The Eigenvalue Problem of the 1-Laplace Operator

Local Perturbation Results and Investigation of Related Vectorial Questions

Dissertation

zur Erlangung des akademischen Grades

Doctor rerum naturalium
(Dr. rer. nat.)

vorgelegt

der Fakultät Mathematik und Naturwissenschaften
der Technischen Universität Dresden

von

Dipl.-Math. Samuel Littig

geboren am 11.07.1984 in Dresden

Gutachter:	Prof. Friedemann Schuricht *Technische Universität Dresden*
	Prof. Jan Kristensen *University of Oxford*
Eingereicht am:	17. 07. 2014
Verteidigt am:	23. 01. 2015

Die Dissertation wurde in der Zeit von April 2010 bis Juli 2014 am Institut für Analysis angefertigt.

Contents

1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded and $p \in (1, \infty)$. One of the most frequently studied nonlinear differential operators is the p-Laplace operator

$$u \mapsto \Delta_p u := -\operatorname{div} |Du|^{p-2} Du \,.$$

Even though nonlinear, it has many nice properties. It turns out to be the normalized duality mapping $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, which is a homeomorphism. Moreover the eigenvalue problem of the p-Laplace operator

$$-\Delta_p u = \lambda |u|^{p-2} u \tag{1.1}$$

proves to be a well posed equation and has been studied intensively from the 80th of the last century. A weak solution u of (1.1) is called eigenfunction of the p-Laplace operator and the associated real number $\lambda \in \mathbb{R}$ is called eigenvalue of the p-Laplace operator.

By homogeneity of (1.1) for any eigenfunction u and any $\alpha \in \mathbb{R}$ the function αu will also be an eigenfunction for the same eigenvalue and we may thus restrict our attention to normalized eigenfunctions u with $\|u\|_p = 1$.

The eigenvalue problem of the p-Laplace operator is related to the following variational problem:

$$\mathcal{E}_p(v) = \frac{1}{p} \int_\Omega |Dv|^p \, \mathrm{d}x \to \operatorname*{Min!}_{v \in W_0^{1,p}(\Omega)} \tag{1.2}$$

subject to

$$\mathcal{G}_p(v) = \frac{1}{p} \int_\Omega |v|^p \, \mathrm{d}x = 1 \,. \tag{1.3}$$

Since both $\mathcal{E}_p$ and $\mathcal{G}_p$ are differentiable, by application of the classical Lagrange multiplier rule a function $u \in W_0^{1,p}(\Omega)$ with $\|u\|_p = 1$ solves (1.1) for some $\lambda \in \mathbb{R}$ if and only if u is a critical point of (1.2), (1.3).

It is not difficult to see that for connected Ω the minimizer $u_{1,p}$ of this variational problem is unique (up to sign) and the associated eigenvalue $\lambda_{1,p}$ is given by

$$\lambda_{1,p} = \mathcal{E}_1(u_{1,p}) = \min_{\substack{v \in W_0^{1,p}(\Omega), \\ \mathcal{G}_p(v)=1}} \mathcal{E}_p(v)\,.$$

Moreover, the first eigenvalue is known to be isolated and positive. These results have been proved without restriction on the boundary of Ω by Lindqvist [37].

Classical critical point theory, in particular Lusternik-Schnirelman theory, verifies the existence of a sequence of eigenvalues of the p-Laplace operator by the construction

$$\lambda_{k,p} := \inf_{S \in \mathscr{S}^{k,p}} \sup_{u \in S} \mathcal{E}_p(u) \tag{1.4}$$

where

$$\mathscr{S}^{k,p} := \{S \subseteq W_0^{1,p}(\Omega)\ ;\ S \text{ symmetric, compact, } \mathcal{G}_p = 1 \text{ on } S \text{ and } \operatorname{gen} S \geq k\}\,.$$

The genus $\operatorname{gen} S$ of a symmetric set S is a topological index measuring loosely speaking the complexity of a subset of a Banach space (cf. Appendix p. 140). The first application of these methods for the p-Laplace operator is due to García Azorero & Peral Alonso [30]. For $p = 2$ it is known that this construction exhausts the whole spectrum of the usual Laplace operator, nevertheless for $p \neq 2$ there is an example of an eigensolution with periodic boundary conditions known that solves the eigenvalue equation, but cannot been obtained via such a minimax procedure (cf. [8]). However, for homogeneous Dirichlet boundary conditions there seems to be no example of a solution for the eigenvalue equation known that is not obtainable by minimax methods.

With the beginning of the 21st century the limit problem $p \to 1$ of the eigenvalue problem of the p-Laplace operator gained attention. This turned out to be a very challenging task. As a first aspect note that in the definition of the eigenvalue problem of the p-Laplace operator (1.1) the expressions $|Du(x)|^{p-2}Du(x)$ and $|u(x)|^{p-2}u(x)$ are interpreted as zero in the sense of continuous continuation, provided $Du(x) = 0$ or $u(x) = 0$, resp. Simply setting $p = 1$ does not answer the question how to interpret the expressions $Du(x)/|Du(x)|$ and $u(x)/|u(x)|$ provided $Du(x) = 0$ or $u(x) = 0$ resp. Even worse, numerical simulations stipulated that the first eigenfunction should be the characteristic function of a certain set, such that we face the foregoing troubles

almost everywhere in Ω.

First results by the limit procedure $p \to 1$ are due to Demengel [23], [24].

The second natural idea, carried out by Kawohl & Schuricht in [36], was to focus on the highly nonsmooth associated variational problem (1.2), (1.3) with $p = 1$ directly. It turns out that $W_0^{1,1}(\Omega)$ is not the suitable limit space. On the one hand it is not reflexive which causes difficulties for the analysis, on the other hand it does not contain functions such as the minimizer we expect, a characteristic function of a certain set. By the results of Federer, Giusti and others from the 80th it is known that $BV(\Omega)$, the space of functions of bounded variation, is the more reasonable space for such problems with 1-homogeneous growth. Indeed the appropriate lower semicontinuous relaxation of the functional (1.2) for $p = 1$ reads as

$$\mathcal{E}_{TV}(v) = \int_\Omega d|Dv| + \int_{\partial\Omega} |v^{\partial\Omega}| \, \mathrm{d}\mathcal{H}^{n-1} \,. \tag{1.5}$$

Here the first integral term denotes the total variation of the function v within Ω and the boundary integral over the trace $v^{\partial\Omega}$ of v relaxes the homogeneous boundary conditions in terms of a penalization.

Thus

$$\mathcal{E}_{TV}(v) \to \underset{v \in BV(\Omega)}{\mathrm{Min!}} \tag{1.6}$$

subject to

$$\mathcal{G}_1(v) = \int_\Omega |v| \, \mathrm{d}x = 1 \tag{1.7}$$

is the suitable variational formulation of the eigenvalue problem for the 1-Laplace operator. Apparently a minimizer u of (1.6), (1.7) is a first eigenfuntion of the 1-Laplace operator and $\lambda_{1,0} = \mathcal{E}_{TV}(u)$ the corresponding eigenvalue.

This problem has been studied intensively by Kawohl & Schuricht [36], Schuricht [52], Milbers & Schuricht [44], [45], [46], Littig & Schuricht [41], Chang [13] and Degiovanni & Magrone [20]. In particular the formal eigenvalue equation of the 1-Laplace operator

$$-\operatorname{div} \frac{Du}{|Du|} = \lambda \frac{u}{|u|} \tag{1.8}$$

was given a well defined meaning in terms of the subgradients of $\mathcal{E}_{TV}$ and $\mathcal{G}_1$. The exis-

tence of a sequence of eigensolutions and a variety of properties of these eigensolutions have been shown. We will review these results in Chapter 2.

Highlights of Own Results and Structure of the Thesis

The contribution of this thesis is to continue and extend the research results for the eigenvalue problem of the 1-Laplace operator in three directions. The first extension concerns perturbation investigations of the eigenvalue problem of the 1-Laplace operator. We will show existence of solutions for these problems and will verify that the eigenvalues of the 1-Laplace operator are bifurcation points for a certain general class of perturbed problems. The second extension treats vector valued problems, both for u taking values in $\mathbb{R}^N$ and, with $N = n$, for the symmetrized 1-Laplace operator. The third extension yields some existence results and properties of associated parabolic problems.

The thesis is organized as follows:

In Chapter 2 we will give the precise framework of the eigenvalue problem of the 1-Laplace operator and review the previously known results of interest in connection with this thesis.

In Chapter 3 we will provide the tools from nonsmooth critical point theory, which are needed for our investigations. This chapter also presents a self-contained derivation of Clarkes generalized gradients for our perturbation functionals (Theorem 5). Even though these ideas are not totally new (cf. [12]), we did not find a reference for this result in the presented form. Note also Theorem 11, which gives existence of an unbounded sequence of critical values of a certain class of variational problems with the aid of genus. This theorem complements previous results that use category as topological index (cf. [21], [45]) and allows to simplify the proof of the existence of an unbounded sequence of eigenvalues of the 1-Laplace operator. This Theorem is the key ingredient to prove existence of eigensolutions of the perturbed 1-Laplace operator, of the vectorial 1-Laplace operator and the symmetrized 1-Laplace operator.

Chapter 4 is devoted to the investigation of the perturbed eigenvalue problem of the 1-Laplace operator. For the p-Laplace operator these questions are usually treated with the aid of the Leray-Schauder mapping degree, which is not available in our highly nonsmooth situation. We thus develop careful estimates of the involved energy functionals and sophisticated scaling arguments which make use of the 1-homogeneity of the functionals $\mathcal{E}_{TV}$ and $\mathcal{G}_1$. This finally provides local perturbation results close

to $u = 0$. These arguments are completely new, to the best knowledge of the author. In two sections we will consider both perturbations of the energy (Section 4.1) and perturbations of the constraint term (Section 4.2). In both cases we ensure for a certain class of perturbations the existence of a sequence of eigensolutions (Theorem 19 and Theorem 25) and verify that the eigenvalues of the 1-Laplace operator are bifurcation values of eigenvalues of the associated perturbed problem (Theorem 22 and Theorem 25). The proofs in Chapter 4 will need certain results in $BV(\Omega)$. We will thus refer to Propositions and Theorems of Chapter 5, where we investigate $BV(\Omega, \mathbb{R}^N)$ and where we can deduce the required statements with $N = 1$. If the reader is not familiar with these basic properties of the space $BV(\Omega)$ and the scalar eigenvalue problem of the 1-Laplace operator, we suggest to study Section 5.1 before reading Chapter 4.

In Chapter 5, which can be read independently of the foregoing chapter, we will investigate the vectorial eigenvalue problem of the 1-Laplace operator in $BV(\Omega, \mathbb{R}^N)$ and the eigenvalue problem of the symmetrized 1-Laplace operator. We demonstrate that the arguments from the scalar case can be transfered to the vectorial situation, but note that a component-wise reduction of the vectorial problem to the scalar one is not possible. The main task is to derive Gauß-Green formulas in a very general situation from that we deduce the subdifferentials of the (vectorial) total variation and the total deformation functional.

Finally, Chapter 6 is devoted to a variety of associated parabolic problems. Our method of choice for the investigation is the notion of gradient and subgradient systems. We will introduce their definition and the central existence and uniqueness results in Section 6.1 and 6.2. The 3rd section contains three applications of the concepts of (sub)gradient system. The first one treats the parabolic problem for the p-Laplace operator. This subsection is considered as an model example of a gradient system, these results are well known. The second subsections treats the Porous medium equation (PME) and the fast diffusion equation (FDE) as gradient system. Basically there are two approaches to the PME/FDE common. The first one is by classical treatment of singular semilinear diffusion equations (cf. Vázquez [57]), the second one is in terms of maximal monotone or accretive operators. However, the treatment as gradient system, which is somewhat in-between the aforementioned ones, is new and was recently published by the author in a joint work with Voigt in [42]. Let us mention that the main ideas like the derivation of the gradient in $H^{-1}(\Omega)$ and its application to prove order preservation and asymptotic behavior are due to

the author. And last but not least we consider the parabolic problem of the (vector valued) 1-Laplace operator and the symmetrized 1-Laplace operator. The problem of perfectly plastic fluids is also considered.

The appendix contains a short review on some results from geometric measure theory, linear algebra and topological indices.

Notation and Conventions

All the function spaces in the thesis are considered as real spaces without mentioning this in the following.

We will denote the eigenvalues of the p-Laplace operator by $\lambda_{k,p}$, the eigenvalues of the perturbed 1-Laplace operator, where the energy is perturbed, by $\lambda_{k,\alpha}$ and the eigenvalues of the 1-Laplace operator with perturbed constraint by $\lambda_{k,\beta}$ (and the eigenvalues of the 1-Laplace operator by $\lambda_{k,0}$). Since we do not intend to compare these eigenvalues with each other, there is no reason to be confused with the cases $p = \alpha$, $p = \beta$ or $\alpha = \beta$. This argument applies for eigenfunctions and critical values in an analogous manner. Moreover we will also use the subscript "v" to point out that we talk about eigenvalues or eigenfunctions of the vectorial 1-Laplace operator and the subscript "s" is used in the notation of eigenvalues and eigenfunctions of the symmetrized 1-Laplace operator.

The total variation functional (1.6) will always be denoted by $\mathcal{E}_{TV}$ and $\mathcal{G}_1$ will always denote the L^1-norm, both for functions with values in $\mathbb{R}$ and $\mathbb{R}^N$. The letter $\mathcal{F}$ stands for an arbitrary function in general and will be fixed from time to time.

We will use the following notation.

List of Notation

$\langle w, u\rangle_{V',V}$	dual pairing of $w \in V'$ and $u \in V$
$\langle w, u\rangle_H$	scalar product of $w, u \in H$
$a \otimes b$	tensor product of a and b, also applied pointwise between two vector valued functions a and b
$a \odot b$	symmetric tensor product of a and b, p. 86
$a \cdot b$	$:= \sum_i a_i b_i$, scalar product of a and b
$a : b$	scalar product of the matrices $a, b \in \mathbb{R}^{N\times n}$, p. 139
$B(x, r)$	open ball in a metric space with radius r around x

$B(A,r)$	open neighborhood $\{y \;;\; d(y,A) < r\}$ of A with radius r
$B_{n-1}(0,r)$	open Euclidean ball in $\mathbb{R}^{n-1}$ around zero with radius r
$BV(\Omega,\mathbb{R}^N)$	space of functions of bounded variation, p. 75
$BD^p(\Omega)$	$:= BD(\Omega) \cap L^p(\Omega,\mathbb{R}^n)$, p. 99
$BD(\Omega)$	space of functions of bounded deformation, p. 75
$BV(\Omega)$	$:= BV(\Omega,\mathbb{R}^1)$, space of scalar functions of bounded variation
$BV^p(\Omega,\mathbb{R}^N)$	$:= BV(\Omega,\mathbb{R}^N) \cap L^p(\Omega,\mathbb{R}^N)$, p. 99
$\operatorname{cat} A$	category of A, p. 140
$C_c(\Omega,\mathbb{R}^N)$	continuous functions $u : \Omega \to \mathbb{R}^N$ with compact support in Ω
$C_0(\Omega,\mathbb{R}^N)$	closure of $C_c(\Omega,\mathbb{R}^N)$ with respect to the supremum norm
$C_c^\infty(\Omega,\mathbb{R}^N)$	arbitrarily often differentiable functions $u : \Omega \to \mathbb{R}^N$ with compact support in Ω
$C_{c,\sigma}^\infty(\Omega,\mathbb{R}^n)$	space of solenoidal test functions, p. 132
χ_A	characteristic function of A, p. 15
$\mathcal{D}'(\Omega,\mathbb{R}^N)$	$\mathbb{R}^N$-valued distributions on Ω
$\partial\mathcal{F}(u)$	convex subdifferential or generalized gradient of $\mathcal{F}$ in u, cf. Subsections 3.1.1 and 3.1.2
$\lvert d\mathcal{F}\rvert(u)$	weak slope of $\mathcal{F}$ in u, p. 38
$\operatorname{div} z$	divergence of z in the sense of distributions, for z with values in $\mathbb{R}^{N\times n}$ the divergence is taken in each row separately, such that $\operatorname{div} z \in \mathbb{R}^N$ pointwise
$\operatorname{div_s} z$	symmetrized divergence of z, p. 86
$\operatorname{dom}(\mathcal{F})$	domain of an operator $\mathcal{F}$, p. 29 and p. 122, or the effective domain of definition of the functional $\mathcal{F}$, p. 28

$D_s u$	symmetrized gradient of u, p. 84
$\mathcal{F}^*$	conjugate function of $\mathcal{F}$, p. 28
$\mathcal{F}^0(u;\, v)$	generalized directional derivative of $\mathcal{F}$ in u in direction v, p. 30
$\operatorname{gen} A$	genus of A, p. 140
$\mathcal{H}^s(A)$	s-dimensional Hausdorff measure of A
I_A	indicator function of A, p. 20
$L^2_\sigma(\Omega)$	solenoidal L^2-functions, p. 132
$L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$	$:= \left\{z \in L^\infty(\Omega, \mathbb{R}^{N\times n})\ ;\ \operatorname{div} z \in L^q(\Omega, \mathbb{R}^N)\right\}$, p. 92
$L^{\infty,q}_{\mathrm{sym}}(\Omega)$	$:= \left\{z \in L^\infty(\Omega, \mathbb{R}^{n\times n})\ ;\ \operatorname{div}_{\mathrm{s}} z \in L^q(\Omega, \mathbb{R}^n)\right\}$, p. 92
$\mathcal{M}(\Omega, \mathbb{R}^N)$	set of (signed), $\mathbb{R}^N$-valued finite Radon measures on Ω, p. 137
$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_{>0}$	the set of strictly positive integers, rational and real numbers, strictly positive numbers
$(u_k)_{k\in\mathbb{N}}$	or short $(u_k)_k$, notation for a sequence
$\mathbf{P}$	$\mathbf{P} : L^2(\Omega, \mathbb{R}^n) \to L^2_\sigma(\Omega)$, the Helmholz projection, p. 133
p'	conjugate exponent of p, i.e. $1/p + 1/p' = 1$
p^*	$:= \frac{np}{n-p}$ for $p < n$ and ∞ otherwise, the Sobolev conjugate of p
$P_C u$	best approximation of the Hilbert space element u on the convex closed set C
$\operatorname{Per}(A)$	Perimeter of a set A in $\mathbb{R}^n$, p. 16
$r^{[m]}$	the signed power of r, p. 124
$\mathbb{R}^{n\times n}_{\mathrm{sym}}$	vector space of symmetric $n \times n$-matrices, p. 84
$\operatorname{Sgn}(x)$	set-valued sign function, p. 18, 29
$\mathbb{S}^k$	k-dimensional sphere, the boundary of the Euclidean unit ball in $\mathbb{R}^{k+1}$
$\operatorname{supp} \varphi$	support of the function φ

A^T	transpose of matrix A
$A^{-\mathsf{T}}$	$:= (A^{-1})^\mathsf{T} = (A^\mathsf{T})^{-1}$ for some invertible matrix A
$u^{\partial\Omega}$	trace of u on $\partial\Omega$
X^* or X'	dual space of the Banach space X
(z, Du)	Radon measure representing the pairing of z and Du, p. 100
$(z, D_s u)$	Radon measure representing the pairing of z and $D_s u$, p. 100
$[z, \nu]^{\partial\Omega}$	normal trace of z, p. 96
$[z, \nu]_s^{\partial\Omega}$	symmetrized normal trace of z, 99

Acknowledgments

Foremost, I would like to thank my supervisor Professor Schuricht for providing this fascinating research topic. Without his encouraged guidance and persistent help this thesis would not have been possible.

In my daily work I have been blessed with a friendly and cheerful group of fellow students. I am grateful to my colleagues Zoja Milbers, Henrik Freymond, Jan Mankau and Moritz Schönherr for fruitful discussions and for sharing a beautiful time with me. Particularly, I am much obliged to Zoja Milbers and Henrik Freymond for diligent and precise proofreading of the manuscript. I want to express my gratefulness to the members of the "Institute of Analysis" and the members of the "Kaffeerunde" for useful advice, fascinating discussions and providing a productive atmosphere to study.

Finally, and most importantly, I owe my deepest graditute to my beloved wife Sandra, who supported me in a way beyound words. Her encouragement, support, quiet patience and ceaseless love were undeniably the bedrock upon which the past eight years of my life have been built.

2 Review of Results for the Eigenvalue Problem of the 1-Laplace Operator

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. The variational form of the eigenvalue problem of the 1-Laplace operator is given by

$$\mathcal{E}_{TV}(v) := \int_\Omega \mathrm{d}|Dv| + \int_{\partial\Omega} |v^{\partial\Omega}| \,\mathrm{d}\mathcal{H}^{n-1} \to \underset{v\in BV(\Omega)}{\mathrm{Min!}} \tag{2.1}$$

subject to

$$\mathcal{G}_1(v) := \int_\Omega |v| \,\mathrm{d}x = \alpha\,. \tag{2.2}$$

Any minimizer of the variational problem (2.1), (2.2) for a fixed $\alpha > 0$ is called first eigenfunction of the 1-Laplace operator. By the compact embedding of $BV(\Omega)$ in $L^1(\Omega)$ and lower semicontinuity of $\mathcal{E}_{TV}$ with respect to $L^1(\Omega)$-convergence, it is not difficult to verify that such a first eigenfunction exists for any $\alpha > 0$ (cf. [36, Theorem 3.2]).

Note that both $\mathcal{E}_{TV}$ and $\mathcal{G}_1$ are 1-homogeneous and, in analogy to the eigenvalue problem of the p-Laplace operator, the first eigenvalue $\lambda_{1,0}$ of the 1-Laplace operator is defined by

$$\lambda_{1,0} := \min_{v\in BV(\Omega)\setminus\{0\}} \frac{\mathcal{E}_{TV}(v)}{\mathcal{G}_1(v)} = \min_{v\in BV(\Omega),\mathcal{G}_1(v)=1} \mathcal{E}_{TV}(v)\,.$$

Kawohl & Fridman [34] stated that the eigenvalue problem of the 1-Laplace operator is connected to the Cheeger problem. Let here and in the following χ_E denote the characteristic function of E, i.e.

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise}\,. \end{cases}$$

Recall that a set $E \subseteq \mathbb{R}^n$ is said to be of finite perimeter provided $\chi_E \in BV(\mathbb{R}^n)$ and

then

$$\mathrm{Per}(E) := |D\chi_E|(\mathbb{R}^n)\,.$$

If E has a Lipschitz boundary we have $\mathrm{Per}(E) = \mathcal{H}^{n-1}(\partial E)$, where ∂E is the topological boundary of E. The Cheeger constant h_C is defined by

$$h_C := \inf \frac{\mathrm{Per}(E)}{|E|}$$

where the infimum is taken over all sets $E \subseteq \Omega$ with finite perimeter and nonvanishing Lebesgue measure $|E|$. Any minimizer $D \subseteq \Omega$ with

$$h_C = \frac{\mathrm{Per}(D)}{|D|}$$

is a Cheeger set of Ω.

The connection to the Cheeger problem is given by the following proposition.

Proposition 1. *It is*

$$h_C = \lambda_{1,0}$$

and a function $u \in BV(\Omega)$ solves

$$\frac{\mathcal{E}_{TV}(v)}{\mathcal{G}_1(v)} \to \underset{v \in BV(\Omega)\setminus\{0\}}{\mathrm{Min!}}$$

if and only if for almost all $t \in \mathbb{R}$ the level sets

$$E_t := \begin{cases} \{u > t\} & \text{for } t > 0 \\ \{u < t\} & \text{for } t < 0 \end{cases}$$

are Cheeger sets of Ω. In particular there always exists a Cheeger set. Moreover, the first eigenfunction of the 1-Laplace operator on Ω is unique (up to scalar multiples) if and only if the Cheeger set D of Ω is unique[1].

It is well known that the characteristic function of a Cheeger set is an eigenfunction of the 1-Laplace operator. The contrary statement that a function is a minimizer if and only if all sublevel sets E_t are Cheeger sets was stated in [10] and refers to [34], where we could not verify the statement in full detail. Thus we present a proof which is due to the author [40] and extends the coarea idea of [34].

[1] Uniqueness means uniqueness of $\chi_D \in L^1(\Omega)$ of course.

Proof of Proposition 1. Obviously $\lambda_{1,0} \leq h_C$, since for each set $E \subseteq \Omega$ of finite perimeter we have $\chi_E \in BV(\Omega)$ by definition. Let now u be a minimizer of $v \mapsto \mathcal{E}_{TV}(v)/\mathcal{G}_1(v)$. By application of the coarea formula (cf. Proposition 35 below) and the Cavalieri principle we have

$$\lambda_{1,0} = \frac{\mathcal{E}_{TV}(u)}{\mathcal{G}_1(u)} = \frac{\int_{\mathbb{R}} \operatorname{Per}(E_t)\,\mathrm{d}t}{\int_\Omega u^+ + u^-\,\mathrm{d}x} = \frac{\int_{\mathbb{R}} \operatorname{Per}(E_t)\,\mathrm{d}t}{\int_{\mathbb{R}} |E_t|\,\mathrm{d}t}\,,$$

thus

$$\int_{\mathbb{R}} \operatorname{Per}(E_t) - \lambda_{1,0}|E_t|\,\mathrm{d}t = 0\,.$$

Since $\lambda_{1,0} \leq h_C \leq \frac{\operatorname{Per}(E_t)}{|E_t|}$ for almost all $t \in \mathbb{R}$ we derive

$$\operatorname{Per}(E_t) - \lambda_{1,0}|E_t| \geq \operatorname{Per}(E_t) - h_C|E_t| \geq 0$$

and thus

$$\operatorname{Per}(E_t) - \lambda_{1,0}|E_t| = 0$$

by the integral representation above for almost all $t \in \mathbb{R}$. But this yields

$$\lambda_{1,0} = \frac{\operatorname{Per}(E_t)}{|E_t|} \geq h_C$$

for almost all $t \in \mathbb{R}$ with $|E_t| \neq 0$, thus

$$h_C = \lambda_{1,0}\,.$$

If on the other hand almost all level sets E_t of an $L^1(\Omega)$-function u are sets of finite perimeter we have again by the Cavalieri principle and the coarea formula

$$\lambda_{0,1}\,\mathcal{G}_1(u) = h_C \int_{\mathbb{R}} |E_t|\,\mathrm{d}t = \int_{\mathbb{R}} \operatorname{Per}(E_t)\,\mathrm{d}t = \mathcal{E}_{TV}(u)\,,$$

i.e. u is a first eigenfunction of the 1-Laplace operator. □

One usually uses the fact that $\mathcal{E}_{TV}(u) \geq \mathcal{E}_{TV}(|u|)$ for all $u \in BV(\Omega)$ and thus restricts the attention to nonnegative u (cf. e.g. [34]). However, in the general case the "only if"-part requires usage of the coarea formula as formulated in Proposition 35.

Let us note that the Cheeger set (and thus the first eigenfunction of the 1-Laplace operator) is unique, provided Ω is convex (cf. [1]). Nevertheless, in contrast to the p-Laplace operator there exist connected Ω with nonunique first eigenfunction (cf. [35]

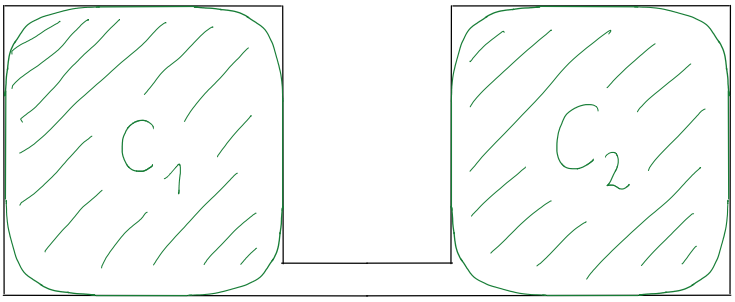

Figure 2.1: The sets C_1 and C_2 and $C_1 \cup C_2$ are Cheeger sets of Ω.

and Figure 2.1).

Since $\mathcal{E}_{TV}$ and $\mathcal{G}_1$ are 1-homogeneous, it is convenient to restrict the attention to normalized eigenfunctions, i.e. eigenfunctions with $\alpha = 1$.

Kawohl & Schuricht [36] investigated the variational formulation of the 1-Laplace operator and derived as necessary condition for a minimizer u of (2.1), (2.2)

$$0 \in \partial\mathcal{E}_1(u) - \lambda\partial\mathcal{G}_1(u)\,,$$

where $\partial\mathcal{E}_1(u)$ and $\partial\mathcal{G}_1(u)$ denote the subdifferentials of $\mathcal{E}_1$ and $\mathcal{G}_1$ in u. This condition is usually stated as single version of the Euler-Lagrange equation. Before we give this precise statement, let us introduce the set-valued sign function

$$\operatorname{Sgn}(r) := \begin{cases} \{-1\} & \text{for } r < 0 \\ [-1,1] & \text{for } r = 0 \\ \{1\} & \text{for } r > 0\,. \end{cases} \tag{2.3}$$

Now the single version of the Euler-Lagrange equation says that there exists $s \in L^\infty(\Omega)$ with

$$s(x) \in \operatorname{Sgn}(u(x))$$

for almost all $x \in \Omega$ and there is a $\lambda \in \mathbb{R}$ and a function $z \in L^\infty(\Omega, \mathbb{R}^n)$ with

$$\|z\|_\infty = 1\,, \quad \operatorname{div} z \in L^n(\Omega) \quad \text{and} \quad -\int_\Omega u \operatorname{div} z \,\mathrm{d}x = \mathcal{E}_{TV}(u)$$

such that the Euler-Lagrange equation

$$-\operatorname{div} z = \lambda s \tag{2.4}$$

holds. Note that s replaces the undetermined expression $u/|u|$ and z replaces the undetermined expression $Du/|Du|$ in the formal equation for the eigenvalue problem of the 1-Laplace operator (1.8). Using u as test function in (2.4) we derive $\lambda = \mathcal{E}_{TV}(u)/\alpha$ and thus obtain a direct correspondence of $\mathcal{E}_{TV}(u)$ and the eigenvalue λ for normalized eigenfunctions.

Moreover Kawohl & Schuricht pointed out that, for a minimizer u of the variational problem (2.1), (2.2), the relation

$$\lambda \partial \mathcal{G}_1(u) \subseteq \partial \mathcal{E}_1(u)$$

is satisfied. In particular this leads to the multiple version of the Euler-Lagrange equation which says that *for any* function $s \in L^\infty(\Omega)$ with $s(x) \in \mathrm{Sgn}(u(x))$ for almost all $x \in \Omega$ a function z with the properties described above exists, such that the Euler-Lagrange equation (2.4) holds.

The question appears how to define higher eigenfunctions of the 1-Laplace operator. The first naive idea might be to use the multiple version of the Euler-Lagrange equation (2.4) to define higher eigenfunctions, but its derivation relies deeply on the energy minimizing property and we may thus get the first eigenfunction only. The second natural idea is to call any solution of the single version of the Euler-Lagrange equation an eigenfunction of the 1-Laplace operator. But this condition turns out to have too many solutions (see [44]). In particular one can show that the single version of the Euler-Lagrange equation is satisfied for any function u of the form $u = \chi_B$, where B is a ball compactly contained in Ω. The number $\lambda = \frac{\mathcal{H}^{n-1}(\partial B)}{|B|}$ would be the associated eigenvalue. Since neither the radius nor the midpoint of B are fixed, we would end up with a continuum of eigenvalues and infinitely many associated normalized eigenfunctions whose structure does not really depend on the geometry of Ω.

Recall that a function u with $\|u\|_p = 1$ satisfies the eigenvalue equation of the p-Laplace operator (1.1) if and only if u is a critical point of the associated constraint variational problem (1.2), (1.3). We thus define higher eigenfunctions of the 1-Laplace operator as critical points of (2.1), (2.2). Since both $\mathcal{E}_{TV}$ and $\mathcal{G}_1$ are not differentiable we need to specify, what is meant by "critical point" in this context. It turned out that the notion of the weak slope is the most powerful concept to define critical points here. To be a bit more precise, we understand, for some $p \in [1, n/(n-1)]$, the function

$\mathcal{E}_{TV}$ to be extended on $L^p(\Omega)$ by

$$\mathcal{E}_{TV}(v) = \infty \quad \text{for } v \in L^p(\Omega) \setminus BV(\Omega) \tag{2.5}$$

and we say that u is an eigenfunction of (2.1), (2.2) provided u is a critical point in the sense of the weak slope of $\mathcal{F} : L^p(\Omega) \to \mathbb{R}$,

$$\mathcal{F}(v) := \mathcal{E}_{TV}(v) + I_{\{\mathcal{G}_1 = \alpha\}}(v)\,. \tag{2.6}$$

Here and in the following I_A denotes the indicator function of the set A which is defined by

$$I_A(v) = \begin{cases} 0 & \text{if } v \in A \\ \infty & \text{otherwise}\,. \end{cases} \tag{2.7}$$

We refer to Subsection 3.1.3 below for details on the weak slope $|d\mathcal{F}|$, which is zero in critical points of $\mathcal{F}$.

Note that the definition of the weak slope depends on the metric chosen in the domain of definition of the functional $\mathcal{F}$, in particular it depends on the choice of $p \in [1, n/(n-1)]$. In contrast to the p-Laplace operator, where $W_0^{1,p}(\Omega)$ induces the natural metric to investigate the eigenvalue problem, for $p = 1$ it turns out that we get "too many" eigensolutions if we consider the weak slope with respect to the metric induced by the $BV(\Omega)$-norm (cf. [46, Section 5.1]).

Milbers & Schuricht [46] derived a further necessary condition for critical points of $\mathcal{F}$ by inner variation and this condition excludes the functions $u = \chi_B$ from above as critical points. This encourages us to use the notion of the weak slope to define higher eigenfunctions of the 1-Laplace operator. However, in our highly nonsmooth situation there seems to be no way to completely characterize higher eigenfunctions in terms of a partial differential equation as e.g. in the case of the p-Laplace operator at the moment.

Milbers & Schuricht [45] verified the existence of a sequence of eigenfunctions $(\pm u_{k,0})_{k\in\mathbb{N}}$ of the 1-Laplace operator with tools from nonsmooth critical point theory (cf. also [13]). The corresponding eigenvalues

$$\lambda_{k,0}$$

are defined with the aid of category and the corresponding eigenfunctions $u_{k,0}$ satisfy

the single version of the Euler-Lagrange equation. This sequence of eigenfunctions $(u_{k,0})_{k\in\mathbb{N}}$ is a sequence of critical points both with respect to the $L^1(\Omega)$- and the $L^p(\Omega)$-metric (cf. [41, Remark 2.3]).

In Littig & Schuricht [41, Corollary 2.2] was verified that the eigenvalues of the 1-Laplace operator constructed with the aid of category coincide with the values constructed with the help of genus as topological index. We thus have

$$\lambda_{k,0} = \inf_{S\in\mathscr{S}_k^1} \sup_{u\in S} \mathcal{E}_{TV}(u) \tag{2.8}$$

with

$$\mathscr{S}_k^1 := \{S \subseteq L^p(\Omega) \text{ compact, symmetric } ;\ \mathcal{G}_1 = 1 \text{ on } S,\ \operatorname{gen}_{L^p} S \geq k\}\,, \tag{2.9}$$

where $\operatorname{gen}_{L^p} S$ denotes the genus of S in $L^p(\Omega)$, a topological index whose definition and properties are given in Appendix 7.3. It turns out that the values $\lambda_{k,0}$ remain the same for any choice $p \in [1, n/(n-1))]$ (with $p < \infty$ for $n = 1$). Littig & Schuricht also proved that, for any $k \in \mathbb{N}$, the variational eigenvalues $\lambda_{k,p}$ of the p-Laplace operator (1.4) converge as $p \to 1$ to the variational eigenvalues $\lambda_{k,0}$ of the 1-Laplace operator (cf. [41, Theorem 2.14]).

Note that the single eigenvalue equation (2.4) is satisfied for $u = 0$ with $s = 0$, $z = 0$ and arbitrary $\lambda \in \mathbb{R}$ and $u = 0$ is the only and trivial critical point of the variational problem (2.1), (2.2) for $\alpha = 0$. From this perspective it is reasonable to consider the eigenvalues of the 1-Laplace operator as bifurcation points: for an eigensolution (λ, u) the eigensolutions $(\lambda, \alpha u)_{\alpha\in\mathbb{R}}$ branch off from the trivial solution curve $(\gamma, 0)_{\gamma\in\mathbb{R}}$. Thus the structure of the eingensolutions of the 1-Laplace operator is given in Figure 2.2

2.1 Perturbation Results for the p-Laplace Operator

Nonlinear eigenvalue problems have a long history and bring together the concepts of critical point theory, partial differential equations and nonlinear functional analysis. Typically bifurcation points of nonlinear eigenvalue problems are related to a homogenized (nonlinear) eigenvalue problem and their investigation provides stability statements. These results find, within others, application in numerical approximation schemes where bifurcations often lead to challenges. A good introduction in the concepts of nonlinear eigenvalue problems are the articles [48] and [49] of Rabinowitz,

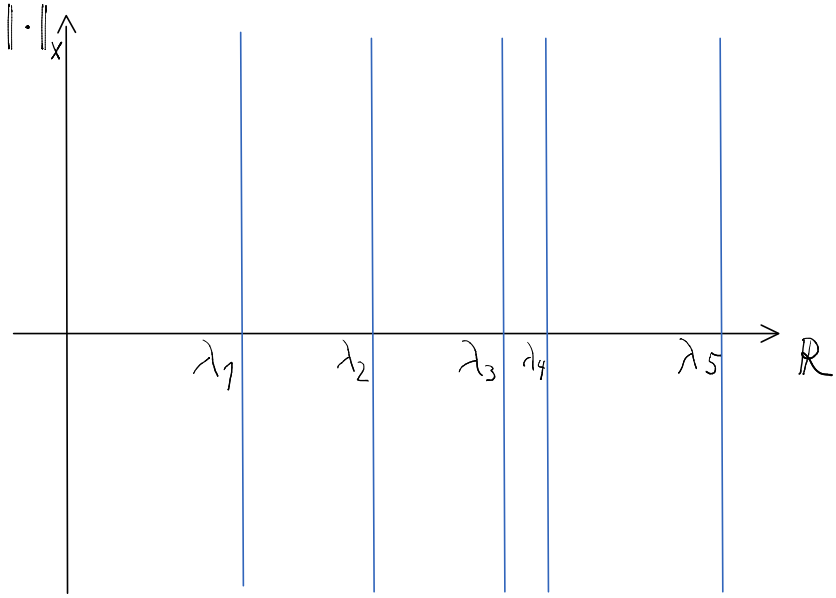

Figure 2.2: Structure of the eigensolutions (λ, u).

where it is shown how the Leray-Schauder mapping degree may be used to investigate global bifurcation results for eigenvalue problems of the form

$$u = G(\lambda, u)$$

where $G : \mathbb{R} \times X \to X$ is a compact and continuous operator mapping to a real Banach space X. Since G is not assumed to be linear in u, the problem does not have the structure of a proper eigenvalue problem in general but is merely of parametric nature. However, we will keep the classical nomenclature of Rabinowitz. A special focus is on the case $G(\lambda, u) = \lambda Lu + H(\lambda, u)$, where L is a compact linear operator and H is $o(\|u\|)$ as $\|u\| \to 0$ uniformly with respect to λ on bounded intervals $I \subseteq \mathbb{R}$. Thus $(\lambda, 0)_{\lambda \in \mathbb{R}}$ is a trivial solution curve. A point $(\mu, 0)$ is called bifuraction point provided any neighborhood of $(\mu, 0)$ contains nontrivial solutions. A necessary condition is that μ^{-1} belongs to the spectrum of L, while the contrary is not true in general; not any number μ^{-1} in the spectrum gives a bifurcation point $(\mu, 0)$ in general [49, p. 162]. However, provided the eigenvalue μ^{-1} is of odd multiplicity, μ will be a bifurcation point [49, Theorem 1.4]. Moreover, in that case there evolves a solution curve which is either unbounded or contains $(\hat{\mu}, 0)$, with $\hat{\mu} \neq \mu$ (cf. [49, Theorem 1.10]).

The first important article on global bifurcation of the eigenvalues of the p-Laplace operator is due to del Pina & Manásevich [25], where the authors proved that the first

eigenvalue of the p-Laplace operator is a bifurcation point of the perturbed eigenvalue equation of the p-Laplace operator on $\Omega \subseteq \mathbb{R}^n$ open, bounded and with $C^{2,\beta}$-smooth boundary. In particular if λ_1 is the first eigenvalue of the p-Laplace operator they show that a branch of solutions (λ, u) of

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u + f(x, u, \lambda) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

evolves from $(\lambda_1, 0)$. Here they assume that for the continuous function f

$$f(x, s, \lambda) = o(|s|^{p-1}) \quad \text{as } s \to 0 \tag{2.10}$$

and, for some $q \in (1, p^*)$

$$f(x, s, \lambda) = o(|s|^{q-1}) \quad \text{as } s \to \infty\,, \tag{2.11}$$

both properties uniformly with respect to a.e. $x \in \Omega$ and with respect to λ on bounded sets. Here and in the following p^* denotes the Sobolev exponent

$$p^* := \begin{cases} \frac{np}{n-p} & \text{if } p < n \\ \infty & \text{if } p \geq n\,. \end{cases}$$

This problem can be studied by investigation of the properties of the operator $G : W_0^{1,p} \to W_0^{1,p}$,

$$G(u) = (-\Delta_p)^{-1} \left(\lambda |u|^{p-2} u + f(x, u, \lambda) \right)\,.$$

In particular one has to calculate the mapping degree of $u \mapsto u + G(u)$. Wherever these methods apply we may think of the eigensolutions of the perturbed p-Laplace operator as in Figure 2.3

A very detailed and self-contained review of the known results of the perturbed eigenvalue problem of the p-Laplace operator on smooth bounded domains $\Omega \subseteq \mathbb{R}^n$ is given in the lecture notes of Peral [47]. While the first chapter of these notes deals with the existence of a sequence of eigensolutions of the p-Laplace operator (making use of the differentiable framework and classical Lusternik-Schnirelman theory) and their properties, the second chapter is devoted to the perturbed eigenvalue problem

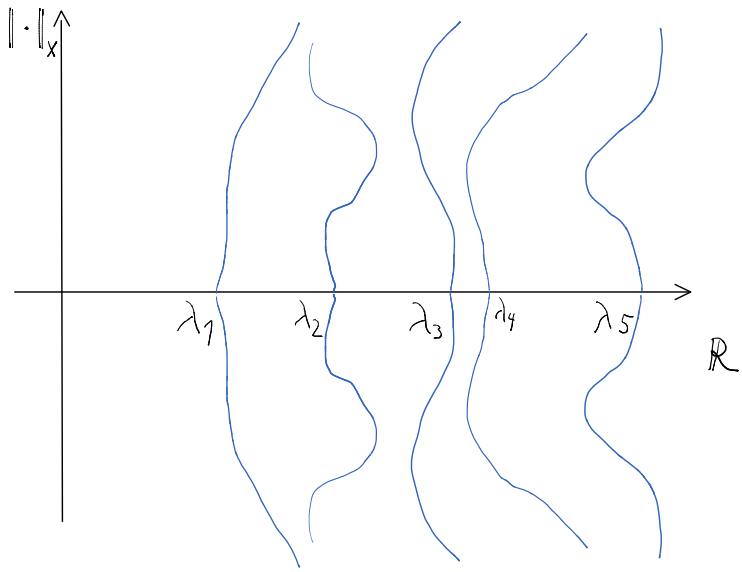

Figure 2.3: Eigensolutions of the perturbed problem.

of the p-Laplace operator

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{r-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

For the parameters it is assumed $1 < q$, $1 < p < r$ and $\lambda > 0$.

However, all the methods to prove these results rely on the nice properties of the p-Laplace operator $W_0^{1,p}(\Omega) \ni u \mapsto -\operatorname{div}|Du|^{p-2}Du \in W^{-1,p'}(\Omega)$ and the nonlinear operator $L^p(\Omega) \ni u \mapsto |u|^{p-2}u \in L^{p'}(\Omega)$. In fact, both operators turn out to be normalized duality mappings and are thus homeomorphisms by the smoothness of the underlying norms. This allows e.g. to use degree theory to treat the solution branches (λ, u) of the perturbed eigenvalue problem of the p-Laplace operator which leads to global bifurcation results. To the best knowledge of the author these mapping degree concepts are not available in our nonsmooth situation.

Note that in contrast to the p-Laplace operator the 1-Laplace operator is far away from being invertible. Any reasonable notion of the 1-Laplace operator in terms of a subdifferential is multi-valued and by 1-homogeneity of $\mathcal{E}_{TV}$ we even have $\partial\mathcal{E}_{TV}(\alpha u) = \partial\mathcal{E}_{TV}(u)$ for any $\alpha > 0$ such that we even cannot expect invertibility of the 1-Laplace operator on 1-dimensional subspaces. Thus we follow a completely different approach and develop estimates of the energy functions involved that allow

to investigate bifurcation results for the 1-Laplace operator at least locally for u close to zero in Chapter 4.

3 Concepts of Nonsmooth Critical Point Theory

3.1 Generalized Concepts of Differentiability

The eigenvalue problem of the 1-Laplace operator is a model example of a variational problem lacking differentiability. Basically we face three major challenges concerning differentiability. The first point is that the integrands of the leading energy functionals (and thus the functionals themselves) are convex but not differentiable. This case is rather standard to treat with the subdifferential of convex analysis. We will summarize the main concepts in Subsection 3.1.1.

The second challenge appears in the perturbation problems that we will investigate, where the associated potentials of the perturbation terms are Lipschitz continuous but again not differentiable in general. The notion of Clarkes generalized gradient is the method of choice to treat this and we will introduce these concepts in the second subsection of this chapter.

