0,T)\max\{T_0^{-1+\kappa}, T^{-1+\kappa}\}r^{-1+\kappa}S(z-y,\alpha(y),r)\\
&\quad + Cr^{-1+\kappa}\max\{T_0^{-1+\kappa}, T^{-1+\kappa}\}g_{\gamma_0^L\wedge\gamma_\infty^L}(z-y)\\
&\le C_G(T_0,T)G(r,z,y)
\end{aligned}
$$

for all $t\in[T_0,T]$, $r\ge 0$ and $y,z\in\mathbb{R}^d$. A very similar calculation shows

$$H^{k-1}(tr,z,y)\le C_{k-1}(T_0,T)H^{k-1}(t,z,y)$$

for a suitable constant $C_{k-1}(T_0,T)$ and $t\in[T_0,T]$. Applying Lemma 4.24, we conclude

$$I_1\le C_G^2\int_0^1\int G(1-s,x,y)G(s,z,y)\,dz\,ds = C_G^2\,(G\circledast G)(1,x,y)\overset{L4.24}{\le} C_G^2H^1(1,x,y)<\infty$$

and

$$I_k\le C_G\,C_{k-1}(H^{k-1}\circledast G)(1,x,y)\overset{L4.24}{\le} C_{k-1}\,C_GH^k(1,x,y)<\infty. \qquad \square$$

A.8 Lemma *(i)* $\sup_{x\in\mathbb{R}^d}\sup_{t\le T} t\int G(t,x,y)\,dy<\infty$

(ii) $t\sup_{x\in\mathbb{R}^d}\int G(t,x,y)\,dy\xrightarrow{t\to 0}0.$

(iii) $\sup_{x\in\mathbb{R}^d}\sup_{t\le T}\int_0^t\int G(s,x,y)\,dy\,ds<\infty.$

(iv) $\int\sup_{|x|\le R}G(t,x,y)\,dy<\infty$ *for all* $R>0$ *and* $t>0$.

(v) $\sup_{t\in[T_0,T]}\sup_{x\in\mathbb{R}^d}\int_{|y-x|\ge R}G(t,x,y)\,dy\xrightarrow{R\to\infty}0$ *for any* $0<T_0<T<\infty$.

(vi) $\sup_{x\in\mathbb{R}^d}\int G(t,x,y)\,dy<\infty$ *for each fixed* $t>0$.

(vii) $\sup_{x\in\mathbb{R}^d}\sup_{r\in(0,1)}\int_0^t\int G(s+r,x,y)\,dy\,ds\xrightarrow{t\to 0}0.$

Proof. Recall the definition of G, cf. (4.43):

$$G(t,x,y) = ct^{-1+\kappa}S(x-y,\alpha(y),t) + ct^{-1+\kappa}g_\gamma(x-y)$$

where $\kappa>0$, $c>0$ is a fixed constant and $\gamma:=\gamma_0^L\wedge\gamma_\infty^L$. Fix $T>0$. We know from Lemma 4.17 that there exists a constant $C_1=C_1(T)>0$ such that

$$\sup_{x\in\mathbb{R}^d}\sup_{t\in(0,T]}\int S(x-y,\alpha(y),t)\,dy\le C_1. \tag{A.1}$$

Moreover, it follows from the definition of g_γ that

$$\int g_\gamma(x-y)\,dy \le 1 + \int_{|x-y|\ge 1} \frac{1}{|x-y|^{d+\gamma}}\,dy.$$

Consequently, there exists a constant $C_2 > 0$ such that

$$\sup_{x\in\mathbb{R}^d} \int g_\gamma(x-y)\,dy \le C_2. \tag{A.2}$$

Combining (A.1) and (A.2) yields

$$\sup_{x\in\mathbb{R}^d} \int G(t,x,y)\,dy \le c(C_1+C_2)t^{-1+\kappa}.$$

Obviously, this implies (i), (ii), and (vi). Moreover, we find

$$\sup_{x\in\mathbb{R}^d} \int_0^t \int G(s+r,x,y)\,dy\,ds \le C_1'\big((t+r)^\kappa - r^\kappa\big) + C_2'\big((t+r)^\kappa - r^\kappa\big)$$

for all $r \in [0,1]$, $t \in [0,T]$ for suitable constants $C_1', C_2' > 0$ depending only on T. For $r = 0$ this gives (iii). Using the (local) Hölder continuity of the mapping $s \mapsto s^\kappa$ we also get (vii). To prove (iv) we fix $R > 0$ and $t > 0$. Without loss of generality, we may assume that $R \ge 1$. Since $(x,y) \mapsto S(x-y,\alpha(y),t)$ is bounded, it follows that $(x,y) \mapsto G(t,x,y)$ is also bounded, say by a constant $M > 0$. Then

$$\int \sup_{|x|\le R} \big(\mathbb{1}_{\{|y-x|\le 3R\}} G(t,x,y)\big)\,dy \le M\lambda^d(B[0,4R]).$$

