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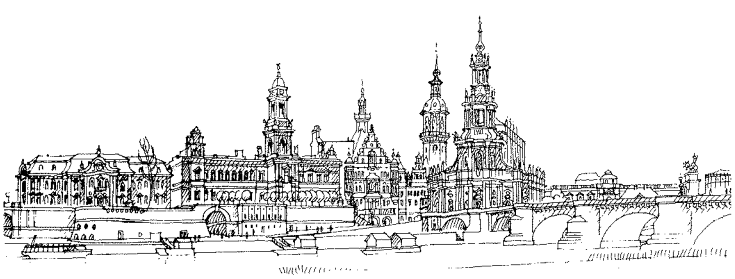

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Decidability Equivalence between the Star Problem and the Finite Power Problem in Trace Monoids*

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Abstract

In the last decade, some researches on the star problem in trace monoids (is the iteration of a recognizable language also recognizable?) has pointed out the interest of the finite power property to achieve partial solutions of this problem. We prove that the star problem is decidable in some trace monoid if and only if in the same monoid, it is decidable whether a recognizable language has the finite power property. Intermediary results allow us to give a shorter proof for the decidability of the two previous problems in every trace monoid without C4-submonoid.

We also deal with some earlier ideas, conjectures, and questions which have been raised in the research on the star problem and the finite power property, e.g., we show the decidability of these problems for recognizable languages which contain at most one non-connected trace.

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Contents

1 Introduction

Free partially commutative monoids, also called trace monoids, were introduced by P. Cartier and D. Foata in 1969 [2]. In 1977, A. Mazurkiewicz proposed these monoids as a potential model for concurrent processes [19]. This marks the beginning of a systematic study of trace monoids by mathematicians and theoretical computer scientists, see e.g., the recent surveys [6, 7]. A part of the research in trace theory deals with examinations of well-known classic results for free monoids in the framework of traces.

One main stream in trace theory is the study of recognizable trace languages, which can be considered as an extension of the well studied concept of regular languages in free monoids. A major step in this research is E. Ochmański's PhD thesis from 1984 [24]. Some of the results concerning regular languages in free monoids can be generalized to recognizable languages in trace monoids. However, there is one major difference: The iteration of a recognizable trace language does not necessarily yield a recognizable language. This fact raises the so called star problem: *Given a recognizable trace language T, is T^* recognizable*?

In general, it is not known whether the star problem is decidable. Sufficient conditions assuring the recognizability of the iteration of a language have been found (e.g. [3, 5, 10, 16, 20]). In the case of finite languages necessary conditions have been given [21, 22]. The decidability of the star problem is also known in the extremal cases of free monoids and free commutative monoids [12, 13]. In 1992, J. Sakarovitch proved the decidability in trace monoids without P3.

In 1990, E. Ochmański introduced the finite power property (for short FPP) to study the star problem [26]: A language T has the finite power property if there exists an integer $n \geq 0$ such that $T^* = T^0 \cup \ldots \cup T^n$. In free monoids, the decidability of the FPP for regular languages was already known due to I. Simon and K. Hashiguchi [30, 15]. Motivated by [26], one asked for the decidability of the FPP for recognizable trace languages.

In 1992, using the decidability of the FPP in free monoids, P. Gastin, E. Ochmański, A. Petit, and B. Rozoy showed the decidability of the star problem in trace monoids of the form $A^* \times b^*$ [11]. In 1994, Y. Métivier and G. Richomme showed the decidability of the FPP for a special class of recognizable trace languages [21, 22]. In the same year, G. Richomme used this decidable case and generalized the proofs of P. Gastin, E. Ochmański, A. Petit, and B. Rozoy. Thereby, he obtained the decidability of the star problem and the FPP in any trace monoid without C4 [29]. The decidability of the star problem and the FPP in any other trace monoid remains open.

Although these works show connections between the star problem and the FPP, the exact correlation was not clear. Here, we show that the star problem is decidable in some trace monoid if and only if the FPP is decidable in the same trace monoid. A crucial role in the proof of this equivalence is played by a new connection between the star problem and the FPP which we show in Section 4: For a particular class of recognizable trace languages the iteration of some language is recognizable if and only if the language has the FPP. In Section 5, we work out several induction steps on independence alphabets. These induction steps allow to give short proofs for the decidability equivalence (Part 5.3), the decidability in trace monoids without C4 (Part 5.4), and a general connection between the star problem and the FPP (Part 5.5).

In Section 6, we work on some conjectures and questions which were discussed in the research on the star problem and the FPP. In Part 6.1, we generalize a result by E. Ochmański [26] by showing that both problems are decidable for languages which contain at most one connected trace. In Part 6.2 to 6.4, we deal with conjectures by E. Ochmański and M. Latteux. In Section 2, we recall notions on semigroups, trace monoids, and recognizability. In Section 3, we present our results in a more precise way. The rest of the paper is devoted to the proofs.

2 Preliminaries

2.1 Generalities

By an *integer*, we mean an element of $\{0, 1, 2, \ldots\}$. We denote by $K \subseteq L$ (resp. $K \subset L$) the fact that K is a subset of L (resp. strict subset of L). If p is an element of some set L, we denote the singleton set which consists of p by p instead of $\{p\}$. For instance, for any $p \in L$ and $K \subseteq L$, we use notations as $K \cup p$ and $K \setminus p$ in a natural way. We denote by $|L|$ the *cardinal* of a finite set L, i.e., the number of elements of L.

A *semigroup* $(S, \cdot_S)$ is an algebraic structure consisting of a set S and a binary associative relation $\cdot_S$ called *operation* or *product*. When no confusion arises, this product is denoted by $\cdot$ or just by juxtaposition. A semigroup $(S, \cdot_S)$ is said *finite* if S is a finite set. A *monoid* is a semigroup equipped with an identity, which is denoted by λ_S or λ.

The product can be extended to subsets $K, L \subseteq S$: KL is the set of elements kl for any $k \in K$ and $l \in L$. Moreover, for any $p \in S$, $K \subseteq S$, and $n \geq 1$, p^n and K^n are defined by $p^1 = p$, $p^{n+1} = p^n p$, $K^1 = K$, and $K^{n+1} = K^n K$. If S is a monoid, we define $k^0 = \lambda$ and $K^0 = \{\lambda\}$.

For every subset $K \subseteq S$ and for any integers $1 \leq n \leq m$, we denote by $K^{n,\ldots,m}$ the union $K^n \cup K^{n+1} \cup \ldots \cup K^m$. Moreover, we denote by $K^{n,\ldots,\infty}$ the union $K^n \cup K^{n+1} \cup \ldots$ As usual, we denote by K^+ the *non-empty iteration* of K, i.e., $K^+ = K^{1,\ldots,\infty}$. Of course, if S is a monoid, we extend the notation $K^{n,\ldots,m}$ to $0 \leq n \leq m$, and we denote the *iteration* of K by $K^* = K^{0,\ldots,\infty}$.

A *homomorphism* is a function h from a semigroup S to a semigroup S' such that for every $k, l \in S$, $h(k) \cdot_{S'} h(l) = h(k \cdot_S l)$. Moreover, if S and S' are monoids, h is a called a *monoid homomorphism* if $h(\lambda_S) = \lambda_{S'}$. For a subset $K \subseteq S$, $h(K)$ is the set of elements $h(p)$ for any $p \in K$. The inverse of h is denoted by h^{-1}. For a subset $L \subseteq S'$, $h^{-1}(L)$ is the set of elements $p \in S$ such that $h(p) \in L$. The homomorphism h is called *surjective* (resp. *injective*) if for any $p \in S'$, the set $h^{-1}(p)$ is non-empty (resp. $|h^{-1}(p)| \leq 1$). It is called an *isomorphism* if it is both surjective and injective. Two semigroups S and S' are *isomorphic* if there exists a bijective homomorphism from S to S'.