The full complexity arises, when we consider constrained variational problems of the type

$$\mathcal{F}(u) + \mathcal{F}_{\mathrm{Per}}(u) \to \mathrm{Min!}$$

subject to

$$\mathcal{G}(u) + \mathcal{G}_{\mathrm{Per}}(u) = \alpha\,.$$

Here α is a given parameter, $\mathcal{F}$ is merely convex and lower semicontinous, $\mathcal{G}$ is convex and continuous (and thus locally Lipschitz continuous, see below) and $\mathcal{F}_{Per}$ and $\mathcal{G}_{\mathrm{Per}}$ are (locally) Lipschitz continuous perturbations. The problem in this form can neither directly be treated with methods from convex analysis nor with the notion of Clarkes generalized gradients. However, we can paraphrase the constrained problem to an unconstrained problem

$$\mathcal{F}(u) + \mathcal{F}_{\mathrm{Per}}(u) + I_{\{\mathcal{G}+\mathcal{G}_{\mathrm{Per}}=\alpha\}}(u) \to \mathrm{Min!} \tag{3.1}$$

where I denotes the indicator function (2.7). This functional (3.1) turns out to be lower semicontinuous and the notion of the weak slope applies. Moreover, due to the special structure of the setting we recall a Lagrange multiplier rule derived in [22] as necessary condition for critical points in the sense of the weak slope. These results will be presented in the third subsection and connect the concepts of the convex subdifferential and Clarkes generalized gradient.

3.1.1 Convex Subdifferential

For this subsection let X be a reflexive Banach space and X^* the associated dual space[1]. Let $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ be convex. The *effective domain of definition* $\mathrm{dom}(\mathcal{F})$ of $\mathcal{F}$ is defined by

$$\mathrm{dom}(\mathcal{F}) := \{u \in X \; ; \; \mathcal{F}(u) < \infty\} \, .$$

In this situation the functional $\mathcal{F}$ is called *proper* provided $\mathrm{dom}(\mathcal{F}) \neq \emptyset$.

The *epigraph* $\mathrm{epi}(\mathcal{F})$ of a functional $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\mathrm{epi}(\mathcal{F}) := \{(u, \beta) \in X \times \mathbb{R} \; ; \; \beta \geq \mathcal{F}(u)\} \, . \tag{3.2}$$

It is not difficult to see that $\mathcal{F}$ is convex if and only if $\mathrm{epi}(\mathcal{F})$ is convex and $\mathcal{F}$ is lower semicontinuous if and only if $\mathrm{epi}(\mathcal{F})$ is closed.

The *conjugate function* $\mathcal{F}^* : X^* \to \mathbb{R} \cup \{\infty\}$ of $\mathcal{F}$ is defined by

$$\mathcal{F}^*(u^*) := \sup_{u \in X} \langle u^*, u \rangle - \mathcal{F}(u) \, .$$

As supremum of affine functions it is easily seen that $\mathcal{F}^*$ is convex and lower semicontinuous. This construction can be iterated and it turns out that $\mathcal{F}^{**} = \mathcal{F}$ for convex, lower semicontinuous functions $\mathcal{F}$.

Provided a convex functional defined on a Banach space is bounded on some open subset of X, it is locally Lipschitz continuous on all of the interior of the effective domain of definition (cf. [16, Prop. 2.2.6, p. 34]).

The *subdifferential* $\partial\mathcal{F}(u)$ of a convex function $\mathcal{F}$ in some point u is defined by

$$\partial\mathcal{F}(u) := \{u^* \in X^* \; ; \; \mathcal{F}(u) \text{ finite and } \forall v \in X : \langle u^*, v - u \rangle + \mathcal{F}(u) \leq \mathcal{F}(v)\}$$

[1] In the following we will sometimes consider the slightly more general situation $X = L^1(\Omega)$. In this context the dual space $X^* = L^\infty(\Omega)$ is considered to be equipped with the weak* topology and the results hold in an analogous manner.

and is a closed convex subset of X^*. The function $\mathcal{F}$ is said to be subdifferentiable at $u \in X$, provided $\partial\mathcal{F}(u) \neq \emptyset$ and the domain of the subdifferential $\mathrm{dom}(\partial\mathcal{F})$ is

$$\mathrm{dom}(\partial\mathcal{F}) := \{u \in X \; ; \; \partial\mathcal{F}(u) \neq \emptyset\} \, .$$

The subdifferential can be characterized via conjugate functions. In particular the Fenchel identity

$$u^* \in \partial\mathcal{F}(u) \quad \text{ if and only if } \quad \mathcal{F}^*(u^*) + \mathcal{F}(u) = \langle u^*, u \rangle \tag{3.3}$$

holds (cf. [56, II, Prop. 5.1]).

We have the following calculus rules for the subdifferential. Let $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ be convex. Then for all $\alpha > 0$ and $u \in X$ we have

$$\partial(\alpha\mathcal{F})(u) = \alpha\partial\mathcal{F}(u) \, .$$

Moreover, for convex functions $\mathcal{F}_1$, $\mathcal{F}_2$ there is always

$$\partial(\mathcal{F}_1 + \mathcal{F}_2)(u) \supseteq \partial\mathcal{F}_1(u) + \partial\mathcal{F}_2(u) \, , \tag{3.4}$$

however equality does not hold in general.

In the following we will need the subdifferential of the $L^1(\Omega, \mathbb{R}^N)$-norm, in Chapter 4 with $N = 1$ and in Chapter 5 for $N \geq 1$. Even though well known for people familiar with convex analysis we will state a proof for completeness.

Proposition 2. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded, $p \in [1, \infty)$ and define $\mathcal{G}_1 : L^p(\Omega, \mathbb{R}^N) \to \mathbb{R}$ by $\mathcal{G}_1(u) = \int_\Omega |u| \, \mathrm{d}x$. Then $\mathcal{G}_1$ is subdifferentiable at all points $u \in L^p(\Omega, \mathbb{R}^N)$ and we have $u^* \in \partial\mathcal{G}_1(u)$ if and only if for almost all $x \in \Omega$ the relation $u^*(x) \in \mathrm{Sgn}(u(x))$ holds, where* Sgn *denotes the subdifferential of the Euclidean norm in $\mathbb{R}^N$, i.e.*

$$\mathrm{Sgn}(y) := \begin{cases} \left\{ \frac{y}{|y|} \right\} & \text{if } y \neq 0 \\ \overline{B(0,1)} & \text{if } y = 0 \end{cases}$$

Note that this notation is consistent with the notation introduced in (2.3).

Proof. To keep the situation simple[2] let us concentrate on the case $p \in (1, \infty)$.

Let us define the closed and convex set

$$S := \{u^* \in L^{p'}(\Omega, \mathbb{R}^N) \; ; \; |u^*(x)| \leq 1 \text{ for a. e. } x \in \Omega\} \, .$$

Thus the indicator functional I_S is a convex lower semicontinuous functional on $L^{p'}(\Omega, \mathbb{R}^N)$. Let now $u \in L^p(\Omega, \mathbb{R}^N)$ and let us calculate the conjugate functional of I_S

$$I_S^*(u) = \sup_{u^* \in S} \int_\Omega u^*(x) \cdot u(x) \, \mathrm{d}x = \int_\Omega |u(x)| \, \mathrm{d}x \, . \tag{3.5}$$

We have thus shown $I_S^* = \mathcal{G}_1$ and the supremum is achieved for any u^* with $u^*(x) \in \mathrm{Sgn}(u(x))$ for almost every $x \in \Omega$. The assertion follows then from the Fenchel identity (3.3). □

3.1.2 Clarkes Generalized Gradient

The notion of Clarkes generalized gradient extends the concept of the gradient to merely locally Lipschitz continuous functionals on a Banach space X. The following tools and properties are basically due to the monograph of Clarke [16]. Given $\mathcal{F} : X \to \mathbb{R}$ locally Lipschitz continuous we define the generalized directional derivative $\mathcal{F}^0(u; v)$ of $\mathcal{F}$ at $u \in X$ in the direction $v \in X$ by

$$\mathcal{F}^0(u; v) := \limsup_{\substack{t \downarrow 0 \\ w \to u}} \frac{\mathcal{F}(w + tv) - \mathcal{F}(w)}{t} \, .$$

Note that letting w tend to u in some sense considers the directional derivatives in a neighborhood of $u \in X$. This is the key point to get a continuity property of Clarkes generalized gradient that turns out to be very fruitful for the analysis. The function $v \mapsto \mathcal{F}^0(u; v)$ is positively homogeneous and subadditive, such that we can define the generalized gradient of $\mathcal{F}$ at $u \in X$ by

$$\partial \mathcal{F}(u) := \{u^* \in X^* \; ; \; \forall v \in X : \; \langle u^*, v \rangle \leq \mathcal{F}^0(u; v)\} \, .$$

[2] For $p = 1$ the arguments concerning the topology in $L^{p'}(\Omega, \mathbb{R}^N) = L^\infty(\Omega, \mathbb{R}^N)$ need to be modified in the sense that one has to work with the weak*-topology in $L^\infty(\Omega, \mathbb{R}^N)$. The calculations are the same.

By an easy application of the Hahn-Banach theorem $\partial\mathcal{F}(u) \neq \emptyset$ and the elements of $\partial\mathcal{F}(u)$ are norm bounded by the Lipschitz constant L that holds for $\mathcal{F}$ in a neighborhood of u. Moreover, the formula

$$\mathcal{F}^0(u;\, v) = \max_{u^* \in \partial\mathcal{F}(u)} \langle u^*, v \rangle \tag{3.6}$$

holds.

Provided a functional $\mathcal{F}$ is continuously differentiable at $u \in X$ with derivative $\mathcal{F}'(u) \in X^*$ we have

$$\partial\mathcal{F}(u) = \{\mathcal{F}'(u)\}\,.$$

However if $\mathcal{F}$ is not continuously differentiable Clarkes generalized gradient might be strictly larger than $\{\mathcal{F}'(u)\}$.

Clarkes generalized gradient is upper semicontinuous, i.e. we have

$$\partial\mathcal{F}(u) = \bigcap_{\delta>0} \overline{\mathrm{co}} \bigcup_{w \in B_\delta(u)} \partial\mathcal{F}(w)\,,$$

where $\overline{\mathrm{co}}A$ denotes the closed convex hull of A (cf. [16, Proposition I 2.1.5]).

Example 3. *The function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $g(0) = 0$ is well known to be differentiable with discontinuous derivative*

$$g'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x^2) & \textit{for } x \neq 0 \\ 0 & \textit{for } x = 0\,. \end{cases}$$

Recalling continuity of g' for $x \neq 0$ and upper semicontinuity of Clarkes generalized gradient we derive

$$\partial g(x) = \begin{cases} \{2x\sin(1/x) - \cos(1/x^2)\} & \textit{for } x \neq 0 \\ [-1, 1] & \textit{for } x = 0\,. \end{cases}$$

Note that convex functions, which are finite at some interior point of the effective domain are locally Lipschitz continuous on all of the interior of the effective domain (cf. [16, p. 34]) and it turns out that the notion of the convex subdifferential and Clarkes generalized gradient coincide. Moreover, in that case $\mathcal{F}^0(u;\, v)$ coincides with the one-sided directional derivative $\mathcal{F}'(u, v) := \lim_{t\downarrow 0} \frac{\mathcal{F}(u+tv)-\mathcal{F}(u)}{t}$ (cf. [16, Proposition 2.2.7, p. 36]).

For the notion of Clarkes generalized gradient, calculus rules similar to those of the subdifferential hold. For scalars $\alpha \in \mathbb{R}$ we have

$$\partial(\alpha\mathcal{F})(u) = \alpha\partial\mathcal{F}(u)$$

and Clarkes generalized gradient of a sum of two functions may be estimated by

$$\partial(\mathcal{F}_1 + \mathcal{F}_2)(u) \subseteq \partial\mathcal{F}_1(u) + \partial\mathcal{F}_2(u)\,, \tag{3.7}$$

while equality in this formula fails in general (cf. [16, p. 38f]).

For the notion of Clarkes generalized gradient a mean value theorem, the Theorem of Lebourg (cf. [16, p. 41]), holds: Let $u, v \in X$ and assume that $\mathcal{F} : X \to \mathbb{R}$ is Lipschitz continuous on an open neigborhood of the line segment $\{tu + (1-t)v \ ; \ t \in [0,1]\}$, then there exists a point $w = tu + (1-t)v$, $t \in (0,1)$, and some $w^* \in \partial\mathcal{F}(w)$, such that

$$\mathcal{F}(u) - \mathcal{F}(v) = \langle w^*, u - v\rangle\,. \tag{3.8}$$

Our major application of the provided tools will be the derivation of Clarkes generalized gradient for functionals $\mathcal{F}$ of Nemytsky type, i.e. $\mathcal{F}(u) = \int_\Omega F(x, u(x))\,\mathrm{d}x$, where $F(x,\cdot)$ is absolutely continuous and satisfies certain growth restrictions. Before we treat this general case let us consider the following simpler proposition (which is [16, Example II, 2.2.5.]).

Proposition 4. *Let $h \in L^\infty(\mathbb{R},\mathbb{R})$ and define $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ by*

$$\mathcal{F}(x) := \int_0^x h(s)\,\mathrm{d}s\,.$$

Define the essential supremum of h at $x \in \mathbb{R}$ by

$$\operatorname*{ess\,sup}_{y\to x} h(y) := \lim_{r\downarrow 0} \operatorname*{ess\,sup}_{y\in B(x,r)} h(y) \tag{3.9}$$

and the essential infimum of h in $x \in \mathbb{R}$ in an analogous manner. Then $\mathcal{F}$ is locally Lipschitz continuous and Clarkes generalized gradient is given by

$$\partial\mathcal{F}(x) = [\operatorname*{ess\,inf}_{y\to x} h(y),\ \operatorname*{ess\,sup}_{y\to x} h(y)]\,.$$

This result is needed in the derivation of Clarkes generalized gradients of the per-

turbation functionals $\mathcal{E}_{\mathrm{Per}}$ and $\mathcal{G}_{\mathrm{Per}}$, which we consider below. In order to present a complete self-contained proof let us thus recall the proof of Proposition 4 from [16, p. 34].

Proof. It is easily seen that $\mathcal{F}$ is globally Lipschitz continuous. Since $\mathcal{F}$ is absolutely continuous it is differentiable almost everywhere with derivative $\mathcal{F}'(x) = h(x)$ for almost all $x \in \mathbb{R}$. For those points of differentiability we have $h(x) \in \partial\mathcal{F}(x)$. Thus, from upper semicontinity we conclude that $\partial\mathcal{F}(x)$ contains all essential cluster points of h at x, that is all cluster points of h at x that persist, after removing arbitrary sets E with measure zero from $\mathbb{R}$. This implies, together with the fact that $\partial\mathcal{F}(x)$ is convex and closed, that $[\operatorname{ess\,inf}_{y\to x} f(y), \operatorname{ess\,sup}_{y\to x} f(y)] \subseteq \partial\mathcal{F}(x)$. To prove the reverse inclusion we recall $\mathcal{F}(y+t) - \mathcal{F}(y) = \int_y^{y+t} h(s)\,\mathrm{d}s$, such that it is easily seen that $\mathcal{F}^0(x;1) \leq \operatorname{ess\,sup}_{y\to x} h(y)$. However, by definition of Clarkes generalized gradient this implies $\operatorname{ess\,sup}_{y\to x} h(y) \geq \mathcal{F}^0(x;1) \geq 1x^*$ for all $x^* \in \partial\mathcal{F}(x)$. Similarly we obtain $\operatorname{ess\,inf}_{y\to x} h(y) \leq x^*$ for all $x^* \in \partial\mathcal{F}(x)$. □

Let us now derive Clarkes generalized gradient of a function which is related to the potential of a Nemytskii type operator with weaker continuity assumptions on the integrand than usual. Note that similar results of this type are known (cf. e.g. [12, Chapter 2]), however the statement under the specific symmetry and growth assumptions is due to the author and the statements on the norm bounds on the functional and Clarks generalized gradients are due to the author. We give our own self-contained proof, which does not require rather abstract measurability arguments.

Theorem 5. *Let $\Omega \subseteq \mathbb{R}^n$ be open and let $p \in (1,\infty)$. Let $f\colon \Omega \times \mathbb{R}$ be integrable on $\Omega \times [-T,T]$ for any $T > 0$ and assume*

(i)

$$f(\xi, s) = -f(\xi, -s) \tag{3.10}$$

for almost all $(\xi, s) \in \Omega \times \mathbb{R}$ and

(ii) there exists $C_{\mathrm{Per}} > 0$ such that

$$|f(\xi, s)| \leq p\, C_{\mathrm{Per}} |s|^{p-1} \tag{3.11}$$

for almost all $\xi \in \Omega$.

Then $\mathcal{F}\colon L^p(\Omega) \to \mathbb{R}$ *given by*

$$\mathcal{F}(u) := \int_\Omega \int_0^{u(x)} f(x,s)\,\mathrm{d}s\,\mathrm{d}x. \tag{3.12}$$

is a well defined even function and $\mathcal{F}(u) = \mathcal{F}(|u|)$ *holds. Moreover,* $\mathcal{F}$ *is Lipschitz continuous on bounded subsets of* $L^p(\Omega)$ *and if* $u^* \in L^{p'}(\Omega)$ *belongs to Clarkes generalized gradient* $\partial\mathcal{F}(u)$ *at* u*, then for almost all* $x \in \Omega$

$$u^*(x) \in \left[\operatorname*{ess\,inf}_{y\to u(x)} f(x,y), \operatorname*{ess\,sup}_{y\to u(x)} f(x,y)\right]. \tag{3.13}$$

In particular we have

$$\|u^*\|_{p'} \le p\, C_{\mathrm{Per}}\, \|u\|_p^{p-1} \tag{3.14}$$

for all $u^* \in \partial\mathcal{F}(u)$*.*

Moreover, for all $u \in L^p(\Omega)$ *and all* $\alpha \ge 0$ *the estimate*

$$|\mathcal{F}(\alpha u)| \le \alpha^p\, C_{\mathrm{Per}} \|u\|_p^p \tag{3.15}$$

holds true.

Apparently provided we assume continuity of f with respect to the second variable Clarkes generalized gradient reduces to a singleton $\partial\mathcal{F}(u) = \{u^*\}$ with

$$u^*(x) = f(x, u(x))$$

for almost all $x \in \Omega$. In this sense $\partial\mathcal{F}(u)$ generalizes the classical Nemytskii operator $u \mapsto f(\cdot, u(\cdot))$.

Proof of Theorem 5. Let us assume that $\mathcal{F}$ is well defined first, then it is easily seen that antisymmetry of f with respect to the second variable (3.10) implies that $\mathcal{F}$ is a symmetric functional, i.e. $\mathcal{F}(u) = \mathcal{F}(-u)$. Moreover, we have

$$\mathcal{F}(u) = \int_\Omega \int_0^{u(x)} f(x,s)\,\mathrm{d}s\,\mathrm{d}x = \int_\Omega \int_0^{|u(x)|} f(x,s)\,\mathrm{d}s\,\mathrm{d}x = \mathcal{F}(|u|)\,.$$

We will thus without loss of generality restrict our attention on f defined on $\Omega \times [0,\infty)$ in the following.

Let us verify that $\mathcal{F}$ is well defined now. In particular we need to justify that

$x \mapsto f(x, u(x))$ is measurable and integrable. Assume $f \geq 0$ on $\Omega \times [0, \infty)$. By assumption (3.11) the function f is integrable on $\Omega \times [0, T]$ for all $T > 0$. Thus the Fubini Theorem states that $f(\cdot, t)$ is measurable for almost all $t \in [0, T]$ and $s \mapsto f(x, s)\chi_{\{s \leq T\}}(x, s)$ is measurable for almost all $x \in \Omega$ on $[0, T]$, but thus also on $[0, \infty)$. Thus

$$f(x, \cdot) = \sup_{T>0} s \mapsto f(x, s)\chi_{\{s \leq T\}}(x, s)$$

is measurable for almost all $x \in \Omega$.

We define $F : \Omega \times [0, \infty) \to \mathbb{R}$ by

$$F(x, t) := \int_0^t f(x, s)\, \mathrm{d}s\,.$$

Obviously $F(x, \cdot)$ is continuous for almost all $x \in \Omega$. Since f is integrable on $\Omega \times [0, T]$ for any $T \in [0, \infty)$ again the Fubini Theorem states that $F(\cdot, t)$ is measurable for almost all $t \in [0, \infty)$. This shows that F is a Carathéodory function and thus for a measurable function u the composition function $x \mapsto F(x, u(x))$ is measurable. The general case follows by separate treatment of the positive and negative part of $f = f^+ - f^-$.

Assume that $\mathcal{F}$ is indeed well defined, i.e. $\mathcal{F}(u)$ finite for $u \in L^p(\Omega)$. Let $u, w \in L^p(\Omega)$, $R > 0$ with $\|u\|_p \leq R$, $\|w\|_p \leq R$. Then by (3.11) and Hölders inequality

$$\begin{aligned}
|\mathcal{F}(u) - \mathcal{F}(w)| &= \left| \int_\Omega \int_{|w(x)|}^{|u(x)|} f(x, s)\, \mathrm{d}s\, \mathrm{d}x \right| \\
&\leq p\, C_{\mathrm{Per}} \int_\Omega \big| |u(x)| - |w(x)| \big| (|u(x)| + |w(x)|)^{p-1}\, \mathrm{d}x \\
&\leq p\, C_{\mathrm{Per}} \|u - w\|_p \|(|u| + |w|)^{p-1}\|_{p'} \\
&\leq p\, C_{\mathrm{Per}} (2R)^{p-1} \|u - w\|_p\,, \qquad (3.16)
\end{aligned}$$

which shows that $\mathcal{F}$ is uniformly Lipschitz continuous on bounded subsets of $L^p(\Omega)$. Here we have used that for $u, w \in L^p(\Omega)$ we have $(|u| + |w|)^{p-1} \in L^{p'}(\Omega)$ with

$$\|(|u| + |w|)^{p-1}\|_{p'}^{p'} = \|(|u| + |w|)\|_p^p \leq (2R)^p.$$

Note that the same calculation (3.16) with $w = 0$, and thus $\mathcal{F}(w) = 0$, also justifies our assumption that $\mathcal{F}$ is finite at all.

Assertion (3.15) is proved by a straightforward calculation

$$\begin{aligned}
|\mathcal{F}(\alpha u)| &\leq \int_\Omega \left| \int_0^{\alpha u(x)} f(x,s)\mathrm{d}s \right| \mathrm{d}x \\
&\leq \int_\Omega \left| \int_0^{|\alpha u(x)|} p\, C_{\mathrm{Per}} s^{p-1}\, \mathrm{d}s \right| \mathrm{d}x \\
&= \int_\Omega C_{\mathrm{Per}} |\alpha u|^p \, \mathrm{d}x \\
&= C_{\mathrm{Per}}\, \alpha^p \|u\|_p^p \,,
\end{aligned}$$

where we used (3.11) again.

It remains to prove the structure of Clarkes generalized gradient for $\mathcal{F}$. To do so let $u \in L^p(\Omega)$ and $u^* \in \partial\mathcal{F}(u)$ and $v \in L^p(\Omega)$. With the notation

$$F_x(u) := F(x,u)\,. \tag{3.17}$$

we derive

$$\begin{aligned}
\int_\Omega u^* v \, \mathrm{d}x \leq \mathcal{F}^0(u;\, v) &= \limsup_{w \to u,\, t \downarrow 0} \frac{\mathcal{F}(w+tv) - \mathcal{F}(w)}{t} \\
&= \limsup_{w \to u,\, t \downarrow 0} \int_\Omega \frac{F_x(w(x)+tv(x)) - F_x(w(x))}{t} \, \mathrm{d}x\,.
\end{aligned}$$

Note that by (3.11) for almost every $x \in \Omega$ the function F_x is the primitive of a locally bounded function, such that Proposition 4 applies and we derive that F_x is locally Lipschitz continuous and Clarkes generalized gradient of F_x is given by

$$\partial F_x(a) = \Big[\operatorname*{ess\,inf}_{y \to a} f(x,y),\, \operatorname*{ess\,sup}_{y \to a} f(x,y) \Big]$$

for $a \in \mathbb{R}$. Now, again by (3.11) the implication

$$a^* \in \partial F_x(a) \Rightarrow |a^*| \leq p\, C_{\mathrm{Per}} |a|^{p-1} \tag{3.18}$$

holds true.

Let us take sequences $(w_k)_k$ in $L^p(\Omega)$ and $t_k \downarrow 0$ such that

$$\mathcal{F}^0(u;\, v) = \lim_{k \to \infty} \int \frac{F_x(w_k(x) + t_k v(x)) - F_x(w_k(x))}{t_k} \, \mathrm{d}x\,. \tag{3.19}$$

Without loss of generality we may assume that $w_k(x) \to u(x)$ a.e. By the Lebourg Theorem (3.8) for almost every $x \in \Omega$ and every $k \in \mathbb{N}$ there is some $\theta \in (0,1)$ and $g_k^*(x) \in \partial F_x\big(w_k(x) + \theta t_k v(x)\big)$ such that

$$\begin{aligned} \big|F_x\big(w_k(x) + t_k v(x)\big) - F_x\big(w_k(x)\big)\big| &= \big|g_k^*(x)\big(w_k(x) + t_k v(x) - w_k(x)\big)\big| \\ &= t_k \big|g_k^*(x) v(x)\big| \\ &\le p\, C_{\mathrm{Per}}(|w_k(x)| + |v(x)|)^{p-1}\, |v(x)|\,, \end{aligned} \tag{3.20}$$

where we used (3.18). Obviously $h_k := |w_k| + |v| \to |u| + |v|$ in $L^p(\Omega)$. The nonlinear operator $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ given by $J_p(u)(x) := |u(x)|^{p-1}\operatorname{sgn}(u(x))$ is a homeomorphism (cf. [15, p. 72]) and thus the functions $h_k^* \in L^{p'}(\Omega)$ given by

$$h_k^* = J_p(h_k)$$

converge in $L^{p'}(\Omega)$ to $J_p(|u| + |v|)$. Whence $g_k := h_k^*\,|v|$ converges in $L^1(\Omega)$ and, by assumption also pointwise a.e., to $g = (|u| + |v|)^{p-1}|v|$. Choosing an appropriate subsequence we may moreover assume that $\sum_{k\in\mathbb{N}} \|g_k - g\|_1 < \infty$, such that $g + \sum_{k\in\mathbb{N}} |f_k|$ is easily seen to be a majorant of all g_k and thus also all integrands in (3.20). We may thus invoke the Fatou Lemma in (3.19) in order to get

$$\int_\Omega u^*(x) v(x)\,\mathrm{d}x \le \mathcal{F}^0(u;\, v) \le \int_\Omega \limsup_{k\to\infty} \frac{F_x(w_k(x) + t_k v(x)) - F_x(w_k(x))}{t_k}\,\mathrm{d}x\,.$$

Note that the integrand on the right hand side of the previous inequality is bounded by $F_x^0(u(x);\, v(x))$ for almost all $x \in \Omega$ and thus varying $v \in L^p(\Omega)$ (choose v of the form $v = a\chi_E$ for appropriate subsets $E \subseteq \Omega$) we derive

$$u^*(x) a \le F^0(u(x);\, a)$$

for all $a \in \mathbb{R}$ and almost all $x \in \Omega$. But this is by definition

$$u^*(x) \in \partial F_x(u(x))$$

for almost all $x \in \Omega$, which proves (3.13). We thus obtain with (3.11)

$$\begin{aligned}\|u^*\|_{p'}^{p'} &\leq \int_\Omega \max\{|f^-(x,u(x))|,\ |f^+(x,u(x))|\}^{p'}\,\mathrm{d}x \\ &\leq \int_\Omega (p\,C_{\mathrm{Per}})^{p'}|u(x)|^p\,\mathrm{d}x \\ &= (p\,C_{\mathrm{Per}})^{p'}\|u\|_p^p\end{aligned}$$

and (3.14) follows by taking the p'-th root of the previous equation.

□

Let us note that under slightly stronger conditions on f one has that (3.13) characterizes Clarkes generalized gradients of $\mathcal{F}$ (cf. [16, p. 83f]).

3.1.3 Weak Slope and a General Lagrange Multiplier Rule

As pointed out before, the energy functions of the problems we will investigate are highly nonsmooth. Thus a very general concept of criticality is needed. For the 1-Laplace operator the notion of the weak slope turned out to be the method of choice. This notion to define critical points of continuous (and to some extent also lower semicontinuous) functionals on metric spaces was introduced by Corvellec, Degiovanni & Marzocchi [18], [17], [21] and, independently, by Ioffe, Katriel and Schwartzmann [32], [33].

Let X be a metric space and $\mathcal{F} : X \to \mathbb{R}$ be continuous. The *weak slope* $|d\mathcal{F}|(u) \in [0,\infty]$ of $\mathcal{F}$ at some point $u \in X$ is defined as

$$\begin{aligned}|d\mathcal{F}|(u) := \sup_{\sigma\in[0,\infty)} \Big\{ &\exists\delta > 0,\ \eta : B(u,\delta)\times[0,\delta] \to X \text{ continuous, such that} \\ &d(\eta(v,t),v) \leq t \ \text{ and } \ \mathcal{F}(\eta(v,t)) \leq \mathcal{F}(v) - \sigma t \\ &\text{for all } (v,t) \in B(u,\delta)\times[0,\delta]\Big\}.\end{aligned}$$

Loosly speeking we look for homotopies η with deformation speed not faster than one, such that the energy $\mathcal{F}$ declines along all paths at least with speed σ (cf. Figure 3.1).

This definition is motivated by critical point theory, where in the classical situation such local deformation mappings η are constructed for noncritical points u with the aid of the derivative at u in order to finally derive the desired Lusternik-Schnirelmann defomations. The weak slope generalizes the norm of the derivative. Indeed it is not difficult to see that, provided X is a Banach space and $\mathcal{F}$ is continuously differentiable

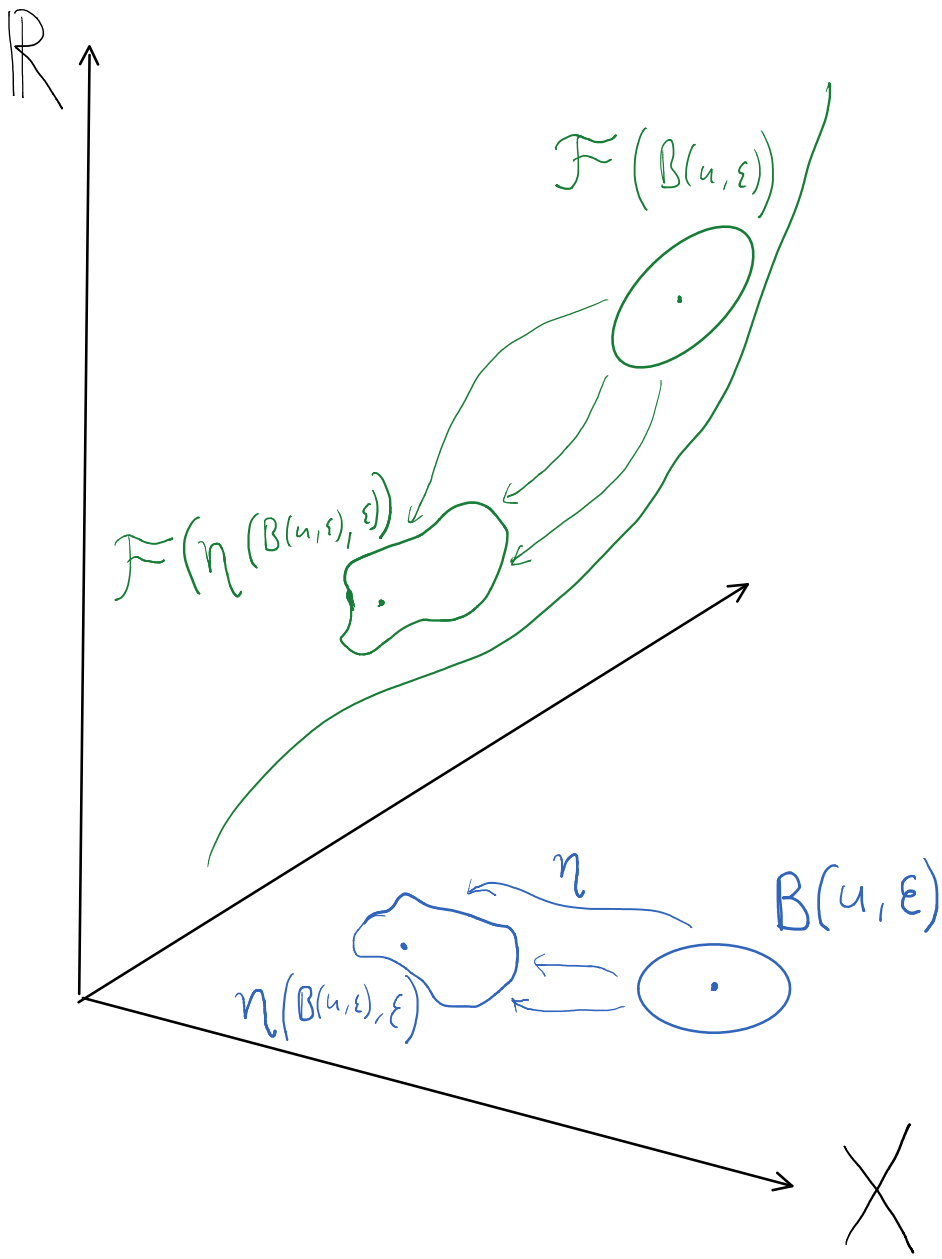

Figure 3.1: The energy $\mathcal{F}$ declines along the homotopy η.

with derivative $\mathcal{F}'(u) \in X^*$, we have $|d\mathcal{F}|(u) = \|\mathcal{F}'(u)\|_{X^*}$. Thus we say that u is a critical point of $\mathcal{F}$ provided

$$|d\mathcal{F}|(u) = 0\,.$$

Let X be a metric space as above and let $\mathcal{F} : X \to \mathbb{R}$ be continuous. We define

$$\mathscr{G}_{\mathcal{F}} : \mathrm{epi}(\mathcal{F}) \to \mathbb{R}, \qquad \mathscr{G}_{\mathcal{F}}(v,t) := t\,,$$

where $\mathrm{epi}(\mathcal{F})$ denotes the epigraph of $\mathcal{F}$ defined in (3.2). We equip $\mathrm{epi}(\mathcal{F})$ with the metric

$$d((v,t),(u,s)) := \sqrt{d(u,v)^2 + (t-s)^2}\,, \tag{3.21}$$

then $\mathscr{G}_{\mathcal{F}}$ turns out to be Lipschitz continuous (with Lipschitz constant 1) and $|d\mathcal{F}|$ and $|d\mathscr{G}_{\mathcal{F}}|$ are connected by the following proposition.

Proposition 6. *Let $\mathcal{F} : X \to \mathbb{R}$ be continuous, then*

$$|d\mathscr{G}_{\mathcal{F}}|(u,t) = \begin{cases} \frac{|d\mathcal{F}|(u)}{\sqrt{1+(|d\mathcal{F}|(u))^2}} & \textit{if } t = \mathcal{F}(u) \textit{ and } |d\mathcal{F}|(u) < \infty \\ 1 & \textit{otherwise}\,. \end{cases}$$

Proof. Cf. [21, Prop. 2.3]. □

This allows to define in a consistent way the weak slope for merely lower semicontinuous functionals $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ on a metric space X by setting

$$|d\mathcal{F}|(u) := \begin{cases} \frac{|d\mathscr{G}_{\mathcal{F}}|(u,\mathcal{F}(u))}{\sqrt{1-\left(|d\mathscr{G}_{\mathcal{F}}|(u,\mathcal{F}(u))\right)^2}} & \text{if } |d\mathscr{G}_{\mathcal{F}}|(u,\mathcal{F}(u)) < 1 \\ \infty & \text{if } |d\mathscr{G}_{\mathcal{F}}|(u,\mathcal{F}(u)) = 1. \end{cases}$$

By application of critical point theory for the continuous function $\mathscr{G}_{\mathcal{F}}$ we then obtain critical points $(u,t) \in \mathrm{epi}(\mathcal{F})$. Although, we may get "artificial" critical points (u,t) with $\mathcal{F}(u) < t$. A natural way to rule out this case is requiring that the (epi)-condition holds for $\mathcal{F}$, which means that for all $b > 0$

$$\inf\{|d\mathscr{G}_{\mathcal{F}}|(u,t)\ ;\ u \in \mathrm{dom}(\mathcal{F}),\ \mathcal{F}(u) < t,\ |t| \le b\} > 0\,. \tag{3.22}$$

Our main application will be the investigation of constraint variational problems of

the form

$$\mathcal{E}(v) \to \operatorname*{Min}_{v \in X}! \tag{3.23}$$

subject to

$$\mathcal{G}(v) = 0\,. \tag{3.24}$$

and we say that u is a critical point of (3.23), (3.24), provided u is a critical point of the functional $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ defined by

$$\mathcal{F}(v) := \mathcal{E}(v) + I_{\{\mathcal{G}=0\}}(v)\,.$$

The epigraphs of the function $\mathcal{F}$ and the restricted function $\mathcal{E}|_G : G \to \mathbb{R} \cup \{\infty\}$, where $G := \{\mathcal{G} = 0\}$ is equipped with the induced metric from X, coincide and thus $|d\mathcal{F}|(u) = \big|d\mathcal{E}|_G\big|(u)$ for $u \in G$. This shows that this approach is consistent with the classical notion of criticality (cf. [45, Lemma 3.1] for details).

In [21, Theorem 2.11] it is pointed out that for convex, lower semicontinuous functionals $\mathcal{F}$ we have

$$|d\mathcal{F}|(u) = \min\{\|u^*\| \;;\; u^* \in \partial\mathcal{F}(u)\}\,.$$

The weak slope for locally Lipschitz continuous functionals may be estimated by

$$|d\mathcal{F}|(u) \geq \min\{\|u^*\| \;;\; u^* \in \partial\mathcal{F}(u)\}$$

(cf. [21, Theorem 2.17]). Nevertheless we are in a more general situation, where we consider the weak slope of the sum of a Lipschitz and a convex, lower semicontinuous function subject to a Lipschitz constraint. This situation has been treated in [22].

Theorem 7 (Lagrange multiplier rule, general form)**.** *Let X be a real Banach space and let $\mathcal{F}_0 : X \to \mathbb{R} \cup \{\infty\}$ be convex and lower semicontinuous and $\mathcal{F}_1, \mathcal{G} : X \to \mathbb{R}$ be locally Lipschitz continuous. Let $u \in X$ be a critical point of $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ given by*

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1 + I_{\{\mathcal{G}=0\}}$$

and assume that there exist $u_1, u_2 \in X$ with $\mathcal{F}(u_1) < \infty$ and $\mathcal{F}(u_2) < \infty$ such that

$$\mathcal{G}^0(u;\, u_1 - u) < 0 \quad \textit{and} \quad \mathcal{G}^0(u;\, u - u_2) < 0\,. \tag{3.25}$$

Then $\partial\mathcal{F}(u) \neq \emptyset$ *and there exist* $\lambda \in \mathbb{R}$, $u_0^* \in \partial\mathcal{F}_0(u)$, $u_1^* \in \partial\mathcal{F}_1(u)$ *and* $w^* \in \partial\mathcal{G}(u)$ *such that*

$$u_0^* + u_1^* = \lambda w^* .$$

Proof. Note that condition (3.25) implies the (epi)-condition by [22, Theorem 3.5]. The theorem is thus an immediate consequence of [22, Corollary 3.6], applied with $f_0 = \mathcal{F}_0$, $f_1 = \mathcal{F}_1$, $g_0 = -1$ and $g_1 = \mathcal{G}$. □

Note that the foregoing theorem only allows to derive the existence of *some* elements u_0^*, u_1^* and w^*. If we additionally know that the critical point u is indeed a minimizer we can assure under slightly stronger assumptions that *for all* $w^* \in \partial\mathcal{G}(u)$ there exists some element in the subdifferential of the energy at u, such that the corresponding Euler-Lagrange equation is satisfied (cf. [36, Proposition 6.4]):

Theorem 8 (Lagrange multiplier rule for minimizers of convex energies)**.** *Let* X *be a real Banach space and let* $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ *be convex and* $\mathcal{G} : X \to \mathbb{R}$ *be convex and continuous. Assume* $u \in X$ *solves*

$$\mathcal{F}(u) = \min_{\substack{v \in X \\ \mathcal{G}(v)=0}} \mathcal{F}(v) < \infty$$

and assume the existence of $w \in X$ *with* $\mathcal{F}(u+w) < \mathcal{F}(u)$, $\mathcal{F}(u-w) < \infty$ *and* $\mathcal{G}(u+w) < 0$. *Then*

$$\partial\mathcal{G}(u) \subseteq \bigcup_{\alpha \geq 0} \alpha\partial\mathcal{F}(u).$$

In other words, for *each* $w^* \in \partial\mathcal{G}(u)$ there exists $\lambda \geq 0$ and $u^* \in \partial\mathcal{F}(u)$ such that

$$u^* = \lambda w^* .$$

3.2 Critical Point Theory

The solutions of many partial differential equations can be considered as critical points of an associated variational problem. Critical point theory treats methods to find such critical points in terms of minimax methods and thus proves existence of solutions of the associated partial differential equation. A good introduction to critical point theory is the monograph of Rabinowitz [50]. To present the basic idea of critical point theory let us state the idea of the existence principle in critical point theory in a casual form.

Existence Principle 9. *Let X be a given space and let $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$. Assume that:*

(a) There exist so called Lusternik-Schnirelmann deformations, i.e. for every non-critical value c and $\tilde{\varepsilon} > 0$ there is $\varepsilon \in (0, \tilde{\varepsilon})$ and a continuous deformation $\eta \in C([0,1] \times X, X)$ with $\eta(0, v) = v$ for all $v \in X$ and

$$\eta(1, \{\mathcal{F} \leq c + \varepsilon\}) \subseteq \{\mathcal{F} \leq c - \varepsilon\},$$

(b) There exists a class $\mathscr{S}$ of, typically compact, subsets of X that is invariant under the Lusternik-Schnirelmann deformations, i.e.

$$S \in \mathscr{S} \quad \Rightarrow \quad \eta(1, S) \in \mathscr{S}.$$

Then

$$c := \inf_{S \in \mathscr{S}} \sup_{u \in S} \mathcal{F}(u),$$

provided it is finite, is a critical value of $\mathcal{F}$.

For a more explicit statement see [50, Theorem A.4], compare also [58, Chapter 44].

In classical theorems of critical point theory it is assumed that $\mathcal{F}$ is continuously differentiable. In that case the derivative $\mathcal{F}'(u)$, provided it is nonzero, can be used to find a direction, where the energy decreases and thus some kind of gradient flow can be constructed to prove existence of the Lusternik-Schnirelmann deformations. Note that the derivative itself does not directly appear in the above existence principle (apart from the definition of "critical value") and it turns out that the property of being a critical value is essentially a topological property.

Example 10. *Consider the function $f_1 : \mathbb{R}^2 \to \mathbb{R}$, $f_1(x, y) = x^2 - y^2$, with zero being the only critical value. Note that the superlevel sets $\{f_1 \geq c\}$ change from connectedness for $c \leq 0$ to disconnetedness for $c > 0$ at the critical level $c = 0$. The graph of the function $f_2 : \mathbb{R}^2 \to \mathbb{R}$, $f_2(x, y) := |x| - |y|$ is also saddle shaped, and the superlevel sets also change their topological property at zero (cf. Figure 3.2). Thus it is reasonable to consider zero as critical point of f_2 despite f_2 is not differentiable in that point (and zero indeed turns out to be a critical point in the sense of the weak slope).*

With increasing technical afford such Lusternik-Schnirelmann deformations have been derived for differentiable (but not continuously differentiable) functionals, locally

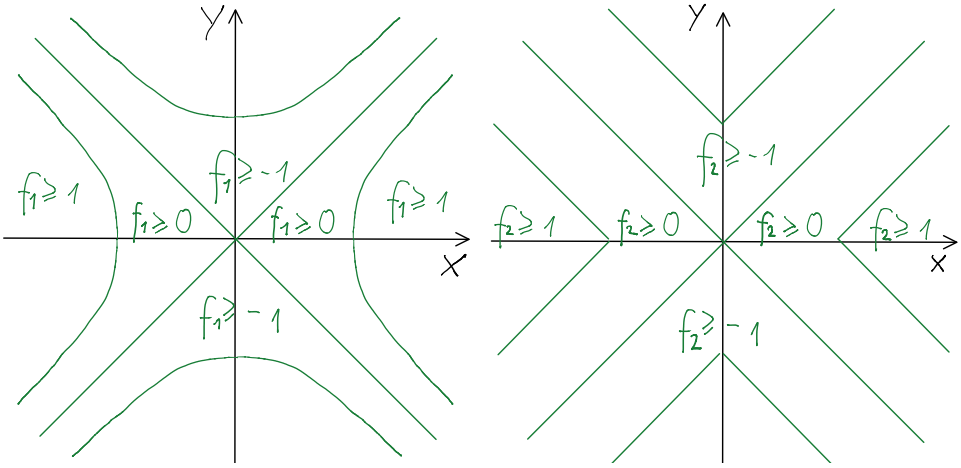

Figure 3.2: Superlevel sets of the functions f_1 and f_2.

Lipschitz continuous functionals and for functionals of the form $\mathcal{F}_1 + \mathcal{F}_2$, where $\mathcal{F}_1$ is convex and $\mathcal{F}_2$ is continuously differentiable.

It was a very big break through in the early 90s, when critical point theory was pushed to merely continuous and even lower semicontinuous functionals on metric spaces. The key idea was the introduction of the notion of the weak slope, which allows the derivation of very general Lusternik-Schnirelmann deformations for noncritical values more or less strait forward from its definition (cf. [21]).