On the other hand, we have

$$\begin{aligned}\sup_{|x|\le R}\big(\mathbb{1}_{\{|y-x|>3R\}} G(t,x,y)\big) &\le c(t^{-1+\kappa}+t^{-1+\kappa}) \sup_{|x|\le R}\left(\mathbb{1}_{\{|y-x|>3R\}} \frac{1}{|x-y|^{d+\gamma}}\right)\\ &\le c2t^{-1+\kappa} \sup_{|x|\le R}\left(\mathbb{1}_{\{|y-x|>3R\}} \frac{1}{(|y|-R)^{d+\gamma}}\right)\\ &\le 2ct^{-1+\kappa} \frac{1}{(|y|-R)^{d+\gamma}} \mathbb{1}_{\{|y|\ge 2R\}}\end{aligned}$$

implying

$$\int \sup_{|x|\le R}\big(\mathbb{1}_{\{|y-x|\le 3R\}} G(t,x,y)\big)\,dy \le C \int_{|y|\ge 2R} \frac{1}{(|y|-R)^{d+\gamma}}\,dy < \infty.$$

If we combine both estimates, we get (iv). Finally, (v) follows directly from the fact that

$$G(t,x,y) \le 2ct^{-1+\kappa} \frac{1}{|x-y|^{d+\gamma}} \le C' \frac{1}{|x-y|^{d+\gamma}}$$

for all $|x-y| \ge R$ sufficiently large and $t \in [T_0,T]$ for some constant $C = C(T_0,T,\kappa)$. □

A.9 Lemma *For any fixed* $0 < T_0 < T < \infty$ *there exists a constant* $C > 0$ *such that the following statements hold true.*

(i) $\sup_{t\in[T_0,T]} G(t,x,y) \le CG(T,x,y)$ *for all* $x,y \in \mathbb{R}^d$.

(ii) $\sup_{t\in[0,T]} \sup_{x,y\in\mathbb{R}^d} G(t+\varepsilon,x,y) < \infty$.

(iii) $C^{-1}G(t,x,y) \le H^1(t,x,y) \le CG(t,x,y)$ *for all* $t \in [T_0,T]$, $x,y \in \mathbb{R}^d$ *and* $k \in \mathbb{N}$.

(iv) $S(x-y,\alpha(y),t) \le CG(t,x,y)$ *for all* $x,y \in \mathbb{R}^d$, $t \in [T_0,T]$.

(v) $\sup_{x\in\mathbb{R}^d} \int H^1(t+\varepsilon,x,y)\,dy \le C(t+\varepsilon)^\kappa$ *for all* $\varepsilon > 0$ *and* $t \in [0,T]$.

Proof. (i) We have seen in the proof of Lemma A.7 that there exists some constant $C = C(T_0,T)$ such that

$$S(z,\alpha,\lambda T) \le CS(z,\alpha,T) \qquad \text{for all } \alpha \in I, z \in \mathbb{R}^d, \lambda \in [T_0/T,1],$$

i. e.

$$S(z,\alpha,t) \le CS(z,\alpha,T) \qquad \text{for all } \alpha \in I, z \in \mathbb{R}^d, t \in [T_0,T].$$

Since $[T_0,T] \ni t \mapsto t^\sigma$ is bounded from above for any $\sigma \in \mathbb{R}$, the claim is a direct consequence of the definition of G, cf. (4.43).

(ii) By (i), there exists a constant $C > 0$ such that

$$\sup_{t\in[0,T]} \sup_{x,y\in\mathbb{R}^d} G(t+\varepsilon,x,y) \le C \sup_{x,y\in\mathbb{R}^d} G(T+\varepsilon,x,y).$$

It is obvious from the definition of G that the expression on the right-hand side is finite.

(iii),(iv) This follows directly from the fact that $[T_0,T] \ni t \mapsto t^\sigma$ is bounded from below and above for any $\sigma > 0$.

(v) Using (A.1) and (A.2), this is a straightforward computation. □

A.10 Lemma *There exists a constant* $C = C(T,\kappa) > 0$ *such that*

$$\int_{t-\delta}^t \int S(t-s+r,x,z)G(s,z,y)\,dz\,ds \le C(S(x-y,\alpha(y),t+r) + (t+r)g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y))\delta^{\kappa\wedge 1}$$

for all $r \ge 0$, $0 < t \le T$, $\delta \in (0,t)$ *and* $x,y \in \mathbb{R}^d$.

Proof. Recall that

$$G(t,x,y) = ct^{-1+\kappa}S(x-y,\alpha(y),t) + ct^{-1+\kappa}g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)$$

for some $\kappa > 0$ and $c > 0$, cf. (4.43). Hence, if we set

$$I_1 := c\int_{t-\delta}^t \int S(x-z,\alpha(z),t-s+r)s^{-1+\kappa}S(z-y,\alpha(y),s)\,dz\,ds$$
$$I_2 := c\int_{t-\delta}^t \int S(x-z,\alpha(z),t-s+r)s^{-1+\kappa}g_{\gamma_0^L\wedge\gamma_\infty^L}(z-y)\,dz\,ds,$$