Assume three semigroups S_1, S_2, S_3 and two homomorphisms $g : S_1 \to S_2$ and $h : S_2 \to S_3$. We denote by $h \circ g : S_1 \to S_3$ the homomorphism obtained by the composition of g and h.

Given two sets S and S', we denote by $S \times S'$ their *cartesian product*. If both S and S' are semigroups (resp. monoids), then $S \times S'$ with the operations $\cdot_S$ and $\cdot_{S'}$ applied componentwise is a semigroup (resp. monoid). To visualize the componentwise operation, we often denote the elements cartesian products by $\binom{p}{q}$. Accordingly, we denote the cartesian product of two subsets K and L of semigroups S and S', resp., by $\binom{K}{L}$.

2.2 Trace Monoids

We recall notions on trace monoids (see e.g. [6, 7] for more information).

An *alphabet* A is a finite set of symbols called *letters*. A word over A is a finite sequence of letters of A. Formally, the set A^* of words over A with the concatenation operation is the free monoid over A. Its identity is the empty word λ.

A binary symmetric and irreflexive relation I over an alphabet A is called an *independence relation* over A. The pair (A, I) is called an *independence alphabet*. Two letters $a, b \in A$ are called *independent* if aIb. Otherwise, they are called *dependent*.

Let $\sim_I$ be the equivalence relation over A^* such that for two words $w_1, w_2 \in A^*$, we have $w_1 \sim_I w_2$ if we can transform w_1 into w_2 by finitely many exchanges of independent adjacent letters. An equivalence class of words is called a *trace*. Clearly, $\sim_I$ is a congruence ($w_1 \sim_I w_2$

and $w_1' \sim_I w_2'$ implies $w_1 w_1' \sim_I w_2 w_2'$). The factorization of the free monoid A^* under $\sim_I$ is called the trace monoid over A and I and denoted by $\mathbb{M}(A,I)$. Its subsets are called *trace languages* or shortly *languages*, or sometimes just *sets*. For any trace monoid $\mathbb{M}(A,I)$, we denote by $\mathbb{M}(A,I)^+$ the semigroup $\mathbb{M}(A,I)\backslash\lambda$.

By $[\,]_I$ or shortly $[\,]$, we denote the homomorphism from A^* to $\mathbb{M}(A,I)$ which assigns every word its congruence class. By $[\,]^{-1}$, we denote its inverse.

Two equivalent words differ only in the order of their letters. Given some trace t, $alph(t)$ denotes the set of letters occurring in t and $|t|$ is the length of t that is the number of letters of t. Further, for any letter a, $|t|_a$ denotes the number of occurrences of the letter a in t. For instance, $alph([aabb]) = \{a,b\}$, $|[aabb]| = 4$, and $|[aabb]|_a = |[aabb]|_b = 2$.

Free monoids are the trace monoids for which the independence relation is empty. If the independence relation is the largest irreflexive relation over A, i.e., any two different letters $a,b \in A$ are independent, then the trace monoid is a free commutative monoid over A.

The cartesian product of two trace monoids can be considered as a trace monoid. Indeed, given two disjoint independence alphabets (A_1, I_1) and (A_2, I_2), the monoid $\mathbb{M}(A_1,I_1) \times \mathbb{M}(A_2,I_2)$ is naturally isomorphic to the trace monoid over $A = A_1 \cup A_2$ and $I = I_1 \cup I_2 \cup A_1 \times A_2 \cup A_2 \times A_1$, i.e., in (A,I), two letters $a,b \in A$ are independent if and only if either they do not belong to the same alphabet or are independent in (A_1,I_1) or in (A_2,I_2).

In particular, given four different letters a,b,c,d, we will denote respectively by P3 and C4 any monoids isomorphic to $\{a,c\}^* \times \{b\}^*$ and $\{a,c\}^* \times \{b,d\}^*$, respectively.

Let $\mathbb{M}(A,I)$ be a trace monoid and B a subset of A, we denote by $\mathbb{M}(B,I)$ the trace monoid $\mathbb{M}(B, I \cap (B \times B))$. We say that $\mathbb{M}(A,I)$ is without P3 (resp. C4) if whatever are the letters a,b,c (resp. a,b,c,d), then $\mathbb{M}(\{a,b,c\},I) \neq \{a,c\}^* \times \{b\}^*$ (resp. $\mathbb{M}(\{a,b,c,d\},I) \neq \{a,c\}^* \times \{b,d\}^*$).

The notion of connected traces plays a central role in recognizability problems. Some trace $t \in \mathbb{M}(A,I)$ is said *connected* if for every non-empty traces t_1, t_2 with $t = t_1 t_2$, there is a letter $a \in alph(t_1)$ and a letter $b \in alph(t_2)$ such that a and b are dependent: equivalently the graph consisting of the letters in $alph(t)$ as vertices and edges between dependent letters is connected. In particular, some trace $\binom{u}{v}$ in P3 or C4 is connected if and only if u or v is the empty word λ. A trace language T is said *connected* if and only if every trace in T is connected. For some trace language T, we denote by $\mathrm{Conn}(T)$ (resp. $\mathrm{NConn}(T)$) the language consisting of the connected (resp. non-connected) traces of T.

We also call a trace monoid $\mathbb{M}(A,I)$ *connected* if the graph which consists of the letters in A as vertices and edges between dependent letters is connected. This does not imply that every trace in $\mathbb{M}(A,I)$ is connected. But, the traces $t \in \mathbb{M}(A,I)$ with $alph(t) = A$ are connected if $\mathbb{M}(A,I)$ is a connected trace monoid.

2.3 Recognizable Languages

We recall the notion of recognizability. We follow [1, 9]. Given a monoid $\mathbb{M}$, an $\mathbb{M}$-*automaton*, or simply *automaton*, is a triple $\mathcal{A} = [Q, h, F]$, where Q is a finite monoid, h is a homomorphism from $\mathbb{M}$ to Q and F is a subset of Q. The set $h^{-1}(F)$ is called the language (or set) of the automaton and denoted by $L(\mathcal{A})$. A subset L of $\mathbb{M}$ is *recognizable* over $\mathbb{M}$ if there exists an $\mathbb{M}$-automaton $[Q,h,F]$ such that $L = L(\mathcal{A})$.

Below, some of the algebraic proofs are simpler if h is a surjection from $\mathbb{M}$ to Q. If h is not a surjection, then we can transform $[Q,h,F]$ into the automaton $[h(\mathbb{M}),\ h,\ F \cap h(\mathbb{M})]$ which defines the same language as $[Q,h,F]$. Consequently, we can assume that h is a surjection.

It is well-known that for any monoid $\mathbb{M}$, the family of recognizable sets over $\mathbb{M}$ contains the empty set, $\mathbb{M}$ itself, and is closed under union, intersection, complement, and inverse homomorphisms (see e.g. [1, 9]). Moreover, for trace monoids, finite languages are recognizable, and the concatenation of two recognizable languages yields a recognizable language [5, 10, 23]. The following result is widely known as Mezei's Theorem (cf. [1, 9].)