Beside the relaxation of the differentiability there are several ways to adapt the Existence Principle 9 to various situations. On the one hand the theory demonstrates its power for functionals with certain symmetries. We will apply the simplest symmetry possible: all our functionals are even, i.e. $\mathcal{F}(u) = \mathcal{F}(-u)$. Nevertheless critical point theory has successfully been extended to situations with much more complex symmetry properties (cf. e.g. the monograph [6] of Bartsch). These concepts are quite cumbersome to write down, since the symmetry not only comes up in the class $\mathscr{S}$, but also in the Lusternik-Schnirelmann deformations which should be invariant with respect to the symmetry.

On the other hand there are various different ways to define the class $\mathscr{S}$. Maybe the most known appears in the mountain pass theorem where $\mathscr{S}$ consists of all paths connecting two given points in X. This construction typically provides existence of *some* critical point. Nevertheless in certain situations it is possible to define, dependent on a topological index, *an ordered sequence* $\mathscr{S}_1 \supset \mathscr{S}_2 \supset \ldots$ and one

can verify that each

$$c_j := \inf_{S \in \mathscr{S}_j} \sup_{u \in S} \mathcal{F}(u)$$

is a critical point of $\mathcal{F}$. In our situation of even functionals one typically makes use of two topological indices, either the genus or the category in the projective space $X_{/\sim}$, where antipodal points of X are identified. Cf. the Appendix 7.3 for details on genus and category.

The Palais-Smale condition ensures the required compactness of the sets of critical points in a level set and in the framework of the weak slope we will apply the following form of the Palais-Smale condition (cf. [22, Definition 2.3]).

Definition. *Let X be a metric space and $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous. The function $\mathcal{F}$ is said to satisfy the Palais-Smale condition* $(\mathrm{PS})_c$ *for the level $c \in \mathbb{R}$, provided any Palais-Smale sequence $(u_j)_j$ for the level c, i.e. $\mathcal{F}(u_j) \to c$ and $|d\mathcal{F}|(u_j) \to 0$, admits a convergent subsequence. The function $\mathcal{F}$ is said to satisfy the Palais-Smale condition* (PS) *provided it satisfies* $(\mathrm{PS})_c$ *for any level $c \in \mathbb{R}$.*

We will repeatedly apply the following existence theorem for critical points in the thesis.

Theorem 11. *Let X be a real Banach space and let $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ satisfy*

(A) $\mathcal{F}$ is lower semicontinuous, even, and $F(0) = \infty$,

(B) $\mathcal{F}$ is bounded from below,

(C) $\mathcal{F}$ satisfies the (PS)*-condition,*

(D) $\mathcal{F}$ satisfies the (epi)*-condition* (3.22).

(E) Assume that for all $k \in \mathbb{N}$ there exists $\Phi : \mathbb{S}^{k-1} \to X$ bijective continuous anti-symmetric (i.e. $\Phi(-x) = -\Phi(x)$) with

$$\sup\{\mathcal{F}(\Phi(x)) \; ; \; x \in \mathbb{S}^{k-1}\} < \infty \, .$$

Here $\mathbb{S}^{k-1}$ denotes the $(k-1)$-dimensional sphere in $\mathbb{R}^k$.

Then there exist infinitely many pairs $\pm u_1, \pm u_2, \ldots$ of critical points of $\mathcal{F}$ and the corresponding critical values $c_k = \mathcal{F}(\pm u_k)$ are given by

$$c_k := \inf_{S \in \mathscr{S}_k} \sup_{v \in S} \mathcal{F}(v) \tag{3.26}$$

where

$$\mathscr{S}_k := \{S \subseteq X \setminus \{0\} \text{ symmetric and compact}\,;\ \operatorname{gen} S \geq k\}$$

for $k \in \mathbb{N}$.

If moreover the sublevel sets $\{\mathcal{F} \leq \beta\}$ *are compact for any* $\beta \in \mathbb{R}$, *then* $c_k \to \infty$ *as* $k \to \infty$.

Let us state some remarks on the theorem before we turn to its proof.

Remark 12. *(a) With minor modification we can apply Theorem 2.5 from [22] in the situation of our Theorem, which provides us the existence of infinitely many critical points. However, it does not provide unboundedness of the sequence* $(c_k)_k$ *and the minimax-characterization is not stated in [22, Theorem 2.5] (and indeed the proof uses a slightly different characterization of the critical values).*

(b) Unboundedness of the sequence of critical values could probably also be derived from [21, Theorem 3.10], but there the category is used as topological index. By classical results of Rabinowitz [49, Theorem 3.7] and Fadell [28, p. 40] it is well known, that the genus of a closed symmetric set equals its category in the projective space where antipodal points are identified. However, since critical point theory for merely lower semicontinuous functionals $\mathcal{F}$ *is reduced to the investigation of the continuous functional* $\mathscr{G}_{\mathcal{F}}$, *some rather technical arguments are needed to verify that the critical values obtained with the concept of category agree with the values obtained by using genus as topological index (cf. [41, Corollary 2.2] and its proof). Thus Theorem 11 will essentially shorten the existence proof of eigensolutions of the perturbed eigenvalue problem of the* 1*-Laplace operator instead of working with the category in a projective space.*

(c) The situation of the theorem might also be covered by the results of [17], but is seems to be very complex to show that all the technical preliminaries of the abstract framework are satisfied and it is also not immediate how to deduce the desired statements.

(d) If condition (E) *is satisfied not for all* $k \in \mathbb{N}$ *but only for some* $k_0 \in \mathbb{N}$ *(which is certainly the case provided* X *is not finite dimensional) it is not difficult to adopt our proof below in order to show that there exist at least* k_0 *pairs of critical points* $\pm u_1, \ldots \pm u_{k_0}$ *and the corresponding critical values are given by* (3.26) *for* $k = 1, \ldots, k_0$.

(e) *Let us finally remark that the argument with the Blaschke Theorem in our proof below, can easily be modified in order to prove that the* inf *in* (3.26) *is indeed a* min.

Proof of Theorem 11. Condition (E) ensures (cf. Property (vi) on p. 141) that the classes $\mathscr{S}_k$ are nonempty and thus, by boundedness of $\mathcal{F}$ from below, the values c_k are finite. Let $k \in \mathbb{N}$.

Note that with

$$\tilde{\mathscr{S}_k} := \{\tilde{S} \subseteq \mathrm{epi}(\mathcal{F})\ ;\ \ \tilde{S} \text{ compact},\ \ \mathrm{gen}_1\, \tilde{S} \geq k,\ \ \forall (u,s) \in \tilde{S} :\ (-u,s) \in \tilde{S},\}$$

where $\mathrm{gen}_1\, \tilde{S}$ denotes the genus of the projection of $\tilde{S}$ on the first coordinate, i.e.

$$\mathrm{gen}_1\, \tilde{S} := \mathrm{gen}\{u \in X\ ;\ (u,s) \in \tilde{S}\}\,,$$

and with

$$\tilde{c}_k := \inf_{\tilde{S} \in \tilde{\mathscr{S}_k}} \sup_{(u,s) \in \tilde{S}} \mathscr{G}_{\mathcal{F}}(u,s)$$

we have

$$c_k = \tilde{c}_k\,.$$

Indeed, invoking the definition of $\mathscr{G}_{\mathcal{F}}$ and $\tilde{S} \subseteq \mathrm{epi}(\mathcal{F})$ we easily see that we can restrict our attention to sets $\tilde{S} \in \tilde{\mathscr{S}_k}$ of the form

$$\tilde{S} = S \times \left\{ \sup_{u \in S} \mathcal{F}(u) \right\}$$

with $S \in \mathscr{S}_k$ without changing the value $\tilde{c}_k$. Note that by (E) we may assume $\sup_{u \in S} \mathcal{F}(u) < \infty$. However, for those sets $\tilde{S}$ the equality is immediate.

We define the set of critical points of $\mathcal{F}$ for the level c_k by

$$K_{c_k} := \{u \in X\ ;\ \mathcal{F}(u) = c_k \text{ and } |d\mathscr{G}_{\mathcal{F}}|(u) = 0\}\,.$$

Let us now assume that c_k is not a critical value, i.e. $K_{c_k} = \emptyset$.

Claim: There is $\tilde{\varepsilon} > 0$, such that there is no critical value of $\mathcal{F}$ in $(c_k - \tilde{\varepsilon}, c_k + \tilde{\varepsilon})$.

Proof of the claim: Assume this is not true, we will then find a sequence of critical points $(u_j)_j$ of $\mathcal{F}$ with $\mathcal{F}(u_j) \to c_k$. Recall that by definition $(u_j, \mathcal{F}(u_j))_j$ is a sequence of critical points of the continuous functional $\mathscr{G}_{\mathcal{F}} : \mathrm{epi}(\mathcal{F}) \to \mathbb{R}$. Since $(u_j)_j$ is a Palais-Smale sequence for $\mathcal{F}$ it admits a convergent subsequence (for simplicity again

denoted by $(u_j)_j$) with limit u and by lower semicontinuity of $\mathcal{F}$ we have $\mathcal{F}(u) \leq c_k$. Since the weak slope is lower semicontinous with respect to the graph metric (see [21, Propostion 2.6]) we obtain that $(u, c_k) = \lim_{j\to\infty}(u_j, \mathcal{F}(u_j))$ is a critical point of $\mathscr{G}_{\mathcal{F}}$. From the (epi)-condition (3.22) we derive that $c_k = \mathcal{F}(u)$. But this amounts to say that $u \in K_{c_k}$, a contradiction.

According to the first part of the proof of Theorem 2.5 from [22] (applied with $f = \mathcal{G}_{\mathcal{F}}$, $X = \mathrm{epi}(\mathcal{F})$, $\Phi(u, s) = (-u, s)$, $\mathcal{O} = \emptyset$) there is $\varepsilon \in (0, \tilde{\varepsilon}]$ and a continuous map $\eta : \mathrm{epi}(\mathcal{F}) \times [0, 1] \to \mathrm{epi}(\mathcal{F})$ such that for all $(u, s) \in \mathrm{epi}(\mathcal{F})$ and $t \in [0, 1]$

$$d\big(\eta((u,s),t),(u,s)\big) \leq t$$

$$s \notin [c_k - \tilde{\varepsilon}, c_k + \tilde{\varepsilon}] \;\Rightarrow\; \eta\big((u,s),t\big) = (u,s)$$

$$\eta\big(\{\mathscr{G}_{\mathcal{F}} \leq c_k + \varepsilon\}, 1\big) \subseteq \{\mathscr{G}_{\mathcal{F}} \leq c_k - \varepsilon\} \tag{3.27}$$

$$\eta\big((-u,s),t\big) = -\eta((u,s),t) \tag{3.28}$$

where d is the epigraph metric (3.21).

By construction (3.26) there is $S_1 \in \mathscr{S}_k$ such that

$$\sup_{u\in S_1} \mathcal{F}(u) \leq c_k + \varepsilon\,.$$

For $a := \sup_{u\in S_1} \mathcal{F}(u)$ define $\eta_1 : X \to X \times \{a\}$ by

$$\eta_1(u) := (u, a) \quad \text{and define} \quad T_1 := \eta_1(S_1) = S_1 \times \{a\}\,.$$

Note that $T_1 \subseteq \mathrm{epi}(\mathcal{F})$. Let η be the Lusternik-Schnirelmann deformation from above and consider

$$T_2 := \eta(T_1, 1) \subseteq \mathrm{epi}(\mathcal{F})\,.$$

Note that by (3.27) for all $(u, s) \in T_2$ we have $s \leq c_k - \varepsilon$. Thus, with the projection $\eta_2 : \mathrm{epi}(\mathcal{F}) \to X$,

$$\eta_2(u, s) := u \quad \text{and with} \quad S_2 := \eta_2(T_2)$$

we have

$$\sup_{u\in S_2} \mathcal{F}(u) \leq \sup_{(u,s)\in T_2} s \leq c_k - \varepsilon\,.$$

We will show that $S_2 \in \mathscr{S}_k$, which will then provide a contradiction to the definition (3.27) of c_k. The set S_2 is obtained as continuous image of S_1 under $\eta_2 \circ \eta \circ \eta_1$ and thus compact. Moreover, due to (3.28) it is not difficult to see that $\eta_2 \circ \eta \circ \eta_1$ is

antisymmetric. Thus by Property (iii), p. 141 of genus we have

$$\operatorname{gen} S_2 \geq \operatorname{gen} S_1$$

as desired.

It remains to prove that the sequence of critical values $(c_k)_k$ is unbounded. The essential tool for this result is a result of Blaschke (cf. [2, Theorem 4.4.15]), which says that the set of nonempty compact subsets $\mathcal{K}$ of a compact metric space (K, d), equipped with the Hausdorff distance d_H,

$$d_H(K, L) := \sup_{x \in K} d(x, L) + \sup_{x \in L} d(x, K)\,,$$

is again a compact space. Moreover, if $K_j \to K$ in Hausdorff distance, then $x \in K$ if and only if for each $j \in \mathbb{N}$ there is $x_j \in K_j$, such that $x_j \to x$ (cf. [2, Proposition 4.4.14]).

Assume that the monotone increasing sequence $(c_k)_k$ converges to some $c < \infty$. Then $\{\mathcal{F} \leq c + 1\}$ is compact by assumption. Moreover, for each $k \in \mathbb{N}$ there is $S_k \in \mathscr{S}_k$ with

$$\sup_{u \in S_k} \mathcal{F} \leq c_k + 1/k\,.$$

Thus the S_k are all compact subsets of the compact set $\{\mathcal{F} \leq c + 1\}$ and there exists a compact subset $S \subseteq \{\mathcal{F} \leq c + 1\}$ such that a subsequence of $(S_k)_k$, for simplicity also denoted $(S_k)_k$, converges to S in Hausdorff metric by the Blaschke's Theorem. By the pointwise characterization of the limit S it is immediate that S is symmetric with respect to the origin and $0 \notin S$, since this would otherwise contradict the lower semicontinuity of $\mathcal{F}$ (recall $\mathcal{F}(0) = \infty$). By property (v), p. 141 of genus there is an open neighborhood U of S, such that $\operatorname{gen} \overline{U} = \operatorname{gen} S$. By convergence of $(S_k)_k$ to S in Hausdorff metric we finally have $S_k \subseteq U$ for all k large enough and thus by Property (ii), p. 141 we have for eventually all $k \in \mathbb{N}$

$$k \leq \operatorname{gen} S_k \leq \operatorname{gen} \overline{U} = \operatorname{gen} S\,,$$

a contradiction to the finiteness of $\operatorname{gen} S$, Property (v), p. 141. $\square$

4 Perturbations of the Eigenvalue Problem in $BV(\Omega)$

In this chapter we will derive perturbation results for the 1-Laplace operator. Note that already the existence of solutions of the perturbed problems is a nontrivial task and beside the investigations of Kawohl & Schuricht [36] and the investigation of one specific perturbation in [20] by Degiovanni & Magrone, our existence proofs for sequences of solutions for a large class of perturbed eigenvalue problems of the 1-Laplace operator are new. Moreover, in contrast to the well studied properties of the p-Laplace operator, bifurcation investigation for the 1-Laplace operator was not accessible up to now. This is not only due to the fact that eigensolutions of the 1-Laplace operator are not elementary to define (and in particular the Euler-Lagrange equation is not suitable to define eigensolutions), but also relies on the fact that both the 1-Laplace operator and the nonlinearity "$u \mapsto u/|u|$" are of multi-valued nature. In particular the concepts developed for the differentiable framework of the p-Laplace eigenvalue problem do not apply. Nevertheless critical point theory gives the powerful tools to obtain a sequence of eigensolutions with a certain robustness inherited from their construction as we will demonstrate below.

In particular we will study the perturbed eigenvalue problem of the 1-Laplace operator which is formally given by

$$-\operatorname{div}\frac{Du}{|Du|} + f(x,u) = \lambda\Big(\frac{u}{|u|} + g(x,u)\Big)\,,$$

where f and g are functions representing the perturbation. As in the unperturbed case this equation is not suitable to define eigenfunctions of the perturbed 1-Laplace operator, and we will thus investigate the associated variational problem

$$\mathcal{E}(v) := \mathcal{E}_{TV}(v) + \mathcal{E}_{\mathrm{Per}}(v) \to \underset{v \in BV(\Omega)}{\mathrm{Min!}} \tag{4.1}$$

subject to

$$\mathcal{G}(v) := \mathcal{G}_1(v) + \mathcal{G}_{\mathrm{Per}}(v) = \beta\,, \tag{4.2}$$

where $\mathcal{E}_{\mathrm{Per}}$ and $\mathcal{G}_{\mathrm{Per}}$ are suitable potentials of f and g resp. In order to keep our calculations short we will split the investigation in two cases, we will assume either $\mathcal{G}_{\mathrm{Per}} = 0$ in Section 4.1 or $\mathcal{E}_{\mathrm{Per}} = 0$ in Section 4.2.

Recall that $u \neq 0$ is defined to be an eigenfunction of the 1-Laplace operator provided, with $\alpha = \|u\|_1$, the function u is a critical point of the variational problem

$$\mathcal{E}_{TV}(v) := \int_\Omega \mathrm{d}|Dv| + \int_{\partial\Omega} |v^{\partial\Omega}|\,\mathrm{d}\mathcal{H}^{n-1} \to \underset{v \in BV(\Omega)}{\mathrm{Min!}} \tag{4.3}$$

subject to

$$\mathcal{G}_1(v) = \int_\Omega |v|\,\mathrm{d}x = \alpha\,, \tag{4.4}$$

where we implicitly assume that $\mathcal{E}_{TV}$ is defined on all of $L^p(\Omega)$ as in (2.5) and criticality is meant in the sense of the weak slope for the functional $\mathcal{F} : L^p(\Omega) \to \mathbb{R} \cup \{\infty\}$ from (2.6).

In the following we will consider eigenfunctions of the 1-Laplace operator with respect to the $L^p(\Omega)$-metric, where p is chosen in such a way that the perturbation functions $\mathcal{E}_{\mathrm{Per}}$ or $\mathcal{G}_{\mathrm{Per}}$ are Lipschitz continuous functions on $L^p(\Omega)$.

By 1-homogeneity of $\mathcal{E}_{TV}$ and $\mathcal{G}_1$ it is easily seen that for any eigenfunction u and any $\beta \in \mathbb{R} \setminus \{0\}$ the function βu will be an eigenfunction as well and the Lagrange multiplier λ_β associated to the critical point βu coincides with the eigenvalue λ of u.

In analogy to the eigenvalue problem of the p-Laplace operator we will, for $\lambda \in \mathbb{R}$, consider $(\lambda, 0)$ to be a trivial solution of the eigenvalue problem of the 1-Laplace operator, such that the same situation as in Figure 2.2 is met.

In particular we can consider any $(\tilde{\lambda}, 0) \in \mathbb{R} \times BV(\Omega)$ as bifurcation point, provided $\tilde{\lambda}$ is an eigenvalue of the 1-Laplace operator, since the solution branches $(\lambda, 0)_{\lambda \in \mathbb{R}}$ and $(\tilde{\lambda}, \beta u)_{\beta \in \mathbb{R}}$ of the eigenvalue problem for the 1-Laplace operator intersect in $(\tilde{\lambda}, 0)$.

Recall that a sequence of eigenvalues $(\lambda_{k,0})_{k \in \mathbb{N}}$ of the 1-Laplace operator is given by

the Lusternik-Schnirelman construction (2.8), (2.9) and by homogeneity of $\mathcal{E}_{TV}$ and $\mathcal{G}_1$ this is equivalent to

$$\lambda_{k,0} = \frac{1}{\alpha} \inf_{S \in \mathscr{S}_k^\alpha} \sup_{u \in S} \mathcal{E}_{TV}(u) \tag{4.5}$$

where

$$\mathscr{S}_k^\alpha := \{S \subseteq L^p(\Omega) \text{ compact } ;\ \mathcal{G}_1 = \alpha \text{ on } S,\ \operatorname{gen}_{L^p} S \geq k\} \tag{4.6}$$

for $\alpha > 0$.

In Section 4.1 we will consider the variational problem

$$\mathcal{E}_{TV}(v) + \mathcal{E}_{\mathrm{Per}}(v) \to \operatorname*{Min!}_{v \in L^p(\Omega)}, \tag{4.7}$$

with $\mathcal{E}_{\mathrm{Per}}$ typically of the form

$$\mathcal{E}_{\mathrm{Per}}(v) = \int_\Omega \int_0^{v(x)} f(x,s)\,\mathrm{d}s\,\mathrm{d}x\,, \tag{4.8}$$

subject to

$$\mathcal{G}_1(v) = \int_\Omega |u|\,\mathrm{d}x = \alpha\,.$$

In a first step we will prove existence of eigensolutions for a certain class of perturbations f and each $\alpha > 0$ sufficiently small. In particular this provides us eigenfunctions $u_{k,\alpha}$ and we will show that those eigenfunctions satisfy the single version of the Euler-Lagrange equation. I.e. for any $\alpha > 0$ and any $k \in \mathbb{N}$ there exists a function $s_{k,\alpha} \in L^\infty(\Omega)$ with

$$s_{k,\alpha}(x) \in \operatorname{Sgn}(u_{k,\alpha}(x))$$

for almost all $x \in \Omega$, a vector field $z_{k,\alpha} \in L^\infty(\Omega, \mathbb{R}^n)$ with

$$\operatorname{div} z_{k,\alpha} \in L^{p'}(\Omega)\,, \quad \|z_{k,\alpha}\|_\infty = 1 \quad \text{and} \quad \mathcal{E}_{TV}(u_{k,\alpha}) = -\int_\Omega u_{k,\alpha} \operatorname{div} z_{k,\alpha}\,\mathrm{d}x\,,$$

a function $u^*_{k,\alpha} \in \partial\mathcal{E}_{\mathrm{Per}}(u_{k,\alpha})$ and a Lagrange multiplier $\lambda_{k,\alpha}$, such that the Euler-Lagrange equation

$$-\operatorname{div} z_{k,\alpha} + u^*_{k,\alpha} = \lambda_{k,\alpha} s_{k,\alpha}$$

is satisfied. We will thus call $\lambda_{k,\alpha}$ eigenvalue of the perturbed eigenvalue problem

of the 1-Laplace operator for the eigenfunction $u_{k,\alpha}$. Note that, in contrast to the homogeneous situation $f = 0$, the eigenvalue $\lambda_{k,\alpha}$ in general depends on α and it is not even clear whether $\lambda_{k,\alpha}$ is uniquely determined for fixed k and α.

In our second result we show that $\lambda_{k,\alpha} \to \lambda_{k,0}$ as $\alpha \to 0$. In other words the eigenvalues of the 1-Laplace operator are bifurcation values of the eigenvalues of the perturbed problem. In this sense the situation of Figure 2.2 holds at least locally for eigenfunctions of small norm.

In Subsection 4.2 we investigate the perturbed eigenvalue problem of the 1-Laplace operator for perturbations of the form

$$-\operatorname{div}\frac{Du}{|Du|} = \lambda\left(\frac{u}{|u|} + g(x,u)\right).$$

As before this equation is not well posed and we thus investigate the variational problem

$$\mathcal{E}_{TV}(v) \to \operatorname*{Min!}_{v\in BV(\Omega)} \tag{4.9}$$

subject to

$$\mathcal{G}_1(v) + \mathcal{G}_{\mathrm{Per}}(v) = \beta \tag{4.10}$$

where $\mathcal{G}_{\mathrm{Per}}$ is typically of the form

$$\mathcal{G}_{\mathrm{Per}}(v) = \int_\Omega \int_0^{v(x)} g(x,s)\,\mathrm{d}s\,\mathrm{d}x\,.$$

Eigenfunctions of this perturbed eigenvalue problem of the 1-Laplace operator are, by definition, critical points (4.9), (4.10) and we will prove the existence of a sequence of eigenfunctions for a suitable class of perturbations $\mathcal{G}_{\mathrm{Per}}$. Again such critical points satisfy the single version of the Euler-Lagrange equation, i.e. for any $\beta > 0$ and any $k \in \mathbb{N}$ there exists a function $s_{k,\beta} \in L^\infty(\Omega)$ with

$$s_{k,\beta}(x) \in \operatorname{Sgn}(u_{k,\beta}(x))$$

for almost all $x \in \Omega$, a vector field $z_{k,\beta} \in L^\infty(\Omega, \mathbb{R}^n)$ with

$$\operatorname{div} z_{k,\beta} \in L^{p'}(\Omega)\,, \quad \|z_{k,\beta}\|_\infty = 1 \quad \text{and} \quad \mathcal{E}_{TV}(u_{k,\beta}) = -\int_\Omega u_{k,\beta}\operatorname{div} z_{k,\beta}\,\mathrm{d}x\,,$$

a function $u^*_{k,\beta} \in \partial\mathcal{G}_{\mathrm{Per}}(u_{k,\beta})$ and a Lagrange multiplier $\lambda_{k,\beta}$, such that the Euler-Lagrange equation

$$-\operatorname{div} z_{k,\beta} = \lambda_{k,\beta}\big(s_{k,\beta} + u^*_{k,\beta}\big)$$

is satisfied and we will thus call $\lambda_{k,\beta}$ eigenvalue of the perturbed eigenvalue problem of the 1-Laplace operator for the eigenfunction $u_{k,\beta}$.

Again as for the perturbations of the first type the eigenvalue $\lambda_{k,\beta}$ depend on β in general and might not be uniquely determined for fixed k and β. However, we can also prove the convergence $\lambda_{k,\beta} \to \lambda_{k,0}$ as $\beta \to 0$ as before. In this sense we verify that the eigenvalues of 1-Laplace operator are bifurcation values of the perturbed eigenvalue problem of the 1-Laplace operator (4.9), (4.10).

In both Sections 4.1 and 4.2 we assume that $\mathcal{E}_{\mathrm{Per}}$ and $\mathcal{G}_{\mathrm{Per}}$ resp. can be bounded by $v \mapsto C\|v\|_p^p$ for some $p \in (1, n/(n-1))$ and the main challenge in our derivation below is to bound the higher order growth of $v \mapsto C\|v\|_p^p$ by the lower order growth of $v \mapsto \mathcal{G}_1(v)$ and $v \mapsto \mathcal{E}_{TV}(v)$. This can be done due to the following proposition and the corollary following it.

Proposition 13. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded and let $u \in W_0^{1,1}(\Omega)$. Let C_{BV} denote the embedding constant of $W_0^{1,1}(\Omega)$ (equipped with the norm $u \mapsto \|Du\|_1$) in $L^{n/(n-1)}(\Omega)$. Then for any $p \in [1, n/(n-1)]$ the estimate*

$$\|u\|_p \le C_{\mathrm{BV}}^{\frac{p-1}{p}n} \|u\|_1^{1-\frac{p-1}{p}n} \|Du\|_1^{\frac{p-1}{p}n}$$

holds for all $u \in W_0^{1,1}(\Omega)$.

Proof. For $u \in L^{n/(n-1)}(\Omega)$ we have

$$\|u\|_p \le \|u\|_1^{\theta} \|u\|_{n/(n-1)}^{1-\theta}$$

with

$$\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{n/(n-1)} \quad \Leftrightarrow \quad \theta = 1 - \frac{p-1}{p}n$$

by the interpolation inequality. Due to the embedding of $W_0^{1,1}(\Omega)$ in $L^{n/(n-1)}(\Omega)$ the term $\|u\|_{n/(n-1)}$ can be estimated by $C_{\mathrm{BV}}\|Du\|_1$, thus the assertion follows. □

In other words this proposition states that we can control the p-norm of u by joint knowledge of $\|u\|_1$ and $\|Du\|_1$. Since $\|Du\|_1 = \mathcal{E}_{TV}(u)$ for $u \in W_0^{1,1}(\Omega)$ the following

statement for BV-functions is unsurprising.

Corollary 14. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary and let $p \in [1, n/(n-1))]$. Let $\mathcal{E}_{TV} \colon BV(\Omega) \to \mathbb{R}$ be defined as in (4.3) and let C_{BV} be the embedding constant of $W_0^{1,1}(\Omega)$ (equipped with the norm $u \mapsto \|Du\|_1$) in $L^{n/(n-1)}(\Omega)$, then*

$$\|u\|_p^p \leq C_{\mathrm{BV}}^{(p-1)n} \|u\|_1^{n-(n-1)p} \mathcal{E}_{TV}(u)^{(p-1)n}$$

for all $u \in BV(\Omega)$.

If additionally $p \leq \frac{n+1}{n}$ we have

$$\|u\|_p^p \leq C_{\mathrm{BV}}^{(p-1)n} \|u\|_1^{n-(n-1)p} \mathcal{E}_{TV}(u) + C_{\mathrm{BV}}^{(p-1)n} \|u\|_1^{n-(n-1)p}. \tag{4.11}$$

Proof. The assertion follows by taking the p-th power of the estimate in the previous proposition and by approximation of $\mathcal{E}_{TV}(u)$ as in Theorem 36. The second estimate follows from the elementary estimate $x^{(p-1)n} \leq 1 + x$ for $x \geq 0$ since $(p-1)n \leq 1$ applied with $x = \mathcal{E}_{TV}(u)$. □

Having this result at hand we will now investigate the two cases of perturbations of the eigenvalue problem of the 1-Laplace operator.

4.1 Perturbation of the Energy

Let $\alpha > 0$, $1 < p < n/(n-1)$ and let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. In this section we will investigate the following perturbed eigenvalue problem of the 1-Laplace operator

$$\mathcal{E}_{TV}(v) + \mathcal{E}_{\mathrm{Per}}(v) \to \underset{v \in L^p(\Omega)}{\mathrm{Min!}} \tag{4.12}$$

$$\mathcal{G}_1(v) = \alpha\,. \tag{4.13}$$

Here $\mathcal{E}_{TV}$ is the total variation functional as in (4.3) and $\mathcal{G}_1(v) = \int_\Omega |v| \,\mathrm{d}x$ is the $L^1(\Omega)$-norm. For the perturbation function $\mathcal{E}_{\mathrm{Per}} : L^p(\Omega) \to \mathbb{R}$ we assume that

(i) $\mathcal{E}_{\mathrm{Per}}$ is locally Lipschitz continuous,

(ii) $\mathcal{E}_{\mathrm{Per}}$ is even, i.e. $\mathcal{E}_{\mathrm{Per}}(v) = \mathcal{E}_{\mathrm{Per}}(-v)$,

(iii) there is a constant $C_{\mathrm{Per}} > 0$, such that

$$|\mathcal{E}_{\mathrm{Per}}(v)| \leq C_{\mathrm{Per}} \|v\|_p^p \tag{4.14}$$

for all $v \in L^p(\Omega)$ and

(iv) for all $v \in L^p(\Omega)$ and $v^* \in \partial\mathcal{E}_{\mathrm{Per}}(v)$ there holds

$$\|v^*\|_{p'} \leq p\, C_{\mathrm{Per}} \|v\|_p^{p-1} \,. \tag{4.15}$$

Note that these conditions are satisfied, provided $\mathcal{E}_{\mathrm{Per}}$ is of the form

$$\mathcal{E}_{\mathrm{Per}}(v) = \int_\Omega \int_0^{v(x)} f(x,s)\,\mathrm{d}s\,\mathrm{d}x \tag{4.16}$$

as in Theorem 5.

Since $\mathcal{E}_{TV}$ is lower semicontinuous on $L^p(\Omega)$, the functional $\mathcal{E}_{TV} + \mathcal{E}_{\mathrm{Per}}$ turns out to be lower semicontinuous on $L^p(\Omega)$. By definition u is a critical point of (4.12), (4.13), if $|d\mathcal{F}|(u) = 0$, where $\mathcal{F} := \mathcal{E}_{TV} + \mathcal{E}_{\mathrm{Per}} + I_{\{\mathcal{G}_1 = \alpha\}}$.

Theorem 15 (Euler-Lagrange Equation)**.** *Let u be a critical point of the variational problem* (4.12), (4.13)*. Then there exists a function $s \in L^\infty(\Omega)$ with*

$$s(x) \in \mathrm{Sgn}(u(x))$$

for almost all $x \in \Omega$, a vector field $z \in L^\infty(\Omega, \mathbb{R}^n)$ with

$$\operatorname{div} z \in L^{p'}(\Omega)\,, \quad \|z\|_\infty = 1 \quad \textit{and} \quad \mathcal{E}_{TV}(u) = -\int_\Omega u \operatorname{div} z\,\mathrm{d}x\,,$$

a function $u^* \in \partial\mathcal{E}_{\mathrm{Per}}(u)$ and a Lagrange multiplier λ, such that the Euler-Lagrange equation

$$-\operatorname{div} z + u^* = \lambda s \tag{4.17}$$

holds.

If $\mathcal{E}_{\mathrm{Per}}$ is of the form $\mathcal{E}_{\mathrm{Per}}(v) = \int_\Omega \int_0^{v(x)} f(x,s)\,\mathrm{d}s\,\mathrm{d}x$ as in Theorem 5, then for $u^* \in \partial\mathcal{E}_{\mathrm{Per}}(u)$ we have

$$u^*(x) \in \Big[\operatorname*{ess\,inf}_{t \to u(x)} f(x,t), \operatorname*{ess\,sup}_{t \to u(x)} f(x,t)\Big] \quad \textit{for almost every } x \in \Omega\,.$$

Proof. This result follows with the same arguments as in the well known unperturbed case. I.e. we need to verify that we can apply Theorem 7 with $\mathcal{F}_0 = \mathcal{E}_{TV}$, $\mathcal{F}_1 = \mathcal{E}_{\text{Per}}$ and $\mathcal{G} = \mathcal{G}_1$. Let $u \in BV(\Omega)$ with $\mathcal{G}_1(u) = \alpha$. As in the unperturbed case we need to show that there are u_1, $u_2 \in X$ such that

$$\mathcal{G}^0(u;\, u_1 - u) < 0 \quad \text{and} \quad \mathcal{G}^0(u;\, u - u_2) < 0\,.$$

Recalling Proposition 2 we derive with (3.6) for $u_1 = 0$ and $u_2 = 2u$

$$\mathcal{G}^0(u;\, u_1 - u) = \mathcal{G}_1^0(u;\, u - u_2) = \mathcal{G}_1^0(u;\, -u) = \max_{s \in \partial \mathcal{G}_1(u)} \langle s, -u \rangle = -\alpha < 0. \tag{4.18}$$

The Euler-Lagrange equation is now a consequence of Theorem 56, Proposition 2 and the properties of Clarkes generalized gradient of $\mathcal{E}_{\text{Per}}$ derived in Theorem 5. □

Remark 16. *Note that in contrast to the differentiable case of the p-Laplace operator we can not expect that the contrary of the previous theorem is true, i.e. a function u that satisfies the single version of the Euler-Lagrange equation need not be a critical point of* (4.12), (4.13). *This is already known for the unperturbed case $f = 0$, where we have solutions of the single version of the Euler-Lagrange equation which are not critical points of the associated problem.*

Let u_α be a critical point of (4.12), (4.13). Using u_α as test function in the Euler-Lagrange equation (4.17) we immediately arrive at

$$\mathcal{E}_{TV}(u_\alpha) + \langle u_\alpha^*, u_\alpha \rangle_{L^{p'}, L^p} = \alpha \lambda_\alpha$$

(subscripts are used to outline the dependence on α) which is equivalent to

$$\lambda_\alpha = \mathcal{E}_{TV}(\tfrac{u_\alpha}{\alpha}) + \langle u_\alpha^*, \tfrac{u_\alpha}{\alpha} \rangle_{L^{p'}, L^p} \tag{4.19}$$

For $u_\alpha^* = 0$ we have an immediate correspondence of λ_α and the critical value $\mathcal{E}_{TV}(u_\alpha)$. However for $\mathcal{F}_{\text{Per}} \neq 0$ we do not have equivalence of the Lagrange multiplier and the critical value any more. Note that Theorem 15 doesn't even state that the eigenvalue λ_α for a critical point u_α is unique. Nevertheless, provided we can control $\langle u_\alpha^*, \frac{u_\alpha}{\alpha} \rangle_{L^{p'}, L^p}$ as $\alpha \to 0$ we get an asymptotic correspondence of λ_α and the (normalized) critical value $\mathcal{E}_{TV}(\frac{u_\alpha}{\alpha})$ as $\alpha \to 0$.

We will verify the existence of a sequence of eigenfunctions $(u_{k,\alpha})_k$ for any $\alpha > 0$ now. The main task is the verification of the Palais-Smale condition in our situation.

Proposition 17 ((PS)-condition). *Let $\mathcal{E}_{TV}$ and $\mathcal{E}_{\mathrm{Per}}$ be as above and $\alpha > 0$. Assume that one of the following conditions holds:*

(a) $\mathcal{E}_{\mathrm{Per}}$ is globally bounded from below or

(b) $p \leq 1 + \frac{1}{n}$ and α is such that for the embedding constant C_{BV} of $W_0^{1,1}(\Omega)$ in $L^{n/(n-1)}(\Omega)$ and the constant C_{Per} from (4.14), (4.15) we have

$$\alpha^{n-(n-1)p} < C_{\mathrm{Per}}^{-1}\, C_{\mathrm{BV}}^{-(p-1)n}\,. \tag{4.20}$$

Then the Palais-Smale condition (PS) *holds for $\mathcal{F} := \mathcal{E}_{TV} + \mathcal{E}_{\mathrm{Per}} + I_{\{\mathcal{G}_1=\alpha\}}$.*

Remark 18. *Condition (a) is in particular satisfied, provided $\mathcal{G}_{\mathrm{Per}}$ is of the form* (4.16) *and the corresponding f in is bounded on $[0,\infty)$ from below (cf. Theorem 5).*

Since with $p \leq 1 + \frac{1}{n}$ we have $n - (n-1)p \geq \frac{1}{n}$ condition (4.20) *can always be achieved for α sufficiently close to zero.*

Proof. Let $c \in \mathbb{R}$ and let $(u_j)_j$ be a Palais-Smale sequence for the functional $\mathcal{F}$, i.e. $\mathcal{F}(u_j) \to c$ and $|d\mathcal{F}|(u_j) \to 0$. Note that $\|u_j\|_1 = \alpha$ for all $j \in \mathbb{N}$ by assumption. If $\mathcal{E}_{\mathrm{Per}}$ is bounded by $L \leq 0$ from below, we eventually have

$$\mathcal{E}_{TV}(u_j) \leq c + 1 - \mathcal{E}_{\mathrm{Per}}(u_j) \leq c + 1 - L\,.$$

Since $\mathcal{E}_{TV}$ is a norm on $BV(\Omega)$, which is equivalent to the standard norm, we can apply the compact embedding of $BV(\Omega)$ in $L^p(\Omega)$ (Theorem 32) to obtain a desired L^p-convergent subsequence $(u_{j_l})_l$ of $(u_j)_j$.

If condition (b) is satisfied, we can estimate, using (4.11),

$$\begin{aligned}
c + 1 &\geq \mathcal{E}_{TV}(u_j) + \mathcal{E}_{\mathrm{Per}}(u_j) \\
&\geq \mathcal{E}_{TV}(u_j) - C_{\mathrm{Per}}\|u_j\|_p^p \\
&\geq \mathcal{E}_{TV}(u_j) - C_{\mathrm{Per}}\Big(C_{\mathrm{BV}}^{(p-1)n}\,\|u_j\|_1^{n-(n-1)p}\,\mathcal{E}_{TV}(u_j) + C_{\mathrm{BV}}^{(p-1)n}\,\|u_j\|_1^{n-(n-1)p}\Big) \\
&= \mathcal{E}_{TV}(u_j) - C_{\mathrm{Per}}C_{\mathrm{BV}}^{(p-1)n}\,\alpha^{n-(n-1)p}\,\mathcal{E}_{TV}(u_j) - C_{\mathrm{Per}}C_{\mathrm{BV}}^{(p-1)n}\,\alpha^{n-(n-1)p} \\
&= \mathcal{E}_{TV}(u_j)\big(1 - C_{\mathrm{Per}}C_{\mathrm{BV}}^{(p-1)n}\,\alpha^{n-(n-1)p}\big) - C_{\mathrm{Per}}C_{\mathrm{BV}}^{(p-1)n}\,\alpha^{n-(n-1)p}\,.
\end{aligned} \tag{4.21}$$

Recalling $c = \lim_{j\to\infty} \mathcal{E}_{TV}(u_j) + \mathcal{E}_{\mathrm{Per}}(u_j)$ we thus obtain by (4.20)

$$\mathcal{E}_{TV}(u_j) \leq \frac{c + 1 + C_{\mathrm{Per}}C_{\mathrm{BV}}^{(p-1)n}\,\alpha^{n-(n-1)p}}{1 - C_{\mathrm{Per}}C_{\mathrm{BV}}^{(p-1)n}\,\alpha^{n-(n-1)p}}\,.$$

Whence as above it turns out that $(u_j)_j$ is bounded in $BV(\Omega)$ and the desired $L^p(\Omega)$-convergent subsequence is obtained by the compact embedding of $BV(\Omega)$ in $L^p(\Omega)$. □

Theorem 19. *Let $\alpha > 0$ and assume either that $\mathcal{E}_{\mathrm{Per}}$ is globally bounded from below or that $p \leq 1 + 1/n$ and that $\alpha > 0$ satisfies (4.20). Then there exists a sequence of eigenfunctions $(\pm u_{k,\alpha})_k$ with $\mathcal{G}_1(\pm u_{k,\alpha}) = \alpha$ of the perturbed eigenvalue problem (4.12), (4.13) with corresponding critical values given by*

$$\hat{c}_{k,\alpha} := \inf_{S \in \mathscr{S}_k^\alpha} \sup_{v \in S} \mathcal{E}_{TV}(v) + \mathcal{E}_{\mathrm{Per}}(v)\,, \tag{4.22}$$

where

$$\mathscr{S}_k^\alpha := \{S \subseteq L^p(\Omega) \text{ compact, symmetric}\,;\ \mathcal{G}_1 = \alpha \text{ on } S, \operatorname{gen}_{L^p} S \geq k\}\,. \tag{4.23}$$

The sequence of eigenvalues $(\hat{c}_{k,\alpha})_k$ is unbounded. Moreover, the Euler-Lagrange equation (4.17) as in Theorem 15 holds for any critical point $u_{k,\alpha}$.

Proof. We will apply Theorem 11 to $\mathcal{F} := \mathcal{E}_{TV} + \mathcal{E}_{\mathrm{Per}} + I_{\{\mathcal{G}_1 = \alpha\}}$. Property (A) is obviously satisfied.

Property (B) is obviously satisfied, provided $\mathcal{E}_{\mathrm{Per}}$ is bounded from below. It thus remains to consider the case $p \leq 1 + 1/n$. We invoke Corollary 14 to derive for $v \in BV(\Omega)$ with $\mathcal{G}_1(v) = \alpha$ similar to (4.21)

$$\begin{aligned}
\mathcal{F}(v) &= \mathcal{E}_{TV}(v) + \mathcal{E}_{\mathrm{Per}}(v) \\
&\geq \mathcal{E}_{TV}(v) - C_{\mathrm{Per}} \|v\|_p^p \\
&\geq \big(1 - C_{\mathrm{Per}} C_{\mathrm{BV}}^{(p-1)n} \alpha^{n-(n-1)p}\big) \mathcal{E}_{TV}(v) - C_{\mathrm{Per}} C_{\mathrm{BV}}^{(p-1)n} \alpha^{n-(n-1)p}\,.
\end{aligned}$$

The latter term is bounded from below provided by (4.20). Moreover, this estimate also shows that the sublevel sets $\{\mathcal{F} \leq \beta\}$ are compact in $L^p(\Omega)$ by Proposition 33 and Proposition 32.

The Palais-Smale condition (PS) holds due to Proposition 17.

The (epi)-condition is satisfied by (4.18) (cf. [22, Theorem 3.4]).

It remains to prove property (E) of Theorem 11. To do so let $k \in \mathbb{N}$ and let $v_1, \dots v_k \in C_c^\infty(\Omega)$ be linearly independent. Then it is easily seen that a desired map

$\Phi : \mathbb{S}^{k-1} \to L^p(\Omega)$ is given by

$$\Phi(x) = \Phi(x_1, \dots, x_k) = \frac{\alpha \sum_{j=1}^n x_j v_j}{\left\| \sum_{j=1}^n x_j v_j \right\|_1} .$$

We have thus justified all preliminaries of Theorem 11 and Theorem 15 applies. □

Let us introduce for $\alpha > 0$, $k \in \mathbb{N}$ the values

$$c_{k,\alpha} := \frac{\hat{c}_{k,\alpha}}{\alpha} .$$

In particular for $\mathcal{E}_{\mathrm{Per}} = 0$ we have $c_{k,\alpha} = \lambda_{k,\alpha}$. Even though $c_{k,\alpha}$ and $\lambda_{k,\alpha}$ do not coincide in general, we will see in a moment that we have coincidence in the limit as $\alpha \to 0$.

Proposition 20. *Let $1 < p \leq \frac{n+1}{n}$, then for any $k \in \mathbb{N}$ it holds*

$$c_{k,\alpha} \to \lambda_{k,0}$$

as $\alpha \to 0$.