then
$$\int_{t-\delta}^{t}\int S(t-s+r,x,z)G(s,z,y)\,dz\,ds = I_1 + I_2.$$
We estimate the terms separately. By Lemma 4.17 there exists a constant $C = C(T) > 0$ such that
$$\int S(x-z,\alpha(z),t-s+r)S(z-y,\alpha(y),s)\,dz\,ds \le C(S(x-y,\alpha(y),t+r)+(t+r)g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)).$$
Consequently,
$$I_1 \le cC(S(x-y,\alpha(y),t+r)+(t+r)g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y))\int_{t-\delta}^{t} s^{-1+\kappa}\,ds.$$
As $[0,\infty)\ni u\mapsto u^\kappa$ is locally Hölder continuous with exponent $\kappa\wedge 1$ for any $\kappa>0$, there exists a constant $C = C(T,\kappa)$ such that
$$\int_{t-\delta}^{t} s^{-1+\kappa}\,ds = \frac{1}{\kappa}(t^\kappa-(t-\delta)^\kappa)\le C'\delta^{\kappa\wedge 1} \tag{$\star$}$$
for all $0\le\delta\le t\le T$. Consequently,
$$I_1 \le C''\delta^{\kappa\wedge 1}(S(x-y,\alpha(y),t+r)+(t+r)g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y)).$$
Similarly, we get from Lemma 4.19 and ($\star$)
$$I_2 \le C'''\delta^{\kappa\wedge 1}g_{\gamma_0^L\wedge\gamma_\infty^L}(x-y).$$
□

A.11 Lemma *For any $f\in C_\infty(\mathbb{R}^d)$ it holds that*
$$(\partial_t - A_x)P_{t,\varepsilon}f(x) = \int(\partial_t - A_x)p_\varepsilon(t,x,y)f(y)\,dy \qquad \text{for all } x\in\mathbb{R}^d.$$

Proof. Because of Lemma 4.38(i),(ii) the expression on the left-hand side is well-defined. In the proof of Lemma 4.38(ii) we have seen that
$$\partial_t P_{t,\varepsilon}f(x) = \int \partial_t p_\varepsilon(t,x,y)f(y)\,dy \qquad \text{for all } x\in\mathbb{R}^d,$$
cf. (4.69). Therefore it suffices to prove
$$A_x P_{t,\varepsilon}f(x) = \int A_x p_\varepsilon(t,x,y)f(y)\,dy, \qquad x\in\mathbb{R}^d. \tag{$\star$}$$
Since the proof of Lemma 4.38(i) shows that we may interchange integration and differentiation, we can restrict ourselves to the non-local part, i. e. we may assume that
$$Ag(x) = \int(g(x+y)-g(x)-\nabla g(x)\cdot y\mathbb{1}_{\{|y|\le 1\}})\,\nu(x,dy), \qquad g\in C_b^2(\mathbb{R}^d), x\in\mathbb{R}^d.$$
Fix $x\in\mathbb{R}^d$. By the (proof of) Lemma 4.38(i) we have
$$\begin{aligned}AP_{t,\varepsilon}f(x) &= \int(P_{t,\varepsilon}f(x+y)-P_{t,\varepsilon}f(x)-\nabla P_{t,\varepsilon}f(x)\cdot y\mathbb{1}_{\{|y|\le 1\}}\,\nu(x,dy)\\ &= \int\left(\int(p_\varepsilon(t,x+y,z)-p_\varepsilon(t,x,z)-\nabla_x p_\varepsilon(t,x,z)\cdot y\mathbb{1}_{\{|y|<1\}})f(z)\,dz\right)\nu(x,dy).\end{aligned}$$

If we can show that Fubini's theorem is applicable, we get $(\star)$ by interchanging the integrals. To check that Fubini's theorem applies, we split up the domain of integration. By Lemma 4.35(ii) and Lemma 4.29(i), there exists $C > 0$ such that

$$\int_{|y|\geq 1}\int |p_\varepsilon(t,x+y,z) - p_\varepsilon(t,x,z)|\,|f(z)|\,dz\,\nu(x,dy) \leq 2C\|f\|_\infty \nu(x,B(0,1)^c) \sup_{y\in\mathbb{R}^d}\int G(t+\varepsilon,y,z)\,dz < \infty.$$

On the other hand, it follows from Taylor's formula that

$$|p_\varepsilon(t,x+y,z) - p_\varepsilon(t,x,z) - \nabla_x p_\varepsilon(t,x,y)\cdot y| \leq \frac{1}{2}|y|^2 \sup_{\lambda\in[0,1]} |\nabla_x^2 p_\varepsilon(t,\lambda x + (1-\lambda)(x+y),z)|.$$

Thus,

$$\int_{|y|\leq 1}\int |(p_\varepsilon(t,x+y,z) - p_\varepsilon(t,x,z) - \nabla_x p_\varepsilon(t,x,z)\cdot y|\,|f(z)|\,dz \leq \frac{1}{2}\|f\|_\infty \int_{|y|\leq 1} |y|^2\,\nu(x,dy) \sup_{|y|\leq 1}\int \sup_{\lambda\in[0,1]} |\nabla_x^2 p_\varepsilon(t,\lambda x+(1-\lambda)(x+y),z)|\,dz.$$