Theorem 2.1 (Mezei's Theorem) *Assume two monoids $\mathbb{M}$ and $\mathbb{M}'$. A set L is recognizable in $\mathbb{M} \times \mathbb{M}'$ if and only if there are an integer n, recognizable sets $L_1, \ldots, L_n \subseteq \mathbb{M}$ and recognizable sets $L_1', \ldots, L_n' \subseteq \mathbb{M}'$, such that $L = L_1 \times L_1' \cup \ldots \cup L_n \times L_n'$.*

Of course, for $i \in \{1, \ldots, n\}$, we can assume $L_i \neq \emptyset$ and $L_i' \neq \emptyset$.

As soon as a trace monoid $\mathbb{M}$ contains two independent letters, the family of recognizable sets of $\mathbb{M}$ is not closed under iteration: If aIb, then $\{ab\}$ is recognizable, but $\{ab\}^*$ is not recognizable.

The class of rational sets of a monoid $\mathbb{M}$ is the smallest class which contains the empty set and every singleton subset of $\mathbb{M}$, and is closed under union, monoid product, and iteration. Kleene's classic result states that in free monoids the recognizable sets and the rational sets coincide. By the non-closureship of recognizable trace languages under iteration, there are rational trace languages which are not recognizable as the previous example shows. However, due to a more general result by J. McKnight [1, 9], every recognizable trace language is rational.

Two decision problems arise: The *star problem*, which means to decide whether the iteration of a recognizable language is recognizable and the *recognizability problem*, which means to decide whether a rational language is recognizable. We say that the star problem (resp. the finite power property, below) is decidable in some trace monoid $\mathbb{M}(A, I)$ if it is decidable for recognizable languages over $\mathbb{M}(A, I)$.

In the extremal cases of free monoids and free commutative monoids, the decidability of the star problem is classically known: In the free monoid it is trivial by Kleene's Theorem and in free commutative monoids its decidability was shown by S. Ginsburg and E. Spanier [12, 13] in 1966.

During the eighties, E. Ochmański [24], M. Clerbout and M. Latteux [3], and Y. Métivier [20] independently proved a special case for the recognizability of the iteration:

Proposition 2.2 *Let $\mathbb{M}$ be a trace monoid and T a recognizable subset of $\mathbb{M}$ such that every trace in T is connected. The iteration T^* is recognizable.*

A related closure property originates from C. Duboc [8, 23]. If $h : \mathbb{M}_1 \to \mathbb{M}_2$ is a connected homomorphism between two trace monoids (namely a morphism such that the images of connected traces are connected), then for every recognizable language $T \subseteq \mathbb{M}_1$, the language $h(T)$ is recognizable. On the other hand, if h is not connected, then there are recognizable languages $T \subseteq \mathbb{M}_1$ such that $h(T)$ is not recognizable [8, 23].

In 1990, E. Ochmański examined connections between the star problem and the finite power property [26]. A trace language T has the *finite power property (for short FPP)* if and only if there is some integer n such that $T^* = T^0 \cup T^1 \cup \ldots \cup T^n$. An obvious connection between the star problem and the FPP is that if some recognizable trace language T has the FPP, then T^* is recognizable by closure properties of recognizable trace languages.

The question whether the finite power property is decidable for recognizable languages in free monoids was already raised by J.A. Brzozowski in 1966, and it took more that 10 years till I. Simon and K. Hashiguchi independently showed its decidability [30, 15]. In 1990, E. Ochmański used this result to show decidability of the star problem for recognizable languages in trace monoids of the form $A^* \times B^*$ which contain at most one non-connected trace [26]. This marks the beginning of the examination of connections between the star problem and the FPP.

In 1992, J. Sakarovitch solved the recognizability problem: given a trace monoid $\mathbb{M}$, it is decidable whether a rational subset of $\mathbb{M}$ is recognizable if and only if $\mathbb{M}$ is without P3. As a conclusion, the star problem is decidable in trace monoids without P3. One conjectured that this characterization can be extended to the star problem. However, just in the same year, P. Gastin, E. Ochmański, A. Petit, and B. Rozoy proved the decidability of the star problem in P3 [11]. The decidability of the FPP in free monoids played a crucial role in their proof.

In 1993, G. Pighizzini proved that for some recognizable trace language T the iteration T^* is recognizable if and only if $\mathrm{NConn}(T^*)$ is recognizable [27].

In 1994, Y. Métivier and G. Richomme showed a decidable case of the FPP [22].

Proposition 2.3 *In any trace monoid, it is decidable whether some connected, recognizable trace language has the finite power property.*

Y. Métivier and G. Richomme showed some connections between the star problem and the FPP: If the star problem is decidable in some trace monoid of the form $\mathbb{M}(A,I) \times b^*$ for some (A,I) and some $b \notin A$, then the FPP is decidable in $\mathbb{M}(A,I)$. Consequently, if the star problem is decidable in any trace monoid, then so is the FPP [22].

G. Richomme generalized the results from P. Gastin, E. Ochmański, A. Petit, and B. Rozoy [11]. In combination with the decidability of the FPP for connected recognizable trace languages, he proved that both the star problem and the finite power problem are decidable in trace monoids without C4 [29].

Recently, D. Kirsten and J. Marcinkowski examined some problems which are related to the star problem [18]: It is decidable whether the intersection $K \cap L^*$ is recognizable for recognizable languages K and L in some trace monoid $\mathbb{M}$ if and only if $\mathbb{M}$ is without P3. If we consider the same problem restricted to finite languages L, then the recognizability of $K \cap L^*$ is decidable in P3 but undecidable in C4.

2.4 Projections and Restrictions

Now, we consider two different ways to transform trace languages: Projections and restrictions. We examine consequences of these constructions on recognizability and the FPP.

Let $\mathbb{M}(A,I)$ be a trace monoid and B be a subset of A. The projection $\Pi_B : \mathbb{M}(A,I) \to \mathbb{M}(B,I)$ is the morphism such that for every trace t, $\Pi_B(t)$ is the trace obtained by erasing the letters of t which do not belong to B. More precisely, the projection Π_B is defined by the image of the letters: $\Pi_B(a) = a$ if $a \in B$, and $\Pi_B(a) = \lambda$ if $a \notin B$.

Consider a language $T \subseteq \mathbb{M}(A,I)$, a subset $B \subseteq A$ and an integer $i \geq 0$. Observe that $\Pi_B(T^i) = \Pi_B(T)^i$ and $\Pi_B(T^*) = \Pi_B(T)^*$.

In general, projections do not preserve recognizability. However, if we consider a trace monoid $\mathbb{M}(A_1,I_1) \times \mathbb{M}(A_2,I_2)$, then both the projection Π_{A_1} and Π_{A_2} are connected homomorphisms and preserve recognizability by Duboc's Theorem.

The notion of restrictions was introduced by G. Pighizzini [27, 28] and also used in [29]. Assume a trace monoid $\mathbb{M}(A,I)$, any subset $B \subseteq A$, and some recognizable language $T \subseteq \mathbb{M}(A,I)$. Let $T_{=B}$ (resp. $T_{\subset B}$ and $T_{\subseteq B}$) denote the subset of traces $t \in T$ with $alph(t) = B$ (resp. $alph(t) \subset B$ and $alph(t) \subseteq B$).