Proof. (1) Initially we show $\limsup_{\alpha\to 0} c_{k,\alpha} \leq \lambda_{k,0}$. For that we essentially use the estimates (4.14) and (4.11) with $\|u\|_1 = 1$ for $u \in S \in \mathscr{S}_k^1$. Thus

$$\begin{aligned}
\limsup_{\alpha\to 0} c_{k,\alpha} &= \limsup_{\alpha\to 0} \inf_{S\in\mathscr{S}_k^\alpha} \sup_{u\in S} \frac{1}{\alpha}\big(\mathcal{E}_{TV}(u) + \mathcal{E}_{\mathrm{Per}}(u)\big) \\
&= \limsup_{\alpha\to 0} \inf_{S\in\mathscr{S}_k^1} \sup_{u\in S} \big(\mathcal{E}_{TV}(u) + \tfrac{1}{\alpha}\mathcal{E}_{\mathrm{Per}}(\alpha u)\big) \\
&\leq \limsup_{\alpha\to 0} \inf_{S\in\mathscr{S}_k^1} \sup_{u\in S} \big(\mathcal{E}_{TV}(u) + \tfrac{1}{\alpha}C_{\mathrm{Per}}\|\alpha u\|_p^p\big) \\
&= \limsup_{\alpha\to 0} \inf_{S\in\mathscr{S}_k^1} \sup_{u\in S} \big(\mathcal{E}_{TV}(u) + \alpha^{p-1}C_{\mathrm{Per}}\|u\|_p^p\big) \\
&\leq \limsup_{\alpha\to 0} \inf_{S\in\mathscr{S}_k^1} \sup_{u\in S} \mathcal{E}_{TV}(u)\big(1 + \alpha^{p-1}C_{\mathrm{BV}}^{(p-1)n}C_{\mathrm{Per}}\big) + \alpha^{p-1}C_{\mathrm{BV}}^{(p-1)n}C_{\mathrm{Per}} \\
&= \inf_{S\in\mathscr{S}_k^1} \sup_{u\in S} \mathcal{E}_{TV}(u) = \lambda_{k,0} .
\end{aligned}$$

(2) The reverse inequality follows by the same estimates

$$\begin{aligned}
\liminf_{\alpha\to 0} c_{k,\alpha} &= \liminf_{\alpha\to 0} \inf_{S\in\mathscr{S}_k^1} \sup_{u\in S} \big(\mathcal{E}_{TV}(u) + \tfrac{1}{\alpha}\mathcal{E}_{\mathrm{Per}}(\alpha u)\big) \\
&\geq \limsup_{\alpha\to 0} \inf_{S\in\mathscr{S}_k^1} \sup_{u\in S} \big(\mathcal{E}_{TV}(u) - \tfrac{1}{\alpha}C_{\mathrm{Per}}\|\alpha u\|_p^p\big)
\end{aligned}$$

$$\geq \limsup_{\alpha\to 0} \inf_{S\in\mathscr{S}_k^1} \sup_{u\in S} \mathcal{E}_{TV}(u)\big(1-\alpha^{p-1}C_{\mathrm{BV}}^{(p-1)n}C_{\mathrm{Per}}\big) - \alpha^{p-1}C_{\mathrm{BV}}^{(p-1)n}C_{\mathrm{Per}}$$
$$= \inf_{S\in\mathscr{S}_k^1} \sup_{u\in S} \mathcal{E}_{TV}(u) = \lambda_{k,0}\,.$$

□

The question is, how the eigenvalues $\lambda_{k,0}$ of the unperturbed problem and the eigenvalues $\lambda_{k,\alpha}$ of the perturbed problem are related to each other. The answer will follow from the next proposition.

Proposition 21. *Let $1 < p \leq \frac{n+1}{n}$ and let $(u_\alpha)_{\alpha>0}$ be a family of critical points of (4.12), (4.13). Let $c_\alpha := \frac{1}{\alpha}\big(\mathcal{E}_{TV}(u_\alpha) + \mathcal{E}_{\mathrm{Per}}(u_\alpha)\big)$ be bounded as $\alpha \to 0$. Then*

$$v_\alpha := u_\alpha/\alpha$$

is bounded in $BV(\Omega)$ as $\alpha \to 0$.

Moreover, $(c_\alpha)_\alpha$ converges to some $c_0 > 0$ as $\alpha \to 0$ if and only if the corresponding family of eigenvalues $(\lambda_\alpha)_\alpha$ converges as $\alpha \to 0$ and in that case $c_0 = \lim_{\alpha\to 0}\lambda_\alpha = \lim_{\alpha\to 0} c_\alpha$.

Proof. Without loss of generality we may assume α small enough such that condition (4.20) is satisfied.

Again we use the estimates (4.14) and (4.11) with $\|u_\alpha\|_1 = \alpha$ and derive

$$\begin{aligned}
c_\alpha &= \tfrac{1}{\alpha}\big(\mathcal{E}_{TV}(u_\alpha) + \mathcal{E}_{\mathrm{Per}}(u_\alpha)\big)\\
&\geq \mathcal{E}_{TV}(v_\alpha) - \tfrac{1}{\alpha}C_{\mathrm{Per}}\|\alpha v_\alpha\|_p^p\\
&= \mathcal{E}_{TV}(v_\alpha) - \alpha^{p-1}C_{\mathrm{Per}}\|v_\alpha\|_p^p\\
&\geq \mathcal{E}_{TV}(v_\alpha) - \alpha^{p-1}C_{\mathrm{Per}}\Big(C_{\mathrm{BV}}^{(p-1)n}\,\|v_\alpha\|_1^{n-(n-1)p}\,\mathcal{E}_{TV}(v_\alpha) + C_{\mathrm{BV}}^{(p-1)n}\,\|v_\alpha\|_1^{n-(n-1)p}\Big)\\
&= \big(1-\alpha^{p-1}C_{\mathrm{Per}}\,C_{\mathrm{BV}}^{(p-1)n}\big)\mathcal{E}_{TV}(v_\alpha) - \alpha^{p-1}C_{\mathrm{Per}}\,C_{\mathrm{BV}}^{(p-1)n}\,.
\end{aligned}$$

For α sufficiently small we obtain

$$\mathcal{E}_{TV}(v_\alpha) \leq \frac{c_\alpha + \alpha^{p-1}C_{\mathrm{Per}}\,C_{\mathrm{BV}}^{(p-1)n}}{1-\alpha^{p-1}C_{\mathrm{Per}}\,C_{\mathrm{BV}}^{(p-1)n}}\,, \tag{4.24}$$

whence $\mathcal{E}_{TV}(v_\alpha)$ is bounded and since $\mathcal{E}_{TV}$ is a norm on $BV(\Omega)$ the first assertion follows.

Analogously to the derivation of (4.24) we obtain

$$\mathcal{E}_{TV}(v_\alpha) \geq \frac{c_\alpha - \alpha^{p-1} C_{\mathrm{Per}} C_{\mathrm{BV}}^{(p-1)n}}{1 + \alpha^{p-1} C_{\mathrm{Per}}\, C_{\mathrm{BV}}^{(p-1)n}}\,. \tag{4.25}$$

We thus derive with (4.19), assumption (4.15) for $u_\alpha^* \in \partial\mathcal{E}_{\mathrm{Per}}(u_\alpha)$ and (4.24)

$$\begin{aligned}
\lambda_\alpha &= \mathcal{E}_{TV}(\tfrac{u_\alpha}{\alpha}) + \alpha^{-1}\langle u_\alpha^*, u_\alpha\rangle_{L^{p'},L^p} \\
&\leq \mathcal{E}_{TV}(\tfrac{u_\alpha}{\alpha}) + \alpha^{-1} p\, C_{\mathrm{Per}} \|u_\alpha\|_p^p \\
&= \mathcal{E}_{TV}(v_\alpha) + \alpha^{p-1} p\, C_{\mathrm{Per}} \|v_\alpha\|_p^p \\
&\leq \Big(1 + \alpha^{p-1} p\, C_{\mathrm{Per}}\, C_{\mathrm{BV}}^{(p-1)n}\Big)\mathcal{E}_{TV}(v_\alpha) + \alpha^{p-1} p\, C_{\mathrm{Per}}\, C_{\mathrm{BV}}^{(p-1)n} \\
&\leq \Big(1 + \alpha^{p-1} p\, C_{\mathrm{Per}}\, C_{\mathrm{BV}}^{(p-1)n}\Big)\frac{c_\alpha + \alpha^{p-1} C_{\mathrm{Per}}\, C_{\mathrm{BV}}^{(p-1)n}}{1 - \alpha^{p-1} C_{\mathrm{Per}}\, C_{\mathrm{BV}}^{(p-1)n}} + \alpha^{p-1} p\, C_{\mathrm{Per}}\, C_{\mathrm{BV}}^{(p-1)n}
\end{aligned}$$

and therefore

$$\limsup_{\alpha\to 0} \lambda_\alpha \leq \liminf_{\alpha\to 0} c_\alpha\,. \tag{4.26}$$

provided $(\lambda_\alpha)_\alpha$ or $(c_\alpha)_\alpha$ is convergent as $\alpha \to 0$.

On the other hand we obtain in a similar manner

$$\begin{aligned}
\lambda_\alpha &= \mathcal{E}_{TV}(\tfrac{u_\alpha}{\alpha}) + \alpha^{-1}\langle u_\alpha^*, u_\alpha\rangle_{L^{p'},L^p} \\
&\geq \mathcal{E}_{TV}(\tfrac{u_\alpha}{\alpha}) - \alpha^{-1} p C_{\mathrm{Per}} \|u_\alpha\|_p^p \\
&\geq \Big(1 - \alpha^{p-1} p C_{\mathrm{Per}}\, C_{\mathrm{BV}}^{(p-1)n}\Big)\frac{c_\alpha - \alpha^{p-1} C_{\mathrm{Per}} C_{\mathrm{BV}}^{(p-1)n}}{1 + \alpha^{p-1} C_{\mathrm{Per}} C_{\mathrm{BV}}^{(p-1)n}} - \alpha^{p-1} p\, C_{\mathrm{Per}} C_{\mathrm{BV}}^{(p-1)n}
\end{aligned}$$

and thus

$$\limsup_{\alpha\to 0} \lambda_\alpha \geq \liminf_{\alpha\to 0} c_\alpha\,. \tag{4.27}$$

provided $(\lambda_\alpha)_\alpha$ or $(c_\alpha)_\alpha$ is convergent as $\alpha \to 0$.

The assertion follows now by combination of (4.26) and (4.27). □

We have thus proved the following theorem.

Theorem 22. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Let $1 < p \leq \frac{n+1}{n}$ and consider some perturbation functional $\mathcal{E}_{\mathrm{Per}}$ as in Theorem 5. Then the eigenvalues $\lambda_{k,\alpha}$ of the eigenfunctions $\pm u_{k,\alpha}$ corresponding to the critical values $\hat{c}_{k,\alpha}$ given in (4.22) are bifurcation points of the perturbed eigenvalue problem (4.12), (4.13) in the sense that for any $\alpha > 0$ with (4.20) there exist a critical points $\pm u_{k,\alpha}$ of (4.12), (4.13) with corresponding critical value $\hat{c}_{k,\alpha} = \mathcal{E}_{TV}(u_{k,\alpha}) + \mathcal{E}_{\mathrm{Per}}(u_{k,\alpha})$ and eigenvalue $\lambda_{k,\alpha}$ and it holds*

$$\lim_{\alpha \to 0} c_{k,\alpha} = \lim_{\alpha \to 0} \frac{\hat{c}_{k,\alpha}}{\alpha} = \lim_{\alpha \to 0} \lambda_{k,\alpha} = \lambda_{k,0}$$

for all $k \in \mathbb{N}$.

4.2 Perturbation of the Constraint

In the foregoing section we considered perturbations of the energy functional $\mathcal{E}_{TV}$, this section is devoted to perturbations of the constraint $\mathcal{G}_1$. While the main challenge in the previous section relayed on the derivation of a suitable compactness argument for the (PS) condition, the perturbation of the constraint causes difficulties for the verification of the (epi) condition. This is also the reason, why we need slightly stronger requirements on $\mathcal{G}_{\mathrm{Per}}$ then on $\mathcal{E}_{\mathrm{Per}}$ in the previous section.

Let $r \in (1, 1 + 1/n)$ and let $\mathcal{G}_{\mathrm{Per}} : L^r(\Omega) \to \mathbb{R}$ be a given functional with

(i) $\mathcal{G}_{\mathrm{Per}}$ is even and locally Lipschitz continuous.

(ii) There exists a constant C_{Per} such that

$$0 \leq \mathcal{G}_{\mathrm{Per}}(v) \leq C_{\mathrm{Per}} \|v\|_r^r \tag{4.28}$$

for all $v \in L^r(\Omega)$.

(iii) For all $v \in L^r(\Omega)$ and all $v^* \in \partial \mathcal{G}_{\mathrm{Per}}(v)$ holds

$$\|v^*\|_{r'} \leq r\, C_{\mathrm{Per}} \|v\|_r^{r-1} \tag{4.29}$$

and

$$\begin{cases} v^*(x) > -1 & \text{for } v(x) > 0 \\ v^*(x) < 1 & \text{for } v(x) < 0 \end{cases} \tag{4.30}$$

for a. e. $x \in \Omega$.

Let us note that condition (4.30) is equivalent to

$$v^*(x)\frac{v(x)}{|v(x)|} > -1$$

for a.e. $x \in \Omega$ with $v(x) \neq 0$, such that (4.30) implies

$$v^*(x)v(x) > -|v(x)| \tag{4.31}$$

for a. e. $x \in \Omega$ with $v(x) \neq 0$.

Note that these conditions are satisfied provided $\mathcal{G}_{\mathrm{Per}}$ is a Nemytskii functional of the form

$$\mathcal{G}_{\mathrm{Per}}(v) = \int_\Omega F_x(v(x))\,\mathrm{d}x$$

with $F_x(t) = \int_0^t f(x,s)\,\mathrm{d}s$ as in in Theorem 5 and additionally satisfies $F_x \geq 0$ for a. e. $x \in \Omega$ in (3.17) and $f > -1$ a.e. on $\Omega \times [0,\infty)$.

It is not too difficult to show that convexity of F_x implies (4.30).

For perturbations $\mathcal{G}_{\mathrm{Per}}$ given above we intend to investigate the following perturbed eigenvalue problem of the 1-Laplace operator

$$\mathcal{E}_{TV}(v) \to \underset{v\in L^r(\Omega)}{\mathrm{Min!}} \tag{4.32}$$

subject to

$$\mathcal{G}_1(v) + \mathcal{G}_{\mathrm{Per}}(v) = \beta\,, \tag{4.33}$$

where $\mathcal{E}_{TV}$ is the total variation functional (4.3) extended by ∞ on $L^r(\Omega) \setminus BV(\Omega)$ and $\mathcal{G}_1(v) = \|v\|_1$ as before.

We say that $u \neq 0$ with $\mathcal{G}_1(u) + \mathcal{G}_{\mathrm{Per}}(u) = \beta$ is an eigenfunction of the perturbed eigenvalue problem (4.32), (4.33) of the 1-Laplace operator, provided u is a critical point of the function $\mathcal{F} : L^r(\Omega) \to \mathbb{R} \cup \{\infty\}$,

$$\mathcal{F} := \mathcal{E}_{TV} + I_{\{\mathcal{G}_1+\mathcal{G}_{\mathrm{Per}}=\beta\}}\,.$$

Before we continue, let us derive some properties of $\mathcal{G}_{\mathrm{Per}}$.

Lemma 23. *Let $u \in L^r(\Omega) \setminus \{0\}$. Then the function*

$$t \mapsto \mathcal{G}_1(tu) + \mathcal{G}_{\mathrm{Per}}(tu)$$

is strictly monotone increasing on $[0, \infty)$.

Moreover, for the directional derivative $(\mathcal{G}_1 + \mathcal{G}_{\mathrm{Per}})^0(u; -u)$ we have

$$(\mathcal{G}_1 + \mathcal{G}_{\mathrm{Per}})^0(u; -u) < 0\,.$$

Proof. For the proof let $\mathcal{G} := \mathcal{G}_1 + \mathcal{G}_{\mathrm{Per}}$. Let $0 \le t_1 < t_2$. By Lebourgs Theorem (3.8) there is $\theta \in (0,1)$ and $w^* \in \partial\mathcal{G}\big((\theta t_1 + (1-\theta)t_2)u\big)$, such that

$$\mathcal{G}(t_2 u) - \mathcal{G}(t_1 u) = \langle w^*, (t_2 - t_1)u\rangle = (t_2 - t_1)\int_\Omega w^*(x)u(x)\,\mathrm{d}x\,. \tag{4.34}$$

By the sum rule for Clarkes generalized gradient (3.7) there are $s \in \partial\mathcal{G}_1(u)$ and $u^* \in \mathcal{G}_{\mathrm{Per}}(u)$ with $w^* = s + u^*$. Note that $s(x) \in \mathrm{Sgn}(u(x))$ for almost every $x \in \Omega$ by Proposition 2. Whence for almost every $x \in \Omega$ with $u(x) \neq 0$ we have

$$w^*(x)u(x) = s(x)u(x) + u^*(x)u(x) = |u(x)| + u^*(x)u(x) > 0$$

by (4.31). But this implies strict monotonicity of $\mathcal{G}$ in (4.34).

The generalized directional derivative can be expressed in terms of Clarkes generalized gradient. By formula (3.6) we can calculate

$$\begin{aligned}
\mathcal{G}^0(u; -u) &= \max_{w^* \in \partial\mathcal{G}(u)} \langle w^*, -u\rangle \\
&\le \max_{s \in \partial\mathcal{G}_1(u)} \langle s, -u\rangle + \max_{u^* \in \partial\mathcal{G}_{\mathrm{Per}}(u)} \langle u^*, -u\rangle \\
&= -\int_\Omega |u(x)|\,\mathrm{d}x + \max_{u^* \in \partial\mathcal{G}_{\mathrm{Per}}(u)} -\int_\Omega u^*(x)u(x)\,\mathrm{d}x \\
&= -\|u\|_1 - \min_{u^* \in \partial\mathcal{G}_{\mathrm{Per}}(u)} \int_\Omega u^*(x)u(x)\,\mathrm{d}x \\
&< -\|u\|_1 - \int_\Omega -|u(x)|\,\mathrm{d}x \\
&= -\|u\|_1 + \|u\|_1 = 0\,,
\end{aligned}$$

where we made use of (4.30) again. □

This technical result allows us to derive the following theorem.

Theorem 24. *Any eigenfunction u of the perturbed eigenvalue problem* (4.32), (4.33) *satisfies the single version of the Euler-Lagrange equation, i.e. there is $s \in L^\infty(\Omega)$ with*

$$s(x) \in \operatorname{Sgn}(u(x))$$

for almost all $x \in \Omega$, a function $z \in L^\infty(\Omega, \mathbb{R}^n)$ with

$$\operatorname{div} z \in L^{r'}(\Omega)\,, \quad \|z\|_\infty = 1 \quad \textit{and} \quad \mathcal{E}_{TV}(u) = -\int_\Omega \operatorname{div} z\, u \,\mathrm{d}x\,,$$

a function $u^* \in \partial\mathcal{E}_{\mathrm{Per}}(u)$ and $\lambda \in \mathbb{R}$ such that the Euler-Lagrange equation

$$-\operatorname{div} z = \lambda(s + u^*) \tag{4.35}$$

holds.

Proof. It is not difficult to see that we can apply Theorem 7 with $\mathcal{F}_0 = \mathcal{E}_{TV}$, $\mathcal{F}_1 = 0$ and $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_{\mathrm{Per}} - \beta$. Note that condition (3.25) is satisfied with $u_1 = 0$ and $u_2 = 2u$ by the preceding lemma. □

We call the Lagrange multiplier λ an eigenvalue of the perturbed eigenvalue problem of the 1-Laplace operator (4.32), (4.33) for the eigenfunction u and the tuple (λ, u) will be called eigensolution of the 1-Laplace operator. Note that as in Section 4.1 it is not clear, whether λ is uniquely determined for each eigenfunction u and obviously it depends on β, provided $\mathcal{G}_{\mathrm{Per}} \neq 0$.

The next theorem summarizes our perturbation results for the perturbed eigenvalue problem of the 1-Laplace operator (4.32), (4.33). In particular it states the existence of eigensolutions of the perturbed eigenvalue problem (4.32), (4.33) and specifies that the eigenvalues of the 1-Laplace operator are bifurcation points of the perturbed eigenvalue problem of the 1-Laplace operator (4.32), (4.33).

Theorem 25. *For each $\beta > 0$ there exists a sequence of pairs of critical points $(\pm u_{k,\beta})_k$ of* (4.32), (4.33) *and the corresponding critical values $\hat{c}_{k,\beta} = \mathcal{E}_{TV}(\pm u_{k,\beta})$ are characterized by*

$$\hat{c}_{k,\beta} = \inf_{S \in \hat{\mathscr{S}}_k^\beta} \sup_{u \in S} \mathcal{E}_{TV}(u)\,, \tag{4.36}$$

with $\hat{\mathscr{S}}_k^\beta$ *given by*

$$\hat{\mathscr{S}}_k^\beta := \{S \subseteq L^p(\Omega) \text{ compact}\,;\ \mathcal{G}_1 + \mathcal{G}_{\mathrm{Per}} = \beta \text{ on } S,\ \mathrm{gen}_{L^p} S \geq k\}\,. \tag{4.37}$$

For each $k \in \mathbb{N}$ *the family of rescaled eigenfunctions* $v_{k,\beta} := u_{k,\beta}/\|u_{k,\beta}\|_1$ *is bounded in* $BV(\Omega)$ *for* β *from bounded sets (and in particular as* $\beta \to 0$*).*

Moreover, the rescaled critical values

$$c_{k,\beta} := \frac{\hat{c}_{k,\beta}}{\beta}$$

and the eigenvalues $\lambda_{k,\beta}$ *for the eigenfunctions* $u_{k,\beta}$ *converge, as* $\beta \to 0$ *to the eigenvalues* $\lambda_{k,0}$ *of the* 1*-Laplace operator (defined in* (2.8)*),*

$$\lim_{\beta\to 0} \hat{c}_{k,\beta} = \lim_{\beta\to 0} \lambda_{k,\beta} = \lambda_{k,0}$$

for all $k \in \mathbb{N}$.

Proof. Let $\beta > 0$ first. Again we intend to apply Theorem 11 for the function $\mathcal{F} := \mathcal{E}_{TV} + I_{\{\mathcal{G}_1+\mathcal{G}_{\mathrm{Per}}=\beta\}}$. Properties (A) and (B) are immediate.

The sublevel sets $\{\mathcal{F} \leq c\}$ are compact in $L^r(\Omega)$ by the compact embedding of $BV(\Omega)$ in $L^r(\Omega)$ (Proposition 32) and the fact that $\mathcal{E}_{TV}$ is an equivalent norm in $BV(\Omega)$ (Theorem 33). Note that any (PS)-sequence is for the level $c \in \mathbb{R}$ is eventually contained in the sublevel set $\{\mathcal{F} \leq c+1\}$ and thus compactness of all sublevel sets implies the (PS)-condition (C).

The (epi)-condition follows from the estimate on the directional derivative in Lemma 23 and [22, Theorem 3.4] applied with $g_0 = -1$, $C = BV(\Omega)$, $g_1 = \mathcal{G}_1 + \mathcal{G}_{\mathrm{Per}} - \beta$, $u_- = 0$ and $u_+ = 2u$.

In order to prove property (E) we need the following technical result.

Lemma 26. *Under the assumptions of Theorem 25, given* $u \in L^r(\Omega) \setminus \{0\}$ *and* $\beta > 0$ *there exists a unique* $t_u > 0$ *such that*

$$\mathcal{G}_1(t_u u) + \mathcal{G}_{\mathrm{Per}}(t_u u) = \beta\,.$$

Moreover, the mapping $u \mapsto t_u$ *is continuous on* $L^r(\Omega) \setminus \{0\}$.

Proof. Let $u \in L^r(\Omega) \setminus \{0\}$. By Lemma 26 the mapping

$$[0,\infty) \ni t \mapsto \|tu\|_1 + \mathcal{G}_{\mathrm{Per}}(tu)$$

is strictly monotone increasing and continuous. From assumption (4.28) we infer $\|0u\|_1 + \mathcal{G}_{\mathrm{Per}}(0u) = 0$ and $\liminf_{t\to\infty} \|tu\|_1 + \mathcal{G}_{\mathrm{Per}}(tu) \geq t\|u\|_1 = \infty$. Thus t_u exists by the intermediate value theorem and is uniquely determined by strict monotonicity. Let now $u_j \to u$ and let (t_j) be the corresponding sequence of numbers such that $\|t_j u_j\|_1 + \mathcal{G}_1(t_j u_j) = \beta$. From $\beta \geq t_j \|u_j\|_1$ and the convergence of u_j in $L^r(\Omega)$ (and thus also in $L^1(\Omega)$) we infer that $(t_j)_j$ must be bounded. By picking a subsequence we may assume without loss of generality that $(t_j)_j$ converges to some $t_0 \geq 0$. By continuity we infer

$$\beta = \lim_{j\to\infty} \mathcal{G}_1(t_j u_j) + \mathcal{G}_{\mathrm{Per}}(t_j u_j) = \|t_0 u\|_1 + \mathcal{G}_{\mathrm{Per}}(t_0 u)\,,$$

thus by uniqueness of t_u we obtain $t_0 = t_u$ and thus continuity of $u \mapsto t_u$. □

By the previous arguments there exists a homeomorphism

$$\Phi_\beta : \{u \in L^r(\Omega)\ ;\ \mathcal{G}_1(u) + \mathcal{G}_{\mathrm{Per}}(u) = \beta\} \to \{u \in L^r(\Omega)\ ;\ \|u\|_1 = 1\}\,,$$

$$u \mapsto u/\|u\|_1$$

with inverse $\Psi_\beta : \{u \in L^r(\Omega)\ ;\ \|u\|_1 = 1\} \to \{u \in L^r(\Omega)\ ;\ \mathcal{G}_1(u) + \mathcal{G}_{\mathrm{Per}}(u) = \beta\}$ given by

$$u \mapsto t_u u$$

where $t_u > 0$ is the uniquely determined number such that $\mathcal{G}(t_u u) + \mathcal{G}_1(t_u u) = \beta$ (Bijectivity and continuity of Φ_β is elementary and continuity of Ψ_β is proven in Lemma 26).

By property (iii) from page 141 and since Φ_β is a homeomorphism we thus have

$$S \in \hat{\mathscr{S}}_k^\beta \quad \Leftrightarrow \quad \Phi_\beta(S) \in \mathscr{S}_k^1\,,$$

where $\mathscr{S}_k^1$ is the set defined in (4.23) with $\alpha = 1$. In the proof of Theorem 19 we verfied that $\mathscr{S}_k^1$ is nonempty and by the observation above we derive the classes $\hat{\mathscr{S}}_k^\beta$ to be nonempty as usually guaranteed by property (E) in Theorem 11. We have thus verified the preliminaries of Theorem 11 and thus obtain that (4.36) defines an unbounded sequence of critical values of the variational problem (4.32), (4.33).

We will now prove the convergence results. Let $k \in \mathbb{N}$.

Using assumption (4.28) and Corollary 14 we derive for any $u \in L^r(\Omega) \cap BV(\Omega)$

with $\mathcal{G}_1(u) + \mathcal{G}_{\mathrm{Per}}(u) = \beta$

$$\begin{aligned}
\|u\|_1 \le \beta = \|u\|_1 + \mathcal{G}_{\mathrm{Per}}(u) \\
&\le \|u\|_1 + C_{\mathrm{Per}}\|u\|_r^r \\
&\le \|u\|_1 + C_{\mathrm{Per}} C_{\mathrm{BV}}^{(r-1)n} \|u\|_1^{n-(n-1)r} \mathcal{E}_{TV}(u)^{(r-1)n} \\
&= \|u\|_1 \big(1 + C_{\mathrm{Per}} C_{\mathrm{BV}}^{(r-1)n} \|u\|_1^{r-1} \mathcal{E}_{TV}(u/\|u\|_1)^{(r-1)n}\big) \\
&\le \|u\|_1 \big(1 + C_{\mathrm{Per}} C_{\mathrm{BV}}^{(r-1)n} \beta^{r-1} \mathcal{E}_{TV}(u/\|u\|_1)^{(r-1)n}\big) .
\end{aligned}$$

This implies

$$\|u\|_1 \ge \frac{\beta}{1 + C_{\mathrm{Per}} C_{\mathrm{BV}}^{(r-1)n} \beta^{r-1} \mathcal{E}_{TV}(u/\|u\|_1)^{(r-1)n}}$$

for any admissible u in the effective domain of definition of the variational problem (4.32), (4.33).

We continue to estimate

$$\begin{aligned}
c_{k,\beta} &= \inf_{S \in \hat{\mathscr{S}}_k^\beta} \sup_{u \in S} \frac{1}{\beta} \mathcal{E}_{TV}(u) \\
&= \inf_{S \in \hat{\mathscr{S}}_k^\beta} \sup_{u \in S} \frac{\|u\|_1}{\beta} \mathcal{E}_{TV}\Big(\frac{u}{\|u\|_1}\Big) \\
&\le \inf_{S \in \hat{\mathscr{S}}_k^\beta} \sup_{u \in S} \frac{\beta}{\beta} \mathcal{E}_{TV}\Big(\frac{u}{\|u\|_1}\Big) \\
&= \inf_{S \in \mathscr{S}_k^1} \sup_{v \in S} \mathcal{E}_{TV}(v) = \lambda_{k,0} . \qquad (4.38)
\end{aligned}$$

To be able to estimate in the reverse direction we need the following lemma.

Lemma 27. *For each $\beta > 0$ there exists a set $S_\beta \in \mathscr{S}_k^\beta$ with $\hat{c}_{k,\beta} = \sup_{u \in S_\beta} \mathcal{E}_{TV}(u)$.*

Proof. Let $\beta > 0$ and consider a sequence $(S_j)_j$ in $\hat{\mathscr{S}}_k^\beta$ with $\sup_{u \in S_j} \mathcal{E}_{TV}(u) \to \hat{c}_{k,\beta}$. Since $\{\mathcal{E}_{TV} \le \hat{c}_{k,\beta} + 1\}$ is compact in $L^r(\Omega)$ (cf. the beginning of this proof), the sequence $(S_j)_j$ admits a subsequence (also denoted by S_j without loss of generality) convergent with respect to Hausdorff-convergence in $L^r(\Omega)$ to a compact $S_\beta \subseteq L^r(\Omega)$ (compare the arguments in the proof of Theorem 11 on page 49). There is some $\delta > 0$ such that $\operatorname{gen} S_\beta = \operatorname{gen} \overline{B(S_\beta, \delta)}$ (cf. Property (v) on page 141). Since $S_j \subseteq B(S_\beta, \delta)$ for j large enough we get $\operatorname{gen} S_\beta \ge \limsup_{j\to\infty} \operatorname{gen} S_j \ge k$. Moreover, by convergence in Hausdorff metric we have $v \in S_\beta$ if and only if there is a sequence $v_j \in S_j$ with $v_j \to v$. Thus we derive by continuity of $\mathcal{G}_1 + \mathcal{G}_{\mathrm{Per}}$ that $S_\beta \in \mathscr{S}_k^\beta$. By lower semicontinuity

of the total variation we conclude

$$\hat{c}_{k,\beta} = \sup_{u \in S_\beta} \mathcal{E}_{TV}(u)\,. \qquad \square$$

By the preceding lemma for $\beta > 0$ we find $S_\beta \in \mathscr{S}_k^\beta$ with

$$c_{k,\beta} = \frac{\hat{c}_{k,\beta}}{\beta} = \sup_{u \in S_\beta} \frac{\mathcal{E}_{TV}(u)}{\beta}\,.$$

For $u \in S_\beta$ we can estimate with (4.28) and Corollary 14

$$\begin{aligned}
\beta &\leq \|u\|_1 + C_{\mathrm{Per}} \|u\|_r^r \\
&\leq \|u\|_1 + C_{\mathrm{Per}}\, C_{BV}^{(r-1)n} \|u\|_1^{n-(n-1)r}\, \mathcal{E}_{TV}(u)^{(r-1)n} \\
&= \|u\|_1 \big(1 + C_{\mathrm{Per}}\, C_{BV}^{(r-1)n} \|u\|_1^{r-1} \mathcal{E}_{TV}(u/\|u\|_1)^{(r-1)n}\big). \qquad (4.39)
\end{aligned}$$

Hence,

$$\begin{aligned}
\lambda_{k,0} \geq c_{k,\beta} &= \sup_{u \in S_\beta} \frac{\mathcal{E}_{TV}(u)}{\beta} \\
&\geq \sup_{u \in S_\beta} \frac{1}{\|u\|_1 \big(1 + C_{\mathrm{Per}}\, C_{BV}^{(r-1)n} \|u\|_1^{r-1} \mathcal{E}_{TV}(u/\|u\|_1)^{(r-1)n}\big)} \mathcal{E}_{TV}(u)\,.
\end{aligned}$$

This implies for all $\beta > 0$ and $u \in S_\beta$

$$\lambda_{k,0} \left(1 + C_{\mathrm{Per}} \beta^{r-1} \mathcal{E}_{TV}\Big(\frac{u}{\|u\|_1}\Big)^{(r-1)n}\right) \geq \mathcal{E}_{TV}\Big(\frac{u}{\|u\|_1}\Big)$$

Since we are interested in the limit $\beta \to 0$ we may assume $\beta \leq \beta_0$ for some $\beta_0 > 0$ and thus derive for those β and $u \in S_\beta$

$$\lambda_{k,0} \left(1 + C_{\mathrm{Per}} \beta_0^{r-1} \mathcal{E}_{TV}\Big(\frac{u}{\|u\|_1}\Big)^{(r-1)n}\right) \geq \mathcal{E}_{TV}\Big(\frac{u}{\|u\|_1}\Big)$$

Since the right hand side of this inequality is of linear growth in $\mathcal{E}_{TV}(u/\|u\|_1)$ and of sublinear growth in $\mathcal{E}_{TV}(u/\|u\|_1)$ on the left hand side (recall $(r-1)n < 1$ by assumption), we conclude that $\mathcal{E}_{TV}(u/\|u\|_1)^{(r-1)n}$ is bounded by some $\tilde{C} > 0$ for all $u \in \bigcup_{0<\beta\leq\beta_0} S_\beta$.

We thus obtain from (4.39) for $\beta \leq \beta_0$

$$\begin{aligned} c_{k,\beta} = \sup_{u \in S_\beta} \frac{\mathcal{E}_{TV}(u)}{\beta} &\geq \sup_{u \in S_\beta} \frac{1}{1 + \beta^{r-1}\, C_{BV}^{(r-1)n}\, C_{\mathrm{Per}}\, \tilde{C}} \mathcal{E}_{TV}\Big(\frac{u}{\|u\|_1}\Big) \\ &\geq \frac{1}{1 + \beta^{r-1}\, C_{BV}^{(r-1)n}\, C_{\mathrm{Per}}\, \tilde{C}} \lambda_{k,0} \end{aligned}$$

From this and (4.38) we derive

$$\lim_{\beta \to 0} c_{k,\beta} = \lambda_{k,0}\,.$$

We have thus shown that the eigenvalues of the 1-Laplace operator are bifurcation values of the critical values of the perturbed problem. Our next goal is to verify that the eigenvalues of the perturbed problem also converge to the eigenvalues or the 1-Laplace operator as $\beta \to 0$.

Let $u_{k,\beta}$ be an eigenfunction of (4.32), (4.33) corresponding to the critical value $\hat{c}_{k,\beta}$. Then $\mathcal{E}_{TV}(u_{k,\beta}) = \beta c_{k,\beta}$ by definition. Testing the Euler-Lagrange Equation (4.35) with $u_{k,\beta}$ and division by β yields

$$c_{k,\beta} = \frac{\mathcal{E}_{TV}(u_{k,\beta})}{\beta} = \lambda_{k,\beta} \frac{\|u_{k,\beta}\|_1 + \langle u^*_{k,\beta}, u_{k,\beta}\rangle}{\beta}\,,$$

where $u^*_{k,\beta}$ is an element of $\partial \mathcal{G}_{\mathrm{Per}}(u_{k,\beta})$.

Since $|\langle u^*_{k,\beta}, u_{k,\beta}\rangle| \leq r\, C_{\mathrm{Per}} \|u_{k,\beta}\|_r^r$ by assumption (4.29) we derive

$$\begin{aligned} c_{k,\beta} &\geq \lambda_{k,\beta} \frac{\|u_{k,\beta}\|_1 - r\, C_{\mathrm{Per}} \|u_{k,\beta}\|_r^r}{\|u_{k,\beta}\|_1 + C_{\mathrm{Per}} \|u_{k,\beta}\|_r^r} \\ &\geq \lambda_{k,\beta} \frac{\|u_{k,\beta}\|_1 - r\, C \mathcal{E}_{TV}(u_{k,\beta})^r}{\|u_{k,\beta}\|_1 + C \mathcal{E}_{TV}(u_{k,\beta})^r} \\ &= \lambda_{k,\beta} \frac{\|u_{k,\beta}\|_1 - r\, C \beta^r \big(c_{k,\beta}\big)^r}{\|u_{k,\beta}\|_1 + C \beta^r \big(c_{k,\beta}\big)^r}\,, \end{aligned}$$

for some constant $C > 0$ due to the embedding $BV(\Omega) \hookrightarrow L^r(\Omega)$ and since $\mathcal{E}_{TV}$ is a norm in $BV(\Omega)$.

Since $(c_{k,\beta})_\beta$ is convergent to a limit strictly greater than zero as $\beta \to 0$ we obtain

$$\limsup_{\beta \to 0} \lambda_{k,\beta} \leq \lim_{\beta \to 0} c_{k,\beta}\,.$$

On the other hand we may estimate

$$\begin{aligned} c_{k,\beta} &\leq \lambda_{k,\beta} \frac{\|u_{k,\beta}\|_1 + r\, C_{\mathrm{Per}} \|u_{k,\beta}\|_r^r}{\beta} \\ &\leq \lambda_{k,\beta} \frac{\|u_{k,\beta}\|_1 + r\, C \mathcal{E}_{TV}(u_{k,\beta})^r}{\|u_{k,\beta}\|_1} \\ &= \lambda_{k,\beta} \frac{\|u_{k,\beta}\|_1 + r\, C \beta^r \big(c_{k,\beta}\big)^r}{\|u_{k,\beta}\|_1} \end{aligned}$$

and again by the convergence properties of $c_{k,\beta}$ as $\beta \to 0$ we derive

$$\liminf_{\beta \to 0} \lambda_{k,\beta} \geq \lim_{\beta \to 0} c_{k,\beta} \,. \qquad \square$$

5 The Eigenvalue Problem in $BV(\Omega, \mathbb{R}^N)$ and $BD(\Omega)$

In this chapter we will investigate the eigenvalue problem of the 1-Laplace operator in $BV(\Omega, \mathbb{R}^N)$ and $BD(\Omega)$. The first two sections contain a review of known properties of the function spaces $BV(\Omega, \mathbb{R}^N)$ and $BD(\Omega)$. Our main results are Theorems 54 and 55, which prove the existence of solutions of the eigenvalue problem in $BV(\Omega, \mathbb{R}^N)$ and $BD(\Omega)$ and certain properties of these solutions. Crucial for the investigation of the eigenvalue problems finally considered in Section 5.4 is the derivation of suitable Gauß-Green formulas in our vectorial framework. This is done in Section 5.3. In the scalar case this was basically done by Anzellotti [5], but the vectorial case can not directly be deduced from the scalar case. Since we could not completely follow the arguments in [5] concerning the construction of the normal trace with the aid of an abstract Hahn-Banach argument and in particular its continuity properties, we will provide an own, alternative proof.

Let Ω be always an open subset of $\mathbb{R}^n$. We will additionally frequently require stronger properties of Ω as a finite measure $|\Omega|$ or regularity of the boundary $\partial\Omega$.

5.1 $BV(\Omega, \mathbb{R}^N)$ and its Properties

The space $BV(\Omega, \mathbb{R}^N)$ consists of those $L^1(\Omega, \mathbb{R}^N)$-functions $u = (u^1, \dots u^N)$ where the distributional gradient

$$Du = \left(\partial_j u^i\right)_{\substack{i=1,\dots,N \\ j=1,\dots,n}}$$

is represented by a finite $\mathbb{R}^{N\times n}$-valued signed Radon measure[1] on Ω. Equipped with the norm

$$u \mapsto \|u\|_{BV} := \int_\Omega |u| \,\mathrm{d}x + |Du|(\Omega)\,,$$

[1] For more details cf. Appendix p. 137

it becomes a Banach space. Here $|Du|$ denotes the total variation measure of the measure Du (cf. Proposition 73) and $|Du|(\Omega)$ is also called the total variation of u in Ω. The $BV(\Omega, \mathbb{R}^N)$-norm is too strong for most applications. Two other notions[2] of convergence are usually applied.

Definition. *A sequence $(u_j)_j$ in $BV(\Omega, \mathbb{R}^N)$ converges weakly* to $u \in BV(\Omega, \mathbb{R}^N)$, if $(u_j)_j$ converges to u in $L^1(\Omega, \mathbb{R}^N)$ and $(Du_j)_j$ converges weakly* in $\mathcal{M}(\Omega, \mathbb{R}^{N\times n})$ to Du. A sequence $(u_j)_j$ in $BV(\Omega, \mathbb{R}^N)$ converges strictly to $u \in BV(\Omega, \mathbb{R}^N)$, if $(u_j)_j$ converges to u in $L^1(\Omega, \mathbb{R}^N)$ and $(|Du_j|(\Omega))_j$ converges to $|Du|(\Omega)$.*

It is easily seen (cf. [3, Proposition 3.13]) that a sequence $(u_j)_j$ converges weakly* if and only if $(u_j)_j$ converges in $L^1(\Omega, \mathbb{R}^N)$ and $(u_j)_j$ is norm bounded in $BV(\Omega, \mathbb{R}^N)$. Thus strict convergence implies weak*-convergence. While $BV(\Omega, \mathbb{R}^N)$ can indeed be considered as dual space of a suitable Banach space (cf. [3, Remark 3.12]), and weak*-convergence is there equivalent to the usual weak*-convergence in Banach spaces. The notion of strict convergence is much more difficult to understand completely. We may metrize the notion of strict convergence with the metric

$$d_s(u, v) := \|u - v\|_1 + \big||Du|(\Omega) - |Dv|(\Omega)\big| .$$

However, this metric turns out to be *not* translation invariant. To give a simple example let $\Omega = (0, 3)$ and let $u := \chi_{[1,2]}$, then $d_s(u, -u) = \|2u\|_1 + |2 - 2| = 2$, $d_s(0, 2u) = \| - 2u\|_1 + |0 - 4| = 6$, such that $u \in B_{d_s}[-u, 2]$, but $2u = u + u \notin B_{d_s}[-u + u, 2] = B_{d_s}[0, 2]$. Thus one has to be very careful with the application of topological vector space arguments (in particular with the application of Hahn-Banach arguments) when considering the strict topology in $BV(\Omega, \mathbb{R}^N)$. A consequence is that the vector space addition is not continuous with respect to strict convergence. To give an example let Ω and u be as above and consider $u_j := \chi_{[1+1/j,2]}$, then $u_j \to u$ strictly. Nevertheless $u - u_j = \chi_{[1,1+1/j]}$ does not converge to $u - u = 0$ strictly.

In $W^{1,p}(\Omega, \mathbb{R}^N)$, $p \in (1, \infty)$, the corresponding notion of strict convergence is equivalent to strong convergence in the usual sense, i.e.

$$u_j \to u \quad \text{in } L^p(\Omega, \mathbb{R}^N) \quad \text{and} \quad \|Du_j\|_{L^p(\Omega,\mathbb{R}^{N\times n})} \to \|Du\|_{L^p(\Omega,\mathbb{R}^{N\times n})}$$

is equivalent to $u_j \to u$ in $W^{1,p}(\Omega, \mathbb{R}^N)$. Indeed, both properties imply that $(u_j)_j$

[2] There is a third, slightly stronger notion of convergence in $BV(\Omega, \mathbb{R}^N)$ – the concept of area-strict convergence, cf. [51] and references therein. However, this slightly more involved techniques are not needed for the investigation of our 1-homogeneous energy functionals.

is bounded in $W^{1,p}(\Omega, \mathbb{R}^N)$ and thus admits a weakly convergent subsequence $(u_{j_l})_l$ with $u_{j_l} \rightharpoonup: \tilde{u}$. However, the limit is unique, $u = \tilde{u}$ by strong convergence of u_j to u in $L^p(\Omega, \mathbb{R}^N)$. Thus $u_j \rightharpoonup u$ in $W^{1,p}(\Omega)$ and in particular $Du_j \rightharpoonup Du$ in $L^p(\Omega, \mathbb{R}^{N\times n})$. However, the latter implies in connection with $\|Du_j\|_p \to \|Du\|_p$ strong convergence $Du_j \to Du$ in $L^p(\Omega, \mathbb{R}^{N\times n})$ by uniform convexity of the L^p-norm (which is not evident for the norm we use, but follows e.g. from [29] with $A(x, \xi) = |\xi|^p$, cf. also [43]). Both properties, reflexivity and uniform convexity of the norm get lost for $p = 1$, which is one reason making $W^{1,1}(\Omega, \mathbb{R}^N)$ unappropriate for many applications.

The embedding

$$W^{1,1}(\Omega, \mathbb{R}^N) \hookrightarrow BV(\Omega, \mathbb{R}^N)$$

and the identity $|Du|(\Omega) = \int_\Omega |Du| \, \mathrm{d}x$ for $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ are immediate from the definition. Moreover, $W^{1,1}(\Omega, \mathbb{R}^N)$ is a proper subset of $BV(\Omega)$ when $\Omega \neq \emptyset$.

For $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ we introduce the *variation* $V(u, \Omega)$ by

$$V(u, \Omega) := \sup \left\{ \int_\Omega u \cdot \operatorname{div} \varphi \, \mathrm{d}x \; ; \; \varphi \in C^1_c(\Omega, \mathbb{R}^{N\times n}), \; \|\varphi\|_\infty \leq 1 \right\} . \tag{5.1}$$

As supremum of continuous linear functionals it is easily seen that $u \mapsto V(u, \Omega)$ is lower semicontinuous as functional on $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Moreover, for $u \in L^1(\Omega, \mathbb{R}^N)$ we have $u \in BV(\Omega, \mathbb{R}^N)$ if and only if $V(u, \Omega) < \infty$ and in that case

$$|Du|(\Omega) = V(u, \Omega)$$

(cf. [3, Remark 3.5, Proposition 3.6]).