Using a similar argument as in the proof of Lemma 4.40(i), cf. (4.67), it is not difficult to see that $|\nabla_y^2 p_\varepsilon(t,y,z)| \leq CG(t+\varepsilon,y,z)$ for all $y,z\in\mathbb{R}^d$ for some constant $C = C(\varepsilon,t)$. Since

$$G(t+\varepsilon,y,z) \leq C' \frac{1}{|y-z|^{d+\gamma_\infty^L\wedge\gamma_0^L}} \qquad \text{for all } |y-z| \gg 1,$$

and $G(t+\varepsilon,\cdot,\cdot)$ is bounded, a straightforward computation gives

$$\sup_{|y|\leq 1}\int \sup_{\lambda\in[0,1]} |\nabla^2 p_\varepsilon(t,\lambda x+(1-\lambda)(x+y),z)|\,dz < \infty$$

for each fixed $x\in\mathbb{R}^d$. Consequently,

$$\int_{|y|\leq 1}\int |(p_\varepsilon(t,x+y,z) - p_\varepsilon(t,x,z) - \nabla_x p_\varepsilon(t,x,z)\cdot y|\,|f(z)|\,dz < \infty. \qquad \square$$

A.12 Lemma

$$A\left(\int_0^t P_{s,\varepsilon}f(\cdot)\,ds\right)(x) = \int_0^t AP_{s,\varepsilon}f(x)\,ds \qquad \textit{for all } x\in\mathbb{R}^d, t\geq 0, f\in C_c^\infty(\mathbb{R}^d).$$

Proof. Let $f\in C_c^\infty(\mathbb{R}^d)$ and $t\in[0,T]$ for some $T>0$. In (the proof of) Lemma 4.38 we have seen that $x\mapsto P_{s,\varepsilon}f(x)$ is twice continuously differentiable for all $s>0$ and that there exists a constant $C = C(\varepsilon,T) > 0$ such that

$$|\partial_{x_j}\partial_{x_i}P_{s,\varepsilon}f(x)| = \left|\int \partial_{x_j}\partial_{x_i}p_\varepsilon(s,x,y)f(y)\,dy\right| \leq C(s+\varepsilon)^{-2/\gamma_\infty^L}\|f\|_\infty \int G(s+\varepsilon,x,y)\,dy$$

for all $i,j\in\{1,\dots,d\}$, $s\in[0,T]$ and $x\in\mathbb{R}^d$. By Lemma 4.34 there exists $C' = C'(\varepsilon,T)$ such that $\|\partial_{x_i}\partial_{x_j}P_{s,\varepsilon}f\|_\infty \leq C'$. On the other hand, it follows from Lemma 4.35(ii) and Lemma 4.34 that there exists $C'' = C''(\varepsilon,T)$ such that

$$\|P_{s,\varepsilon}f\|_\infty \leq C''\|f\|_\infty \qquad \text{for all } s\in[0,T].$$

If we define $h(x) := \int_0^t P_{s,\varepsilon} f(x)\, ds$, then $h \in C_b^2(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ by the differentiation lemma. Moreover,

$$\begin{aligned} Ah(x) &= \int (h(x+y) - h(x) - \nabla h(x) \cdot y \mathbb{1}_{\{|y| \leq 1\}})\, \nu(x,dy) \\ &= \int \left[\int_0^t \left(P_{s,\varepsilon} f(x+y) - P_{s,\varepsilon} f(x) - \nabla P_{s,\varepsilon} f(x) \cdot y \mathbb{1}_{\{|y|\leq 1\}} \right) ds \right] \nu(x,dy) \end{aligned}$$

for all $x \in \mathbb{R}^d$. Therefore, the claim follows if we can show that Fubini's theorem is applicable. This, however, is a direct consequence of Taylor's formula and the upper bounds for $P_{s,\varepsilon} f$ and its derivatives which we have just derived; see also the proof of Lemma A.11. □

A.13 Lemma *The derivative*

$$I_j := \frac{\partial}{\partial x} \int_{\Omega_j(t)} \int_{\mathbb{R}^k} p_0 \left(t - \sum_{i=1}^k s_i, x, z_1 \right) \prod_{i=1}^{k-1} F(s_i, z_i, z_{i+1}) F(s_k, z_k, y)\, dz\, ds$$

exists for all $t > 0$, $x, y \in \mathbb{R}$ and $j \in \{0, \ldots, k\}$. Moreover, the identities (4.86) *and* (4.87) *hold true.*

Proof. Throughout this proof, we fix $t > 0$ and $y \in \mathbb{R}$. Since, by Lemma 4.15,

$$\sup_{z \in \mathbb{R}^d} \sup_{s \in [0,t]} \int S(x - z, \alpha(z), s)\, dx < \infty \tag{A.3}$$

an easy calculation shows that

$$\sup_{z \in \mathbb{R}^d} \int_0^t \int G(s,x,z)\, dx\, ds < \infty. \tag{A.4}$$