G. Pighizzini proved that restrictions preserve recognizability. This can easily be verified using the closure properties of the family of recognizable trace languages. Since $T_{\subseteq B} = T \cap \mathbb{M}(B,I)$, $T_{\subseteq B}$ is recognizable. From $T_{\subset B} = \cup_{C \subset B} T_{\subseteq C}$ and $T_{=B} = T_{\subseteq B} \setminus T_{\subset B}$ and the closure properties of the family of recognizable trace languages, it follows that $T_{\subset B}$ and $T_{=B}$ are recognizable.

Observe that for every integer $i \geq 0$, $(T^i)_{\subseteq B} = (T_{\subseteq B})^i$. Consequently, the restriction $T_{\subseteq B}$ preserves the FPP. Further, $(T^*)_{\subseteq B} = (T_{\subseteq B})^*$. We denote the languages $(T^i)_{\subseteq B} = (T_{\subseteq B})^i$ and $(T^*)_{\subseteq B} = (T_{\subseteq B})^*$ by $T^i_{\subseteq B}$ and $T^*_{\subseteq B}$, respectively.

These facts cannot be generalized to $T_{=B}$ or $T_{\subset B}$: In the free monoid $\{a,b\}^*$, the language $T = (b^+a^+)^* \cup a^* \cup b^* \cup ab$ satisfies $T^* = T^3 = \{a,b\}^*$, but the restrictions $T_{=\{a,b\}} = (b^+a^+)^* \cup ab$ and $T_{\subset\{a,b\}} = a^* \cup b^*$ do not have the FPP.

3 Main Results, Conclusions, and Future Steps

In this section, we state the main results of this paper. We present also the plan of the rest of the paper which is completely devoted to the proofs.

3.1 Decidability Equivalence and Decidable Cases

Our main result claims the decidability equivalence between the star problem and the FPP:

Theorem 3.1 *Let $\mathbb{M}(A,I)$ be a trace monoid. The star problem is decidable in $\mathbb{M}(A,I)$ if and only if the finite power problem is decidable in $\mathbb{M}(A,I)$.*

To prove this theorem, we proceed in several steps. At first, we show a close connection between the star problem and the FPP for a special class of languages.

Proposition 3.2 *Let $\mathbb{M}(A_1,I_1)$ and $\mathbb{M}(A_2,I_2)$ be two disjoint trace monoids. Assume a recognizable language $T \subseteq (\mathbb{M}(A_1,I_1)\setminus\lambda) \times (\mathbb{M}(A_2,I_2)\setminus\lambda)$. Then, T^* is recognizable if and only if T has the finite power property.*

This proposition was already announced in [17]. Its proof is done in Section 4 using the notions of ideals and left ideals of semigroups which are recalled in Part 4.1. In Section 5, we achieve several results by inductions on independence alphabets. In Part 5.1, we perform an induction step for non-connected trace monoids:

Proposition 3.3 *Let $\mathbb{M}(A_1,I_1)$ and $\mathbb{M}(A_2,I_2)$ be two disjoint trace monoids. Assume both the star problem and the finite power problem are decidable in both $\mathbb{M}(A_1,I_1)$ and $\mathbb{M}(A_2,I_2)$. Then, the following four assertions are equivalent:*

1. *The star problem is decidable in $\mathbb{M}(A_1,I_1) \times \mathbb{M}(A_2,I_2)$.*
2. *The star problem is decidable for recognizable subsets of $(\mathbb{M}(A_1,I_1)\setminus\lambda) \times (\mathbb{M}(A_2,I_2)\setminus\lambda)$.*
3. *The finite power problem is decidable for recognizable subsets of $(\mathbb{M}(A_1,I_1)\setminus\lambda) \times (\mathbb{M}(A_2,I_2)\setminus\lambda)$.*
4. *The finite power problem is decidable in $\mathbb{M}(A_1,I_1) \times \mathbb{M}(A_2,I_2)$.*

We give a stronger result in the case that one of the monoids is a free monoid over a singleton.

Proposition 3.4 *Let $\mathbb{M}(A,I)$ be a trace monoid with a decidable star problem and a decidable finite power problem. Assume a letter $b \notin A$. Then, both the star problem and the finite power problem are decidable in $\mathbb{M}(A,I) \times b^*$.*

Besides other results, this proposition was already stated in [29] and its presented proof used techniques and results from P. Gastin, E. Ochmański, A. Petit, and B. Rozoy [11]. However, we can shorten its proof by applying Proposition 3.2. In Part 5.2, we give an induction step for connected monoids.

Proposition 3.5 *Let $\mathbb{M}(A,I)$ be a connected trace monoid.*

1. *The star problem is decidable in $\mathbb{M}(A,I)$ if and only if for every strict subset $B \subset A$, the star problem is decidable in $\mathbb{M}(B,I)$.*
2. *The finite power problem is decidable in $\mathbb{M}(A,I)$ if and only if for every strict subset $B \subset A$, the finite power problem is decidable in $\mathbb{M}(B,I)$.*

Of course, this result is related to Proposition 2.2 and 2.3. In Part 5.3, we use Proposition 3.3 and 3.5 to prove Theorem 3.1. In Part 5.4, we use Proposition 3.4 and 3.5 to prove the following theorem, which was already announced in [29].

Theorem 3.6 *The star problem and the finite power property problem are decidable in any trace monoid without C4.*

3.2 A General Characterization

In [11], P. Gastin, E. Ochmański, A. Petit, and B. Rozoy showed that for some recognizable trace language T in any trace monoid the iteration T^* is recognizable if the set $\mathrm{Conn}(T)^* \cup \mathrm{NConn}(T)$ has the finite power property. They implicitly used the fact that this sufficient condition is necessary in trace monoids $A^* \times b^*$. They asked whether this condition is necessary in any trace monoid. In [22], Y. Métivier and G. Richomme showed that this condition is not necessary in the trace monoid over $A = \{a,b,c\}$ and $I = \{(a,c),(c,a)\}$. In Part 5.5, we give a similar condition which is sufficient and necessary:

Proposition 3.7 *Let T be a recognizable set of traces. The set T^* is recognizable if and only if $\mathrm{Conn}(T^*) \cup \mathrm{NConn}(T)$ has the finite power property, i.e., every trace of T^* can be decomposed in a bounded (the bound depends on T) concatenation of connected traces of T^* and non-connected traces of T.*

This condition generalizes Proposition 3.2. Let remark that in his PhD Thesis, G. Pighizzini had given another general characterization: for recognizable trace languages T, the iteration T^* is recognizable if and only if $\mathrm{NConn}(T^*)$ is recognizable [27].

3.3 On Some Ideas to Solve the Star Problem

Within the researches on the star problem and the FPP, many restricted cases and conjectures have been discussed, in particular in [26]. We give some answers using materials from Section 5.

At first, we give an improvement of a result by E. Ochmański: In [26], he proved that the star problem is decidable for recognizable languages in $A^* \times B^*$ which contain at most one non-connected trace. In Section 6.1, we show:

Proposition 3.8 *In any trace monoid, both the star problem and the finite power property are decidable for recognizable languages containing at most one non-connected trace.*

In [21, 22], Y. Métivier and G. Richomme proved that the star problem is decidable for finite sets containing at most two connected traces. This result combined with the previous proposition allows to see that the star problem is decidable for languages containing at most four traces (result also in [21, 22]): Such a language contains at most two connected traces or at most one connected trace.