Let us quote Theorem 3.9 from [3], which states that C^∞-functions are dense in $BV(\Omega, \mathbb{R}^N)$ in terms of strict convergence.

Proposition 28. *Let $u \in L^1(\Omega, \mathbb{R}^N)$. Then $u \in BV(\Omega, \mathbb{R}^N)$ if and only if there is a sequence $(u_j)_j$ in $C^\infty(\Omega, \mathbb{R}^N) \cap L^1(\Omega, \mathbb{R}^N)$, such that $u_j \to u$ in $L^1(\Omega, \mathbb{R}^N)$ and*

$$\limsup_{j\to\infty} \int_\Omega |Du_j| \, \mathrm{d}x < \infty.$$

Moreover, if $u \in BV(\Omega, \mathbb{R}^N)$, then there is a sequence $(u_j)_j$ in $C^\infty(\Omega, \mathbb{R}^N) \cap L^1(\Omega, \mathbb{R}^N)$, converging to u in $L^1(\Omega, \mathbb{R}^N)$ and

$$\lim_{j\to\infty} \int |Du_j| \, \mathrm{d}x = |Du|(\Omega).$$

Before we continue let us recall the following classical Gauß-Green Theorem.

Proposition 29 (Classical Gauß-Green Theorem). *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary and let $u \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$, $z \in C^1(\Omega, \mathbb{R}^{N\times n}) \cap C(\overline{\Omega}, \mathbb{R}^{N\times n})$. Then*

$$\int_\Omega u \cdot \operatorname{div} z \,\mathrm{d}x + \int_\Omega Du : z \,\mathrm{d}x = \int_{\partial\Omega} (u \otimes \nu) : z \,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial\Omega} u \cdot (z\,\nu) \,\mathrm{d}\mathcal{H}^{n-1}, \quad (5.2)$$

where ν is the outer unite normal on $\partial\Omega$ and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

The validity of this formula follows from the usual Gauß Theorem applied to $\operatorname{div}(u \cdot z)$. In Section 5.3 we will extend this formula to functions u and z belonging to larger classes of function spaces.

We will now state the trace theorem in $BV(\Omega, \mathbb{R}^N)$.

Theorem 30. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary and outer unit normal ν. There exists a linear trace operator $\cdot^{\partial\Omega} : BV(\Omega, \mathbb{R}^N) \to L^1(\partial\Omega, \mathbb{R}^N)$, continuous with respect to strict convergence in $BV(\Omega, \mathbb{R}^N)$ such that*

$$\int_\Omega u \cdot \operatorname{div} z \,\mathrm{d}x + \int_\Omega z : \mathrm{d}Du = \int_{\partial\Omega} u^{\partial\Omega} \cdot (z\nu) \,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial\Omega} z : (u^{\partial\Omega} \otimes \nu) \,\mathrm{d}\mathcal{H}^{n-1} \quad (5.3)$$

for all $z \in C^1(\Omega, \mathbb{R}^{N\times n}) \cap C(\overline{\Omega}, \mathbb{R}^{N\times n})$ and $u \in BV(\Omega, \mathbb{R}^N)$. In particular $u^{\partial\Omega}$ can be obtained as continuous extension of u on $\partial\Omega$, provided $u \in C(\overline{\Omega}, \mathbb{R}^N)$ and for $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ it coincides with the usual trace of Sobolev functions.

Proof. This result is well known and proofs are given in [27, Theorem 1, p. 177] (for $N = 1$) and [3, Theorem 3.88] ($N > 1$)[3]. We will need to refer to the technique of the proof and therefore sketch the procedure from [27].

Since Ω has a Lipschitz boundary we can assume $u \in BV(\Omega \cap V, \mathbb{R}^N)$ and u has compact support in $\overline{\Omega} \cap V$, where $V := B_{n-1}(0, r) \times (-h, h)$ and there is a Lipschitz function $\gamma : B_{n-1}(0, r) \to [-h/2, h/2]$ such that $\partial\Omega \cap V = \{(x, \gamma(x)), x \in B_{n-1}(0, r)\}$ and $\Omega \cap V = \{(x, y) \ ; \ x \in B_{n-1}(0, r), \gamma(x) < y < h\}$ (cf. [27, p. 177]) first. This situation is sketched in Figure 5.1.

For our further treatment we introduce the following notation. For a function $g : \Omega \cap V \to X$ (X is arbitrary at the moment) and $0 < \tau < h/2$ we define $g^\tau : \partial\Omega \cap V \to X$

[3] Note that the technique of the proof for the scalar case $BV(\Omega, \mathbb{R})$ can be transfered to the vectorial case, however a reduction of the vectorial to the scalar case arguing component-wise is not possible.

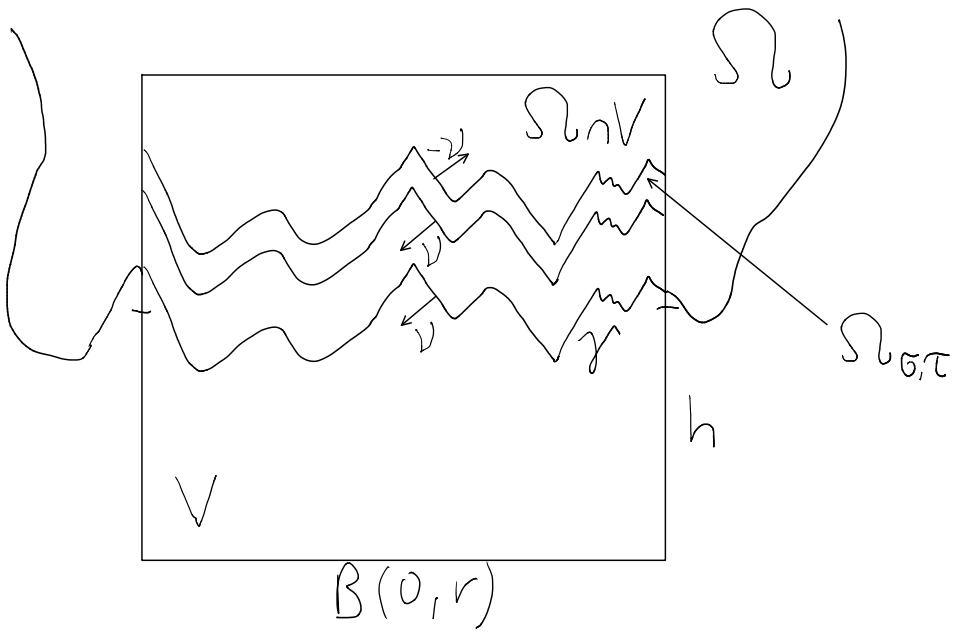

Figure 5.1: Local representation of the Lipschitz boundary of Ω.

by setting

$$g(x, \gamma(x))^\tau := g(x, \gamma(x) + \tau)\,, \qquad (x, \gamma(x)) \in \partial\Omega \cap V \tag{5.4}$$

where we made use of the standard parametrization $\Phi : B_{n-1}(0, r) \to \partial\Omega$, $x \mapsto (x, \gamma(x))$ of $\partial\Omega \cap V$.

The first step to prove the trace theorem for $BV(\Omega, \mathbb{R}^N)$-functions is to show that the trace $u^{\partial\Omega}$ for $u \in C^\infty(\Omega \cap V, \mathbb{R}^N) \cap BV(\Omega \cap V, \mathbb{R}^N)$[4] is (locally) obtained as $L^1(\partial\Omega \cap V, \mathbb{R}^N)$-limit of $(u^\tau)_\tau$ as $\tau \to 0$. Then formula (5.3) is established for $u \in BV(\Omega, \mathbb{R}^N) \cap C^\infty(\Omega, \mathbb{R}^N)$ using a partition of unity argument. General $u \in BV(\Omega, \mathbb{R}^N)$ is then approximated by functions $u_j \in BV(\Omega, \mathbb{R}^N) \cap C^\infty(\Omega, \mathbb{R}^N)$ in terms of strict convergence and, using similar estimates as in the first step, one verifies that $(u_j^{\partial\Omega})_j$ converges in $L^1(\partial\Omega, \mathbb{R}^N)$ as $j \to \infty$ to a limit $u^{\partial\Omega}$ (which turns out to be independent of the approximating sequence). $\square$

A straightforward calculation and the application of the preceding theorem yields the following extension theorem for the situation as in Figure 5.2

Theorem 31. *Assume that Ω_1 and Ω_2 are disjoint open subsets of $\mathbb{R}^n$ and let $\Omega =$*

[4] still assumed to be compactly supported in $\overline{\Omega} \cap V$

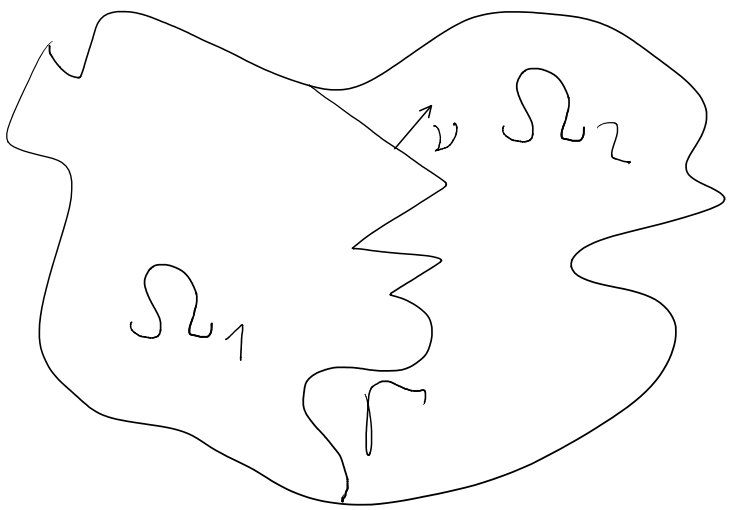

Figure 5.2: The sets Ω_1 and Ω_2 with joint boundary Γ.

$\Omega_1 \cup \Omega_2 \cup \Gamma$, where $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is the joint boundary of Ω_1 and Ω_2 which is assumed to be Lipschitz. Let ν be the unit normal on Γ pointing from Ω_1 to Ω_2. Let $u_1 \in BV(\Omega_1, \mathbb{R}^N)$ and $u_2 \in BV(\Omega_2, \mathbb{R}^N)$. By Theorem 30 the traces $u_1^{\partial\Omega}$ of u_1 and $u_2^{\partial\Omega}$ of u_2 on Γ exist. The function $u : \Omega \to \mathbb{R}^N$ defined by

$$u(x) = \begin{cases} u_1(x) & \text{for } x \in \Omega_1 \\ u_2(x) & \text{for } x \in \Omega_2 \end{cases}$$

is in $BV(\Omega, \mathbb{R}^N)$. Viewing Du_1 and Du_2 as measures on Ω (extended by the zero measure on $\Omega \setminus \Omega_1$ and $\Omega \setminus \Omega_2$ resp.), the weak derivative of Du is given by

$$Du = Du_1 + Du_2 + (u_1 - u_2) \otimes \nu \mathcal{H}^{n-1}|_\Gamma \,,$$

where the latter term denotes the $(n-1)$-dimensional Hausdorff measure on Γ with density $(u_1 - u_2) \otimes \nu$. In particular for the total variation we have

$$|Du|(\Omega) = |Du_1|(\Omega_1) + |Du_2|(\Omega_2) + \int_\Gamma |u_1 - u_2| \, \mathrm{d}\mathcal{H}^{n-1} \,.$$

Proof. Let us refer to [3, Corollary 3.89] and [27, p. 183f] for these statements. □

Note that in view of the previous theorem for Ω bounded with Lipschitz boundary

we have $u \in BV(\Omega, \mathbb{R}^N)$ if and only if the extended function $\tilde{u} : \mathbb{R}^n \to \mathbb{R}^N$, given by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{for } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

is in $BV(\mathbb{R}^n, \mathbb{R}^N)$. Thus by an abuse of notion we will also write u for the extended function $\tilde{u}$, such that for $u \in BV(\Omega, \mathbb{R}^N)$ the identity

$$|Du|(\mathbb{R}^n) = |Du|(\Omega) + \int_{\partial\Omega} |u^{\partial\Omega}| \, \mathrm{d}\mathcal{H}^{n-1} \tag{5.5}$$

holds. A further consequence of the trace theorem is the following embedding result.

Proposition 32. *Let $N, n \in \mathbb{N}$ and let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then the embedding*

$$BV(\Omega, \mathbb{R}^N) \hookrightarrow L^p(\Omega, \mathbb{R}^N)$$

is continuous for $p = n/(n-1)$ and even compact for $p \in [1, n/(n-1))$.

Proof. Cf. [3, Corollary 3.49]. □

Proposition 33. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then the functional $\mathcal{E}_{TV} : BV(\Omega, \mathbb{R}^N) \to \mathbb{R}$ given by*

$$\mathcal{E}_{TV}(u) := |Du|(\mathbb{R}^n) = \int_\Omega \mathrm{d}|Du| + \int_{\partial\Omega} |u^{\partial\Omega}| d\mathcal{H}^{n-1}$$

is a norm on $BV(\Omega, \mathbb{R}^N)$, equivalent to the $BV(\Omega, \mathbb{R}^N)$-norm. Moreover, $\mathcal{E}_{TV}$ is lower semicontinuous with respect to $L^1(\Omega, \mathbb{R}^N)$-convergence.

Proof. It is not difficult to verify that $\mathcal{E}_{TV}$ is indeed a norm.

Obviously $|Du|(\Omega) \leq \mathcal{E}_{TV}(u)$. By the Poincaré inequality in BV (cf. [3, Theorem 3.47]) we can bound the L^1-norm of u by $\mathcal{E}_{TV}(u)$ times a constant which depends only on the space dimensions n and N. Thus it remains to show that $\mathcal{E}_{TV}$ can be bounded by the usual BV-norm. Since $|Du|(\Omega) \leq \|u\|_{BV}$ it suffices to bound the trace term. Let $u \in BV(\Omega, \mathbb{R}^N)$. There exists a sequence $(u_j)_j$ in $W^{1,1}(\Omega, \mathbb{R}^N)$ converging strictly to u (cf. Proposition 28). Since the trace is continuous in $W^{1,1}(\Omega, \mathbb{R}^N)$ (cf. [27, p. 133]), there is a constant $C_{W^{1,1}} > 0$ such that $\|v^{\partial\Omega}\|_{L^1(\partial\Omega, \mathbb{R}^N)} \leq C_{W^{1,1}} \|v\|_{W^{1,1}}$ for all $v \in W^{1,1}(\Omega, \mathbb{R}^N)$. By strict convergence of $(u_j)_j$ and the continuity of the trace

operator with respect to strict convergence we thus obtain

$$\|u^{\partial\Omega}\|_{L^1} = \lim_{j\to\infty} \|u_j^{\partial\Omega}\|_{L^1} \leq \lim_{j\to\infty} C_{W^{1,1}} \|u_j\|_{W^{1,1}} = C_{W^{1,1}} \|u\|_{BV}$$

as desired.

By Proposition 28 and the remarks preceding it the total variation $u \mapsto V(u, \mathbb{R}^n)$ is lower semicontinuous with respect to $L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^N)$-convergence and in particular for L^1-convergence of functions with support on Ω, i.e. $L^1(\Omega, \mathbb{R}^N)$-convergence. Note that $V(u, \mathbb{R}^n) = \mathcal{E}_{TV}(u)$ for any $u \in BV(\Omega, \mathbb{R}^N)$, where we used the convention after Theorem 31. □

We finish our review of properties of $BV(\Omega, \mathbb{R}^N)$ with a statement which holds only for $N = 1$, the coarea formula. A measurable set $E \subseteq \Omega$ is said to be of finite perimeter in Ω, provided $\chi_E \in BV(\Omega)$ and its perimeter $\text{Per}(E, \Omega)$ in Ω is defined as

$$\text{Per}(E, \Omega) := |D\chi_E|(\Omega)\,.$$

We write $\text{Per}(E) := \text{Per}(E, \mathbb{R}^n)$. The next theorem states that almost all superlevel sets of a $BV(\Omega)$-function are sets of finite perimeter and its total variation can be obtained as integral over the perimeter of the superlevel sets.

Proposition 34 (Coarea Formula (1)). *Let $u \in BV(\Omega)$, then for almost all $t \in \mathbb{R}$ the superlevel sets*

$$\tilde{E}_t := \{u > t\} \tag{5.6}$$

are sets of finite perimeter in Ω and the total variation of u is given by

$$|Du|(\Omega) = \int_{\mathbb{R}} \text{Per}(\tilde{E}_t, \Omega)\,\mathrm{d}t\,. \tag{5.7}$$

If on the other hand for an $L^1(\Omega)$-function almost all superlevel sets $\tilde{E}_t$ as in (5.6) *are sets of finite perimeter and the function $t \mapsto \text{Per}(\tilde{E}_t, \Omega)$ is integrable on $\mathbb{R}$, then $u \in BV(\Omega)$ and the total variation of u is given by* (5.7).

Proof. See [27, p. 185]. □

We will need the following variant of the coarea formula

Proposition 35 (Coarea Formula (2)). *The statements of the foregoing theorem remain unchanged, provided "$>$" in the definition of the superlevel sets is replaced by "$\geq$" or provided we use the alternative definition*

$$E_t := \begin{cases} \{u > t\} & \text{for } t > 0 \\ \{u < t\} & \text{for } t < 0 \end{cases}$$

for the sub-/superlevel sets.

Proof. The starting point of the proof of Proposition 34 is to write the positive part u^+ of u as

$$u^+(x) = \int_0^\infty \chi_{\{u>t\}}(x)\,\mathrm{d}t = \int_0^\infty \chi_{\tilde{E}_t}(x)\,\mathrm{d}t\,.$$

And in this formula replacing $>$ by $\geq$ does not make a difference.

The validity of the proposition with the sublevel sets $E_t = \Omega \setminus \{u \geq t\}$ for $t < 0$ (for $t > 0$ there is nothing to do) follows from Proposition 34, provided we can show

$$\mathrm{Per}(E, \Omega) = \mathrm{Per}(\Omega \setminus E, \Omega) \tag{5.8}$$

for all sets $E \subseteq \Omega$ of finite perimeter. To see this let $\varphi \in C_c^\infty(\Omega)$ with $\|\varphi\|_\infty \leq 1$, then

$$\int_E \operatorname{div}\varphi\,\mathrm{d}x = \int_\Omega \chi_E \operatorname{div}\varphi\,\mathrm{d}x = \int_\Omega (1 - \chi_{\Omega\setminus E}) \operatorname{div}\varphi\,\mathrm{d}x = -\int_{\Omega\setminus E} \operatorname{div}\varphi\,\mathrm{d}x$$

since $\int_\Omega \operatorname{div}\varphi\,\mathrm{d}x = 0$ by Gauß' theorem. Taking the supremum over all $\varphi \in C_c^\infty(\Omega)$ equation (5.8) follows. □

Last but not least let us provide a Theorem that supplements the approximation result in Proposition 28.

Theorem 36. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary and let $u \in BV(\Omega) \cap L^p(\Omega)$ for some $p \in [1, \infty)$. Then there is a sequence $(u_k)_k$ in $C_c^\infty(\Omega)$ such that, for any $q \in [1, p]$,*

$$u_k \to u$$

in $L^q(\Omega)$ and

$$|Du_k|(\mathbb{R}^n) \to |Du|(\mathbb{R}^n) = \int_\Omega \mathrm{d}|Du| + \int_{\partial\Omega} |u^{\partial\Omega}|\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

Proof. Cf. [41, Theorem 3.2]. □

Note that in contrast to the standard approximation from Proposition 28 the approximating functions are assumed to be compactly supported in Ω and we do not only approximate the total variation in Ω, but also the boundary term.

5.2 $BD(\Omega)$ and its Properties

We will introduce spaces occurring in the theory of plasticity now. Basically one considers some body $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$ that is deformed by a mapping $u : \Omega \to \mathbb{R}^n$. Thus appropriate spaces for deformations contain maps $u : \Omega \to \mathbb{R}^n$ where Ω has the same dimension as its image. The energy of the deformation can usually be expressed in terms of the symmetrized gradient of u. In particular for a distribution $u = (u^1, \dots u^n) \in \mathcal{D}'(\Omega, \mathbb{R}^n)$ we define the *symmetrized gradient* of u by

$$D_s u := \frac{1}{2}\left(Du + (Du)^\mathsf{T}\right) = \frac{1}{2}\left(\partial_j u^i + \partial_i u^j\right)_{i,j=1,\dots,n} .$$

Definition. *The space $BD(\Omega)$ is defined to be the space of those functions $u \in L^1(\Omega, \mathbb{R}^n)$, for which the symmetrized gradient*

$$D_s u$$

is represented by some finite $\mathbb{R}^{n\times n}$-valued Radon measure on Ω, i.e. $D_s u \in \mathcal{M}(\Omega, \mathbb{R}^{n\times n})$. It becomes a Banach space with the norm

$$\|u\|_{BD} := \int_\Omega |u| \,\mathrm{d}x + |D_s u|(\Omega)$$

where $|D_s u|$ denotes the total variation measure of $D_s u \in \mathcal{M}(\Omega, \mathbb{R}^{n\times n})$.

If $u \in BD(\Omega)$, we may write $D_s u = \sigma |D_s u|$, where $\sigma \in L^1(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}}; |D_s u|)$ and $|\sigma| = 1$ $|D_s u|$-a.e. by an application of Proposition 73. Here and in the following $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ denotes the vector space of symmetric $n \times n$-matrices.

Remark 37. *Note that the definition of the symmetrized gradient is independent of the orthonormal frame chosen in $\mathbb{R}^n$. In fact an equivalent definition of Temam & Strang [56, p. 9] reads: $u \in L^1(\Omega, \mathbb{R}^n)$ is in $BD(\Omega)$ if and only if*

$$(\alpha \cdot D)(\alpha \cdot u) \in \mathcal{M}(\Omega)$$

for any $\alpha \in \mathbb{R}^n$.

Note that α *in this characterization acts both on* Ω *(i.e. in the differential operator* $\alpha \cdot D$ *on* Ω*) as well as on the image* $u(\Omega)$*, i.e.* $\alpha \cdot u$.

We carried out a transformation formula in $BD(\Omega)$. It will not be applied within our further results of the thesis, but might be of general interest for approximation results in $BD(\Omega)$.

Proposition 38. *Let* $\Omega \subseteq \mathbb{R}^n$ *be open and of finite measure and let* $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ *be an affine transformation of* $\mathbb{R}^n$ *represented by*

$$\Phi(x) = Ax + y = \tilde{x}$$

for some invertible matrix $A \in \mathbb{R}^{n\times n}$ *and some* $y \in \mathbb{R}^n$ *and let* $\tilde{\Omega} := \Phi(\Omega)$. *Then* $u \in BD(\Omega)$ *if and only if the function* $\tilde{u} : \tilde{\Omega} \to \mathbb{R}^n$,

$$\tilde{u}(\tilde{x}) := A^{-\mathsf{T}} u(A^{-1}(\tilde{x} - y))$$

is in $BD(\tilde{\Omega})$ *and there holds*

$$\int_{\tilde{\Omega}} |\tilde{u}(\tilde{x})| \,\mathrm{d}\tilde{x} = |\det A|^{1-1/n} \int_{\Omega} |u(x)| \,\mathrm{d}x$$

and

$$D_s\tilde{u}(\tilde{x}) = A^{-\mathsf{T}}\big((D_s u)(A^{-1}(\tilde{x} - y))\big)A^{-1} = A^{-\mathsf{T}} D_s u(x) A^{-1}\,,$$

where $x = \Phi^{-1}(\tilde{x})$, *such that*

$$|D_s\tilde{u}|(\tilde{\Omega}) = |\det A|^{1-2/n} |D_s u|(\Omega)\,.$$

Proof. By the transformation theorem we have

$$\int_{\tilde{\Omega}} |\tilde{u}(\tilde{x})| \,\mathrm{d}\tilde{x} = \int_{\Omega} |\det A|^1 |A^{-\mathsf{T}} u(x)| \,\mathrm{d}x = \int_{\Omega} |\det A|^{1-1/n} |u(x)| \,\mathrm{d}x\,,$$

where we have used that by the invariance of the Frobenius norm under unitary transformations $|A^{-\mathsf{T}} u| = |\det A^{-\mathsf{T}}|^{1/n} \left| \frac{A^{-\mathsf{T}}}{|\det A^{-\mathsf{T}}|^{1/n}} u \right| = |\det A|^{-1/n} |u|$.

The transformation of the derivative follows from the calculation

$$\begin{aligned} 2D_s\tilde{u} &= D\big(A^{-\mathsf{T}}(u \circ \Phi^{-1})\big) + \big(D\big(A^{-\mathsf{T}}(u \circ \Phi^{-1})\big)\big)^{\mathsf{T}} \\ &= A^{-\mathsf{T}}(Du) \circ \Phi^{-1} A^{-1} + \big(A^{-\mathsf{T}}(Du) \circ \Phi^{-1} A^{-1}\big)^{\mathsf{T}} \\ &= 2A^{-T}(D_s u) \circ \Phi^{-1} A^{-1} \end{aligned}$$

in the sense of distributions. Obviously the left hand side is a Radon measure if and only if the right hand side is a Radon measure. Applying the transformation formula and the invariance of the Frobenius norm with respect to unitary transformations we obtain

$$\begin{aligned} |D_s\tilde{u}|(\tilde{\Omega}) = \int_{\tilde{\Omega}} \mathrm{d}|D_s\tilde{u}| &= \int_{\Omega} |\det A| \, \mathrm{d}|A^{-\mathsf{T}} D_s u A^{-1}| \\ &= \int_{\Omega} |\det A|^{1-2/n} \, \mathrm{d}\Big| |\det A|^{1/n} A^{-\mathsf{T}} D_s u \, |\det A|^{1/n} A^{-1} \Big| \\ &= |\det A|^{1-2/n} |D_s u|(\Omega) \,. \end{aligned}$$

□

Note that by the transposition operation in the definition of the symmetrized gradient it is necessary to perform a transformation both in the domain of definition of u and in the range of u. In particular in contrast to $BV(\Omega, \mathbb{R}^N)$-functions (cf. [31, Lemma 10.1]) we did not succeed to perform a transformation formula with respect to arbitrary diffeomorphic deformations of Ω and conjecture that it does not exist in general.

Definition. *For $\Omega \subseteq \mathbb{R}^n$ open and $z = (z_{i,j})_{i,j=1,\dots,n} \in \mathcal{D}'(\Omega, \mathbb{R}^{n\times n})$ we define the symmetrized divergence* $\operatorname{div_s} z \in \mathcal{D}'(\Omega, \mathbb{R}^n)$ *by*

$$\operatorname{div_s} z := \frac{1}{2}\Big(\operatorname{div}(z + z^{\mathsf{T}})\Big) := \frac{1}{2}\left(\sum_{i=1}^{n} \partial_i (z^i_j + z^j_i)\right) .$$

The symmetrized tensor product $a \odot b$ for vectors $a, b \in \mathbb{R}^n$ is defined by

$$a \odot b := \frac{a \otimes b + b \otimes a}{2} .$$

Crucial for the investigation of functions of bounded deformation will be the following symmetrized Gauß-Green formula.

Proposition 39. *Let $\Omega \subseteq \mathbb{R}^n$ be open with Lipschitz boundary and let $u \in C^1(\overline{\Omega}, \mathbb{R}^n)$*

and $z \in C^1(\overline{\Omega}, \mathbb{R}^{n\times n})$*, then*

$$\int_\Omega u \cdot \operatorname{div_s} z \,\mathrm{d}x + \int_\Omega D_s u : z \,\mathrm{d}x = \int_{\partial\Omega} (u \odot \nu) : z \,\mathrm{d}\mathcal{H}^{n-1}$$
$$= \frac{1}{2} \int_{\partial\Omega} u^\mathsf{T} z \,\nu + \nu^\mathsf{T} z u \,\mathrm{d}\mathcal{H}^{n-1} . \tag{5.9}$$

Proof. Let $u \in C^1(\overline{\Omega}, \mathbb{R}^n)$ and $z \in C^1(\overline{\Omega}, \mathbb{R}^{n\times n})$. Adding the Gauß-Green formula (5.2) to the same equation with z replaced by z^T we get

$$\int_\Omega u \cdot (\operatorname{div} z + \operatorname{div}(z^\mathsf{T})) \,\mathrm{d}x + \int_\Omega Du : (z + z^\mathsf{T}) \,\mathrm{d}x = \int_{\partial\Omega} (u \otimes \nu) : (z + z^\mathsf{T}) \,\mathrm{d}\mathcal{H}^{n-1} .$$

The first integral gives two times the first term in the proposition by definition. For the second and third integrals it suffices to recall that the decomposition of matrices in their symmetric and antisymmetric parts is orthogonal with respect to the matrix product ":" (cf. equation (7.1)) and thus only two times the symmetric parts $D_s u$ of Du and $u \odot \nu$ of $u \otimes \nu$ persist. □

Thus by definition for $u \in BD(\Omega)$ (with the decomposition $D_s u = \sigma |D_s u|$ of the derivative) and $z \in C_c^\infty(\Omega, \mathbb{R}^{n\times n})$ the partial integration formula

$$\int_\Omega u \cdot \operatorname{div_s} z \,\mathrm{d}x = -\int_\Omega z : \sigma \,\mathrm{d}|D_s u| \tag{5.10}$$

holds.

In analogy to the variation of a function, we introduce for $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ the *deformation*

$$D(u, \Omega) := \sup \left\{ \int_\Omega u \cdot \operatorname{div_s} \varphi \,\mathrm{d}x \;;\; \varphi \in C_c^1(\Omega, \mathbb{R}^{n\times n}), \|\varphi\|_\infty \le 1 \right\} .$$

Instead of using the symmetrized divergence of general tensor fields it is common to use the usual divergence of symmetric tensor fields in specialist literature. This is justified by the following lemma. We will keep our approach with the symmetric divergence operator in the following to point out the analogy to the $BV(\Omega, \mathbb{R}^N)$-situation.

Lemma 40. *For any* $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ *we have*

$$D(u, \Omega) = \sup \left\{ \int_\Omega u \cdot \operatorname{div} \varphi \,\mathrm{d}x \;;\; \varphi \in C_c^1(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}}), \|\varphi\|_\infty \le 1 \right\} .$$

Proof. Note that for symmetric distributions u in $\mathcal{D}'(\Omega, \mathbb{R}^{n\times n})$ we have $\operatorname{div_s} u = \operatorname{div} u$ and for skew symmetric distributions we have $\operatorname{div_s} u = 0$. Thus the unique (and pointwise orthogonal with respect to the Frobenius norm) decomposition

$$\varphi = \frac{1}{2}(\varphi + \varphi^{\mathsf{T}}) + \frac{1}{2}(\varphi - \varphi^{\mathsf{T}})$$

of $\varphi \in C_c^1(\Omega, \mathbb{R}^{n\times n})$ shows that it suffices to consider the classical divergence of symmetric tensor fields φ only. □

Proposition 41. *The deformation is convex and lower semicontinuous with respect to the $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$-topology.*

Proof. By definition the deformation is the supremum of continuous linear functionals on $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. □

The following proposition similar to the situation in $BV(\Omega, \mathbb{R}^N)$ holds.

Proposition 42. *A function $u \in L^1(\Omega, \mathbb{R}^n)$ is in $BD(\Omega)$ if and only if $D(u, \Omega) < \infty$ and in that case*

$$D(u, \Omega) = |D_s u|(\Omega)\,,$$

where $|D_s u|$ denotes the total variation of the measure $D_s u \in \mathcal{M}(\Omega, \mathbb{R}^{n\times n})$.

Proof. Let $u \in BD(\Omega)$ and $\varphi \in C_c^1(\Omega, \mathbb{R}^{n\times n})$. By definition we have

$$\int_\Omega u \cdot \operatorname{div_s} \varphi \,\mathrm{d}x = \int_\Omega \varphi : \sigma \,\mathrm{d}|D_s u| \le \|\varphi\|_\infty \int_\Omega \mathrm{d}|D_s u| < \infty,$$

thus $D(u, \Omega) \le |D_s u|(\Omega) < \infty$ for $u \in BD(\Omega)$.

Let now $u \in L^1(\Omega, \mathbb{R}^n)$ and assume $D(u, \Omega) < \infty$, then by homogeneity in φ we obtain

$$\left| \int_\Omega u \cdot \operatorname{div_s} \varphi \,\mathrm{d}x \right| \le D(u, \Omega) \|\varphi\|_\infty$$

for all $\varphi \in C_c^1(\Omega, \mathbb{R}^{n\times n})$, which is a dense subspace of $C_0(\Omega, \mathbb{R}^{n\times n})$. Thus there is a unique continuous linear functional $L : C_c^1(\Omega, \mathbb{R}^{n\times n}) \to \mathbb{R}$ with

$$L(\varphi) = \int_\Omega u \cdot \operatorname{div_s} \varphi \, dx$$

for $\varphi \in C_0(\Omega, \mathbb{R}^{n\times n})$ and with $\|L\| \le D(u, \Omega)$. By Riesz' Theorem 74 L is represented

by a finite $\mathbb{R}^{n\times n}$-valued Radon measure $\mu = (\mu_{ij})_{i,j=1,\dots,n}$, such that

$$L(\varphi) = \int_\Omega \varphi : \mathrm{d}\mu$$

for all $\varphi \in C_0(\Omega, \mathbb{R}^{n\times n})$. Considering $\varphi \in C_c^\infty(\Omega, \mathbb{R}^{n\times n})$ it is immediate that $u \in BD(\Omega)$ and $-\mu = D_s u$ and

$$|D_s u|(\Omega) = |\mu|(\Omega) = \|L\| \leq D(u, \Omega)\,. \qquad \square$$

Similar to $BV(\Omega, \mathbb{R}^N)$ the norm topology of $BD(\Omega)$ is too strong for many applications and we often use strict convergence instead. A sequence $(u_j)_j$ in $BD(\Omega)$ is said to converge strictly[5] to $u \in BD(\Omega)$, provided $u_j \to u$ in $L^1(\Omega, \mathbb{R}^n)$ and $|D_s u_j|(\Omega) \to |D_s u|(\Omega)$. Smooth functions are dense in $BD(\Omega)$ with respect to strict convergence.

Proposition 43. *Let $u \in L^1(\Omega, \mathbb{R}^n)$. Then $u \in BD(\Omega)$ if and only if there is a sequence $(u_j)_j$ in $C^\infty(\Omega, \mathbb{R}^n) \cap L^1(\Omega, \mathbb{R}^n)$ converging to u in $L^1(\Omega, \mathbb{R}^n)$ with*

$$\limsup_{j\to\infty} D(u_j, \Omega) < \infty\,.$$

Moreover, the latter implies weak-convergence of the measures $D_s u_j$ to $D_s u$ as $j \to \infty$, and $u_j \to u$ in $L^p(\Omega, \mathbb{R}^N)$ as $j \to \infty$ for all $p \in [1, n/(n-1))$ provided Ω is bounded with Lipschitz boundary. If $u \in BD(\Omega)$ we can find a sequence $(u_j)_j$ with $\limsup_{j\to\infty} D(u_j, \Omega) = D(u, \Omega)$ or, in other words, u can be approximated in terms of strict convergence by smooth functions.*

Proof. The "if"-part is a consequence of Propositions 41 and 42. The approximation by smooth functions follows from [55, Theorem 3.2, p. 162]. There the theorem is stated for Ω with smooth boundary. It is however easily seen that one can adapt the technique from [3, Theorem 3.9] to derive the approximation by functions in $C^\infty(\Omega, \mathbb{R}^n)$. The convergence in $L^p(\Omega, \mathbb{R}^n)$ is a consequence of Theorem 44 below. $\square$

Similar to the $BV(\Omega, \mathbb{R}^N)$-case we can prove the continuous embedding of $BD(\Omega)$ in $L^{n/(n-1)}(\Omega, \mathbb{R}^n)$ and the compact embedding in $L^1(\Omega, \mathbb{R}^n)$ (and thus compact in $L^p(\Omega, \mathbb{R}^n)$ for all $p \in [1, n/(n-1))$ by interpolation), provided Ω is bounded.

[5] In [55] the terminus "intermediate topology" is used to denote the topology induced by strict convergence here.

Theorem 44. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then the embedding*

$$BD(\Omega) \hookrightarrow L^p(\Omega, \mathbb{R}^n)$$

is continuous for $p = n/(n-1)$ and even compact for $p \in [1, n/(n-1))$.

Proof. Combine the statements from [55, Theorem 2.2 p. 152] and [55, Theorem 2.4 p. 153]. □

An essential tool in the proof of Theorem 44 is the following trace theorem in $BD(\Omega)$.

Theorem 45. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. There exists a linear trace operator $\cdot^{\partial\Omega} : BD(\Omega) \to L^1(\partial\Omega, \mathbb{R}^n)$, continuous with respect to strict convergence in $BD(\Omega, \mathbb{R}^n)$ such that*

$$\int_\Omega z : \mathrm{d}D_s u + \int_\Omega u \cdot \mathrm{div_s}\, z \,\mathrm{d}x = \int_{\partial\Omega} z : (u^{\partial\Omega} \odot \nu) \,\mathrm{d}x$$

for all $z \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$. In particular $u^{\partial\Omega}$ is obtained by continuous extension of u on $\partial\Omega$, provided $u \in C(\overline{\Omega}, \mathbb{R}^n)$ and for $u \in W^{1,1}(\Omega, \mathbb{R}^n)$ it coincides with the usual trace for Sobolev functions.

Proof. Existence of the trace operator is shown in [55, Theorem 2.1, p. 148]. Note that it is stated for C^1-boundaries there only. Since Lipschitz boundaries are almost everywhere differentiable by Rademachers theorem and the continuity of the normal ν on $\partial\Omega$ is not needed in the proof of [55], the proof is easily adapted to Ω with Lipschitz boundary. □

Similar to the BV-case we have an extension property as in the situation of Figure 5.2.

Proposition 46. *Assume that Ω_1 and Ω_2 are disjoint open subsets of $\mathbb{R}^n$ and let $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, where $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is the joint boundary[6] of Ω_1 and Ω_2 which is assumed to be Lipschitz. Let ν be the unit normal on Γ pointing from Ω_1 to Ω_2. Let $u_1 \in BD(\Omega_1)$ and $u_2 \in BD(\Omega_2)$ and let $u_1^{\partial\Omega}$ denote the trace of u_1 and $u_2^{\partial\Omega}$ the trace of u_2 on Γ, resp. The function $u : \Omega \to \mathbb{R}^n$ defined by*

$$u(x) = \begin{cases} u_1(x) & \text{for } x \in \Omega_1 \\ u_2(x) & \text{for } x \in \Omega_2 \end{cases}$$

[6] The equation $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ should be interpreted in the $\mathcal{H}^{n-1}$-a.e. sense of course.

is in $BD(\Omega)$. Interpreting $D_s u_1$ and $D_s u_2$ as measures on Ω (extended by the zero measure on $\Omega \setminus \Omega_1$ and $\Omega \setminus \Omega_2$ resp.), the weak derivative of $D_s u$ is given by

$$D_s u = D_s u_1 + D_s u_2 + (u_1 - u_2) \odot \nu \mathcal{H}^{n-1}|_\Gamma$$

where the latter term denotes the $n-1$-dimensional Hausdorff measure on Γ with density $(u_1 - u_2) \odot \nu$. In particular for the total variation of the measures $D_s u$ we have

$$|D_s u|(\Omega) = |D_s u_1|(\Omega_1) + |D_s u_2|(\Omega_2) + \int_\Gamma |(u_1 - u_2) \odot \nu| \, \mathrm{d}\mathcal{H}^{n-1} \,. \tag{5.11}$$

Proof. See [55, Propostion 2.1, p. 151] and [55, Remark 2.3, p. 151]. □

Note that, in contrast to Theorem 31, we can not neglect the normal ν in the boundary term in 5.11. This is due to the algebraic fact that for $|b| = 1$

$$|a \otimes b| = |a| \,,$$

but merely

$$|a \odot b| \leq |a|$$

and strict inequality occures in general.

In the light of the foregoing Proposition we will identify functions $u \in BD(\Omega)$ with their extension by zero on $\mathbb{R}^n \setminus \Omega$, which is then in $BD(\mathbb{R}^n)$, in the following.

We finish this review with a proposition stating that the function $\mathcal{E}_{TD}$ is a norm on $BD(\Omega)$.

Proposition 47. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then the functional $\mathcal{E}_{TD} : BD(\Omega) \to \mathbb{R}$ given by*

$$\mathcal{E}_{TD}(u) := |D_s u|(\mathbb{R}^n) = \int_\Omega \mathrm{d}|D_s u| + \int_{\partial\Omega} |u^{\partial\Omega} \odot \nu| d\mathcal{H}^{n-1}$$

is a norm on $BD(\Omega)$, equivalent to the usual norm and lower semicontinuous with respect to $L^1(\Omega, \mathbb{R}^n)$-convergence.

Proof. The function $\mathcal{E}_{TD}$ is a norm due to [55, Proposition 2.4, p. 155] and the subsequent remark.

It remains to prove lower semicontinuity. Let $(u_j)_j$ be a sequence in $BD(\Omega)$ with $\liminf_{j\to\infty} \mathcal{E}_{TD}(u_j) < \infty$ and $u_j \to u$ in $L^1(\Omega, \mathbb{R}^n)$. Identifying u_j and u with their

extensions by zero outside Ω we have $u_j \in BD(K, \mathbb{R}^n)$, where $\overline{\Omega} \subseteq K$ is an open ball containing $\overline{\Omega}$. Since the total deformation is lower semicontinuous with respect to $L^1(K, \mathbb{R}^n)$-convergence (Proposition 41) we have $u \in BD(K)$ and

$$\mathcal{E}_{TD}(u) = \int_K \mathrm{d}|D_s u| \leq \liminf_{j\to\infty} \int_K \mathrm{d}|D_s u_j| = \liminf_{j\to\infty} \mathcal{E}_{TD}(u_j)\,,$$

which verifies the assertion. □

5.3 Spaces with Integrable Divergence Fields

A key point in the derivation of the subdifferentials of the total variation and the total deformation functional is to derive suitable Gauß-Green formulas. We thus follow the ideas of Anzelotti [5] and introduce the spaces

$$L^{\infty,q}(\Omega, \mathbb{R}^{N\times n}) := \left\{ z \in L^\infty(\Omega, \mathbb{R}^{N\times n})\ ;\ \operatorname{div} z \in L^q(\Omega, \mathbb{R}^N) \right\}$$

and

$$L^{\infty,q}_{\mathrm{sym}}(\Omega) := \left\{ z \in L^\infty(\Omega, \mathbb{R}^{n\times n})\ ;\ \operatorname{div}_{\mathrm{s}} z \in L^q(\Omega, \mathbb{R}^n) \right\}$$

for $q \in (1, \infty)$.

The key idea is to show that the normal trace of functions in $L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ exists and to verify that suitable Gauß-Green formulas for functions from $BV(\Omega, \mathbb{R}^N)$ and $L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ hold. An analogous procedure will be carried out between $BD(\Omega)$ and $L^{\infty,q}_{\mathrm{sym}}(\Omega)$.

We may not expect uniform approximation of the functions in $L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ and $L^{\infty,q}_{\mathrm{sym}}(\Omega)$ by smooth functions. Thus let us introduce the following notion of convergence in $L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ and $L^{\infty,q}_{\mathrm{sym}}(\Omega)$.

Definition. *A sequence $(z_j)_j$ in $L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ is said to be $L^{\infty,q}$-convergent to $z \in L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$, provided $z_j \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega, \mathbb{R}^{N\times n})$ and $\operatorname{div} z_j \to \operatorname{div} z$ in $L^q(\Omega, \mathbb{R}^{N\times n})$.*

Similarly a sequence $(z_j)_j$ in $L^{\infty,q}_{\mathrm{sym}}(\Omega)$ is said to be $L^{\infty,q}_{\mathrm{sym}}$-convergent to $z \in L^{\infty,q}_{\mathrm{sym}}(\Omega)$, provided $z_j \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega, \mathbb{R}^{n\times n})$ and $\operatorname{div}_{\mathrm{s}} z_j \to \operatorname{div}_{\mathrm{s}} z$ in $L^q(\Omega, \mathbb{R}^n)$.

Smooth functions are dense in $L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ in the sense of $L^{\infty,q}$-convergence:

Proposition 48. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then $C^\infty(\Omega, \mathbb{R}^{N\times n}) \cap L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ is dense in $L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ in the following sense: For any $z \in L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ there is a sequence $(z_j)_j$ in $C^\infty(\Omega, \mathbb{R}^{N\times n}) \cap L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ such that*

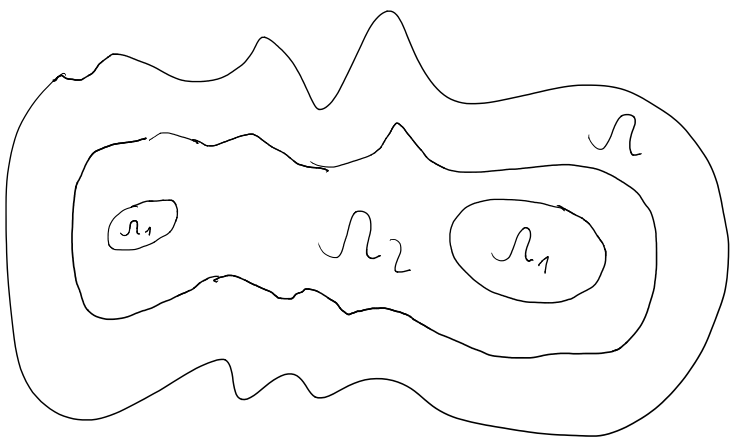

Figure 5.3: Exhaustion of Ω by Ω_k.