First, we check that I_0 is well-defined, i. e. that the derivative exists. Recall that

$$\Omega_0(t) = \{ s \in \mathbb{R}^k; \sum_{i=1}^k s_i \leq t/2 \}.$$

By Theorem 4.12 and (4.11), we find constants such that

$$\begin{aligned} \left| \frac{\partial}{\partial x} p_0 \left(t - \sum_{i=1}^k s_i, x, z_1 \right) \right| &\leq C_0 \left(t - \sum_{i=1}^k s_i \right)^{-1/\gamma_\infty^L} S\left(x - z_1, \alpha(z_1), t - \sum_{i=1}^k s_i \right) \\ &\leq C_0' \left(\frac{t}{2} \right)^{-1/\gamma_\infty^L} =: C_0'' \end{aligned}$$

for all $s \in \Omega_0(t)$ and $x, z_1 \in \mathbb{R}$. Consequently,

$$w(s,z) := C_0'' \prod_{i=1}^{k-1} G(s_i, z_i, z_{i+1}) G(s_k, z_k, y), \qquad s \in \Omega_0(t), z \in \mathbb{R}^k$$

satisfies

$$\sup_{x \in \mathbb{R}} \left| \partial_x \left[p_0 \left(t - \sum_{i=1}^k s_i, x, z_1 \right) \prod_{i=1}^{k-1} F(s_i, z_i, z_{i+1}) F(s_k, z_k, y) \right] \right| \leq w(s,z).$$

Since we also have by Tonelli's theorem and (A.4)

$$\int_{\Omega_0(t)} \int |w(s,z)|\, dz\, ds \le C'' \left(\sup_{z\in\mathbb{R}} \int_0^t \int_{\mathbb{R}} G(s,\eta,z)\, d\eta\, ds \right)^k < \infty,$$

this means that we may apply the differentiation lemma for parametrized integrals to conclude that I_0 is well-defined and that (4.86) holds.
The proof of the differentiability for $j \in \{1,\dots,k\}$ is more involved. We have already seen in the proof of Lemma 4.77 that

$$\begin{aligned}
&\int_{\Omega_j(t)} \int p_0 \left(t - \sum_{i=1}^k s_i, x, z_1 \right) \prod_{i=1}^{k-1} F(s_i, z_i, z_{i+1}) F(s_k, z_k, y)\, dz\, ds \\
&= \int_{\Omega_j(t)} \int p_0 \left(t - \sum_{i=1}^k s_i, x, z_1 \right) P_{j,\varnothing}(s,x,z,y)\, dz\, ds.
\end{aligned}$$

Therefore, it suffices to check that the right-hand side is differentiable with respect to x and the derivative satisfies (4.87). To this end, let us first establish that

$$U(s,x) := \int \underbrace{p_0 \left(t - \sum_{i=1}^k s_i, x, z_1 \right) P_{j,\varnothing}(s,x,z,y)}_{=:u(s,x,z)}\, dz \tag{A.5}$$

is differentiable for each fixed $s \in \Omega'_j(t) := \{ s \in \Omega_j(t); \forall i \in \{1,\dots,k\} : s_i > 0, \sum_{i=1}^k s_i < t \}$ and that we may interchange differentiation and integration. We define another auxiliary function by

$$Q_j(s,x,z) := \prod_{i=1}^{j-1} G(s_i, x - z_i, x - z_{i+1}) G(s_j, x - z_j, z_{j+1}) \prod_{i=j+1}^{k-1} G(s_k, z_i, z_{i+1}) G(s_k, z_k, y);$$

note that $|P_{j,\varnothing}(\cdot,\cdot,\cdot,y)| \le Q_j$. Obviously,

$$\begin{aligned}
\partial_x u(s,x,z) &= \left[\partial_x p_0 \left(t - \sum_{i=1}^k s_i, x, x - z_1 \right) \right] P_{j,\varnothing}(s,x,z,y) \\
&\quad + \sum_{\ell=1}^{k-1} p_0 \left(t - \sum_{i=1}^k s_i, x, x - z_1 \right) P_{j,\{\ell\}}(s,x,z,y) \partial_x F(s_\ell, x - z_\ell, x - z_{\ell+1}) \\
&\quad + p_0 \left(t - \sum_{i=1}^k s_i, x, x - z_1 \right) P_{j,\varnothing}(s,x,z,y) \frac{\partial_x F(s_j, x - z_j, z_{j+1})}{F(s_j, x - z_j, z_{j+1})}.
\end{aligned} \tag{A.6}$$

Using (4.88) (for the first term on the right-hand side), Lemma 4.47 (for the second term) and Lemma 4.46 (for the third term) and the fact that $|F| \le G$, we get for all $s \in \Omega'_j(t)$

$$\begin{aligned}
|\partial_x u(s,x,z)| &\le C_1 k S \left(z_1, \alpha(z_1 - x), t - \sum_{i=1}^k s_i \right) Q_j(s,x,z,y) \\
&\quad + C_1 S \left(z_1, \alpha(z_1 - x), t - \sum_{i=1}^k s_i \right) s_j^{-1/\gamma_\infty^L} Q_j(s,x,z,y) \\
&\overset{(4.85)}{\le} C_1' S \left(z_1, \alpha(z_1 - x), t - \sum_{i=1}^k s_i \right) Q_j(s,x,z,y)
\end{aligned} \tag{A.7}$$

for a constant $C_1' = C_1'(t)$ which does not depend on s. As $t - \sum_{i=1}^k s_i > 0$ is fixed, there exists, by (4.11), a constant $C_2 = C_2(s,t)$ such that $|\partial_x u(s,x,z)| \le C_2 Q_j(s,x,z)$. Since