In [26], E. Ochmański also announced two conjectures. The first one says: *Given a non-empty, finite language T in any trace monoid, if T^* is recognizable, then there exists a trace $t \in T$ such that $(T \setminus t)^*$ is recognizable.* We show the following proposition in Part 6.2:

Proposition 3.9 *Assume some trace monoid* $\mathbb{M}$. *The following assertion is true if and only if* $\mathbb{M}$ *does not contain a P3: If for some non-empty, finite language* $T \subset \mathbb{M}$ *the iteration* T^* *is recognizable, then there exists some* $t \in T$ *such that* $(T \backslash t)^*$ *is recognizable.*

The second conjecture announced by E. Ochmański is quite similar to the first one: *Given a finite trace language* T *with at least two traces, if* T^* *is not recognizable, then there exists a trace* $t \in T$ *such that* $(T \setminus t)^*$ *is not recognizable.* We do not know whether this conjecture is true or not, but it is verified in monoids without C4 (see Part 6.3):

Proposition 3.10 *Assume some finite language* T *in a trace monoid without C4. If* T *contains at least two traces and if* T^* *is not recognizable, then there exists a trace* $t \in T$ *such that* $(T \setminus t)^*$ *is not recognizable.*

In a talk given by G. Richomme at the Laboratoire d'Informatique Fondamentale de Lille (LIFL), M. Latteux raised the question whether the following conjecture is true: *For every trace monoid* $\mathbb{M}$ *there is some integer* $n_0 > 0$ *such that a recognizable language* T *has the FPP if and only if there exists an integer* $0 < n \leq n_0$ *such that* $[T^{0,\ldots,n}]^{-1}$ *has the FPP.* The idea is derived from a closure property which says that some trace language T is recognizable if and only if $[T]^{-1}$ is recognizable. Note that the integer n_0 depends on the monoid $\mathbb{M}$, but, n_0 does not depend on T, otherwise the result is immediate. Because $[T^{0,\ldots,n}]^{-1}$ is a recognizable language in a free monoid, we can decide whether it has the FPP.

The conjecture is obviously true in free monoids with $n_0 = 1$. Unfortunately, it is false in any other trace monoid. In Part 6.4, we show the following proposition:

Proposition 3.11 *Assume some trace monoid* $\mathbb{M}$ *which is not a free monoid. For every integer* $n_0 > 0$, *there is some recognizable language* $T \subseteq \mathbb{M}$ *such that* T *has the FPP, but for* $n \in \{1, \ldots, n_0\}$ $[T^{0,\ldots,n}]^{-1}$ *does not have the FPP.*

3.4 Conclusions and Future Steps

From now, the star problem and the finite power problem can be viewed as one and the same problem. We know that they are decidable in trace monoids without C4. We do not know whether they are decidable in other trace monoids. If one can show that one of these problems is undecidable in the trace monoid C4, then in all the remaining trace monoids, both problems are undecidable.

Proposition 3.2 raises a decision problem: Let $\mathbb{M}(A_1, I_1)$ and $\mathbb{M}(A_2, I_2)$ be two disjoint trace monoids with a decidable star problem and finite power property. Assume some recognizable language $T \subseteq (\mathbb{M}(A_1, I_1) \backslash \lambda) \times (\mathbb{M}(A_2, I_2) \backslash \lambda)$. Can we decide whether T^* is recognizable, i.e., can we decide whether T has the FPP?

Provided that the answer of this question is "yes", we can show the decidability of the star problem and the FPP as follows: We can improve Proposition 3.3 by showing that the four assertions are true. This is also an improvement of Proposition 3.4. Then, we obtain the decidability of the star problem and the FPP in any trace monoid by a straightforward adaptation of the proofs of Theorem 3.1 and 3.6.

Another open question is whether the second conjecture by E. Ochmański (cf. Proposition 3.10) is true in trace monoids with C4. Further, one could try to modify M. Latteux' Conjecture in order to solve the star problem by solving the FPP.

4 An Important Special Case

In this section, we prove Proposition 3.2: *Given two disjoint trace monoids* $\mathbb{M}_1$ *and* $\mathbb{M}_2$*, and given a recognizable language* $T \subseteq \mathbb{M}_1^+ \times \mathbb{M}_2^+$*, the iteration* T^* *is recognizable if and only if* T *has the FPP.*

All the notions and results presented here are only used within this section. Hence, the reader can skip this section and still understand the rest of the paper.

As it was already mentioned in Section 2.3, if some recognizable language T has the finite power property, then T^* is recognizable. Thus, just the "only if" part of Proposition 3.2 has to be proved.

This section is organized as follows: We recall the notion of generators of a semigroup. Based on this notion, we state Proposition 4.1 and use it to prove Proposition 3.2. Then, Part 4.1 to 4.5 are exclusively devoted to the proof of Proposition 4.1. In Part 4.1, we recall notions from ideal theory to give a classification of non-empty, finite semigroups. In Part 4.2, we present product automata to recognize subsets of a cartesian product of two trace monoids. In Part 4.3, using these automata and the previous classification, we prove a special case of Proposition 4.1. In Part 4.4, we prove the remaining cases of Proposition 4.1 by an induction on the ideal structure of the semigroups in product automata. Finally, in Part 4.5, we summarize the results to complete to proof of Proposition 4.1.

Within this whole section, we assume two disjoint independence alphabets (A_1, I_1) and (A_2, I_2). We abbreviate $\mathbb{M}(A_1, I_1)$, $\mathbb{M}(A_2, I_2)$, $\mathbb{M}(A_1, I_1)\setminus\lambda$, and $\mathbb{M}(A_2, I_2)\setminus\lambda$ by $\mathbb{M}_1$, $\mathbb{M}_2$, $\mathbb{M}_1^+$, and $\mathbb{M}_2^+$, respectively. The traces in $\mathbb{M}_1^+ \times \mathbb{M}_2^+$ are exactly the traces in $\mathbb{M}_1 \times \mathbb{M}_2$ which contain at least one letter in A_1 and at least one letter in A_2.

Some trace language T is called *concatenation closed* if $T^2 \subseteq T$. Then, T is a semigroup, i.e., $T = T^+$. Assume a concatenation closed trace language T with $\lambda \notin T$. The *set of generators* of T is defined by $\mathrm{Gen}(T) = T \setminus T^2$. Of course, $\mathrm{Gen}(T) \subseteq T$ and $\mathrm{Gen}(T)^+ \subseteq T$. Moreover, it is easy to prove by an induction on the length of a trace $t \in T$ that t can be decomposed into $t_1 \ldots t_n$ with $t_i \in \mathrm{Gen}(T)$, since if $t \in T$ and $t \notin \mathrm{Gen}(T)$, then $t = t_1 t_2$ with $t_1, t_2 \in T$. Thus, $T^+ \subseteq \mathrm{Gen}(T)^+$ and more precisely, $T^+ = \mathrm{Gen}(T)^+$. Now, if we consider some trace language L such that $L^+ = T$, then $\mathrm{Gen}(T) = L^+ \setminus (L^+L^+) = \cup_{i\geq 1} L^i \setminus \cup_{i\geq 2} L^i = L \setminus (LL^+)$ and thus, $\mathrm{Gen}(T) \subseteq L$.

For example, consider the concatenation closed, recognizable language $T = \binom{a^+}{b^+} \subseteq a^* \times b^*$. Note that $\lambda \notin T$. We have $\mathrm{Gen}(T) = \binom{a}{b^+} \cup \binom{a^+}{b}$. Observe that $T = \mathrm{Gen}(T) \cup \mathrm{Gen}(T)^2$. Hence, for any (not necessarily recognizable) trace language L with $L^+ = T$, we have $\mathrm{Gen}(T) \subseteq L$, and consequently, L has the FPP: $L^* = \{\lambda\} \cup L \cup L^2$. In general, we have:

Proposition 4.1 *Assume some recognizable, concatenation closed language* $T \subseteq \mathbb{M}_1^+ \times \mathbb{M}_2^+$*. The set of generators of* T *has the FPP.*

We close this introduction by deriving Proposition 3.2 from Proposition 4.1.