(i) $z_j \to z$ *in* $L^r(\Omega, \mathbb{R}^{N\times n})$ *for all* $r \in [1, \infty)$,

(ii) $z_j \overset{*}{\rightharpoonup} z$ *in* $L^\infty(\Omega, \mathbb{R}^{N\times n})$,

(iii) $z_j(x) \to z(x)$ *for almost every* $x \in \Omega$,

(iv) $|z_j(x)| \le \|z\|_\infty$ *for all* $x \in \Omega$, *and*

(v) $\operatorname{div} z_j \to \operatorname{div} z$ *in* $L^q(\Omega, \mathbb{R}^N)$

as $j \to \infty$. *In particular all* $z \in L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ *can be approximated by* $L^{\infty,q}$-*convergent sequences in* $C^\infty(\Omega, \mathbb{R}^{N\times n}) \cap L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$.

Proof. Set $\Omega_0 := \emptyset$ and let $(\Omega_k)_{k\ge1}$ be the standard exhaustion (cf. Figure 5.3) of Ω defined by

$$\Omega_k := \{x \in \Omega \,;\, \operatorname{dist}(x, \partial\Omega) > 1/k\}\,.$$

Take a smooth partition of unity of Ω,

$$0 \le \rho_k \le 1\,, \quad \rho_k \in C_c^\infty(\Omega)\,, \quad \sum_{k\in\mathbb{N}} \rho_k = 1\,,$$

such that $\operatorname{supp}\rho_1 \subseteq \Omega_1$ and $\operatorname{supp}\rho_k \subseteq \Omega_{k+1} \setminus \Omega_{k-1}$ $(k \ge 1)$. For $z \in L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ we define

$$z_k := \rho_k z\,,$$

such that $z_k \in L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ with $\operatorname{div} z_k = \rho_k \operatorname{div} z + zD\rho_k$. Note that all z_k are compactly supported in Ω, such that the convolution (in each component of z_k)

is well defined, provided the mollification parameter is sufficiently small. Let $\eta \in C_c^\infty(B(0,1))$, $0 \leq \eta \leq 1$, $\|\eta\|_1 = 1$ be the standard mollifier and for $\delta > 0$ let $\eta_\delta \in C_c^\infty(B(0,\delta))$ be defined by

$$\eta_\delta(x) := \delta^{-n}\eta\Big(\frac{x}{\delta}\Big)\,.$$

Let $\varepsilon > 0$, take a sequence $\varepsilon_k > 0$, such that the convolutions (performed componentwise)

$$v_k := z_k * \eta_{\varepsilon_k}$$

are still compactly supported in $\Omega_{k+1} \setminus \Omega_{k-1}$ $(k \geq 1)$ and such that

$$\|z_k - v_k\|_1 \leq \frac{\varepsilon}{2^k}$$

and

$$\|\operatorname{div} z_k - (\operatorname{div} z_k) * \eta_{\varepsilon_k}\|_q \leq \frac{\varepsilon}{2^k}\,. \tag{5.12}$$

Since for all $x \in \Omega$

$$|v_k(x)| \leq \|\eta_{\varepsilon_k}\|_1 \|z_k\|_{L^\infty(B(x,\varepsilon_k),\mathbb{R}^{N\times n})} = \|z_k\|_{L^\infty(B(x,\varepsilon_k),\mathbb{R}^{N\times n})}$$

and $|z_k(y)| \leq \|z\|_\infty \rho_k(y)$ for almost all $y \in \mathbb{R}^n$ we may, by equicontinuity of ρ_k, choose the ε_k small enough to obtain

$$|v_k(x)| \leq \|z_k\|_\infty \rho_k(x) + \varepsilon/2 \tag{5.13}$$

for all $x \in \Omega_{k+1} \setminus \Omega_{k-1}$.

Define $z_\varepsilon := \sum_{k\in\mathbb{N}} v_k \in C^\infty(\Omega, \mathbb{R}^{N\times n})$.

By integration and (locally finite) summation we obtain

$$\|z_\varepsilon - z\|_1 = \int_\Omega \Big|\sum_{k\in\mathbb{N}} z_k - v_k\Big|\,\mathrm{d}x \leq \int_\Omega \sum_{k\in\mathbb{N}} \Big|z_k - v_k\Big|\,\mathrm{d}x \leq \varepsilon\,,$$

such that $z_\varepsilon \to z$ in $L^1(\Omega, \mathbb{R}^{N\times n})$ as $\varepsilon \to 0$.

Moreover, $(z_\varepsilon)_\varepsilon$ is bounded in $L^\infty(\Omega, \mathbb{R}^{N\times n})$: Since for each x there are at most two summands $v_k(x)$ different from zero, we obtain from (5.13) and the partition of

unity property

$$\|z_\varepsilon\|_\infty \le \operatorname*{ess\,sup}_{x\in\Omega} \sum_{k\in\mathbb{N}} |v_k(x)| \le \|z\|_\infty(1+\varepsilon)\,.$$

Thus, by a standard application of Hölders' theorem the convergence $z_\varepsilon \to z$ is also strong in $L^r(\Omega,\mathbb{R}^{N\times n})$ (for all $r\in(1,\infty)$) and weak* in $L^\infty(\Omega,\mathbb{R}^{N\times n})$.

It remains to prove strong convergence $\operatorname{div} z_\varepsilon \to \operatorname{div} z$. Again, using that in a neighborhood of each $x\in\Omega$ there are at most two $\rho_k(x)$ different from zero we derive

$$\begin{aligned}
\|\operatorname{div} z - \operatorname{div} z_\varepsilon\|_q &= \Big\|\operatorname{div} z - \operatorname{div}\sum_{k\in\mathbb{N}} v_k\Big\|_q \\
&= \Big\|\operatorname{div} z - \sum_{k\in\mathbb{N}} \operatorname{div} v_k\Big\|_q \\
&= \Big\|\sum_{k\in\mathbb{N}} \operatorname{div} z_k - \operatorname{div}(z_k * \eta_{\varepsilon_k})\Big\|_q \\
&= \Big\|\sum_{k\in\mathbb{N}} \operatorname{div} z_k - (\operatorname{div} z_k) * \eta_{\varepsilon_k}\Big\|_q \\
&\le \sum_{k\in\mathbb{N}} \|\operatorname{div} z_k - (\operatorname{div} z_k) * \eta_{\varepsilon_k} \operatorname{div}\|_q \\
&\le \varepsilon
\end{aligned}$$

by (5.12).

A sequence with the claimed convergence properties as in the theorem (in particular with $|z_j|_\infty \le \|z\|_\infty$) is then obtained by choosing a suitable pointwise almost everywhere convergent subsequence from $z_\varepsilon := \frac{1}{1+\varepsilon} z_\varepsilon$. □

Proposition 49. *Let $\Omega\subseteq\mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then $C^\infty(\Omega,\mathbb{R}^{n\times n})\cap L^{\infty,q}_{\mathrm{sym}}(\Omega)$ is dense in $L^{\infty,q}_{\mathrm{sym}}(\Omega)$ in the following sense: For any $z\in L^{\infty,q}_{\mathrm{sym}}(\Omega)$ there is a sequence $(z_j)_j$ in $C^\infty(\Omega)\cap L^{\infty,q}_{\mathrm{sym}}(\Omega)$ such that*

(i) $z_j \to z$ *in* $L^r(\Omega,\mathbb{R}^{n\times n})$ *for all* $r\in[1,\infty)$,

(ii) $z_j \overset{*}{\rightharpoonup} z$ *in* $L^\infty(\Omega,\mathbb{R}^{n\times n})$,

(iii) $z_j(x)\to z(x)$ *for almost every* $x\in\Omega$,

(iv) $|z_j(x)|\le\|z\|_\infty$ *for all* $x\in\Omega$, *and*

(v) $\operatorname{div_s} z_j \to \operatorname{div_s} z$ *in* $L^q(\Omega,\mathbb{R}^N)$

as $j\to\infty$. *In particular all* $z\in L^{\infty,q}_{\mathrm{sym}}(\Omega)$ *can be approximated by* $L^{\infty,q}_{\mathrm{sym}}$*-convergent sequences in* $C^\infty(\Omega,\mathbb{R}^n)\cap L^{\infty,q}_{\mathrm{sym}}(\Omega)$.

Proof. The proof is analogous to the proof of the proposition above, where div is replaced by $\operatorname{div}_{\mathrm{s}}$. □

It is well known that integrability of the divergence of a vector field leads to integrability of the normal component of this vector field. In particular the ideas for the following results are due to Anzellotti [5]. But note that our results may not be deduced from the scalar valued theorems in [5] and it is necessary to repeat the proofs in the vectorial situation. Note that basically one faces the following difficulty: The naive idea is to define the normal trace for continuous functions first and to extend it then by a Hahn-Banach argument to the whole space. Nevertheless since $L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ is not separable with the norm $\|z\|_\infty + \|\operatorname{div} z\|_q$ the extension might be not unique. Since we could not follow the arguments concerning the continuity of the Hahn-Banach argument in [5] completely, we decided to give an own proof. Our idea is to derive the trace as unique weak*-limit of traces of continuous functions and we will show that this limit is continuous with respect to $L^{\infty,q}$-convergent (or $L^{\infty,q}_{\mathrm{sym}}$-convergent) sequences. This defines the normal trace uniquely by Propositions 48 and 49.

Proposition 50. *Let Ω be open and bounded with Lipschitz boundary. There exists a linear trace operator $[\cdot,\nu]^{\partial\Omega} : L^{\infty,q}(\Omega, \mathbb{R}^{N\times n}) \to L^\infty(\partial\Omega, \mathbb{R}^N)$, such that*

$$\int_\Omega z : D\varphi \,\mathrm{d}x + \int_\Omega \operatorname{div} z \cdot \varphi \,\mathrm{d}x = \int_\Omega [z,\nu]^{\partial\Omega} \cdot \varphi \,\mathrm{d}\mathcal{H}^{n-1}$$

for all $\varphi \in C^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$. In particular for $z \in C^1(\Omega, \mathbb{R}^{N\times n}) \cap C(\overline{\Omega}, \mathbb{R}^{N\times n})$ we have

$$[z,\nu]^{\partial\Omega}(x) = z(x)\nu(x)\,.$$

Moreover, the mapping $z \mapsto [z,\nu]^{\partial\Omega}$ is sequentially continuous with respect to $L^{\infty,q}$-convergent sequences and there holds

$$\|[z,\nu]^{\partial\Omega}\|_{L^\infty(\partial\Omega,\mathbb{R}^N)} \le \|z\|_{L^\infty(\Omega,\mathbb{R}^{N\times n})}$$

for all $z \in L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$.

Proof. Let us assume $z \in C^\infty(\Omega, \mathbb{R}^{N\times n}) \cap L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$ first. Let us moreover assume z to be compactly supported in $\overline{\Omega} \cap V$ with V as in the proof of Theorem 30.

Using the parametrization and notation (5.4) we define $[z,\nu]^\tau \in L^\infty(\partial\Omega, \mathbb{R}^N)$ by

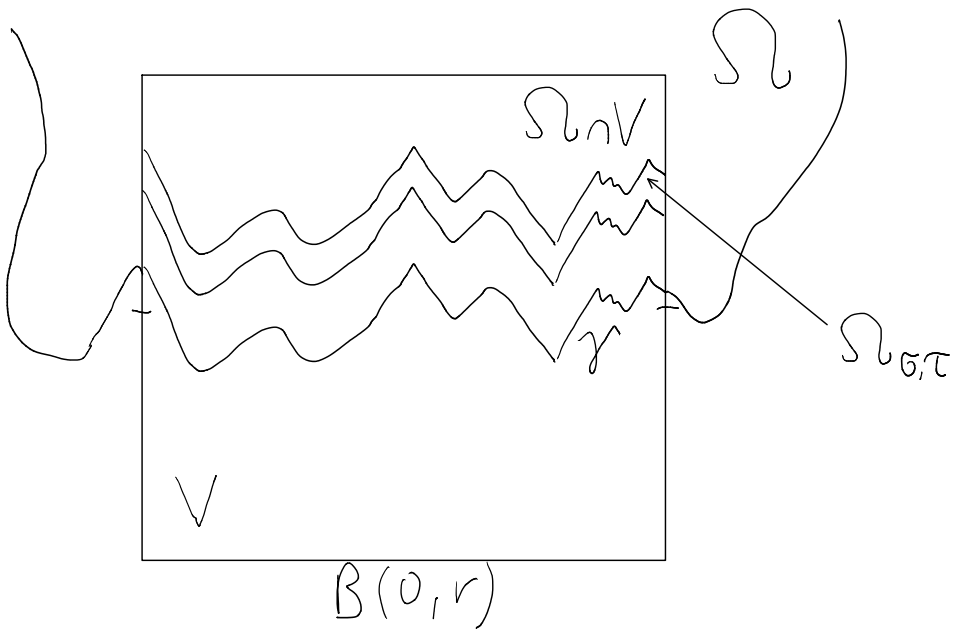

Figure 5.4: The set $\Omega_{\sigma,\tau}$.

setting

$$[z,\nu]^\tau := z^\tau \nu$$

where ν is the outer unit normal at $\partial\Omega$. Since $z \in C^\infty(\Omega, \mathbb{R}^{N\times n}) \cap L^{\infty,q}(\Omega, \mathbb{R}^{N\times n})$, it is easily seen that $[z,\nu]^\tau$ is bounded in $L^\infty(\partial\Omega, \mathbb{R}^N)$.

Let $\tilde{\varphi} : \partial\Omega \to \mathbb{R}^N$ be Lipschitz continuous. We define $\varphi : \overline{\Omega} \cap V \to \mathbb{R}^N$, by

$$\varphi(x, \gamma(x) + y) := \tilde{\varphi}(x, \gamma(x)) \quad (x, \gamma(x) + y) \in \overline{\Omega} \cap V \subseteq B_{n-1}(0, r) \times [-h, h],$$

Note $\varphi^\tau = \tilde{\varphi}$ for all $0 < \tau < h/2$. It is easily seen that φ is Lipschitz continuous and is thus in $W^{1,\infty}(\Omega \cap V, \mathbb{R}^N) \cap C(\overline{\Omega} \cap V, \mathbb{R}^N)$ (cf. [27, Theorem 5, p. 131]). Let us write $\partial\Omega$ for $\partial\Omega \cap V$ in the rest of the proof for simplicity. Recalling the Gauß-Green formula (cf. Theorem 30) and the support of z we derive for $0 < \sigma < \tau < h/2$

$$\begin{aligned}\left| \int_{\partial\Omega} ([z,\nu]^\tau - [z,\nu]^\sigma) \cdot \tilde{\varphi} \, \mathrm{d}\mathcal{H}^{n-1} \right| &= \left| \int_{\partial\Omega} (\varphi^\tau)^\mathsf{T} z^\tau \nu - (\varphi^\sigma)^\mathsf{T} z^\sigma \nu \, \mathrm{d}\mathcal{H}^{n-1} \right| \\ &= \left| \int_{\Omega_{\sigma,\tau}} \operatorname{div}(\varphi^\mathsf{T} z) \, \mathrm{d}x \right|\end{aligned}$$

$$
\begin{aligned}
&= \left| \int_{\Omega_{\sigma,\tau}} \operatorname{div} z \cdot \varphi \,\mathrm{d}x + \int_{\Omega_{\sigma,\tau}} z : D\varphi \,\mathrm{d}x \right| \\
&\leq \| \operatorname{div} z \|_{L^q(\Omega_{\sigma,\tau},\mathbb{R}^N)} \, \|\varphi\|_{L^{q'}(\Omega_{\sigma,\tau},\mathbb{R}^N)} \\
&\quad + \|z\|_{L^\infty(\Omega_{\sigma,\tau},\mathbb{R}^{N\times n})} \|D\varphi\|_{L^1(\Omega_{\sigma,\tau},\mathbb{R}^{N\times n})} \,,
\end{aligned}
$$

where $\Omega_{\sigma,\tau} := \{(x,\gamma(x)+y) \in \Omega \cap V \;;\; \sigma < y < \tau\}$. Note that σ and τ can be choosen arbitrarily small. Moreover the calculation above can be performed for each Lipschitz continuous $\tilde{\phi}$ from a countable set of the unit ball in $L^1(\partial\Omega,\mathbb{R}^N)$. We thus obtain that $([z,\nu]^\tau)_\tau$ is a Cauchy sequence for the metric

$$
d_*(w_1,w_2) := \sum_{j\in\mathbb{N}} 2^{-j} \min\left\{1, \; \left| \int_{\partial\Omega} (w_1 - w_2) \cdot \tilde{\varphi}_j \,\mathrm{d}\mathcal{H}^{n-1} \right| \right\},
$$

where $\{\tilde{\varphi}_j \;;\; j \in \mathbb{N}\}$ is a dense subset of Lipschitz functions in the unite ball of $L^1(\Omega,\mathbb{R}^N)$. Since $L^1(\partial\Omega,\mathbb{R}^N)^* = L^\infty(\partial\Omega,\mathbb{R}^N)$ is complete and the weak* topology in $L^\infty(\partial\Omega,\mathbb{R}^N)$ can be metricized by the metric d_*, there is a unique limit $[z,\nu]^{\partial\Omega} \in L^\infty(\partial\Omega,\mathbb{R}^N)$ with $[z,\nu]^\tau \overset{*}{\rightharpoonup} [z,\nu]^{\partial\Omega}$ as $\tau \to 0$.

Letting $\sigma \to 0$ in the previous inequality we derive the estimate

$$
\begin{aligned}
\left| \int_{\partial\Omega} ([z,\nu]^\tau - [z,\nu]^{\partial\Omega}) \cdot \tilde{\varphi} \,\mathrm{d}\mathcal{H}^{n-1} \right| &\leq \| \operatorname{div} z \|_{L^q(\Omega_{0,\tau},\mathbb{R}^N)} \, \|\varphi\|_{L^{q'}(\Omega_{0,\tau},\mathbb{R}^N)} \\
&\quad + \|z\|_{L^\infty(\Omega_{0,\tau},\mathbb{R}^{N\times n})} \|D\varphi\|_{L^1(\Omega_{0,\tau},\mathbb{R}^{N\times n})} \,,
\end{aligned}
$$

which also shows that the limit does not depend on the chosen orthonormal frame of the cylinder V. For $\varphi \in C^1(\overline{\Omega} \cap V, \mathbb{R}^N)$ we derive by dominated convergence

$$
\begin{aligned}
\int_\Omega \operatorname{div} z \cdot \varphi + z : D\varphi \,\mathrm{d}x &= \lim_{\tau\to 0} \int_{\Omega\setminus\Omega_{0,\tau}} \operatorname{div} z \cdot \varphi + z : D\varphi \,\mathrm{d}x \\
&= \lim_{\tau\to 0} \int_{\partial\Omega} [z,\nu]^\tau \cdot \varphi^\tau \,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial\Omega} [z,\nu]^{\partial\Omega} \varphi^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1} \,,
\end{aligned}
$$

where we used $\varphi^\tau \to \varphi^{\partial\Omega}$ in $L^1(\partial\Omega,\mathbb{R}^N)$.[7]

General z (still assumed to be compactly supported in $\overline{\Omega} \cap V$) are approximated by a sequence $(z_j)_j$ as in Proposition 48. Using the previous formulas we derive for

[7] Note that φ is uniformly continuous and thus $\varphi^\tau \to \varphi^{\partial\Omega}$ uniformly.

$\varphi \in C^1(\Omega \cap V, \mathbb{R}^N) \cap C(\overline{\Omega} \cap V, \mathbb{R}^N)$

$$\begin{aligned}\left| \int_{\partial\Omega} \big([z_j,\nu]^{\partial\Omega} - [z_l,\nu]^{\partial\Omega}\big) \cdot \varphi \,\mathrm{d}\mathcal{H}^{n-1} \right| &= \left| \int_\Omega (\operatorname{div} z_j - \operatorname{div} z_l) \cdot \varphi + (z_j - z_l) : D\varphi \,\mathrm{d}x \right| \\ &\leq \| \operatorname{div} z_j - \operatorname{div} z_l \|_{L^q(\Omega,\mathbb{R}^N)} \, \|\varphi\|_{L^{q'}(\Omega,\mathbb{R}^N)} \\ &\quad + \left| \int_\Omega (z_j - z_l) : D\varphi \,\mathrm{d}x \right|\end{aligned}$$

Since z_j is $L^{\infty,q}$-convergent we obtain arguing as above that $\big([z_j,\nu]^{\partial\Omega}\big)_j$ is a weak*-convergent sequence in $L^\infty(\partial\Omega,\mathbb{R}^N)$ and the limit $[z,\nu]^{\partial\Omega}$ has the desired properties.

We have thus defined the normal trace $[z,\nu]$ locally on $\partial\Omega$. The general case follows using compactness of $\partial\Omega$ and a standard partition of unity argument. □

In the symmetric case we have the following result.

Proposition 51. *Let Ω be open and bounded with Lipschitz boundary. There exists a linear trace operator $[\cdot,\nu]_s^{\partial\Omega} : L^{\infty,q}_{\mathrm{sym}}(\Omega) \to L^\infty(\partial\Omega,\mathbb{R}^n)$, such that*

$$\int_\Omega z : D_s\varphi \,\mathrm{d}x + \int_\Omega \operatorname{div}_{\mathrm{s}} z \cdot \varphi \,\mathrm{d}x = \int_{\partial\Omega} [z,\nu]_s^{\partial\Omega} \cdot \varphi \,\mathrm{d}\mathcal{H}^{n-1}$$

for all $\varphi \in C^1(\Omega,\mathbb{R}^n) \cap C(\overline{\Omega},\mathbb{R}^n)$. In particular for $z \in C^1(\Omega,\mathbb{R}^{n\times n}) \cap C(\overline{\Omega},\mathbb{R}^{n\times n})$ we have

$$[z,\nu]_s^{\partial\Omega}(x) = \tfrac{1}{2}\big(z(x) + z(x)^{\mathsf{T}}\big)\nu(x)\,.$$

Moreover, the mapping $z \mapsto [z,\nu]_s^{\partial\Omega}$ is sequentially continuous with respect to $L^{\infty,q}_{\mathrm{sym}}$-convergence and there holds

$$\|[z,\nu]_s^{\partial\Omega}\|_{L^\infty(\partial\Omega,\mathbb{R}^n)} \leq \|z\|_{L^\infty(\Omega,\mathbb{R}^{n\times n})}$$

for all $z \in L^{\infty,q}_{\mathrm{sym}}(\Omega)$.

Proof. The proof is similar to the previous one, one has to replace the BV- and $L^{\infty,q}$-arguments by their symmetrized counterparts. □

5.3.1 Gauß-Green Formulas

We finally prove the desired Gauß-Green formulas for $BV(\Omega,\mathbb{R}^N)$ and $L^{\infty,q}(\Omega,\mathbb{R}^{N\times n})$ functions (and $BD(\Omega)$ and $L^{\infty,q}_{\mathrm{sym}}(\Omega)$ functions resp.).

Let $p \in [1, \infty)$ and $p' \in (1, \infty]$ with $1/p + 1/p' = 1$ and define

$$BV^p(\Omega, \mathbb{R}^N) := BV(\Omega, \mathbb{R}^N) \cap L^p(\Omega, \mathbb{R}^N)$$

and

$$BD^p(\Omega) := BD(\Omega) \cap L^p(\Omega, \mathbb{R}^n)\,.$$

For $u \in BV^p(\Omega, \mathbb{R}^N)$ and $z \in L^{\infty,p'}(\Omega, \mathbb{R}^{N\times n})$ we define a distribution $(z, Du) : C_c^\infty(\Omega) \to \mathbb{R}$ by setting

$$\langle (z, Du), \varphi \rangle := -\int_\Omega \varphi\, u \cdot \operatorname{div} z \,\mathrm{d}x - \int_\Omega z : (u \otimes D\varphi)\,\mathrm{d}x.$$

Analogously for $u \in BD^p(\Omega)$, $z \in L^{\infty,p'}_{\mathrm{sym}}(\Omega)$ we define a distribution $(z, D_s u) : C_c^\infty(\Omega) \to \mathbb{R}$ by setting

$$\langle (z, D_s u), \varphi \rangle := -\int_\Omega \varphi\, u \cdot \operatorname{div_s} z \,\mathrm{d}x - \int_\Omega z : (u \odot D\varphi)\,\mathrm{d}x.$$

Of course if Du and $D_s u$ or z resp. are sufficiently regular (e.g. $u \in W^{1,1}$) we have

$$\langle (z, Du), \varphi \rangle = \int_\Omega \varphi\, z : Du \,\mathrm{d}x \quad \text{and} \quad \langle (z, D_s u), \varphi \rangle = \int_\Omega \varphi\, z : D_s u \,\mathrm{d}x\,.$$

However, the products $z : Du$ and $z : D_s u$ might not be defined pointwise for general z and u. Nevertheless the distribution above has a representation as Radon measure:

Theorem 52. *For the distribution defined above we have the estimate*

$$|\langle (z, Du), \varphi \rangle| \le \|z\|_\infty \, \|\varphi\|_\infty \int_\Omega \mathrm{d}|Du|$$

and thus the distribution (z, Du) has a representation as finite signed Radon measure on Ω.

Moreover, the measure (z, Du) and its total variation measure $|(z, Du)|$ are absolutely continuous with respect to $|Du|$.

Likewise for the BD-case we have

$$|\langle (z, D_s u), \varphi \rangle| \le \|z\|_\infty \, \|\varphi\|_\infty \int_\Omega \mathrm{d}|D_s u|$$

such that again $(z, D_s u)$ has a representation as finite signed Radon measure on Ω and we have that $(z, D_s u)$ and its total variation measure $|(z, D_s u)|$ are absolutely

continuous with respect to $|D_s u|$.

Proof. Let $u \in BV^p(\Omega, \mathbb{R}^N)$, $z \in L^{\infty,p'}(\Omega, \mathbb{R}^{N\times n})$ and $A \subseteq \Omega$ be open. Take $\varphi \in C_c^\infty(A)$. Approximate u by a strictly convergent sequence $(u_k)_k$ in $C^\infty(\Omega, \mathbb{R}^N) \cap BV^p(\Omega, \mathbb{R}^N)$ and z by a sequence $(z_j)_j$ as in Proposition 48, then

$$\begin{aligned}
|\langle (z, Du), \varphi\rangle| &= \left| -\int_\Omega \varphi\, u \cdot \operatorname{div} z \,\mathrm{d}x - \int_\Omega z : (u \otimes D\varphi)\,\mathrm{d}x \right| \\
&= \lim_{k\to\infty} \lim_{j\to\infty} \left| -\int_\Omega \varphi\, u_k \cdot \operatorname{div} z_j \,\mathrm{d}x - \int_\Omega z_j : (u_k \otimes D\varphi)\,\mathrm{d}x \right| \\
&= \lim_{k\to\infty} \lim_{j\to\infty} \left| \int_\Omega \varphi\, z_j : Du_k \,\mathrm{d}x \right| \\
&\le \limsup_{k\to\infty} \limsup_{j\to\infty} \|\varphi\|_\infty \, \|z_j\|_\infty \int_{\operatorname{supp}\varphi} |Du_k| \,\mathrm{d}x \\
&\le \|z\|_\infty \|\varphi\|_\infty |Du|(\operatorname{supp}\varphi) \\
&\le \|z\|_\infty \|\varphi\|_\infty |Du|(A)\,.
\end{aligned}$$

Here we applied the Gauß-Green formula (5.2) and the boundary term vanishes by the support properties of φ.

The first estimate of the theorem follows now with $A = \Omega$ and since A was arbitrary, absolute continuity of (z, Du) with respect to $|Du|$ is proved.

The BD-case follows analogously with the symmetric Gauß-Green formula (5.9). □

Theorem 53 (Gauß-Green formulas)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded and with Lipschitz boundary. Then for $u \in BV^p(\Omega, \mathbb{R}^N)$ and $z \in L^{\infty,p'}(\Omega, \mathbb{R}^{N\times n})$ the generalized Gauß-Green formula*

$$\int_\Omega \mathrm{d}(z, Du) + \int_\Omega u \cdot \operatorname{div} z \,\mathrm{d}x = \int_{\partial\Omega} [z, \nu]^{\partial\Omega} \cdot u^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1}$$

holds. Moreover, for $u \in BD^p(\Omega)$ and $z \in L^{\infty,p'}_{sym}(\Omega)$ the generalized symmetrized Gauß-Green formula

$$\int_\Omega \mathrm{d}(z, D_s u) + \int_\Omega u \cdot \operatorname{div_s} z \,\mathrm{d}x = \int_{\partial\Omega} [z, \nu]_s^{\partial\Omega} \cdot u^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1}$$

holds.

Proof. By compactness of $\partial\Omega$ we can cover the boundary $\partial\Omega$ by a finite number of cylinders $V_1, \ldots, V_M$, which are, upon a rotation and translation of the coordinate

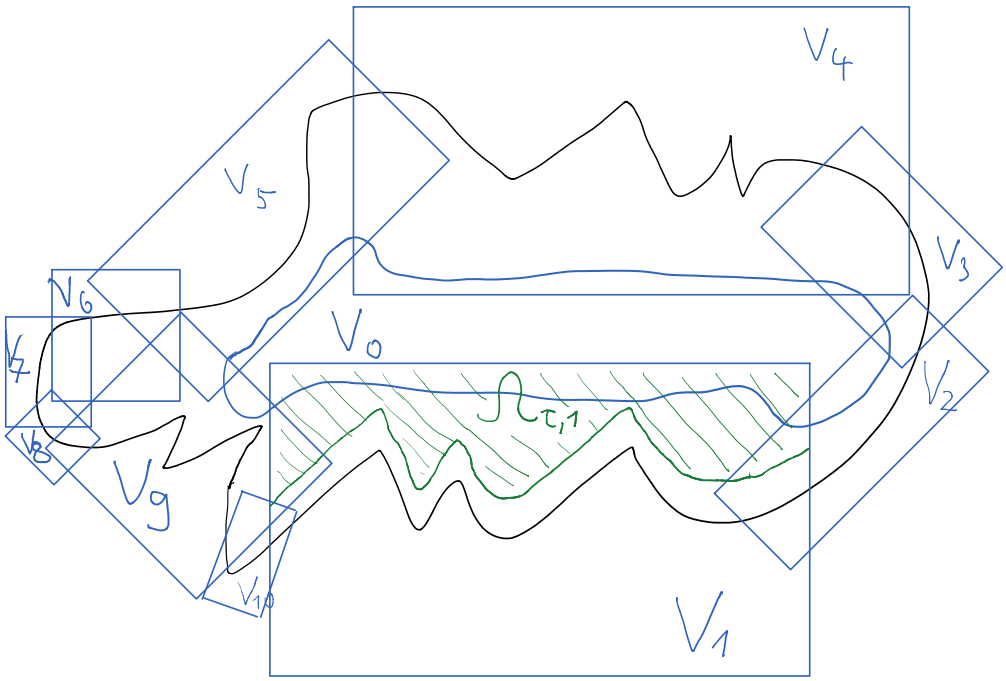

Figure 5.5: The sets $V_0, \dots V_{10}$ cover Ω.

system such as V in the proofs of Theorem 30, Theorem 45, Proposition 50 and Proposition 51. We find $V_0 \subseteq \Omega$ open with $\overline{V_0} \subseteq \Omega$, such that $\bigcup_{k=0}^M V_k$ covers $\overline{\Omega}$. Choose a partition of unity, $0 \le \rho_k \le 1$ $(k = 0, \dots, M)$, $\sum_{k=0}^M \rho_k = 1$ on $\overline{\Omega}$ and ρ_k has compact support on V_k $(k = 0, \dots, M)$.

In the local coordinates of each cylinder $V_1, \dots V_M$, for $0 < \tau < h_k/2$, we define

$$\Omega_{\tau,k} := \{(x_1, \dots, x_n) \in V_k \; ; \; x_n > \gamma_k(x_1, \dots, x_{n-1}) + \tau\}$$

for $k = 1, \dots, M$. Cf. Figure 5.5. For each such $\tau > 0$ sufficiently small we find a cutoff function $\varphi_\tau \in C_c^\infty(\Omega)$ with $0 \le \tilde{\varphi}_\tau \le 1$ and $\varphi = 1$ on $V_0 \cup \bigcup_{k=1}^M \Omega_{\tau,k}$. Thus φ_τ approximates χ_Ω as $\tau \to 0$.

Let us carry out the proof for the $BV(\Omega, \mathbb{R}^N)$-case first. To do so let $u \in BV^p(\Omega, \mathbb{R}^N)$ and $z \in L^{\infty,p'}(\Omega, \mathbb{R}^{N\times n})$. By majorized convergence we have

$$\begin{aligned}\int_\Omega \mathrm{d}(z, Du) &= \lim_{\tau\to 0} \int_\Omega \varphi_\tau \sum_{k=0}^M \rho_k \, \mathrm{d}(z, Du) \\ &= \lim_{\tau\to 0} \sum_{k=0}^M \int_\Omega \varphi_\tau \rho_k \, \mathrm{d}(z, Du)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{M} \lim_{\tau\to 0} \int_\Omega \varphi_\tau \rho_k \,\mathrm{d}(z, Du) \\
&= \sum_{k=0}^{M} \lim_{\tau\to 0} \left(- \int_\Omega (\varphi_\tau \rho_k)\, u \cdot \operatorname{div} z \,\mathrm{d}x - \int_\Omega z : (u \otimes D(\varphi_\tau \rho_k)) \,\mathrm{d}x \right) \\
&= \lim_{\tau\to 0} - \int_\Omega \varphi_\tau\, u \cdot \operatorname{div} z \,\mathrm{d}x - \sum_{k=0}^{M} \lim_{\tau\to 0} \int_\Omega z : (u \otimes D(\varphi_\tau \rho_k)) \,\mathrm{d}x \\
&= - \int_\Omega u \cdot \operatorname{div} z \,\mathrm{d}x - \sum_{k=0}^{M} \lim_{\tau\to 0} \int_\Omega \rho_k\, z : (u \otimes D\varphi_\tau) \,\mathrm{d}x \\
&\qquad\qquad - \lim_{\tau\to 0} \int_\Omega \sum_{k=0}^{M} \varphi_\tau\, z : (u \otimes D\rho_k) \,\mathrm{d}x \\
&= - \int_\Omega u \cdot \operatorname{div} z \,\mathrm{d}x - \sum_{k=0}^{M} \lim_{\tau\to 0} \int_\Omega \rho_k\, z : (u \otimes D\varphi_\tau) \,\mathrm{d}x \,,
\end{aligned}
$$

where we applied the definition of the distribution (z, Du), majorized convergence for the first integral and $\sum_{k=0}^{M} D\rho_k = 0$ by the partition of unity property.

Thus it remains to prove

$$
- \sum_{k=0}^{M} \int_\Omega \rho_k\, z : (u \otimes D\varphi_\tau) \,\mathrm{d}x \to \int_{\partial\Omega} [z, \nu]^{\partial\Omega} \cdot u^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1}
$$

as $\tau \to 0$. We will show

$$
- \int_\Omega \rho_k\, z : (u \otimes D\varphi_\tau) \,\mathrm{d}x \to \int_{\partial\Omega} \rho_k\, [z, \nu]^{\partial\Omega} \cdot u^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1}
$$

for each $k \in \{0, \dots, M\}$, in which the case $k = 0$ is elementary. Thus let $k \in \{1, \dots, M\}$ and approximate u by $(u_l)_l$ strictly and z by $(z_j)_j$ as in Proposition 48. In the following we neglect the dependence of the representation of the boundary of Ω on k and introduce

$$
\Omega_{0,\tau} := (\Omega \cap V_k) \setminus \Omega_{\tau,k} \,.
$$

With $D\varphi_\tau = 0$ on $\Omega_{\tau,k}$ and by application of the classical Gauß-Green theorem we derive

$$
\begin{aligned}
&\left| \int_\Omega \rho_k\, z : (u \otimes D\varphi_\tau) \,\mathrm{d}x + \int_{\partial\Omega} \rho_k\, [z, \nu]^{\partial\Omega} \cdot u^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1} \right| \\
&\quad = \lim_{j\to\infty} \lim_{l\to\infty} \left| \int_{\Omega_{0,\tau}} \rho_k\, z_j : (u_l \otimes D\varphi_\tau) \,\mathrm{d}x + \int_{\partial\Omega} \rho_k\, [z_j, \nu]^{\partial\Omega} \cdot u_l^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1} \right|
\end{aligned}
$$

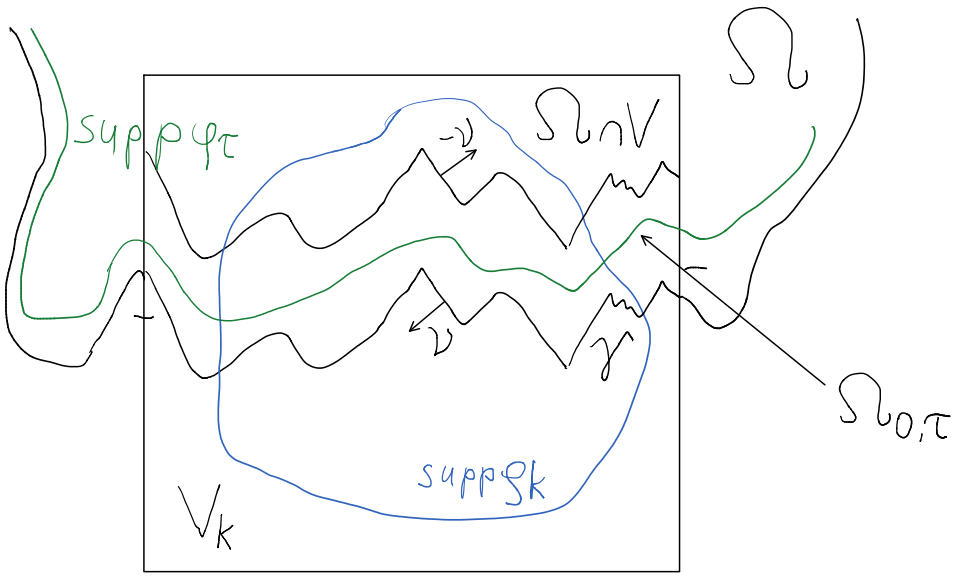

Figure 5.6: The set $\Omega_{0,\tau}$ and the support of ρ_k.

$$
\begin{aligned}
&= \lim_{j\to\infty} \lim_{l\to\infty} \Bigg| -\int_{\Omega_{0,\tau}} \varphi_\tau \operatorname{div}\left(\rho_k u_l^\mathsf{T} z_j\right) \mathrm{d}x \\
&\qquad + \int_{\partial\Omega} \rho_k^\tau \left[z_j, -\nu\right]^\tau \cdot u_l^\tau \,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial\Omega} \rho_k \left[z_j, \nu\right]^{\partial\Omega} \cdot u_l^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1} \Bigg| \\
&= \lim_{j\to\infty} \lim_{l\to\infty} \Bigg| -\int_{\Omega_{0,\tau}} \varphi_\tau \Big(z_j : (u_l \otimes D\rho_k) + \rho_k \left(\operatorname{div} z_j\right) \cdot u_l + \rho_k\, z_j : Du_l\Big) \,\mathrm{d}x \qquad (5.14) \\
&\qquad - \int_{\partial\Omega} \rho_k^\tau [z_j, \nu]^\tau \cdot u_l^\tau \,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial\Omega} \rho_k [z_j, \nu]^{\partial\Omega} \cdot u_l^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1} \Bigg| \\
&\le \limsup_{j\to\infty} \limsup_{l\to\infty} \Bigg(\|z_j\|_{L^\infty(\Omega_{0,\tau},\mathbb{R}^{N\times n})} \|u_l\|_{L^1(\Omega_{0,\tau},\mathbb{R}^N)} \|D\rho_k\|_\infty \\
&\qquad + \|\operatorname{div} z_j\|_{L^{p'}(\Omega_{0,\tau},\mathbb{R}^N)} \|u_l\|_{L^p(\Omega_{0,\tau},\mathbb{R}^N)} \\
&\qquad + \|z_j\|_{L^\infty(\Omega_{0,\tau},\mathbb{R}^{N\times n})} \|Du_l\|_{L^1(\Omega_{0,\tau},\mathbb{R}^{N\times n})} \\
&\qquad + \Bigg| -\int_{\partial\Omega} \rho_k^\tau [z_j, \nu]^\tau \cdot u_l^\tau \,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial\Omega} \rho_k [z_j, \nu]^{\partial\Omega} \cdot u_l^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1} \Bigg| \Bigg) \\
&\le \|z\|_{L^\infty(\Omega,\mathbb{R}^{N\times n})} \|u\|_{L^1(\Omega_{0,\tau},\mathbb{R}^N)} \|D\rho_k\|_\infty \\
&\quad + \|\operatorname{div} z\|_{L^{p'}(\Omega,\mathbb{R}^N)} \|u\|_{L^p(\Omega_{0,\tau},\mathbb{R}^N)} \\
&\quad + \|z\|_{L^\infty(\Omega,\mathbb{R}^{N\times n})} |Du|(\overline{\Omega_{0,\tau}} \cap \Omega) \\
&\quad + \limsup_{j\to\infty} \limsup_{l\to\infty} \Bigg| -\int_{\partial\Omega} \rho_k^\tau [z_j, \nu]^\tau \cdot u_l^\tau \,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial\Omega} \rho_k [z_j, \nu]^{\partial\Omega} \cdot u_l^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1} \Bigg| . \qquad (5.15)
\end{aligned}
$$

Here we used the notation (5.4) and took into account that $|\rho_k| \leq 1$ and $|\varphi_\tau| \leq 1$. Note that the first three terms can be made arbitrarily small, provided τ is chosen small enough, it thus remains to prove that the $\limsup$-term converges to zero as $\tau \to 0$.

Again by the classical Gauß-Green theorem and the support properties of ρ_k we calculate

$$-\int_{\partial\Omega} \rho_k^\tau [z_j,\nu]^\tau \cdot u_l^\tau \,\mathrm{d}\mathcal{H}^{n-1} = \int_{\Omega_{0,\tau}} z_j : (u_l \otimes D\rho_k) + \rho_k\big(\operatorname{div} z_j\big) \cdot u_l + \rho_k z_j : Du_l \,\mathrm{d}x$$
$$- \int_{\partial\Omega} \rho_k\, [z_j,\nu]^{\partial\Omega} \cdot u_l^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1}\,.$$

Recalling the convergence properties $u_l \to u$ and $z_j \to z$ the integral over $\Omega_{0,\tau}$ can be assumed to be arbitrarily small provided τ is chosen small enough with the estimates similar to that after (5.14). But this proves that the limsups in (5.15) tend to zero as $\tau \to 0$, which finishes the proof of the BV-case.

Replacing D, div and $[\cdot,\nu]^{\partial\Omega}$ by their symmetrized counterparts, we get the proof for the $BD^p(\Omega)$-$L^{\infty,p'}_{sym}(\Omega)$-setting.

□

5.4 Investigation of the Eigenvalue Problem in $BV(\Omega,\mathbb{R}^N)$ and $BD(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. In this section we will investigate the vectorial and the symmetrized eigenvalue problem of the 1-Laplace operator, formally given by

$$\begin{cases} -\operatorname{div} \frac{Du}{|Du|} = \lambda \frac{u}{|u|} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.16}$$

for u with values in $\mathbb{R}^N$ and

$$\begin{cases} -\operatorname{div} \frac{D_s u}{|D_s u|} = \lambda \frac{u}{|u|} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.17}$$

for u with values in $\mathbb{R}^n$.

As in the well studied case $N = 1$ these problems are not well defined, since it is

not clear how to interpret expressions 0/0. Nevertheless the associated variational problems $\mathcal{E}_{TV} : BV(\Omega, \mathbb{R}^N) \to \mathbb{R}$,

$$\mathcal{E}_{TV}(v) = \int_\Omega \mathrm{d}|Dv| + \int_\Omega |v^{\partial\Omega} \otimes \nu| \, \mathrm{d}\mathcal{H}^{n-1} \tag{5.18}$$

for the BV-case and of $\mathcal{E}_{TD} : BD(\Omega) \to \mathbb{R}$,

$$\mathcal{E}_{TD}(v) = \int_\Omega \mathrm{d}|D_s v| + \int_\Omega |v^{\partial\Omega} \odot \nu| \, \mathrm{d}\mathcal{H}^{n-1} \tag{5.19}$$

for the BD case, both subject to the constraint

$$\mathcal{G}_1(v) = \int_\Omega |v| \, \mathrm{d}x = \alpha \,, \tag{5.20}$$

are well posed for $\alpha > 0$. We define u to be an eigensolution of the vectorial 1-Laplace operator or the symmetrized 1-Laplace operator provided u is a critical point of the variational problem (5.18), (5.20) or (5.19), (5.20) resp. As before criticality is meant in the sense of the weak slope on the metric space $L^p(\Omega, \mathbb{R}^N)$ or $L^p(\Omega, \mathbb{R}^n)$: That is we take $p \in (1, n/(n-1))$ and extend $\mathcal{E}_{TV}$ to $L^p(\Omega, \mathbb{R}^N)$ and $\mathcal{E}_{TD}$ to $L^p(\Omega, \mathbb{R}^n)$ by setting

$$\mathcal{E}_{TV}(u) = \infty \quad \text{for } u \in L^p(\Omega, \mathbb{R}^N) \setminus BV(\Omega, \mathbb{R}^N)$$

and

$$\mathcal{E}_{TD}(u) = \infty \quad \text{for } u \in L^p(\Omega, \mathbb{R}^n) \setminus BD(\Omega) \,.$$

A function u is said to be an eigenfunction of the vectorial 1-Laplace operator or the symmetrized 1-Laplace operator provided u is a critical point, $|d\mathcal{F}|(u) = 0$, of the function $\mathcal{F} : L^p(\Omega, \mathbb{R}^N) \to \mathbb{R} \cup \{\infty\}$,

$$\mathcal{F} := \mathcal{E}_{TV} + I_{\{\mathcal{G}_1 = \alpha\}}$$

or $\mathcal{F} : L^p(\Omega, \mathbb{R}^n) \to \mathbb{R} \cup \{\infty\}$, $\mathcal{F} := \mathcal{E}_{TD} + I_{\{\mathcal{G}_1 = \alpha\}}$ in the BD-case.