$$\sup_{z_{i+1}\in\mathbb{R}} \int_{\mathbb{R}} \sup_{|x|\le R} |G(s_i, x - z_i, x - z_{i+1})| \, dz_i + \sup_{z_j \in\mathbb{R}} \int_{\mathbb{R}} \sup_{|x|\le R} |G(s_j, x - z_j, z_{j+1})| \, dz_{j+1} < \infty$$

for each fixed $R > 0$ and $i \in \{0, \dots, j-1\}$, see Lemma A.14 below, it is not difficult to see from Lemma 4.29 that

$$\sup_{|x|\le R} |Q_j(s,x,\cdot)| \in L^1(\mathbb{R}^k).$$

Consequently, the differentiation lemma applies, and we find that

$$\frac{\partial}{\partial x} U(s,x) = \int \partial_x u(s,x,z) \, dz \tag{A.8}$$

for all $s \in \Omega_j'(t)$. If we can show that

$$\frac{\partial}{\partial x} \int_{\Omega_j'(t)} U(s,x) \, ds = \int_{\Omega_j'(t)} \frac{\partial}{\partial x} U(s,x) \, ds, \tag{A.9}$$

then it follows that from the above considerations and the fact that $\Omega_j(t) \backslash \Omega_j'(t)$ is a (Lebesgue) null set that

$$\begin{aligned}
&\frac{\partial}{\partial x} \int_{\Omega_j(t)} \int_{\mathbb{R}^k} p_0\left(t - \sum_{i=1}^k s_i, x, z_1\right) \prod_{i=1}^{k-1} F(s_i, z_i, z_{i+1}) F(s_k, z_k, y) \, dz \, ds \\
&\overset{(A.7)}{=} \frac{\partial}{\partial x} \int_{\Omega_j(t)} U(s,x) \, ds \quad = \quad \frac{\partial}{\partial x} \int_{\Omega_j'(t)} U(s,x) \, ds \\
&\overset{(A.9)}{=} \int_{\Omega_j'(t)} \frac{\partial}{\partial x} U(s,x) \, ds \overset{(A.8)}{=} \int_{\Omega_j'(t)} \int \frac{\partial}{\partial x} u(s,x,z) \, dz \, ds
\end{aligned}$$

This yields the differentiability as well as identity (4.87) (see (A.6)). It remains to prove (A.9). To this end, we show that the assumptions of the differentiation lemma are satisfied. Using (A.7) and performing a change of variables ($z_i \rightsquigarrow x + z_i$ for $i = 1, \dots, j$), we get

$$\left|\frac{\partial}{\partial x} U(s,x)\right| \le C_3 \int S\left(x - z_1, \alpha(z_1), t - \sum_{i=1}^k s_i\right) \prod_{i=1}^{k-1} G(s_i, z_i, z_{i+1}) G(s_k, z_k, y) \, dz.$$

By Lemma 4.17 and Lemma 4.19, there exists a constant $C_4 = C_4(t)$ such that

$$\int_{\mathbb{R}} S\left(x - z_1, \alpha(z_1), t - \sum_{i=1}^k s_i\right) G(s_1, z_1, z_2) \, dz_1 \le C_4 s_1^{-1+\kappa} \left[S\left(x, z_2, t - \sum_{i=2}^k s_i\right) + g_\gamma(x - z_2) \right]$$

for all $s \in \Omega_j'(t)$ and $z_2 \in \mathbb{R}$. Hence,

$$\begin{aligned}
&\left|\frac{\partial}{\partial x} U(s,x)\right| \\
&\quad \le C_3 C_4 \int_{\mathbb{R}^{k-1}} \left[S\left(x, z_2, t - \sum_{i=2}^k s_i\right) + g_\gamma(x - z_2) \right] \prod_{i=2}^{k-1} G(s_i, z_i, z_{i+1}) G(s_k, z_k, y) \, d(z_2, \dots, z_k).
\end{aligned}$$

The convolution estimates obtained in Section 4.2 also show that

$$\int_{\mathbb{R}} \left[S\left(t - \sum_{i=\ell}^k s_\ell, x, z_\ell\right) + g_\gamma(x - z_\ell) \right] G(s_\ell, z_\ell, z_{\ell+1}) \, dz_\ell$$

$$\leq C_5 s_\ell^{-1+\kappa}\left[S\left(x, z_{\ell+1}, t - \sum_{i=\ell+1}^{k} s_i\right) + g_\gamma(x - z_{\ell+1})\right]$$

for $\ell \in \{2,\dots,k\}$, $z_{\ell+1} \in \mathbb{R}$. Applying this estimate iteratively, it is not difficult to see that

$$\left|\frac{\partial}{\partial x}U(s,x)\right| \leq C_6 \prod_{i=1}^{k} s_i^{-1+\kappa}(S(t,x,y) + g_\gamma(x-y)).$$

If we define

$$w(s) := C_6 \prod_{i=1}^{k} s_i^{-1+\kappa} \sup_{x\in\mathbb{R}^d}(S(t,x,y) + g_\gamma(x-y)),$$

then w is a dominating function and $\int_{\Omega_j(t)} w(s)\,ds < \infty$. Therefore, (A.9) is a direct consequence of the differentiation lemma for parametrized integrals. □

A.14 Lemma *(i)* $\sup_{z\in\mathbb{R}^d}\int \sup_{|x|\leq R} G(t, x-y, x-z)\,dy < \infty$ *for any* $R > 0$, $t > 0$.