Proof of Proposition 3.2. Assume some recognizable language $L \subseteq \mathbb{M}_1^+ \times \mathbb{M}_2^+$. If L has the FPP, then L^* is recognizable because of the closure properties of recognizable trace languages.

Conversely, assume that L^* is recognizable. Then, so is $L^+ = L^* \setminus \lambda$. Furthermore, we have $L^+ \subseteq \mathbb{M}_1^+ \times \mathbb{M}_2^+$. By Proposition 4.1 applied on L^+, there is some integer $l \geq 1$ such that we have $\mathrm{Gen}(L^+)^{1,\ldots,l} = \mathrm{Gen}(L^+)^+ = L^+$.

Since $\mathrm{Gen}(L^+) \subseteq L$, we have $\mathrm{Gen}(L^+)^{1,\ldots,l} \subseteq L^{1,\ldots,l}$. Thus, $L^+ \subseteq L^{1,\ldots,l}$, i.e., $L^+ = L^{1,\ldots,l}$. Consequently, $L^* = L^{0,\ldots,l}$, i.e., L has the FPP. $\square$

4.1 A Classification of Non-Empty, Finite Semigroups

In this part, we classify non-empty, finite semigroups using ideals and left ideals. This classification plays a crucial role in the proof of Proposition 4.1.

Ideal theory originates from J.A. Green and other pioneers in semigroup theory. We recall some notions in a way that the reader does not require previous knowledge in semigroup theory (see e.g. [4, 14] for more information).

As already said, a *semigroup* is a set together with a binary associative operation. A subset H of a semigroup S is a *subsemigroup* of S if and only if $H^2 \subseteq H$. We call a subset $U \subseteq S$ a *left ideal* of S if and only if $SU \subseteq U$. We call a subset $J \subseteq S$ an *ideal* of S if and only if $JS \subseteq J$ and $SJ \subseteq J$. Every semigroup has itself and the empty set as ideals. Every ideal is a left ideal and every left ideal is a subsemigroup. We call a left ideal $U \subseteq S$ (ideal $J \subseteq S$) *proper* if and only if U (resp. J) is non-empty and different from S. The intersection and the union of two left ideals (resp. ideals) yield left ideals (resp. ideals).

Now, we introduce a notion and a lemma which will help us to prove the completeness of the classification. Assume some non-empty, finite semigroup Q. Assume some ideal $J \subseteq Q$ with $J \neq Q$. We call a left ideal $U \subseteq Q$ *J-minimal* if and only if we have $J \subset U$ and there is not any left ideal U' with $J \subset U' \subset U$. The intersection of two different J-minimal left ideals U and V contains J. If J is properly contained in the left ideal $U \cap V$, then, one of the left ideals U or V is not J-minimal, because $J \subset (U \cap V) \subset U$ or $J \subset (U \cap V) \subset V$. Consequently, $U \cap V = J$. If $J = \emptyset$, we shortly say *minimal* instead of $\emptyset$-minimal.

Lemma 4.2 *Assume a non-empty, finite semigroup Q and an ideal $J \neq Q$. Then, the union of all J-minimal left ideals yields an ideal of Q.*

Proof. There is at least one left ideal properly containing J, namely Q itself. Hence, there is also some smallest left ideal which contains J properly.

Let J' be the union of all J-minimal left ideals. Then, J' is a left ideal with $J \subset J'$. We have to show $J'Q \subseteq J'$. It is sufficient to prove that for every J-minimal left ideal L and for every element $q \in Q$, the set $J \cup Lq$ yields J or some J-minimal left ideal (and thus, $J \cup Lq \subseteq J'$). Just assume $J \subset (J \cup Lq)$.

Because L is a left ideal, we have $QL \subseteq L$. Thus, we have $QLq \subseteq Lq$. Therefore, Lq and $J \cup Lq$ are left ideals of Q.

Now, we show by a contradiction that $J \cup Lq$ is J-minimal. Just assume a left ideal K such that we have $J \subset K \subset (J \cup Lq)$. We define a set K' by $K' := \{x \in L \mid xq \in K\}$. We show the proper inclusions $J \subset K' \subset L$.

We have $J \subseteq L$ and $Jq \subseteq J \subset K$. Hence, we have $J \subseteq K'$. We show that the inclusion $J \subseteq K'$ is strict: There is some $p \in K \setminus J$. Then, $p \in Lq$. Hence, there is some $p' \in L$ with $p = p'q$. We have $p' \notin J$, because J is an ideal and $p = p'q \notin J$. However, $p' \in K'$.

The inclusion $K' \subseteq L$ is obvious. There is some $r \in (J \cup Lq) \setminus K$. Then, we have $r \in Lq \setminus J$. Thus, there is some $r' \in L$ with $r'q = r$. Then, $r' \notin K'$, i.e., we have $K' \subset L$.

We show that K' is a left ideal. Just assume some $x \in K'$ and some $y \in Q$. We have $yx \in L$, because x belongs to L which is a left ideal. Further, we have $yxq \in K$, because xq belongs to the left ideal K. Thus, we have $yx \in K'$.

Hence, the set K' is a left ideal with $J \subset K' \subset L$, i.e., L is not J-minimal. This is a contradiction, such that the assumed left ideal K does not exist. Thus, $J \cup Lq$ is a J-minimal left ideal. $\square$

Now, we can give the classification of finite semigroups:

Proposition 4.3 *Every non-empty, finite semigroup Q satisfies one of the following assertions:*

(A) *Q has not any proper left ideal.*

(B) *Q has two proper left ideals U, V such that $U \cup V = Q$ and $U \cap V$ is an ideal of Q.*

(C) *Q has an ideal J such that $Q \backslash J$ yields a singleton $\{r\}$ with $r^2 \in J$.*

(D) *Q has a proper ideal J and a subsemigroup H such that $J \cap H = \emptyset$ and $J \cup H = Q$.*

Proof. Assume that Q does not have any proper ideal. If Q does not have a proper left ideal, it satisfies assertion (A). If Q has a proper left ideal, we apply Lemma 4.2 with $J = \emptyset$. The union of all minimal left ideals of Q yields an ideal of Q. Because Q does not have proper ideals, the union of all minimal left ideals of Q yields Q itself. Now, assume that Q has exactly one minimal left ideal. Then, this minimal left ideal is Q itself. Thus, the semigroup Q does not have proper left ideals, which is a contradiction. Hence, Q has at least two minimal left ideals. Let U be a minimal left ideal and let V be the union of all other minimal left ideals of Q. Then, U and V are two disjoint left ideals and their union yields Q. Thus, Q satisfies assertion (B), because the empty set is an ideal.

Now, assume Q has a proper ideal. Let J be a proper ideal of Q such that there is not any ideal J' with $J \subset J' \subset Q$. Such an ideal exists because Q is finite and Q has at least one proper ideal. We show that Q and J satisfy assertion (B), provided that they contradict assertion (C) and (D).

Since J is proper, there is some $r \in Q \backslash J$. Then, $Q \backslash J = \{r\}$ implies assertion (C) or (D), depending on whether $r^2 \in J$ or $r^2 = r$. Hence, $Q \backslash J$ contains at least two elements.