For an eigenfunction u we define $\lambda := \mathcal{E}_{TV}(u)/\mathcal{G}_1(u)$ or $\lambda := \mathcal{E}_{BD}(u)/\mathcal{G}_1(u)$ to be the corresponding eigenvalue of the eigenfunction u. This is justified by Theorem 54 and Theorem 55 below. As before we call eigenfunctions u with $\mathcal{G}_1(u) = 1$ *normalized*.

5.4.1 Existence of Eigensolutions and the Euler-Lagrange Equation

The next two theorems are the main results of this chapter concerning the eigenvalue problem of the 1-Laplace operator in $\mathbb{R}^N$ and the symmetrized 1-Laplace operator. In particular we recover the results already known for the scalar 1-Laplace operator.

Theorem 54. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary.*

Then each eigenfunction u of the vectorial 1-Laplace operator satisfies the single version of the Euler-Lagrange equation, i.e. there is a function $s \in L^\infty(\Omega, \mathbb{R}^N)$ with

$$s(x) \in \operatorname{Sgn}(u(x))$$

for almost all $x \in \Omega$ and a function $z \in L^\infty(\Omega, \mathbb{R}^{N\times n})$ with

$$\|z\|_\infty = 1\,, \quad \operatorname{div} z \in L^{p'}(\Omega, \mathbb{R}^N) \quad \text{and} \quad \mathcal{E}_{TV}(u) = -\int_\Omega \operatorname{div} z \cdot u \,\mathrm{d}x$$

such that for the eigenvalue $\lambda = \frac{\mathcal{E}_{TV}(u)}{\mathcal{G}_1(u)}$ the Euler-Lagrange equation

$$-\operatorname{div} z = \lambda s \tag{5.21}$$

is satisfied.

There exists a sequence of pairs $(\pm u_k)_k$ of normalized eigenfunctions of the vectorial 1-Laplace operator.

Moreover, the corresponding sequence of eigenvalues[8] *$(\lambda_{k,v})_k$, $\lambda_{k,v} = \mathcal{E}_{TV}(\pm u_k)$ is unbounded and characterized by*

$$\lambda_{k,v} := \inf_{S \in \mathscr{S}_k} \sup_{w \in S} \mathcal{E}_{TV}(w)$$

where

$$\mathscr{S}_k := \{S \subseteq L^p(\Omega, \mathbb{R}^N) \text{ compact and symmetric}, \mathcal{G}_1 = 1 \text{ on } S \text{ and } \operatorname{gen} S \geq k\}\,.$$

In particular the smallest eigenvalue $\lambda_{1,v}$ is given by

$$\lambda_{1,v} = \min_{\substack{w \in BV(\Omega,\mathbb{R}^N) \\ \|w\|_1 = 1}} \mathcal{E}_{TV}(w)$$

[8] The subscript "v" is used to point out that we mean the eigenvalues of the vectorial 1-Laplace operator.

and for each eigenfunction u corresponding to $\lambda_1 = \mathcal{E}_{TV}(u)$ the multiple version of the Euler-Lagrange equation is statisfied, i.e. for each *function $s \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|s\|_\infty = 1$ and $s(x) \in \mathrm{Sgn}(u(x))$ for almost all $x \in \Omega$ there exists a function z with the properties described above such that the corresponding Euler-Lagrange equation* (5.21) *is satisfied.*

The proof will be given in the next subsection.

Note that the functions s and z are well defined replacements for the undefined expressions $u/|u|$ and $\frac{Du}{|Du|}$ in the formal Euler-Lagrange equation (5.16). In contrast to the scalar case we can not characterize the first eigenfunctions in terms of Cheeger sets, since the coarea formula is not available in $BV(\Omega, \mathbb{R}^N)$.

In $BD(\Omega)$ we derive an analogous theorem for the symmetrized 1-Laplace operator.

Theorem 55. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then each eigenfunction u of the symmetrized 1-Laplace operator satisfies the single version of the Euler-Lagrange equation, i.e. there is a function $s \in L^\infty(\Omega, \mathbb{R}^n)$ with*

$$s(x) \in \mathrm{Sgn}(u(x))$$

for almost all $x \in \Omega$ and a function $z \in L^\infty(\Omega, \mathbb{R}^{n\times n})$ with

$$\|z\|_\infty = 1\,, \quad \mathrm{div_s}\, z \in L^{p'}(\Omega, \mathbb{R}^n) \quad \textit{and} \quad \mathcal{E}_{TD}(u) = -\int_\Omega \mathrm{div_s}\, z \cdot u \,\mathrm{d}x$$

such that for the eigenvalue $\lambda = \frac{\mathcal{E}_{TD}(u)}{\mathcal{G}_1(u)}$ the Euler-Lagrange equation

$$-\mathrm{div_s}\, z = \lambda s \tag{5.22}$$

is satisfied.

There exists a sequence of pairs $(\pm u_{k,s})_k$ of normalized eigenfunctions of the symmetrized 1-Laplace operator.

Moreover, the corresponding sequence of eigenvalues[9] *$\lambda_{k,s} = \mathcal{E}_{TD}(u_{k,s})$ is unbounded and characterized by*

$$\lambda_{k,s} := \inf_{S \in \mathscr{S}_k} \sup_{v \in S} \mathcal{E}_{TV}(v)\,,$$

[9] The subscript "s" is used to point out that we mean the eigenvalues of the symmetrized 1-Laplace operator.

where

$$\mathscr{S}_k := \{S \subseteq L^p(\Omega, \mathbb{R}^n) \text{ compact and symmetric, } \mathcal{G}_1 = 1 \text{ on } S \text{ and } \operatorname{gen} S \geq k\}\,.$$

In particular the smallest eigenvalue $\lambda_{1,s}$ is given by

$$\lambda_{1,s} = \min_{\substack{v \in BD(\Omega) \\ \|v\|_1 = 1}} \mathcal{E}_{TD}(v)$$

and for each first eigenfunction u corresponding to $\lambda_{1,s} = \mathcal{E}_{TD}(u)$ the multiple version of the Euler-Lagrange equation is statisfied, i.e. for each *function $s \in L^\infty(\Omega, \mathbb{R}^n)$ with $\|s\|_\infty \leq 1$ and $s(x) \in \operatorname{Sgn}(u(x))$ for almost all $x \in \Omega$ there exists a function z with the properties described above such that the corresponding Euler-Lagrange equation* (5.22) *is satisfied.*

The proof is given in the next subsection, too.

Note as above that the function s in (5.22) is a well defined replacement for the formal expression $u/|u|$ in (5.17). Moreover, we see that for any function z in (5.22) we may also consider the symmetric part $z_s := \frac{z+z^\mathsf{T}}{2}$, then $\operatorname{div_s} z = \operatorname{div_s} z_s = \operatorname{div} z_s$ and $\|z_s\|_\infty \leq \|z\|_\infty \leq 1$, such that z_s also satisfies the Euler-Lagrange equation (5.22) and is thus a well defined replacement for the formal expression $\frac{D_s u}{|D_s u|}$ in (5.17).

5.4.2 Proof of Theorems 54 and 55

By Proposition 33 the function $\mathcal{E}_{TV}$ is convex and lower semicontinuous on $L^p(\Omega, \mathbb{R}^N)$. Similarly by Proposition 47 the function $\mathcal{E}_{TD}$ turns out to be convex and lower semicontinuous on $L^p(\Omega, \mathbb{R}^n)$. As preparation of the proof of Theorem 54 and Theorem 55 we need to calculate the subdifferentials of $\mathcal{E}_{TV}$ and $\mathcal{E}_{BD}$.

Theorem 56 (Characterization of subdifferentials)**.** *Let $u \in BV^p(\Omega, \mathbb{R}^N)$. For $v^* \in L^{p'}(\Omega, \mathbb{R}^N)$ we have $v^* \in \partial\mathcal{E}_{TV}(u)$ if and only if there is $z \in L^{\infty,p'}(\Omega, \mathbb{R}^{N\times n})$ with $\|z\|_\infty \leq 1$, $\operatorname{div} z = v^*$ and*

$$\mathcal{E}_{TV}(u) = -\int_\Omega \operatorname{div} z \cdot u \,\mathrm{d}x\,. \tag{5.23}$$

Similary for $u \in BD^p(\Omega)$ we have $v^* \in \partial\mathcal{E}_{TD}(u)$ if and only if there exists $z \in L^{\infty,p}_{\mathrm{sym}}(\Omega)$

with $\|z\|_\infty \leq 1$, $\operatorname{div}_s z = v^*$ *and*

$$\mathcal{E}_{TD}(u) = -\int_\Omega \operatorname{div}_s z \cdot u \, \mathrm{d}x \,. \tag{5.24}$$

Remark 57. *(a) When* $u \neq 0$ *the requirement* $\|z\|_\infty \leq 1$ *in connection with* (5.23) *or* (5.24) *turns out to be equivalent to* $\|z\|_\infty = 1$. *Let us sketch the argument in the symmetrized case (the vectorial case is analogue). Indeed, if we assume the Euler- Lagrange equation is satisfied for some* z *with* $\|z\| < \infty$, *we find* $\gamma > 1$, *such that for* $\tilde{z} := \gamma z$ *we have* $\|\tilde{z}\|_\infty = 1$ *and*

$$\int_\Omega \operatorname{div}_s \tilde{z} \cdot u \, \mathrm{d}x = \gamma \int_\Omega \operatorname{div}_s z \cdot u \, \mathrm{d}x = \gamma \mathcal{E}_{TD}(u) > \mathcal{E}_{TD}(u) \,.$$

It is not difficult to see, that this finally contradicts the relation (5.25) *that we will verify below.*

(b) Provided $u \in W_0^{1,1}(\Omega)$ *we may integrate by parts in* (5.23) *to derive*

$$\mathcal{E}_{TV}(u) = \int_\Omega |Du| \, \mathrm{d}x = \int z : Du \, \mathrm{d}x \,.$$

The latter is certainly satisfied provided $z(x) = \frac{Du(x)}{|Du(x)|}$, *where* $Du(x) \neq 0$. *In this sense the subgradients in* $\partial \mathcal{E}_{TV}(u)$ *are well defined replacements for the undetermined symbol* $-\operatorname{div} \frac{Du}{|Du|}$ *of the* 1*-Laplace operator.*

A similar calculation shows that the subgradients in $\partial \mathcal{E}_{BD}(u)$ *are a well defined replacements for the symbol* $-\operatorname{div} \frac{D_s u}{|D_s u|}$ *of the symmetrized* 1*-Laplace operator.*

Proof of Theorem 56. We extend the proof of [36] to the vectorial setting. Since the case in $BV^p(\Omega, \mathbb{R}^N)$ and $BD^p(\Omega)$ is very similar, we will treat the second case only. Let $u \in BD^p(\Omega)$ and define the convex set

$$M^* := \{v^* \in L^{p'}(\Omega, \mathbb{R}^n) \; ; \; \exists z \in L^{\infty,p}_{\mathrm{sym}}(\Omega) \text{ with } \|z\|_\infty \leq 1 \text{ and } v^* = -\operatorname{div}_s z\} \,.$$

It is straightforward to see that M^* is weakly closed in $L^{p'}(\Omega, \mathbb{R}^{n\times n})$: By separability of $L^p(\Omega, \mathbb{R}^n)$ it suffices to show that M^* is weak-sequentially closed. Thus take some sequence $(v_k^*)_k$ in M^* with $v_k^* \rightharpoonup: v^*$. There exists a sequence $(z_k)_k$ in $L^{\infty,p}_{\mathrm{sym}}$ with $v_k^* = -\operatorname{div}_s z_k$, $\|z_k\|_\infty \leq 1$ and by the Banach-Alaoglu Theorem we may thus assume (by picking a subsequence) that $z_k \overset{*}{\rightharpoonup}: z$ in $L^\infty(\Omega, \mathbb{R}^{n\times n})$. Note that by weak-lower

semicontinuity of the norm we obtain $\|z\|_\infty \leq 1$. Moreover

$$\begin{aligned}\int_\Omega z : D_s\varphi \,\mathrm{d}x &= \lim_{k\to\infty} \int_\Omega z_k : D_s\varphi \,\mathrm{d}x \\ &= \lim_{k\to\infty} \int_\Omega -\operatorname{div_s} z_k \cdot \varphi \,\mathrm{d}x \\ &= \lim_{k\to\infty} \int_\Omega v_k^* \cdot \varphi \,\mathrm{d}x \\ &= \int_\Omega v^* \cdot \varphi \,\mathrm{d}x\end{aligned}$$

for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^n)$ and thus $-\operatorname{div_s} z = v^*$ in the sense of distributions. This verifies $v^* \in M^*$.

Following the proof of [36] we intend to show

$$\mathcal{E}_{TD} = \left(I_{M^*}\right)^*, \tag{5.25}$$

the assertion will then follow from the Fenchel identity (3.3).

In order to show (5.25) we take $v^* = -\operatorname{div_s} z \in M^*$ and $w \in BD^p(\Omega)$ and calculate with the Gauß-Green formula from Theorem 53

$$\begin{aligned}\int_\Omega w \cdot v^* \,\mathrm{d}x &= -\int_\Omega w \cdot \operatorname{div_s} z \,\mathrm{d}x \\ &= \int_\Omega \mathrm{d}(z, Dw) - \int_{\partial\Omega} [z, \nu]^{\partial\Omega} \cdot w^{\partial\Omega} \,\mathrm{d}\mathcal{H}^{n-1} \\ &\leq \|z\|_\infty \left(\int_\Omega \mathrm{d}|Dw| + \int_{\partial\Omega} |w^{\partial\Omega}| \,\mathrm{d}\mathcal{H}^{n-1} \right) \\ &\leq \mathcal{E}_{TV}(w),\end{aligned}$$

which verifies $I_{M^*}^*(w) \leq \mathcal{E}(w)$ for all $w \in L^p(\Omega, \mathbb{R}^n)$.

For the reversed inequality we take $w \in BD^p(\Omega)$ and a large ball K containing Ω. By Proposition 46 we derive

$$\begin{aligned}\mathcal{E}_{TD}(w) &= \int_K \mathrm{d}|D_s w| \\ &= \sup\left\{ \int_K w \cdot \operatorname{div_s} z \,\mathrm{d}x \;;\; z \in C_c^\infty(K, \mathbb{R}^{n\times n}), \|z\|_\infty \leq 1 \right\} \\ &= \sup\left\{ \int_\Omega w \cdot \operatorname{div_s} z \,\mathrm{d}x \;;\; z \in C_c^\infty(K, \mathbb{R}^{n\times n}), \|z\|_\infty \leq 1 \right\}\end{aligned}$$

$$\leq \sup\Big\{\int_\Omega w\cdot \operatorname{div}_s z\,\mathrm{d}x\ ;\ z\in L^{\infty,p'}_{sym}(\Omega), \|z\|_\infty\leq 1\Big\}$$
$$= \sup\Big\{\int_\Omega w\cdot v\,\mathrm{d}x\ ;\ v\in M^*\Big\} = \big(I_{M^*}\big)^*(u)\,.$$

Now the proof is complete. □

Proof of Theorem 54. For the verification of the single version of the Euler-Lagrange equation we intend to apply Theorem 7 with $\mathcal{F}_0=\mathcal{E}_{TV}$, $\mathcal{F}_1=0$ and $\mathcal{G}=\mathcal{G}_1-\alpha$. Note that the subdifferential of $\mathcal{E}_{TV}$ is characterized in Theorem 56 above and the subdifferential of $\mathcal{G}_1$ was given in Proposition 2. It remains to prove (3.25). For convex functions the subdifferential and Clarkes generalized gradient coincide and obviously $\partial\mathcal{G}=\partial\mathcal{G}_1$. We thus calculate for $u\in BV(\Omega,\mathbb{R}^N)$ with $\mathcal{G}_1(u)=\alpha$, $u_+:=2u$ and $u_-:=0$

$$\mathcal{G}^0(u;\,u-u^+)=\mathcal{G}^0(u;\,u_--u)=\mathcal{G}^0_1(u;\,-u)=\max_{u^*\in\partial\mathcal{G}(u)}\langle u^*,-u\rangle=-\alpha<0 \tag{5.26}$$

by formula (3.6).

Thus the single version of the Euler-Lagrange equation (5.21) follows for some $\lambda\in\mathbb{R}$ by Theorem 7 and we get $\lambda=\mathcal{E}_{TV}(u)/\mathcal{G}_1(u)$ by testing the Euler-Lagrange equation with the eigenfunction u.

The existence of a sequence of eigenfunctions follows from Theorem 11 with $\mathcal{F}=\mathcal{E}_{TV}+I_{\{\mathcal{G}_1=\alpha\}}$: The preliminary (A) was already verified and (B) is immediate since $\mathcal{F}\geq 0$. For any $\beta\in\mathbb{R}$ the sublevel sets $\{\mathcal{F}\leq\beta\}$ are compact in $L^p(\Omega,\mathbb{R}^N)$ by Proposition 32 and Proposition 33, which also implies preliminary (C). The estimate (5.26) implies the (epi)-condition (D) by [22, Theorem 3.4].

We finally need to show (E), i.e. that for $k\in\mathbb{N}$ there is some odd function $\psi:\mathbb{S}^{k-1}\to L^p(\Omega,\mathbb{R}^N)$ with $\sup\{\mathcal{F}(\psi(x))\ ;\ x\in\mathbb{S}^{k-1}\}<\infty$. To do so we take $v_1,\dots,v_k\in C^\infty_c(\Omega,\mathbb{R}^N)$ linearly independent and using Euclidean coordinates $x=(x_1,\dots,x_k)$ for $x\in\mathbb{S}^{k-1}$ we define $\psi:\mathbb{S}^{k-1}\to W^{1,1}_0(\Omega,\mathbb{R}^N)\subseteq L^p(\Omega,\mathbb{R}^N)$ by

$$\psi(x_1,\dots,x_k):=\frac{\sum_{j=1}^k x_jv_j}{\Big\|\sum_{j=1}^k x_jv_j\Big\|_1}\,.$$

Since enumerator and denominator of ψ are continuous in x and by compactness of $\mathbb{S}^{k-1}$ we thus observe the desired property for ψ.

The multiple version of the Euler-Lagrange equation for normalized eigenfunctions

minimizing the energy follows from Theorem 8 with $\mathcal{F} = \mathcal{E}_{TV}$ and $\mathcal{G} = \mathcal{G}_1 - 1$. □

Proof of Theorem 55. By taking into account Propositions 41, 42, Theorem 44, Proposition 47 and Theorem 56, the proof is analogous to the proof of Theorem 54. □

Remark 58.

- *With minor modifications concerning the weak/weak*-convergence arguments the investigations above can be extended to $BV^1(\Omega, \mathbb{R}^N)$ and $BD^1(\Omega)$. Similarly we may also consider the situation in $BV^p(\Omega, \mathbb{R}^N)$ and $BD^p(\Omega)$ for $p \in [n/(n-1), \infty)$.*

- *Note that the derivation of the subdifferentials and the necessary Gauß-Green formulas in Section 5.3 was the main task in the foregoing investigations.*

- *With the arguments from [41, Remark 2.12] it is not too difficult to see that the eigenvalues $\lambda_{k,v}$ and $\lambda_{k,s}$ remain unchanged, provided we replace the requirement "compact" by "closed" in the definition of $\mathscr{S}_k$ or provided we switch from the $L^p(\Omega, \mathbb{R}^N)$-topology to the $L^r(\Omega, \mathbb{R}^N)$-topology with $p \neq r$ and $r \in [1, n/(n-1))$ in the determination of* $\operatorname{gen} S$.

6 Associated Parabolic Problems

In the final chapter we will treat some nonlinear parabolic equations related to the p- and the 1-Laplace Operator. In particular we will consider the parabolic problems of the p-, the vectorial and the symmetrized 1-Laplace operator, the problem of perfectly plastic fluids, the Porous Medium Equation (PME) and Fast Diffusion Equation (FDE). The fundamental idea of approach to these problems is in terms of (sub)gradient systems. The author learned these methods in the 13th International Internet Seminar (ISEM), a lecture series organized from Chill and Fašangová in 2009/2010. In the 'self-study phase' of ISEM 2009/2010 the author discovered that the FDE and the PME can be considered as gradient systems in the Hilbert space $H^{-1}(\Omega)$. This finally lead to the article "Porous Medium Equation and Fast Diffusion Equation as Gradient Systems" published by the autor in a joint work with Voigt, TU Dresden.

Note that existence results for the FDE/PME are known for a long time (in terms of maximal monotone or accretive operators), however the higher regularity framework, in terms of what we will call gradient system below, was new. Moreover, the approach allows to treat the PME/FDE without any restriction on the parameter $m \in \mathbb{R}_{>0}$, without any regularity assumption on $\partial\Omega$ and with weak assumptions on Ω (which need not be bounded or of finite volume) – in contrast to former treatments.

The problem of perfectly plastic fluids came to the focus of the author in a cooperation with Naumann who investigated this problem by a limiting procedure of the related parabolic symmetrized p-Laplace problem as $p \to 1$. The author applied the subgradient system framework to derive an existence result without the limiting procedure and was even able to include the incompressibility constraint in the framework using tools from convex analysis and Fredholm decomposion techniques.

6.1 Subgradient Systems

In this chapter let H be a Hilbert space. A subgradient system[1] is an abstract ordinary differential inclusion of the form

$$\dot{u}(t) + \partial\mathcal{E}\big(u(t)\big) \ni f(t)\,, \quad t \in I\,. \tag{6.1}$$

The solution u takes values in the Hilbert space[2] H, $\mathcal{E}$ is a given convex lower semi-continuous energy functional on H, $\partial\mathcal{E}$ denotes the convex subdifferential and f is assumed to be in $L^2_{\text{loc}}(I,H)$. The function $\mathcal{E} : H \to \mathbb{R}\cup\{\infty\}$ is called elliptic, provided there is $\omega > 0$, such that $u \mapsto \mathcal{E}(u) + \frac{\omega}{2}\|u\|_H^2$ is coercive. This is obviously satisfied, provided $\mathcal{E}$ is bounded from below.

A solution of the subgradient system (6.1) is a function

$$\begin{aligned}
&u \in W^{1,2}_{\text{loc}}(I;H), \text{ with}\\
&u(t) \in D(\partial\mathcal{E}) := \{u \in H\ ;\ \partial\mathcal{E}(u) \neq \emptyset\} \text{ for almost all } t \in I, \text{ and}\\
&\text{equation (6.1) holds for almost all } t \in I.
\end{aligned}$$

Since $W^{1,2}_{\text{loc}}(I;H)$-functions have continuous representatives, initial conditions

$$u(0) = u_0 \tag{6.2}$$

are well posed and according to [14] we have the following existence and uniqueness result.

Theorem 59. *Let H be a Hilbert space and $\mathcal{E} : H \to \mathbb{R}\cup\{\infty\}$ be convex and elliptic. Then for any $u_0 \in \operatorname{dom}(\mathcal{E})$ and any $f \in L^2(0,T;H)$ there exists a unique solution*

[1] Note that in literature subgradient systems are commonly just denoted "gradient systems". However, in our context the notion "gradient system" refers to evolution equations that have higher regularity as pointed out below.

[2] Our framework of gradient systems is based on the Hilbert space structure. There are several more general approaches to nonlinear evolutionary equations (e.g. in terms of accretive operators or in terms of gradient systems in metric spaces). However these concepts are more sophisticated and one can not expect the higher regularity results of solutions as in the case of (sub)gradient systems. Note that on the other hand the focus on Hilbert spaces is no essential restriction. We will see below that the Hilbert space structure does not lead to a restriction on the partial differential equation/inclusion we can solve, but on the regularity of initial data and the regularity of the right hand side f of the associated evolution equation that can be treated.

$u \in W^{1,2}(0,T;H)$ of the initial value problem

$$\begin{cases} \dot{u}(t) + \partial\mathcal{E}\big(u(t)\big) \ni f(t)\,, & t \in I \\ u(0) = u_0\,. \end{cases}$$

For the proof we refer to [14, Theorem 14.1]. Let us mention that the solution is obtained by an implicit Euler scheme. Moreover, the proof shows that for $f = 0$ the solution is obtained by the exponential formula

$$u(t) = \lim_{k\to\infty} J_{\frac{t}{k}}^k u_0$$

uniformly for $t \in [0,T]$, where the operator J_h is defined by

$$J_h g := \operatorname*{argmin}_{v\in H}\Big(\mathcal{E}(v) + \frac{\|v-g\|_H^2}{2h}\Big)\,. \tag{6.3}$$

Note that this is well defined due to the strict convexity of the functional on the right hand side and the ellipticity assumption on $\mathcal{E}$.

6.1.1 Invariance of Convex Sets

A set $C \subseteq H$ is called invariant under a subgradient system, provided for all $u_0 \in C$ and the corresponding solution u we have $u(t) \in C$ for any $t > 0$. Many properties of certain gradient systems can be reduced to the question, whether a closed, convex set C is invariant under a subgradient system. Fortunately there is an often easy to check condition, if a given closed, convex set C is invariant under a gradient system. For a closed convex set $C \subseteq H$ let $P_C u$ denote the best approximation (sometimes also called metric projection or orthogonal projection) of $u \in H$ on C.

Theorem 60. *Let $\mathcal{E} : H \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous, proper, convex, coercive function and $C \subseteq H$ be closed and convex. If*

$$\mathcal{E}(P_C u) \le \mathcal{E}(u)$$

for any $u \in H$, then the set C is invariant under the gradient system

$$\dot{u}(t) + \partial\mathcal{E}\big(u(t)\big) \ni 0\,. \tag{6.4}$$

If on the other hand $C \subseteq \overline{\operatorname{dom}(\mathcal{E})}$ and C is invariant under the gradient system (6.4),

then $\mathcal{E}(P_C u) \leq \mathcal{E}(u)$.

This result is due to Brézis [9, Proposition 4.5, p. 131]. A proof of the first statement is also given in [14, Theorem 15.3].

In many applications the Hilbert space is $L^2(\Omega)$. Let us give some examples of convex sets and the associated best approximation on those sets.

Positivity preserving systems

Let $H := L^2(\Omega)$ and let $C := \{u \in H \; ; \; u \geq 0 \text{ a.e.}\}$. If C is invariant under a subgradient system on H, then the system is called positivity preserving, i.e. nonnegative initial values lead to nonnegative solutions. In many diffusion problems u models a density and thus positivity preservation is a very natural physical property.

It is well known that the metric projection in H on C is given by

$$P_C u = u^+ \,,$$

where $u^+ := u\chi_{\{u \geq 0\}}$ denotes the positive part of u.

Thus it suffices to check if

$$\mathcal{E}(u^+) \leq \mathcal{E}(u)$$

for all $u \in H$ to verify positivity preservation of the gradient system (6.4) (cf. [14, Corollary 15.5]).

Order preserving systems

A subgradient system is called order preserving, provided pointwise ordered initial data have pointwise ordered solutions for all times $t > 0$. It is not immediate how this property can be expressed in terms of an invariant convex set. The key is to duplicate the gradient system and to add the two energy functionals.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two convex lower semicontinuous, elliptic functions defined on $L^2(\Omega)$. We can then define

$$\mathbf{E}(\mathbf{u}) = \mathcal{E}_1(u_1) + \mathcal{E}_2(u_2) \quad \text{for } \mathbf{u} = (u_1, u_2) \in \mathbf{H} := L^2(\Omega) \times L^2(\Omega)\,.$$

It is easily seen that $\partial\mathbf{E}(u_1, u_2) = \partial\mathcal{E}_1(u_1) \times \partial\mathcal{E}_2(u_2)$ and thus $\mathbf{u} = (u_1, u_2)$ is a solution

of the gradient system with inital value

$$\begin{cases} \dot{\mathbf{u}}(t) + \partial \mathbf{E}\big(\mathbf{u}(t)\big) \ni 0\,, \quad t \in [0,T] \\ \mathbf{u}(0) = (u_{0,1}, u_{0,2}) \end{cases}$$

if and only if u_j is a solution of the gradient system with inital value

$$\begin{cases} \dot{u}_j(t) + \partial \mathcal{E}_j\big(u_j(t)\big) \ni 0\,, \quad t \in [0,T] \\ u_j(0) = u_{0,j} \end{cases}$$

for $j = 1, 2$.

Thus order preservation of a subgradient system with energy $\mathcal{E} : L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ is equivalent to verify that the convex closed set

$$C := \{\mathbf{u} = (u_1, u_2) \in \mathbf{H}\ ;\ u_1 \leq u_2 \text{ a.e.}\}$$

is invariant under the gradient system

$$\dot{\mathbf{u}}(t) + \partial \mathbf{E}\big(u(t)\big) \ni 0\,, \quad t \in [0,T]\,,$$

with $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$.

Using a well known characterization of the best approximation in Hilbert spaces it is not too difficult to verify that the best approximation on C is given by

$$P_C(u_1, u_2) = (u_1 \wedge u_2, u_1 \vee u_2)\,,$$

where $\wedge$ and $\vee$ denote (pointwise) minima and maxima resp. (cf. [14, p. 169]).

Thus order preservation can be proved by verifying the inequality

$$\mathcal{E}(u_1 \wedge u_2) + \mathcal{E}(u_1 \vee u_2) \leq \mathcal{E}(u_1) + \mathcal{E}(u_2) \tag{6.5}$$

for any $(u_1, u_2) \in \mathbf{H}$.

Note that the projection above essentially relies on the metric in $L^2(\Omega)$.

Let us finally note that certain other properties of subgradient systems like contraction properties can also be proved in terms of invariance of closed convex sets. For a detailed description we refer to [14, Lecture 15].

6.2 Gradient Systems

In many applications a higher a priori regularity of subgradient systems is satisfied. Let us consider the following Gelfant triple framework (cf. also [60, Chapter 23] where the framework is called evolution triple):

Let H be a Hilbert space and let V be a reflexive separable Banach space and assume V is continuously and densely embedded in H. It is well known that in this case H', which is identified with H by the Riesz representation Theorem, embeds continuosly and densely in V'.

Moreover, let $\mathcal{E} : V \to \mathbb{R}$ be a continuously differentiable functional. Note that the derivative $\mathcal{E}'$ is a map from V to V' in general. The gradient of $\mathcal{E}$ in H is the mapping $\mathcal{E}'$ restricted in the image to $H' = H$, i.e.

$$\operatorname{dom}(\nabla_H \mathcal{E}) := \{u \in V \; ; \; \exists v =: \nabla_H \mathcal{E}(u) \in H \; \forall w \in V : \; \langle v, w\rangle_H = \langle \mathcal{E}'(u), w\rangle_{V',V}\} \, .$$

It is not difficult to see that for convex $\mathcal{E}$ and for the extended, convex, lower semicontinuous functional $\tilde{\mathcal{E}} : H \to \mathbb{R}$,

$$\tilde{\mathcal{E}}(u) := \begin{cases} \mathcal{E}(u) & \text{for } u \in V \\ \infty & \text{otherwise} \, , \end{cases} \tag{6.6}$$

the following connection between the subdifferential of $\tilde{\mathcal{E}}$ and the gradient of $\mathcal{E}$ holds:

$$\operatorname{dom}(\nabla_H \mathcal{E}) = \operatorname{dom}(\partial \tilde{\mathcal{E}}) \quad \text{and} \quad \partial \tilde{\mathcal{E}}(u) = \{\nabla_H \mathcal{E}(u)\} \text{ for all } u \in \operatorname{dom}(\nabla_H \mathcal{E}) \, .$$

A gradient system is an H-valued evolution equation of the form

$$\dot{u}(t) + \nabla_H \mathcal{E}(u(t)) = f(t), \; t \in I \, , \tag{6.7}$$

where I is some time interval and f is a given $L^2_{\text{loc}}(I; H)$-function.

A solution of (6.7) is a measurable function $u : I \to V$, such that

$$\begin{aligned} &u \in W^{1,2}_{\text{loc}}(I; H) \cap L^\infty(I; V) \, , \\ &u(t) \in D(\nabla_H \mathcal{E}) \text{ for almost all } t \in I, \text{ and} \\ &\text{equation (6.7) holds for almost all } t \in I. \end{aligned}$$

Thus solutions of gradient systems are obviously solutions of the associated subgradient system, where the energy functional $\tilde{\mathcal{E}}$ is given by (6.6). Moreover, due to

the embedding of $W^{1,2}(I;H)$ in $C(I;H)$ initial value problems for gradient systems are well defined. Before we state the following existence and uniqueness theorem for gradient systems we need to introduce a notion. A convex function $\mathcal{E} : V \to \mathbb{R}$ is said to be H-elliptic, provided there is an $\omega > 0$, such that $V \ni u \mapsto \mathcal{E}(u) + \frac{\omega}{2}\|u\|_H$ is coercive with respect to the norm in H.

Theorem 61. *Let V be a separable reflexive Banach space that is continuously and densely embedded into a Hilbert space H, and suppose that $\mathcal{E} : V \to \mathbb{R}$ is a convex, H-elliptic, continuously differentiable function and that $\mathcal{E}'$ maps bounded sets of V to bounded sets of V'. Then for all $T > 0$, $f \in L^2(0,T;H)$ and $u_0 \in V$ the gradient system with initial value*

$$\begin{cases} \dot{u} + \nabla_H \mathcal{E}(u) = f\,, \\ u(0) = u_0 \end{cases} \tag{6.8}$$

admits a unique solution $u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$. The solution can be chosen as a weakly continuous function $u\colon [0,T] \to V$, and for this function one has the energy inequality

$$\int_s^t \|\dot{u}(\tau)\|_H^2 \,\mathrm{d}\tau + \mathcal{E}\big(u(t)\big) \le \mathcal{E}\big(u(s)\big) + \int_s^t \langle f(\tau), \dot{u}(\tau)\rangle_H \,\mathrm{d}\tau\,, \tag{6.9}$$

for all $0 \le s \le t \le T$.

This is essentially Theorem 6.1 from [14]. Compare also [42] for the proof

6.3 Applications

Having this general framework at hand we will give three examples of (sub)gradient systems. The first one is the well known example of the p-Laplace evolution. The second example covers the PME and FDE without any restriction on the parameter m and no restriction on the regularity on the boundary of Ω. Even though constructions of solution for the PME/FDE are known for a long time, the higher regularity context as gradient system is new to the best knowledge of the author and was recently published in [42]. The third class of examples covers certain problems with 1-homogeneous energies that occure for example in the treatment of perfectly plastic fluids. This topic was recently also treated by Bildhauer, Naumann & Wolf, [7], however our approach is different to the construction of the aforementioned, who use an

approximation procedure of associated symmetrized p-Laplacian evolutions and the limiting process $p \to 1$.

6.3.1 The p-Laplace Evolution

Let $\Omega \subseteq \mathbb{R}^n$ be bounded[3] and let $H := L^2(\Omega)$. Let $p \geq \frac{2n}{n+2}$ and let $V := W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|Du\|_p}$, where $\|Du\|_p^p = \int_\Omega |Du|^p \,\mathrm{d}x$. Due to the usual Sobolev embedding we have $V \hookrightarrow H$ and the embedding is dense, since $C_c^\infty(\Omega)$-functions are also dense in $L^2(\Omega)$. Moreover, $W_0^{1,p}(\Omega)$ is separable. The dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,p'}(\Omega)$, where $1/p + 1/p' = 1$. $W^{-1,p'}(\Omega)$ can be characterized as the set of divergences of $L^{p'}(\Omega, \mathbb{R}^n)$-vector fields.

Let us define

$$\mathcal{E}_p(u) := \frac{1}{p} \int_\Omega |Du|^p \,\mathrm{d}x.$$

It is not difficult to verify that $\mathcal{E}_p$ is convex and continuously differentiable on V and the derivative of $\mathcal{E}_p$ in $u \in V$ is given by

$$\langle \mathcal{E}_p'(u), v \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} = \int_\Omega Du|Du|^{p-2} \cdot Dv \,\mathrm{d}x$$

for all $v \in W_0^{1,p}(\Omega)$. In particular the derivative of $\mathcal{E}_p$ in u is just $-\Delta_p u$ and since the p-Laplace operator is the normalized duality mapping between $W^{-1,p}(\Omega)$ and $W_0^1(\Omega)$ the derivative $\mathcal{E}_p'$ maps bounded sets on bounded sets (cf. [15, Proposition 4.12]). A detailed and elementary proof of the aforementioned results is given in [14, Chapter 4.3]. Since $\mathcal{E}_p$ is nonnegative, it is obviously H-elliptic.

Let us finally characterize the gradient $\nabla_H \mathcal{E}_p$ of $\mathcal{E}_p$ in $L^2(\Omega)$. By definition we have

$$\begin{aligned} \mathrm{dom}(\nabla_H \mathcal{E}_p) = \Big\{ u \in W_0^{1,p}(\Omega) \; ; \; \exists w =: \nabla_H \mathcal{E}_p(u) \in L^2(\Omega) : \\ \int_\Omega |Du|^{p-2} Du \cdot Dv \,\mathrm{d}x = \int_\Omega wv \,\mathrm{d}x \quad \text{for all } v \in W_0^{1,p}(\Omega) \Big\} \end{aligned}$$

In other words the effective domain of the gradient is the set of those $W_0^{1,p}(\Omega)$-functions, where $-\Delta_p u$, which is merely a distribution in general, has a representation as $L^2(\Omega)$-function.

Thus by application of Theorem 61 we obtain the following existence and uniqueness result.

[3] Note that there is no boundary regularity needed for this approach, in particular the embedding $W_0^{1,p}(\Omega)$ in $L^2(\Omega)$ need not be compact.

Theorem 62. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded and $T > 0$. For any $f \in L^2(0,T;L^2(\Omega))$ and any $u_0 \in W_0^{1,p}(\Omega)$ the gradient system with initial value*

$$\begin{cases} \dot{u}(t) - \Delta_p\, u(t) = f(t)\,, \\ u(0) = u_0 \end{cases} \tag{6.10}$$

admits a unique solution $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W_0^{1,p}(\Omega))$. The solution is weakly continuous as mapping $u\colon [0,T] \to W_0^{1,p}(\Omega)$, and for this function one has the energy inequality

$$\int_s^t \|\dot{u}(\tau)\|_2^2 \,\mathrm{d}\tau + \tfrac{1}{p}\|Du(t)\|_p^p \le \tfrac{1}{p}\|Du(s)\|_p^p + \int_s^t \langle f(\tau), \dot{u}(\tau)\rangle_{L^2}\,\mathrm{d}\tau\,, \tag{6.11}$$

for all $0 \le s \le t \le T$.

Let us finally note that the energy inequality can be used to prove asymptotic decay properties of the parabolic p-Laplace operator and the validity of order preservation can be verified with the arguments from Subsection 6.1.1.

6.3.2 The Porous Medium and the Fast Diffusion Equation

For $m \in \mathbb{R}_{>0}$ the porous medium equation/fast diffusion equation is given by

$$\dot{u}(t,x) - \Delta\, u(t,x)^m = f(t,x) \quad \text{in } (0,T) \times \Omega\,.$$

For $m = 1$ we obtain the well known linear heat equation, for $m > 1$ the equation is called porous medium equation and for $m < 1$ it is called fast diffusion equation. In applications u usually models some density or other nonegative quantity, however below we will define and consider signed solutions u, too.

The calculation

$$-\Delta\, u^m = -\operatorname{div} m u^{m-1} Du$$

shows that in this model the diffusion coefficient is given by mu^{m-1}. In particular the diffusivity depends monotonously on the density u. For $m > 1$ the coefficient is small, when the density is small and large, when the density is large. This is a reasonable assumption e.g. in porous media where, due to sorption powers, the diffusion of some infiltrating fluid is smaller as long as it can still cover the surface of the porous media and it gets higher, the more saturated the surface of the porous medium is.

For $m < 1$ we expect the opposite behavior, the smaller the density, the larger

the diffusion. This can e.g. be observed in plasma physics, where molecular powers suppress diffusion, the higher the density of the plasma is.

Note that due to the fact that the nonlinearity $u \mapsto u^m$ is applied before the differential operator, it is not immediate, which is a suitable space where one should seek for solutions. Moreover, the PME and FDE are prototypes of semilinear differential operators, which are either, degenerate for $u = 0$ and singular at infintiy for $m > 1$ or singular for $u = 0$ and degenerate at infinity in the FDE-case. Thus the classical approach requires different techniques to construct solutions in the PME and the FDE case and the consideration of signed solutions is more complicated (cf. [57, Chapter 5.5]). A different approach to the PME/FDE is in terms of accretive or maximal monotone operators in $H^{-1}(\Omega)$, however these approaches usually neglect some of the structure of the equation and one usually assumes some regularity of the boundary of Ω and boundedness of Ω to get a suitable compact Sobolev embedding. This results in restrictions on the parameter m from below[4]. It was an observation of the author that the PME/FDE can be considered as gradient system in the Hilbert space $H^{-1}(\Omega)$. This allows to derive very elegant existence and uniqueness results for the PME/FDE in a simultaneous fashion, without regularity assumptions on $\partial\Omega$, with very weak assumptions on Ω and without technical restrictions on the parameter m. Furthermore we will also allow signed solutions and thus introduce for $r \in \mathbb{R}$ the following notation

$$r^{[m]} := \operatorname{sgn}(r)|r|^m ,$$

i.e. $r \mapsto r^{[m]}$ is the odd extension of the mapping $[0, \infty) \ni r \mapsto r^m$ on $\mathbb{R}$.

In the following let $\Omega \subseteq \mathbb{R}^n$ be open and assume that the Poincaré inequality holds on Ω, i.e. there is a constant $C_P > 0$, such that

$$\|u\|_2 \leq C_P \|Du\|_2$$

for all $u \in C_c^\infty(\Omega)$. This is satisfied, provided Ω does not contain, loosely speaking, arbitrary large balls. In particular it is sufficient that

$$\begin{aligned}\rho(\Omega) := \sup\{R > 0\,;\ &\text{there exists a ball } B \subseteq \mathbb{R}^n \text{ with radius } R,\\ &\text{such that } B \cap (\mathbb{R}^n \setminus \Omega) \text{ contains no interior point}\}\end{aligned}$$

is finite (see [54]). As usual define $H_0^1(\Omega)$ to be the closure of $C_c^\infty(\Omega)$ with respect to

[4] Usually $m > \frac{n-2}{n+2}$

the $H^1(\Omega)$-norm $\|u\|_2 + \|Du\|_2$. However due to the Poincaré inequality we will use the equivalent norm $\|Du\|_2$ in the following. It is well known that this is a Hilbert space norm for the scalar product

$$\langle u, v\rangle_{H_0^1} = \int_\Omega Du \cdot Dv \,\mathrm{d}x\,. \tag{6.12}$$

The space $H^{-1}(\Omega)$ is defined to be the dual space of $H_0^1(\Omega)$. Usual characterizations state that it can be considered as set of distributions being the sum of a regular distribution induced from an $L^2(\Omega)$-function and the distributional divergence of an $L^2(\Omega, \mathbb{R}^n)$ vector field (cf. [26, p. 283, Theorem 1]). However, for our alternative norm we have a slightly different characterization.

Lemma 63. *The mapping $-\Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism (and in fact the Riesz mapping) between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. In particular, $H^{-1}(\Omega)$ can be identified with the set of all distributions of the form $-\Delta v$, for $v \in H_0^1(\Omega)$.*

Cf. [42, Lemma 1.3] for a proof of the Lemma.

In the following let

$$G := (-\Delta)^{-1}$$

denote the inverse of $-\Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega)$. Then the scalar product in $H^{-1}(\Omega)$ can be written as

$$\langle u, v\rangle_{H^{-1}} = \langle Gu, Gv\rangle_{H_0^1} = \int_\Omega \nabla Gu \cdot \nabla Gv \,\mathrm{d}x\,,$$

because G is an isometry.

For $m \in (0, \infty)$ define[5] $\mathcal{E} : L^{m+1}(\Omega) \to \mathbb{R}$ by

$$\mathcal{E}(u) := \frac{1}{m+1} \int_\Omega |u|^{m+1} \,\mathrm{d}x\,.$$

It is easy to see (cf. i.e. [42, Proposition 1.4]) that $\mathcal{E}$ is convex, continuously differentiable and the derivative is characterized by

$$\langle \mathcal{E}'(u), v\rangle = \int_\Omega u^{[m]}\, v \,\mathrm{d}x = \langle u^{[m]}, v\rangle_{L^{1+1/m}, L^{m+1}} \tag{6.13}$$

[5] The case when $m = 0$ turns out to be not differentiable, however, if fits in the framework of a subgradient system and can be treated in a similar manner.

for all $v \in L^{m+1}(\Omega)$. Moreover, the identity

$$\|\mathcal{E}'(u)\|_{1+1/m} = \Big(\int_\Omega |u^{[m]}|^{\frac{m+1}{m}} \Big)^{\frac{m}{m+1}} = \|u\|_{m+1}^m \tag{6.14}$$

shows that $\mathcal{E}'$ maps bounded sets of $L^{m+1}(\Omega)$ to bounded sets of $L^{1+1/m}(\Omega)$.

The following proposition (which is stated in this form in [42] as Proposition 1.5) answers the question when $L^{m+1}(\Omega)$ embeds to $H^{-1}(\Omega)$.