(ii) $\sup_{z\in\mathbb{R}^d}\int \sup_{|x|\leq R} G(t, x-y, z)\,dy < \infty$ *for any* $R > 0$, $t > 0$.

Proof. Since $S(y-z, \alpha(x-z), t) \leq t^{-d/\gamma_\infty(\alpha(x-z))} \leq C_1 t^{-d/\gamma_\infty^L}$ and

$$S(y-z,\alpha(x-z),t) \leq \frac{t}{|y-z|^{d+\gamma_0(\alpha(x-z))\wedge\gamma_\infty(\alpha(x-z))}} \leq \frac{t}{|y-z|^{d+\gamma_0^L\wedge\gamma_\infty^L}} \qquad \text{for all } |y-z| \geq 1$$

we have

$$\begin{aligned}\int \sup_{|x|\leq R} S(y-z,\alpha(x-z),t)\,dy &\leq C_1 t^{-d/\gamma_\infty^L}\lambda^d(B(0,1)) + \int_{|y-z|\geq 1} \sup_{|x|\leq R} S(y-z,\alpha(x-z),t)\,dy \\ &\leq C_1 t^{-d/\gamma_\infty^L}\lambda^d(B(0,1)) + t\int_{|w|\geq 1}\frac{1}{|w|^{d+\gamma_0^L\wedge\gamma_\infty^L}}\,dw < \infty\end{aligned}$$

and the bound on the right-hand side does not depend on z. Moreover, we have

$$\sup_{z\in\mathbb{R}^d}\int g_{\gamma_0^L\wedge\gamma_\infty^L}(y-z)\,dy < \infty$$

by the very definition of g. Hence,

$$\begin{aligned}&\sup_{z\in\mathbb{R}^d}\int \sup_{|x|\leq R} G(t,x-y,x-z)\,dy \\ &\overset{(4.43)}{=} C_2 \sup_{z\in\mathbb{R}^d}\int \sup_{|x|\leq R}\left(S(y-z,\alpha(x-z),t) + g_{\gamma_0^L\wedge\gamma_\infty^L}(y-z)\right)dy < \infty\end{aligned}$$

for a suitable constant $C_2 = C_2(t)$. This proves the first assertion. Similarly, we find

$$\begin{aligned}&\int \sup_{|x|\leq R} S(x-y-z,\alpha(z),t)\,dy \\ &\leq C_1 t^{-d/\gamma_\infty^L}\lambda^d(B(0,R+1)) + \int_{|y-z|\geq R+1}\sup_{|x|\leq R} S(x-y-z,\alpha(z),t)\,dy \\ &\leq C_1 t^{-d/\gamma_\infty^L}\lambda^d(B(0,R+1)) + t\int_{|y-z|\geq R+1}\sup_{|x|\leq R}\frac{1}{|x-y-z|^{d+\gamma_0^L\wedge\gamma_\infty^L}} \\ &\leq C_1 t^{-d/\gamma_\infty^L}\lambda^d(B(0,R+1)) + t\int_{|w|\geq R+1}\frac{1}{(|w|-R)^{d+\gamma_0^L\wedge\gamma_\infty^L}}\,dw < \infty\end{aligned}$$

and the bound on the right-hand side does not depend on z; here we have used that $|x-y-z| \geq 1$ for any $|y-z| \geq R+1$ and $|x| \leq R$ and therefore

$$S(x-y-z, \alpha(z), t) \leq \frac{t}{|x-y-z|^{d+\gamma_0^L \wedge \gamma_\infty^L}} \qquad \text{for all } \; |y-z| \geq R+1, |x| \leq R.$$

Since also

$$\sup_{z \in \mathbb{R}^d} \int \sup_{|x| \leq R} g_{\gamma_0^L \wedge \gamma_\infty^L}(x-y-z)\, dy = \int \sup_{|x| \leq R} g_{\gamma_0^L \wedge \gamma_\infty^L}(x-y)\, dy < \infty$$

(see the proof of Lemma A.8(iv) for details), this proves (ii). □

A.15 Lemma *The mapping*

$$\arctan : \mathbb{C} \backslash \{z \in \mathbb{C}; |\operatorname{Im} z| \geq 1\} \to \mathbb{C}, z \mapsto \int_0^z \frac{1}{1+\zeta^2}\, d\zeta$$

is holomorphic and satisfies

$$|\operatorname{Im}(\arctan z)| \leq 4 |\sin(\arg z)|$$

for all $z \in \Omega(0, \pi/4) = \{z \in \mathbb{C} \backslash \{0\}; |\arg z \mod \pi| < \pi/4\}$ *(see* (3.1) *for the definition of* Ω*). Moreover, for any* $\varepsilon > 0$ *there exist* $R > 0$ *and* $\vartheta \in (0, \pi/2)$ *such that*

$$\frac{\pi}{2}(1-\varepsilon) \leq |\operatorname{Re}(\arctan z)| \leq \frac{\pi}{2}(1+\varepsilon) \qquad \text{for all } \; z \in \Omega(0, \vartheta), |z| \geq R.$$