Because $Q \backslash J$ is not a subsemigroup of Q, there are $p, q \in Q \backslash J$ such that $pq \in J$. We have $J \cup Qq = J \cup \Big(J \cup \{p\} \cup Q \backslash J \backslash \{p\}\Big)q = J \cup Jq \cup \{pq\} \cup \Big(Q \backslash J \backslash \{p\}\Big)q$. The sets Jq and $\{pq\}$ are contained in J such that we have $J \cup Qq = J \cup \Big(Q \setminus J \setminus \{p\}\Big)q$.

Now, we have $\Big|J \cup Qq\Big| = \Big|J \cup (Q \setminus J \setminus \{p\})q\Big| \leq \Big|J\Big| + \Big|(Q \setminus J \setminus \{p\})q\Big| \leq \Big|J\Big| + \Big|Q \setminus J \setminus \{p\}\Big|$. We have $p \in Q \setminus J$, and thus, $\Big|J\Big| + \Big|Q \setminus J \setminus \{p\}\Big| < \Big|J\Big| + \Big|Q \setminus J\Big| = \Big|Q\Big|$. Hence, we have $\Big|J \cup Qq\Big| < \Big|Q\Big|$. Therefore, we have the proper inclusion $J \cup Qq \subset Q$.

We show the existence of some left ideal U' of Q with $J \subset U' \subset Q$. Assume that Qq is not a subset of J. Then, the union $J \cup Qq$ yields the desired left ideal U'. Assume that $Qq \subseteq J$. Then, the set $J \cup \{q\}$ is the desired left ideal U'. The inclusion $(J \cup \{q\}) \subset Q$ is proper since $Q \setminus J$ contains at least two different elements.

Now, we can apply Lemma 4.2. The union of all J-minimal left ideals of Q yields an ideal. This ideal properly contains J. The only ideal properly containing J is Q itself. Hence, the union of all J-minimal left ideals yields Q itself.

Assume there is exactly one J-minimal left ideal. Then, this J-minimal left ideal is Q itself. However, Q cannot be a J-minimal left ideal, because we have shown that there is some left ideal U' with $J \subset U' \subset Q$. Therefore, there are at least two different J-minimal left ideals.

Now, let U be a J-minimal left ideal and let V be the union of all other J-minimal left ideals. Then, U and V are the desired left ideals in assertion (B). □

Every proper ideal is also a proper left ideal. Thus, if some non-empty, finite semigroup Q satisfies one of the assertions (B), (C), or (D), then it cannot satisfy assertion (A). However, the assertions (B), (C), and (D) are not exclusive.

4.2 Product Automata

In this part, we deal with a special kind of automata. We adapt the notion of $\mathbb{M}$-automata from Part 2.3. We use ideas from the proof of Mezei's Theorem (cf. [1, 9]).

Assume four semigroups S_1, S_2, S_1', and S_2'. Assume two homomorphisms $g : S_1 \to S_1'$ and $h : S_2 \to S_2'$. We define a homomorphism $\binom{g}{h}$ from $S_1 \times S_2$ to $S_1' \times S_2'$ componentwise: For every $\binom{p}{q} \in S_1 \times S_2$, $\binom{g}{h}\binom{p}{q}$ yields $\binom{g(p)}{h(q)}$. The homomorphism $\binom{g}{h}$ is a surjection from $S_1 \times S_2$ to $S_1' \times S_2'$ if and only if both g and h are surjective homomorphisms from S_1 to S_1' and S_2 to S_2', respectively. Whenever we deal with a cartesian product of two semigroups S_1 and S_2, we denote the canonical projections by $\Pi_1 : S_1 \times S_2 \to S_1$ and $\Pi_2 : S_1 \times S_2 \to S_2$. As an exercise, one can verify that the homomorphisms $h \circ \Pi_2$ and $\Pi_2 \circ \binom{g}{h}$ from $S_1 \times S_2$ to S_2' are identical.

We still assume the trace monoids $\mathbb{M}_1$ and $\mathbb{M}_2$ from the beginning of Section 4. Note that the projection Π_1 (resp. Π_2) from $\mathbb{M}_1 \times \mathbb{M}_2$ to $\mathbb{M}_1$ (resp. $\mathbb{M}_2$) is Π_{A_1} (resp. Π_{A_2}).

A *product automaton* $\mathcal{A}$ over $\mathbb{M}_1 \times \mathbb{M}_2$ is a quintuple $[P, R, g, h, F]$, where

- P and R are non-empty, finite semigroups,
- g and h are surjective homomorphisms $g : \mathbb{M}_1 \to P$, $h : \mathbb{M}_2 \to R$,
- F is a subset of $P \times R$.

We can regard every product automaton $[P, R, g, h, F]$ as an $\mathbb{M}_1 \times \mathbb{M}_2$-automaton $[P \times R, \binom{g}{h}, F]$. A product automaton $\mathcal{A}$ defines a recognizable language by $L(\mathcal{A}) = \binom{g}{h}^{-1}(F)$. This means that a trace $t \in \mathbb{M}_1 \times \mathbb{M}_2$ belongs to $L(\mathcal{A})$ if and only if we obtain a pair in F when we apply g and h on the first and second compound of t, respectively. Let us assume that $L(\mathcal{A})$ is closed under concatenation. Then, because $\binom{g}{h}$ is a surjective homomorphism, F is a subsemigroup of $P \times R$. Similarly, $\Pi_1(F)$ and $\Pi_2(F)$ are subsemigroups of P and R, respectively.

We are going to use product automata to prove assertions on recognizable languages in $\mathbb{M}_1 \times \mathbb{M}_2$. Therefore, we have to show that every recognizable language $T \subseteq \mathbb{M}_1 \times \mathbb{M}_2$ is the language of some product automaton.

Lemma 4.4 *Assume a recognizable language $T \subseteq \mathbb{M}_1 \times \mathbb{M}_2$. There is a product automaton for T.*

Proof. By Mezei's Theorem, there is some integer n and recognizable languages $T_1, \ldots, T_n \subseteq \mathbb{M}_1$ and $T_1', \ldots, T_n' \subseteq \mathbb{M}_2$ such that

$$T = \binom{T_1}{T_1'} \cup \ldots \cup \binom{T_n}{T_n'}$$

For $i \in \{1, \ldots, n\}$, let $[P_i, g_i, F_i]$ (resp. $[R_i, h_i, F_i']$) be an automaton for T_i (resp. T_i'). We can freely assume $P_i = P_j$, $g_i = g_j$, $R_i = R_j$, and $h_i = h_j$ for any $1 \le i \le j \le n$. Further, we can assume that g_1 and h_1 are surjective homomorphisms from $\mathbb{M}_1$ to P_1 and $\mathbb{M}_2$ to R_1, respectively. Then, T is the language of the product automaton $[P_1, R_1, g_1, h_1, F]$ with $F = \binom{F_1}{F_1'} \cup \ldots \cup \binom{F_n}{F_n'}$. $\square$

We examine connections between product automata and ideal theory. Assume a recognizable language $T \subseteq \mathbb{M}_1 \times \mathbb{M}_2$ which is closed under concatenation. Assume further a product automaton $\mathcal{A} = [P, R, g, h, F]$ for T. Let us denote $\Pi_2(F)$ by Q. Then, Q is a subsemigroup of R. We can verify that $Q = h \circ \Pi_2(T) = \Pi_2 \circ \binom{g}{h}(T)$. Assume some subset $W \subseteq Q$. We define a language T_W by

$$T_W = \{\, t \in T \mid h \circ \Pi_2(t) \in W \,\}$$

We obviously have $T_W \subseteq T$. Some trace $t \in \mathbb{M}_1 \times \mathbb{M}_2$ belongs to T_W if and only if we have $\binom{g}{h}(t) \in F \cap (P \times W)$.