Proposition 64. *Let $\Omega \subseteq \mathbb{R}^n$ be open and assume that the Poincaré inequality* (6.12) *holds on Ω. Then:*

(i) For all

$$m \in (0,1] \cap \left[\tfrac{n-2}{n+2}, \infty\right)$$

the space $L^{m+1}(\Omega)$ embeds continuously and densely into $H^{-1}(\Omega)$.

(ii) If moreover the measure of Ω is finite the embedding $L^{m+1}(\Omega) \hookrightarrow H^{-1}(\Omega)$ is continuous and dense even for

$$m \in (0,\infty) \cap \left[\tfrac{n-2}{n+2}, \infty\right).$$

For the proof we refer to [42, Proposition 1.5]

Note that for bounded Ω the continuous embeddings above (except for $m = \frac{n-2}{n+2}$) are even compact, a fact that we will not use. However this is needed in certain alternative solution techniques for the PME/FDE and is the reason for the restriction on the parameter m in these treatments.

It remains to ask, whether we may get rid of the restriction on the parameter m at all? In fact this can be done by considering the Banach space (the intersection is well defined, since both spaces embed in the space of distributions)

$$V := L^{m+1} \cap H^{-1}(\Omega) := L^{m+1}(\Omega) \cap H^{-1}(\Omega)$$

with the norm

$$\|u\|_V := \|u\|_{m+1} + \|u\|_{H^{-1}}$$

This space is separable and reflexive because it is isomorphic to the closed subspace $\{(u,u)\ ;\ u \in L^{m+1} \cap H^{-1}(\Omega)\}$ of the product space $L^{m+1}(\Omega) \times H^{-1}(\Omega)$. Moreover, the obvious embedding $L^{m+1} \cap H^{-1}(\Omega) \hookrightarrow H^{-1}(\Omega)$ is dense since $C_c^\infty(\Omega)$-functions

are contained in $L^{m+1} \cap H^{-1}(\Omega)$ and dense in $H^{-1}(\Omega)$. Thus the restriction of $\mathcal{E}$ to $L^{m+1} \cap H^{-1}(\Omega)$ is continuously differentiable and

$$\big(\mathcal{E}|_{L^{m+1}\cap H^{-1}}\big)' : L^{m+1} \cap H^{-1}(\Omega) \to \big(L^{m+1} \cap H^{-1}(\Omega)\big)'$$

maps bounded sets to bounded sets by 6.13 and 6.14.

Without danger of confusion we will use the same letter $\mathcal{E}$ for the functional defined on $V = L^{m+1} \cap H^{-1}(\Omega)$ below and recall the identiy $L^{m+1} \cap H^{-1}(\Omega) = L^{m+1}(\Omega)$ for the cases given in Proposition 64.

Let us finally calculate the gradient of $\mathcal{E}$ in the Hilbert space $H^{-1}(\Omega)$.

By definition for $u \in \operatorname{dom}(\nabla_H \mathcal{E})$ and all $v \in L^{m+1} \cap H^{-1}(\Omega)$ we have

$$\begin{aligned}\langle \mathcal{E}'(u), v\rangle_{V',V} = \int_\Omega u^{[m]} v \,\mathrm{d}x &= \langle \nabla_H \mathcal{E}(u), v\rangle_{H^{-1}} \\ &= \int_\Omega DG\nabla_H \mathcal{E}(u) \cdot DGv \,\mathrm{d}x\,.\end{aligned}$$

Considering $v \in C_c^\infty(\Omega)$ we immediately obtain $G\nabla_H \mathcal{E}(u) = u^{[m]}$ in the sense of distributions:

Proposition 65. *There holds $u \in \operatorname{dom}(\nabla_H \mathcal{E})$ if and only if $u^{[m]} \in H_0^1(\Omega)$. Moreover in that case $\nabla_H \mathcal{E}(u) = -\Delta\, u^{[m]}$ in the sense of distributions.*

Thus, by application of Theorem 61 we derive the following existence and uniqueness result for the PME/FDE.

Theorem 66. *Let $\Omega \subseteq \mathbb{R}^n$ be open and such that a Poincaré inequality (6.12) holds on Ω, and let $m \in (0,\infty)$. Then for all $T > 0$, $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^{m+1} \cap H^{-1}(\Omega)$ the PME/FDE gradient system*

$$\begin{cases} \dot u - \Delta\, u^{[m]} = f \\ u(0,\cdot) = u_0 \end{cases} \tag{6.15}$$

admits a unique solution $u \in W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;L^{m+1} \cap H^{-1}(\Omega))$. The solution can be chosen as a weakly continuous mapping $u\colon [0,T] \to L^{m+1} \cap H^{-1}(\Omega)$, and for this mapping one has the energy inequality

$$\int_s^t \|\dot u(\tau)\|_{H^{-1}}^2 \,\mathrm{d}\tau \;+\; \mathcal{E}\big(u(t)\big) \;\le\; \mathcal{E}\big(u(s)\big) \;+\; \int_s^t \langle f(\tau), \dot u(\tau)\rangle_{H^{-1}} \,\mathrm{d}\tau\,, \tag{6.16}$$

for all $0 \le s \le t \le T$. *(Recall that* $L^{m+1} \cap H^{-1}(\Omega)$ *reduces to* $L^{m+1}(\Omega)$*, provided the embedding* $L^{m+1}(\Omega) \hookrightarrow H^{-1}(\Omega)$ *holds; cf. Proposition 64.)*

Remark 67. *(a) The property that* $u(t) \in \operatorname{dom}(\nabla_H \mathcal{E})$*, or equivalently, that* $u(t)^{[m]} \in H_0^1(\Omega)$*, is a weak replacement for* $u(t)|_{\partial\Omega} = 0$.

(b) Note that elements of the domain of $\nabla_H \mathcal{E}$ *are regular distributions and thus have a reasonable physical interpretation. In particular we can consider* u *as element of* $L^1_{\mathrm{loc}}([0,T] \times \Omega)$.

(c) For a more detailed analysis it might be helpful to get a better understanding of the structure of $\operatorname{dom}(\nabla_H \mathcal{E})$. *Note that by the invertibility of* $-\Delta \colon H_0^1(\Omega) \to H^{-1}(\Omega)$ *we obtain that*

$$\begin{aligned} \operatorname{dom}(\nabla_H \mathcal{E}) &= \left\{ u \in L^{m+1} \cap H^{-1}(\Omega) \; ; \; -\Delta\, u^{[m]} \in H^{-1}(\Omega) \right\} \\ &= \left\{ u \in L^{m+1} \cap H^{-1}(\Omega) \; ; \; u^{[m]} \in H_0^1(\Omega) \right\}. \end{aligned} \tag{6.17}$$

Moreover, recall that $u \mapsto u^{[m]}$ *is the duality map* $L^{m+1} \to L^{(m+1)'} = L^{\frac{m+1}{m}}$. *This continuous nonlinear map is bijective and we have that*

$$\langle u, u^{[m]} \rangle_{L^{m+1}, L^{\frac{m+1}{m}}} = \|u\|_{m+1}^{m+1} \quad \text{and} \quad \|u^{[m]}\|_{\frac{m+1}{m}} = \|u\|_{m+1}^{m},$$

and the inverse of $u \mapsto u^{[m]}$ *is given by* $v \mapsto v^{[1/m]}$. *(For details on duality mappings we refer to [15, Chapter II, Sec. 4].) If* Ω *and* m *are such that the embedding* $L^{m+1}(\Omega) \hookrightarrow H^{-1}(\Omega)$ *holds (cf. Proposition 64), this allows to rewrite*

$$\begin{aligned} \operatorname{dom}(\nabla_H \mathcal{E}) &= \left\{ u \in L^{m+1}(\Omega) \; ; \; u^{[m]} \in H_0^1(\Omega) \right\} \\ &= \left\{ v^{[1/m]} \; ; \; v \in L^{\frac{m+1}{m}}(\Omega) \cap H_0^1(\Omega) \right\}. \end{aligned}$$

Note that $L^{m+1}(\Omega) \hookrightarrow H^{-1}(\Omega)$ *if and only if* $H_0^1(\Omega) \hookrightarrow L^{\frac{m+1}{m}}(\Omega)$ *(see the proof of Proposition 64), and then*

$$\operatorname{dom}(\nabla_H \mathcal{E}) = \left\{ v^{[1/m]} \; ; \; v \in H_0^1(\Omega) \right\}.$$

Littig & Voigt showed in [42], how the energy inequality (6.16) can be used to derive the asymptotic decay of solutions of the PME/FDE. In particular we assume $f = 0$ and that the embedding $L^{m+1}(\Omega) \hookrightarrow H^{-1}(\Omega)$ holds (cf. Proposition 64). Let μ_1 denote the square root of the inverse of the embedding constant from $L^{m+1}(\Omega) \hookrightarrow H^{-1}(\Omega)$. Then for any solution u of the initial value problem (6.15) the energy $\mathcal{E}(u(t))$, $t > 0$

is dominated by

$$G_m(t) := \begin{cases} \left(\left(\mathcal{E}(u_0)^{\frac{1-m}{1+m}} - \mu_1 (1-m)(m+1)^{-\frac{1-m}{1+m}} t\right)^+\right)^{\frac{1+m}{1-m}} & \text{if } m < 1\,, \\ \mathcal{E}(u_0)\, e^{-2\mu_1 t} & \text{if } m = 1\,, \\ \left(\mu_1 (m-1)(m+1)^{\frac{m-1}{m+1}} t + \mathcal{E}(u_0)^{-\frac{m-1}{m+1}}\right)^{-\frac{m+1}{m-1}} & \text{if } m > 1\,. \end{cases}$$

The $(\cdot)^+$ - notation in the case $m < 1$ indicates that we take the positive part, and therefore $G_m(t) = 0$ for all

$$t \geq t_{\max} := \frac{(m+1)^{\frac{1-m}{1+m}}}{\mu_1 (1-m)} \mathcal{E}(u_0)^{\frac{1-m}{1+m}}\,.$$

In particular we obtain a decay of the energy of the solution

(1) polynomially with decay rate $-\frac{m+1}{m-1}$, for the PME,

(2) exponentially with decay rate $-2\mu_1$, for the heat equation

(3) in finite time, in case of fast diffusion.

In [42] it is also shown that the PME/FDE is order preserving. Remarkably this was done by straightforward calculations, but not by application of Theorem 60 in favor of Subsection 6.1.1. The challenge is that the best approximation in $H^{-1}(\Omega)$ on, lets say the set of positive distributions for simplicity, is very hard to calculate and an open question at the moment.

6.3.3 Subgradient Systems with 1-Homogeneous Energies

For this subsection let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Let $N \in \mathbb{N}$ and define

$\mathcal{E}_{TV} : L^2(\Omega, \mathbb{R}^N) \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{E}_{TV}(u) = \begin{cases} \int_\Omega \mathrm{d}|Du| + \int_\Omega |u^{\partial\Omega} \otimes \nu| \,\mathrm{d}\mathcal{H}^{n-1} & u \in BV(\Omega, \mathbb{R}^N) \cap L^2(\Omega, \mathbb{R}^N) \\ \infty & \text{otherwise} \end{cases} \tag{6.18}$$

and $\mathcal{E}_{TD} : L^2(\Omega, \mathbb{R}^n) \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{E}_{TD}(u) = \begin{cases} \int_\Omega \mathrm{d}|D_s u| + \int_\Omega |u^{\partial\Omega} \odot \nu| \,\mathrm{d}\mathcal{H}^{n-1} & u \in BD(\Omega) \cap L^2(\Omega, \mathbb{R}^n) \\ \infty & \text{otherwise} \end{cases} \tag{6.19}$$

By Proposition 33 and Proposition 47 the functions $\mathcal{E}_{TV}$ and $\mathcal{E}_{TD}$ are convex and lower semicontinuous on $L^1(\Omega, \mathbb{R}^N)$ and $L^1(\Omega, \mathbb{R}^n)$ resp., and by boundedness of Ω also on $L^2(\Omega, \mathbb{R}^N)$ and $L^2(\Omega, \mathbb{R}^n)$ resp.

The subdifferentials of $\mathcal{E}_{TV}$ and $\mathcal{E}_{TD}$ have been derived in Theorem 56:

$$w^* \in \partial\mathcal{E}_{TV}(u) \Leftrightarrow \exists z \in L^\infty(\Omega, \mathbb{R}^{N\times n}) \text{ with } \|z\|_\infty \leq 1, \quad w^* = -\operatorname{div} z \in L^2(\Omega, \mathbb{R}^N)$$
$$\text{and} \int_\Omega w^* u \,\mathrm{d}x = \mathcal{E}_{TV}(u)\,.$$

Thus we may apply Theorem 59 with $H = L^2(\Omega)$ to obtain

Theorem 68. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded with Lipschitz boundary and let $\mathcal{E}_{TV} : L^2(\Omega, \mathbb{R}^N) \to \mathbb{R} \cup \{\infty\}$ be defined as in* (6.18)*. Then for all $u_0 \in BV(\Omega, \mathbb{R}^N)$ and all $f \in L^2(0, T; L^2(\Omega, \mathbb{R}^N))$ there is a unique solution $u \in W^{1,2}(0, T; L^2(\Omega, \mathbb{R}^N))$ of the initial value problem*

$$\begin{cases} \dot{u}(t) + \partial\mathcal{E}_{TV}(u(t)) \ni f(t)\,, & t \in [0, T] \\ u(0) = u_0\,. \end{cases} \tag{6.20}$$

Recall that the subdifferential of $\mathcal{E}_{TV}$ is a well defined substitute for the 1-Laplace operator. Thus, in other words our theorem proves the existence of unique solutions for the Dirichlet problem of the vectorial total variation flow. Obviously for $N = 1$ we recover the existence result for the scalar total variation flow (cf. [4, Chapter 5]). To give an application of the subgradient system notion let us state and prove the comparison principle for the total variation flow, which seems to be new.

Theorem 69. *The scalar (i.e. $N = 1$) total variation flow from Theorem 68 is order preserving, i.e. if u_{01}, $u_{02} \in BV(\Omega) \cap L^2(\Omega)$ with $u_{01} \leq u_{02}$ and if u_1 and u_2 are the solutions of* (6.20) *with $f = 0$ for the initial values u_{01} and u_{02} resp., then*

$$u_1(t) \leq u_2(t)$$

for $t \in [0, T]$.

Proof. Let u_1, $u_2 \in BV(\Omega)$. According to (6.5) we need to verify

$$\mathcal{E}_{TV}(u_1 \wedge u_2) + \mathcal{E}_{TV}(u_1 \vee u_2) \leq \mathcal{E}_{TV}(u_1) + \mathcal{E}_{TV}(u_2) \tag{6.21}$$

By the coarea formula from Proposition 35 the above condition is equivalent to

$$\int_{\mathbb{R}} \mathrm{Per}(\hat{E}_{1,2,t}) + \mathrm{Per}(\check{E}_{1,2,t})\,\mathrm{d}t \leq \int_{\mathbb{R}} \mathrm{Per}(E_{1,t}) + \mathrm{Per}(E_{2,t})\,\mathrm{d}t\,, \tag{6.22}$$

with

$$E_{j,t} := \begin{cases} \{u_j > t\} & \text{for } t > 0 \\ \{u_j < t\} & \text{for } t < 0\,, \end{cases}$$

for $j = 1, 2$,

$$\begin{aligned} \hat{E}_{1,2,t} :&= \begin{cases} \{u_1 \wedge u_2 > t\} & \text{for } t > 0 \\ \{u_1 \wedge u_2 < t\} & \text{for } t < 0 \end{cases} \\ &= \begin{cases} E_{1,t} \cap E_{2,t} & \text{for } t > 0 \\ E_{1,t} \cup E_{2,t} & \text{for } t < 0\,, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \check{E}_{1,2,t} :&= \begin{cases} \{u_1 \vee u_2 > t\} & \text{for } t > 0 \\ \{u_1 \vee u_2 < t\} & \text{for } t < 0 \end{cases} \\ &= \begin{cases} E_{1,t} \cup E_{2,t} & \text{for } t > 0 \\ E_{1,t} \cap E_{2,t} & \text{for } t < 0\,. \end{cases} \end{aligned}$$

For almost every $t \in \mathbb{R}$ the sets $E_{1,t}$ and $E_{2,t}$ (and thus also $E_{1,t} \cap E_{2,t}$ and $E_{1,t} \cup E_{2,t}$) are sets of finite perimeter. We may thus estimate for almost every $t \in \mathbb{R}$

$$\mathrm{Per}(E_{1,t} \cap E_{2,t}, \mathbb{R}^n) + \mathrm{Per}(E_{1,t} \cup E_{2,t}, \mathbb{R}^n) \leq \mathrm{Per}(E_{1,t}, \mathbb{R}^n) + \mathrm{Per}(E_{2,t}, \mathbb{R}^n)$$

by [31, Lemma 15.1] which proves the assertion in (6.22). □

Analogously to Theorem 68 we get the following existence theorem for the total deformation flow.

Theorem 70. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded with Lipschitz boundary and let $\mathcal{E}_{TD} : L^2(\Omega, \mathbb{R}^N) \to \mathbb{R} \cup \{\infty\}$ be defined as in (6.19). Then for all $u_0 \in BD(\Omega)$ and all $f \in L^2(0, T; L^2(\Omega, \mathbb{R}^n))$ there is a unique solution $u \in W^{1,2}(0, T; L^2(\Omega, \mathbb{R}^n))$ of the*

initial value problem

$$\begin{cases} \dot{u}(t) + \partial\mathcal{E}_{TD}(u(t)) \ni f(t)\,, \quad t \in I \\ u(0) = u_0\,. \end{cases} \tag{6.23}$$

Since the subdifferential of $\mathcal{E}_{TD}$ is a well defined replacement for the formal symmetrized 1-Laplace operator $-\operatorname{div}\frac{D_s u}{|D_s u|}$ this theorem states the existence of unique solutions for the Dirichlet problem of the symmetrized total variation flow.

Let us remark that in many problems of plasticity one additionally assumes incompressibility of the material, which is expressed by the additional constraint

$$\operatorname{div} u = 0\,.$$

In classical approaches this constraint is quite tricky to handle for analytical reasons. Many approximation techniques can not ensure that the requirement $\operatorname{div} u = 0$ is satisfied for approximating functions (cf. [7] for certain results in this direction). Nevertheless there is a very elegant trick to include this constraint in the subgradient models above with the aid of the Helmholz decomposition and a standard idea from Convex Analysis.

Before we continue let us recall the following results from Sohr [53].

Let $n \geq 2$ in the following. The space of solenoidal test functions is defined by

$$C^\infty_{c,\sigma}(\Omega,\mathbb{R}^n) := \{\varphi \in C^\infty_c(\Omega,\mathbb{R}^n)\ ;\ \operatorname{div}\varphi = 0\} \tag{6.24}$$

and

$$L^2_\sigma(\Omega) := \overline{C^\infty_{c,\sigma}(\Omega,\mathbb{R}^n)}^{\|\cdot\|_2}$$

is the closure of this space in $L^2(\Omega,\mathbb{R}^n)$. Furthermore we define

$$G(\Omega) := \{u \in L^2(\Omega,\mathbb{R}^n)\ ;\ \exists \mathbf{p} \in L^2_{\mathrm{loc}}(\Omega):\ u = D\mathbf{p}\}\,,$$

where D denotes the usual gradient operator and the equality is interpreted in the weak sense. These spaces are orthogonal to each other,

$$G(\Omega) = \{u \in L^2(\Omega,\mathbb{R}^n)\ ;\ \langle u, v\rangle_{L^2} = 0 \text{ for all } v \in L^2_\sigma(\Omega)\}\,,$$

and each $u \in L^2(\Omega,\mathbb{R}^n)$ has a unique decomposition $u = u_0 + D\mathbf{p}$ with $u_0 \in L^2_\sigma(\Omega)$,

$D\mathbf{p} \in G(\Omega)$ and $\langle u_0, D\mathbf{p}\rangle_{L^2} = 0$, such that $\|u\|_2^2 = \|u_0\|_2^2 + \|D\mathbf{p}\|_2^2$ (cf. [53, p. 81]). The operator $\mathbf{P} : L^2(\Omega, \mathbb{R}^n) \to L^2_\sigma(\Omega)$,

$$\mathbf{P}u = u_0$$

is called the Helmholz projection.

The previous results hold for arbitrary open $\Omega \subseteq \mathbb{R}^n$. However, provided Ω is bounded with Lipschitz boundary we have a more detailed characterization of $L^2_\sigma(\Omega)$ and $G(\Omega)$ (cf. [53, p. 83]).

Lemma 71. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then*

$$L^2_\sigma(\Omega) = \{u \in L^2(\Omega, \mathbb{R}^n)\ ;\ \operatorname{div} u = 0,\ [u \cdot \nu]^{\partial\Omega} = 0\}\,,$$

where $[u \cdot \nu]^{\partial\Omega}$ denotes the normal trace of u on $\partial\Omega$ (which exists in $W^{-1/2,2}(\partial\Omega)$, cf. [53, p. 50f], and

$$G(\Omega) = \{u \in L^2(\Omega, \mathbb{R}^n)\ ;\ \exists \mathbf{p} \in L^2(\Omega) \text{ with } u = D\mathbf{p}\}\,.$$

Thus we intend to include the constraint by adding $I_{L^2_\sigma}$ to the energy functional, i.e. we consider the convex, nonnegative, lower semicontinuous energy functional

$$\mathcal{E}_{TD,\sigma}(u) := \mathcal{E}_{TD}(u) + I_{L^2_\sigma(\Omega)}(u)$$

for $u \in L^2(\Omega, \mathbb{R}^n)$. In order to derive the subdifferential of $\mathcal{E}_{TD,\sigma}$ let us calculate the subdifferential of the indicator functional $I_{L^2_\sigma(\Omega)}$ first. Let $u^* \in L^2(\Omega, \mathbb{R}^n)$, then

$$\begin{aligned}
I^*_{L^2_\sigma}(u^*) &= \sup_{u \in L^2(\Omega,\mathbb{R}^n)} \langle u^*, u\rangle_{L^2} - I_{L^2_\sigma}(u) \\
&= \sup_{u \in L^2_\sigma(\Omega,\mathbb{R}^n)} \langle u^*, u\rangle_{L^2} - I_{L^2_\sigma}(u) \\
&= \begin{cases} 0 & \text{if } u^* \in G(\Omega) \\ \infty & \text{if } \mathbf{P}u^* \neq 0 \end{cases} \\
&= I_{G(\Omega)}(u^*).
\end{aligned}$$

In particular by the Fenchel identity (3.3) we derive that $\partial I_{L^2_\sigma(\Omega)}(u) = G(\Omega)$ for any $u \in L^2_\sigma(\Omega) \setminus \{0\}$ by orthogonality of the spaces $G(\Omega)$ and $L^2_\sigma(\Omega)$. Note that by the

sum rule of convex analysis we have

$$\partial\mathcal{E}_{TD,\sigma}(u) \supseteq \partial\mathcal{E}_{TD}(u) + \partial I_{L^2_\sigma(\Omega)}(u)$$

for all $u \in L^2(\Omega, \mathbb{R}^n)$. However, since neither $I_{L^2_\sigma}$ nor $\mathcal{E}_{TD,\sigma}$ is continuous on $L^2(\Omega, \mathbb{R}^n)$ equality in the above inclusion is not to expect and in order to characterize which differential inclusion is solved by Theorem 72 below, we are actually interested in a superset of $\partial\mathcal{E}_{TD,\sigma}$. Nevertheless, we can at least characterize the subdifferential of $\mathcal{E}_{TD,\sigma}$ in terms of a closure of the Minkowski sum of certain sets. In order to do so we define

$$M^* := \{v^* \in L^2(\Omega, \mathbb{R}^n)\ ;\ \exists z \in L^\infty(\Omega, \mathbb{R}^{n\times n}),\ \|z\|_\infty \le 1,\ v^* = -\operatorname{div} z\}$$

and have

$$\mathcal{E}_{TD}(u) = I^*_{M^*}(u) = \sup_{u^*\in M^*} \langle u^*, u\rangle_{L^2}$$

for any $u \in L^2(\Omega, \mathbb{R}^n)$ by (5.25). We thus derive

$$\begin{aligned}\mathcal{E}_{TD,\sigma}(u) &= \sup_{v^*\in M^*} \langle v^*, u\rangle_{L^2} + \sup_{w^*\in G(\Omega)} \langle w^*, u\rangle_{L^2}\\ &= \sup_{w^*\in M^*+G(\Omega)} \langle w^*, u\rangle_{L^2}\\ &= I_{M^*+G(\Omega)}(u)\,.\end{aligned}$$

The Minkowski sup $M^* + G(\Omega)$ is obviously convex but not necessarily closed in $L^2(\Omega, \mathbb{R}^n)$. However, with the closed and convex set

$$N^* := \overline{M^* + G(\Omega)}^{L^2(\Omega,\mathbb{R}^n)}$$

we thus derive

$$\mathcal{E}_{TD,\sigma} = I_{N^*}\,.$$

From that we obtain by the Fenchel identity (3.3)

$$u^* \in \partial\mathcal{E}_{TD,\sigma}(u) \quad\Leftrightarrow\quad u^* \in N^*,\ u \in BD(\Omega)\cap L^2_\sigma(\Omega),\ \text{and}\ \langle u^*, u\rangle_{L^2} = \mathcal{E}_{TD}(\Omega)$$

for all u, $u^* \in L^2(\Omega, \mathbb{R}^n)$.

Summarizing the previous results we obtain the following existence and uniqueness result for the flow of incompressible perfectly plastic fluids.

Theorem 72. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded and with Lipschitz boundary and let $T > 0$. Then for all $u_0 \in L^2_\sigma(\Omega) \cap BD(\Omega)$ and all $f \in W^{1,2}(0,T;L^2(\Omega,\mathbb{R}^n))$ there exists a unique solution of the gradient system with initial value*

$$\begin{cases} \dot{u}(t) + \partial\mathcal{E}_{TD,\sigma}(u(t)) \ni f(t)\,, \quad t \in I \\ u(0) = u_0\,. \end{cases} \tag{6.25}$$

Let us note that we have thus found a precise existence and uniqueness result for the initial value problem formally given by

$$\begin{cases} \partial_t u(t,x) - \operatorname{div} \frac{D_s u(t,x)}{|D_s u(t,x)|} = f(t,x) & \text{for } (t,x \in (0,T) \times \Omega) \\ u(0,x) = u_0(x) & \text{for } x \in \Omega \\ \operatorname{div} u(t,x) = 0 & \text{for } (t,x) \in (0,T) \times \Omega\,. \end{cases} \tag{6.26}$$

7 Appendix

The final chapter summarizes some general results needed for our derivations. In three sections we will concentrate on basics from measure theory, linear algebra and topological indices.

7.1 Basics from Measure Theory

Let X be a locally compact separable metric space and μ is a $\mathbb{R}^m$-valued measure on X (cf.[3, p. 2] for definition).

Proposition 73. *(a) The* total variation *functional $|\mu|$ on the Borel sets of X, defined by*

$$|\mu|(E) := \sup\left\{\sum_{k=1}^{\infty}|\mu(E_k)|\ ;\ E_k \subseteq X \text{ Borel, pairwise disjoint, } E = \bigcup_{k=1}^{\infty} E_k\right\},$$

is a positive finite measure on X and μ is absolutely continuous with respect to $|\mu|$, i.e.

$$|\mu|(B) = 0 \Rightarrow \mu(B) = 0\,.$$

(b) There is a unique $\mathbb{S}^{m-1}$-valued function $f \in L^1(X, \mathbb{R}^m, |\mu|)$, such that

$$\mu = f|\mu|$$

(c) If μ is additionally Radon regular, then for all open sets $A \subseteq X$ we have

$$|\mu|(A) = \sup\left\{\int_X u \cdot \mathrm{d}\mu\ ;\ u \in C_c(A)^m, \|u\|_\infty \leq 1\right\}$$

Proof. Combine the statements of [3, Theorem 1.6], [3, Corollary 1.29] and [3, Proposition 1.47] . □

The following theorem states that the set of finite Radon measures $\mathcal{M}(X, \mathbb{R}^m)$ on X is the dual space of

$$C_0(X, \mathbb{R}^m) := \overline{C_c(X, \mathbb{R}^m)}^{\|\cdot\|_\infty},$$

where $\|\varphi\|_\infty := \sup_{x \in X} \left(\sum_{i=1}^m \varphi_i(x)^2 \right)^{1/2}$.

Be aware that the subscript c is used to denote compactly supported functions, while the subscript 0 is used to denote the closure of those functions with respect to the supremum norm. Moreover note that X is not assumed to be open and thus $C_0(X, \mathbb{R}^m)$-functions need not vanish at the boundary.

Theorem 74. *Let X be a locally compact separable metric space and suppose that $L : C_0(X, \mathbb{R}^m) \to \mathbb{R}$ is an additive bounded functional, i.e.*

$$\forall u, v \in C_0(X, \mathbb{R}^m): \quad L(u + v) = L(u) + L(v)$$

and

$$\|L\| := \sup_{\substack{u \in C_0(X, \mathbb{R}^m) \\ \|u\|_\infty \leq 1}} L(u) < \infty.$$

Then there is a unique $\mathbb{R}^m$-valued finite Radon measure μ on X, such that

$$\forall u \in C_0(X, \mathbb{R}^m): \quad L(u) = \int_X u \cdot \mathrm{d}\mu,$$

and

$$\|L\| = |\mu|(X)$$

where $|\mu|$ denotes the total variation of the measure μ.

Proof. Cf. [3, Theorem 1.54]. □

Note that $\|L\|$ and $|\mu|$ depend on the choice of the norm in $\mathbb{R}^m$ (which is the Euclidean norm $|\cdot|$ in the setting above). Choosing a different norm (which is e.g. the case, when we equip $\mathbb{R}^m = \mathbb{R}^{N \times n}$ with some operator norm) yields different values of $|\mu|(X)$ and $\|L\|$ in general[1]. However, the statement remains true for $\mathbb{R}^{N \times n}$-valued spaces equipped with the Frobenius norm

$$|u| := \sqrt{\sum_{i,j} u_{ij}^2}$$

[1] In particular it is not too difficult to show that equality of $\|L\|$ and $|\mu|(X)$ holds if and only if the underlying norms in $\mathbb{R}^m$ in the definitions of $\|L\|$ and $|\mu|$ are dual to each other.

for all $u = (u_{ij})_{i,j} \in \mathbb{R}^{N\times n}$, since the Frobenius norm is the norm induced from the Euclidean norm in $\mathbb{R}^m \cong \mathbb{R}^{N\times n}$.

Moreover, our preference of the Euclidean and Frobenius norm is stipulated from applications. Those norms are invariant under orthonormal transformations of $\mathbb{R}^m$ and thus are reasonable to model isotropic problems. In cases of non-isotropic materials l^p-norms with $p \neq 2$ might be a better choice. The derivation of our results would be similar. However, one has to be aware of choosing the p-norm or the conjugate p'-norm at the right situation.

A small analytic drawback of this approach is that we can not reduce questions of higher dimensions to the one-dimensional case. Instead we need to treat all dimensions simultaneously.

7.2 Linear Algebra

In this section we will summarize some results for the scalar product of matrices and the corresponding Frobenius norm. For matrices $A = (a_{ij})_{i,j=1,\dots n}, B = (b_{ij})_{i,j=1,\dots,n} \in \mathbb{R}^{n\times n}$ we consider the scalar product

$$A : B \ := \sum_{i,j} a_{ij} b_{ij}$$

and the induced norm, denoted by $|\cdot|$, is called Frobenius norm. The identity

$$|A| = \sqrt{\operatorname{trace}(A^\mathsf{T} A)}$$

holds. We will use the property that the Frobenius norm is invariant under rotations of the coordinate system.

Lemma 75. *For any orthogonal matrix $S \in O(n)$ and any $A \in \mathbb{R}^{n\times n}$ we have*

$$|SA| = |AS| = |A|$$

Proof. Since obviously $|A| = |A^\mathsf{T}|$, we show $|SA| = |A|$ only. Let $A = (a_{ij})_{i,j=1,\dots,n} \in$

$\mathbb{R}^{n\times n}$ and $S = (s_{ij})_{i,j=1,\dots n} \in O(n)$, then

$$\begin{aligned}|SA|^2 &= \sum_{i,l}\Big(\sum_j s_{ij}a_{jl}\Big)^2 = \sum_{i,l}\sum_{j,k} s_{ij}a_{jl}s_{ik}a_{kl}\\ &= \sum_{j,k,l} a_{jl}a_{kl}\sum_i s_{ij}s_{ik} = \sum_{j,k,l} a_{jl}a_{kl}\delta_{jk} = \sum_{j,l} a_{jl}^2 = |A|^2\end{aligned}$$

where we have used that the columns of S are an orthonormal basis of $\mathbb{R}^n$. □

A crucial property is that the decomposition of matrices $A \in \mathbb{R}^{n\times n}$ in their symmetric and antisymmetric parts $A = \frac{A+A^\mathsf{T}}{2} + \frac{A-A^\mathsf{T}}{2}$ is orthogonal with respect to the scalar product of matrices, such that for a symmetric matrix $A \in \mathbb{R}^{n\times n}$ and any $B \in \mathbb{R}^{n\times n}$ we have

$$A : B = A : \left(\tfrac{B+B^\mathsf{T}}{2}\right) \tag{7.1}$$

and in particular $A : B = 0$, provided B is antisymmetric.

7.3 Topological Indices

Topologial tools are useful concepts to classify families of certain subsets of a topological space for the constructions in critical point theory. Let us introduce the notion of genus and category and their basic properties here.

Let X be a Banach space. A subset $S \subseteq X$ is called symmetric, provided $u \in X$ implies $-u \in X$. The *genus of* a symmetric set $S \subseteq X \setminus \{0\}$, denoted

$$\operatorname{gen} S\,,$$

is defined to be the least $k \in \mathbb{N}$ such that there exists an odd continuous map $\Phi : S \to \mathbb{R}^k \setminus \{0\}$. Note that it is not difficult to see that this notion is equivalent to require that Φ maps to $\mathbb{S}^{k-1} := \{x \in \mathbb{R}^k, |x| = 1\}$. If no such k exists at all we set $\operatorname{gen} S = \infty$. Moreover, we define $\operatorname{gen}\emptyset := 0$ for technical reasons. Frequently we will also write $\operatorname{gen}_X S$ wherever it is helpful to highlight that the genus of S is determined in the space X.

The genus has the following properties. Let A, B be nonempty symmetric subsets of $X \setminus \{0\}$. Then

(i) if A is finite, then $\operatorname{gen} A = 1$. In particular $\operatorname{gen}\{\pm u\} = 1$ for any $u \in X \setminus \{0\}$.

(ii) $A \subseteq B \;\Rightarrow\; \operatorname{gen} A \leq \operatorname{gen} B$.

(iii) If there exists a continuous antisymmetric (i.e. $\Phi(-x) = -\Phi(x)$) map $\Phi : A \to B$, then $\operatorname{gen} A \leq \operatorname{gen} B$.

(iv) $\operatorname{gen}(A \cup B) \leq \operatorname{gen} A + \operatorname{gen} B$

(v) For each compact symmetric set A the genus is finite and there is some open neighborhood $U \supseteq A$, such that $\operatorname{gen} \overline{U} = \operatorname{gen} A$.

(vi) If there is an antisymmetric homeomorphism $\Phi : A \to \mathbb{S}^{k-1}$, then $\operatorname{gen} A = k$.

Properties (i)-(iv) are elementary to prove. The proof of property (v) makes use of the Tietze extension theorem. Property (vi) is more delicate to show and requires usage of the Borsuk-Ulam theorem (cf. [58, Chapter 44.3]).

A second widely used topological index is the category. A subset A of a metric space X is said to be of *category* 1, $\operatorname{cat} A = 1$, provided it is contractible within X. The *category* $\operatorname{cat} A$ of an arbitrary subset $A \subseteq X$ is defined as the least number of sets of first category covering A. If no such number exists, then $\operatorname{cat} A := \infty$.

In contrast to the genus, which can be directly applied within a Banach space X, this is not suitable for the category, since *every* subset of a Banach space is contractible. Thus one goes to the projective space $X_{/\sim}$ which is the quotient space of $X \setminus \{0\}$, where antipodal points (i.e. u and $-u$) are identified. It is well known that the genus of compact symmetric sets $A \subset X$ and the category of the corresponding set

$$\tilde{A} := \{\pm u \; ; \; u \in A\}$$

in the quotient space $X_{/\sim}$ coincide (cf. [49, Theorem 3.7] for compact sets A or [28, p. 40] for the case of merely closed sets A).

Bibliography

[1] F. Alter and V. Caselles. Uniqueness of the Cheeger set of a convex body *Nonlinear Anal.*, 70:32–44, 2009.

[2] L. Ambrosio and P. Tilli. *Topics on Analysis in Metric Spaces.* Oxford University Press, Oxford, 2004.

[3] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Clarendon Press, Oxford, 2000.

[4] F. Andreu-Vaillo, V. Caselles, and J. M. Mazón. *Parabolic Quasilinear Equations Minimizing Linear Growth Functionals*, volume 223 of *Progress in Mathematics.* Birkhäuser, Boston, 2004.

[5] G. Anzellotti. Pairings between measures and bounded functions and compensated compactness. *Ann. Mat. Pura Appl.*, 135:293–318, 1983.

[6] T. Bartsch. *Topological Methods for Variational Problems with Symmetries.* Springer, Berlin, 1993.

[7] M. Bildhauer, J. Naumann, and J. Wolf. An approximation theorem for vector fields in BD_{div}. http://www.math.uni-sb.de/service/preprints/preprint311.pdf.

[8] P. A. Binding and B. P. Rynne. Variational and non-variational eigenvalues of the p-Laplacian. *J. Differential Equations*, 244:24–39, 2008.

[9] H. Brézis. Monotonicity Methods in Hilbert Spaces and Some Applications to Nonlinear Partial Differential Equations, *Contributions to Nonlinear Functional Analysis*, E. Zarantonello (Ed.), Academic Press, New York, 1971.

[10] V. Caselles, J. Novaga, and A. Chambolle. Some remarks on uniqueness and regularity of Cheeger sets. *Rend. Semin. Mat. Univ. Padova*, 123:191–202

[11] T. Champion and L. de Pascale. Asymptotic behaviour of nonlinear eigenvalue problems involving p-Laplacian-type operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 137(06):1179–1195, 2007.

[12] K. C. Chang. Variational methods for non-differentiable functionals and their applications to partial differential equations. *J. Math. Anal. Appl.*, 80:102–129, 1981.

[13] K. C. Chang. The spectrum of the 1-laplace operator. *Commun. Contemp. Math.*, 11(5):865–894, 2009.

[14] R. Chill and E. Fašangová. Gradient systems. Matfyzpress, Publishing House of the Faculty of Mathematics and Physics, Charles University in Prague, 2010.

[15] I. Cioranescu. *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems.* Kluwer Academic Publishers, Dordrecht, 1990.

[16] F. Clarke. *Optimization and Nonsmooth Analysis (Classics in Applied Mathematics).* Society for Industrial and Applied Mathematics, 1987.

[17] J.-N. Corvellec. A general approach to the min-max principle. *Z. Anal. Anwend.*, 16(2):405–433, 1997.

[18] J.-N. Corvellec, M. Degiovanni, and M. Marzocchi. Deformation properties for continuous functionals and critical point theory. *Topol. Methods Nonlinear Anal.*, 1:151–171, 1993.

[19] M. Degiovanni. On topological and metric critical point theory. *J. Fixed Point Theory Appl.*, 7:85–102, 2010.

[20] M. Degiovanni and P. Magrone. Linking solutions for quasilinear equations at critical growth involving the "1-Laplace" operator. *Calc. Var. Partial Differential Equations*, 36:591–609, 2009.

[21] M. Degiovanni and M. Marzocchi. A critical point theory for nonsmooth functionals. *Ann. Mat. Pura Appl. (4)*, 167:73–100, 1994.

[22] M. Degiovanni and F. Schuricht. Multiplicity results for free and constrained nonlinear elastic rods based on nonsmooth critical point theory. *Math. Ann.*, 331:675–728, 1998.

[23] F. Demengel. Functions locally almost 1-harmonic. *Appl. Anal.*, 83(9):865–896, 2004.

[24] F. Demengel. Some existence's results for noncoercive "1-Laplacian" operator. *Asymptot. Anal.*, 43(4):287–322, 2005.

[25] M. A. del Pino and R. F. Manásevich. Global bifurcation of the p-Laplacian. *J. Differential Equations*, 92:226–251, 1991.

[26] L. C. Evans. *Partial Differential Equations*, Volume 19 of *Graduade Studies in Mathemtaics*. American Mathematical Society, Providence, 1998.

[27] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions (Studies in Advanced Mathematics)*. CRC-Press, Boca Raton, 1992.

[28] E. Fadell. The relationship between Ljusternik-Schnirelman Category and the Concept of Genus. *Pacific J. Math.*, 89(1):33–42, 1980.

[29] X. Fan and C.-X. Guan. Uniform convexity of Musielak-Orlicz-Sobolev spaces and applications. *Nonlinear Anal.*, 73:163–175, 2010.

[30] J. P. García Azorero and I. Peral Alonso. Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues. *Commun. in Partial Differential Equations*, 12:1389–1430, 1987.

[31] E. Giusti. *Minimal Surfaces and Functions of Bounded Variation*. Birkhäuser, Basel, 1984.

[32] A. Ioffe and E. Schwartzman. Metric Critical Point Theory 1. Morse Regularity and Homotopic Stability of a Minimum. *J. Math. Pures Appl.*, 75:125–153, 1996.

[33] G. Katriel. Mountain pass theorems and global homeomorphism theorems. *Ann. Inst. H. Poincaré Anal. Non Linéare*, 11:189–209, 1994.

[34] B. Kawohl and V. Fridman. Isoperimetric estimate for the first eigenvalue of the p-Laplace operator and the Cheeger constant. *Comment. Math. Univ. Carolin.*, 44(4):659–667, 2003.

[35] B. Kawohl and T. Lachand-Robert. Characterization of Cheeger sets for convex subsets of the plane. *Pacific J. Math.*, 225(1):103–118, 2006.

[36] B. Kawohl and F. Schuricht. Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem. *Commun. Contemp. Math.*, 9(4):513–543, August 2007.

[37] P. Lindqvist. On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. *Proc. Amer. Math. Soc.*, 109:157–164, 1990.

[38] P. Lindqvist. On non-linear Rayleigh quotients. *Potential Anal.*, 2(3):199–218, 1993.

[39] P. Lindqvist. On a nonlinear eigenvalue problem, 2000. URL `http://www.math.helsinki.fi/~analysis/GraduateSchool/lindqvist/nonlin.pdf`.

[40] S. Littig. Über das Eigenwertproblem des 1-Laplace operators. Master's thesis, TU Dresden, 2010.

[41] S. Littig and F. Schuricht. Convergence of the eigenvalues of the p-Laplace operator as p goes to 1. *Calc. Var. Partial Differential Equations*, 49:707–727, 2014.

[42] S. Littig and J. Voigt. Porous medium equation and fast diffusion equation as gradient systems. Czechoslovak Math. J., to appear

[43] M. Lucia and F. Schuricht. A class of degenerate elliptic eigenvalue problems. Adv. Nonlinear Anal. 2:91–125, 2013.

[44] Z. Milbers and F. Schuricht. Some special aspects related to the 1-Laplace operator. *Adv. Calc. Var.*, 4:101–126, 2010.

[45] Z. Milbers and F. Schuricht. Existence of a sequence of eigensolutions for the 1-Laplace operator. *J. Lon. Math. Soc. (2)*, 82:74–88, 2010.

[46] Z. Milbers and F. Schuricht. Necessary condition for eigensolutions of the 1-Laplace operator by means of inner variations. *Math. Ann.*, 356:147–177, 2012.

[47] I. Peral. Multiplicity of solutions for the p-Laplacian, 21 April - 9 May 1997. Second School of Nonlinear Functional Analysis and Applications to Differential Equations.

[48] P. Rabinowitz. Some global results for nonlinear eigenvalue problems. *J. Funct. Anal.*, 7:487–513, 1971.

[49] P. Rabinowitz. Some aspects of nonlinear eigenvalue problems. *Rocky Mountain J. Math.*, 3:161–202, 1973.

[50] P. H. Rabinowitz. *MiniMax Methods in Critical Point Theory with Applications to Differential Equations.* Number 65. American Mathematical Society, 1986.

[51] F. Rindler. Lower semicontinuity for integral functionals in the space of functions of bounded deformation via rigidity and Young measures. *Arch. Ration. Mech. Anal.*, 202(1):63–113, 2011.

[52] F. Schuricht. An alternative derivation of the eigenvalue equation for the 1-Laplace operator. *Arch. Math. (Basel)* 87:572–577, 2006

[53] H. Sohr. *The Navier-Stokes Equation.* Birkhäuser, Basel, 2001.

[54] P. Souplet. Geometry of unbounded domains, Poincaré inequalities and stability in semilinear parabolic equations. *Commun. Partial Differential Equations*, 24: 545–551, 1999.

[55] R. Temam. *Mathematical problems in plasticity.* Modern Applied Analysis, Wiley, 1985.

[56] R. Temam and G. Strang. Functions of bounded deformation. *Arch. Ration. Mech. Anal.*, 75:7–21, 1980.

[57] J. L. Vázquez. *The Porous Medium Equation, Mathematical Theory.* Clarendon Press, Oxford, 2007.

[58] E. Zeidler. *Nonlinear Functional Analysis and its Applications: Part 3: Variational Methods and Optimization.* Springer, Berlin, 1984.

[59] E. Zeidler. *Nonlinear Functional Analysis and its Applications: Part 1: Fixed-Point Theorems.* Springer, Berlin, corrected printing 1993 edition, 1985.

[60] E. Zeidler. *Nonlinear Functional Analysis and its Applications II/A.* Springer, Berlin, 1990.