Proof. Since $\zeta \mapsto \frac{1}{1+\zeta^2}$ is holomorphic on $\mathbb{C} \backslash \{\zeta \in \mathbb{C}; |\operatorname{Im} \zeta| \geq 1\}$, it follows easily (e. g. from the Cauchy-Riemann equations) that arctan is holomorphic on $\mathbb{C} \backslash \{z \in \mathbb{C}; |\operatorname{Im} z| \geq 1\}$. Fix $z \in \Omega(0, \pi/4)$ and set $u := \operatorname{Re} z$, $v := \operatorname{Im} z$. Using

$$\frac{1}{1+t^2 z^2} = \frac{1+t^2(u^2-v^2)}{1+t^4(u^2+v^2)^2 + 2t^2(u^2-v^2)} - i \frac{2uvt^2}{1+t^4(u^2+v^2)^2+2t^2(u^2-v^2)},$$

we find

$$\operatorname{Im}(\arctan z) = \operatorname{Im}\left(z \int_0^1 \frac{1}{1+t^2 z^2}\, dt\right) = v \int_0^1 \frac{1-t^2(u^2+v^2)}{1+t^4(u^2+v^2)^2+2t^2(u^2-v^2)}\, dt$$

and

$$\operatorname{Re}(\arctan z) = u \int_0^1 \frac{1+t^2(u^2+v^2)}{1+t^4(u^2+v^2)^2+2t^2(u^2-v^2)}\, dt. \tag{$\star$}$$

Since $u = \sqrt{u^2+v^2} \cos(\arg z)$ and $v = \sqrt{u^2+v^2} \sin(\arg z)$, the first assertion follows if we can show that

$$I := \int_0^1 \frac{1+t^2(u^2+v^2)}{1+t^4(u^2+v^2)^2+2t^2(u^2-v^2)}\, dt \leq 4 \frac{1}{\sqrt{u^2+v^2}}.$$

To this end, we note $u^2 - v^2 \geq 0$ for any $z = u + iv \in \Omega(0, \pi/4)$. Thus,

$$I \leq \int_0^1 \frac{1+t^2(u^2+v^2)}{1+t^4(u^2+v^2)^2}\, dt = \int_0^1 f(t^2(u^2+v^2))\, dt$$

where $f(x) := (1+x)/(1+x^2)$. Now the elementary estimate

$$f(x) \le 2\min\{1, x^{-1}\}, \qquad x \ge 0,$$

implies

$$I \le 2\int_0^{\frac{1}{\sqrt{u^2+v^2}}\wedge 1} 1\,dt + \int_{\frac{1}{\sqrt{u^2+v^2}}\wedge 1}^{1} \frac{1}{t^2(u^2+v^2)}\,dt \le 4\frac{1}{\sqrt{u^2+v^2}}.$$

It remains to prove the last assertion. To this end, fix $\vartheta \in (0,\pi/2)$ and $z = u + iv \in \Omega(0,\vartheta)$. Then $|v| \le |u| \arctan\vartheta$, and therefore

$$1 - \arctan^2\vartheta \le \frac{u^2 - v^2}{u^2} \le \frac{u^2+v^2}{u^2} \le 1 + \arctan^2\vartheta.$$

Performing a change of variables ($s := |u|t$) in ($\star$), we obtain

$$\begin{aligned}\int_0^{|u|} &\frac{1+s^2(1-\arctan^2\vartheta)}{1+s^4(1-\arctan^2\vartheta)^2 + 2s^2(1+\arctan^2\vartheta)}\,ds \\ &\le |\operatorname{Re}\arctan z| \le \int_0^{|u|} \frac{1+s^2(1+\arctan^2\vartheta)}{1+s^4(1+\arctan^2\vartheta)^2+2s^2(1-\arctan^2\vartheta)}\,ds.\end{aligned}$$

Letting $|u| \to \infty$ we find that the lower and upper bound converge to

$$\int_0^\infty \frac{1+s^2(1-\arctan^2\vartheta)}{1+s^4(1-\arctan^2\vartheta)^2+2s^2(1+\arctan^2\vartheta)}\,ds$$

and

$$\int_0^\infty \frac{1+s^2(1+\arctan^2\vartheta)}{1+s^4(1+\arctan^2\vartheta)^2+2s^2(1-\arctan^2\vartheta)}\,ds,$$

respectively. Since both expressions converge to $\pi/2$ as $\vartheta \to 0$, this finishes the proof. □

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Affirmation

Hereby I affirm that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken directly or indirectly from other sources are indicated as such. This thesis has not been presented to any other examination board in this or a similar form, neither in Germany nor in any other country.

I have written this dissertation at the Institute of Mathematical Stochastics at Technische Universität Dresden under the scientific supervision of Prof. Dr. René L. Schilling from June 2014 to July 2016.

I accept the requirements for obtaining a PhD (Promotionsordnung) of the Faculty of Science of the TU Dresden, issued February 23, 2011.

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Die vorgelegte Dissertation wurde in der Zeit von Juni 2014 bis Juli 2016 am Institut für Mathematische Stochastik der Technischen Universität Dresden unter der wissenschaftlichen Betreuung von Prof. Dr. René L. Schilling angefertigt.

Ich erkenne die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der TU Dresden vom 23. Februar 2011 an.