Proposition 4.5 *Assume a non-empty, concatenation closed language* $T \subseteq \mathbb{M}_1 \times \mathbb{M}_2$. *Assume a product automaton* $\mathcal{A} = [P, R, g, h, F]$ *for* T. *Let* Q *denote* $\Pi_2(F)$. *For every subset* $W \subseteq Q$, *the product automaton* $\mathcal{A}_W = [P, R, g, h, F \cap (P \times W)]$ *defines* T_W. *If* W *is a non-empty subset (resp. subsemigroup, left ideal, ideal) of* Q, *then the language* T_W *is a non-empty subset (resp. subsemigroup, left ideal, ideal) of* T.

Proof. The quintuple $\mathcal{A}_W$ is a product automaton. For every $t \in T_W$, we have $\binom{g}{h}(t) \in F$ and $\binom{g}{h}(t) \in P \times W$. Thus, we have $\binom{g}{h}(t) \in F \cap (P \times W)$. Hence, $T_W \subseteq L(\mathcal{A}_W)$.

Conversely, let $t \in L(\mathcal{A}_W)$. Then, we have $\binom{g}{h}(t) \in F$ and $\binom{g}{h}(t) \in (P \times W)$. Hence, $t \in T$ and $h \circ \Pi_2(t) \in W$, i.e., $t \in T_W$. Thus, $L(\mathcal{A}_W) \subseteq T_W$.

Let $f : T \to Q$ be the restriction of $h \circ \Pi_2$ to T. Then, f is a surjection from T to Q and $T_W = f^{-1}(W)$. If W is a non-empty subset (resp. subsemigroup, left ideal, ideal) of Q, so is its preimage T_W under f. □

4.3 A Special Case of Proposition 4.1

In the following three parts, we prove Proposition 4.1: *Assume some concatenation closed, recognizable language* $T \subseteq \mathbb{M}_1^+ \times \mathbb{M}_2^+$. *The set of generators of* T *has the FPP.*

Proposition 4.1 is obviously true if the language T is empty. Thus, we just need to prove it for non-empty languages T. The general structure of the proof is the following: By Lemma 4.4, there is a product automaton $\mathcal{A} = [P, R, g, h, F]$ for T. We denote $\Pi_2(F)$ by Q. Because T is non-empty, Q is non-empty. We apply Proposition 4.3 on Q. Therefore, the proof of Proposition 4.1 consists of four cases. In this part, we deal with the case that Q does not have proper left ideals. After that, in Part 4.4, we deal with the cases that Q fulfills one of the assertions (B), (C), or (D) in Proposition 4.3. We will do this by an induction on the number of elements of Q. In Part 4.5, we summarize the results to prove Proposition 4.1. Now, we consider case (A):

Proposition 4.6 *Assume a non-empty, concatenation closed language* $T \subseteq \mathbb{M}_1^+ \times \mathbb{M}_2^+$ *which is recognized by a product automaton* $[P, R, g, h, F]$, *such that the semigroup* $\Pi_2(F)$ *does not have proper left ideals. Then,* $\mathrm{Gen}(T)$ *has the FPP. Moreover, we have* $T = \mathrm{Gen}(T)^{1,\ldots,|\Pi_2(F)|+1}$.

At first, we need a technical result on finite semigroups without proper left ideals:

Lemma 4.7 *Assume a non-empty, finite semigroup* Q *which has not any proper left ideal. Then, for every elements* $p, p', q \in Q$, *the equality* $pq = p'q$ *implies* $p = p'$.

Proof. Just assume $p, p', q \in Q$ such that $pq = p'q$ and $p \neq p'$. We have $QQ \subseteq Q$, and thus, $Q\,Qq \subseteq Qq$ such that Qq is a left ideal. Further, Qq yields a proper left ideal of Q, because the result of the product pq "occurs twice", such that at least one element of Q cannot occur in Qq. □

Now, we introduce the notion of the most oblique cut. We assume a language T as in Proposition 4.6. Assume some traces $t, t_1, s_1 \in T$. We call the pair (t_1, s_1) a *most oblique cut* of t if and only if $t = t_1 s_1$ and for every traces $t_1', s_1' \in T$ with $t = t_1' s_1'$ we have either

- $|\Pi_1(t_1')| > |\Pi_1(t_1)|$ or
- $|\Pi_1(t_1')| = |\Pi_1(t_1)|$ and $|\Pi_2(t_1')| \leq |\Pi_2(t_1)|$.

Intuitively, we can understand the definition as follows. We try to factorize $t \in T$ into two traces $t_1, s_1 \in T$. We try to do this in a way that the first compound of t_1 is small, but, the second compound of t_1 is big. A most oblique cut of some trace $t \in T$ exists if and only if $t \notin \mathrm{Gen}(T)$.

Lemma 4.8 *Assume $t, t_1, s_1 \in T$ such that (t_1, s_1) is a most oblique cut of t. Then, $t_1 \in \mathrm{Gen}(T)$.*

Proof. Just assume that $t_1 \notin \mathrm{Gen}(T)$. Then, there are two traces $t_{1a}, t_{1b} \in T$ such that $t = t_{1a}t_{1b}$. We can factorize t into t_{1a} and $t_{1b}s_1$. We have t_{1a}, $t_{1b}s_1 \in T$. Further, $\Pi_1(t_{1a})$ contains properly less letters than $\Pi_1(t_1)$, since $\Pi_1(t_{1b}) \neq \lambda$. This contradicts that (t_1, s_1) is a most oblique cut. $\square$

We can factorize every trace $t \in T$ into generators by successive most oblique cuts. We factorize t into a generator t_1 and a trace s_1 in T. Then, we factorize s_1 by a most oblique cut and so on, until a most oblique cut yields two generators. This iterative factorization terminates, because "the remaining part of t" becomes properly shorter in every most oblique cut.

Proof of Proposition 4.6. Assume some trace $t \in T$. We denote by Q the semigroup $\Pi_2(F)$. We show that a factorization of t by successive most oblique cuts yields a factorization of t into at most $|Q| + 1$ generators of T.

We factorize t into generators of T by successive most oblique cuts. We obtain an integer $n \geq 0$ and generators $t_1, \ldots, t_n$ of T such that $t_1 \ldots t_n = t$. For every $i \in \{1, \ldots, n-1\}$, the pair $(t_i, t_{i+1} \ldots t_n)$ is a most oblique cut of $t_i \ldots t_n$.

We introduce two notations. For every $i \in \{1, \ldots, n\}$, we define $u_i = \Pi_1(t_i)$ and $v_i = \Pi_2(t_i)$, i.e., we have $t_i = \binom{u_i}{v_i}$. For every $i \in \{1, \ldots, n\}$, we have $h(v_i) \in Q$, because $t_1, \ldots, t_n \in T$.

We show by a contradiction that $n \leq |Q| + 1$. Assume $n > |Q| + 1$.

By $h(v_{i+1} \ldots v_n) = h(v_{i+1}) \ldots h(v_n) \in Q$ for $1 \leq i < n$ and $n-1 > |Q|$, we get the existence of $1 \leq i < j < n$ such that $h(v_{i+1} \ldots v_n) = h(v_{j+1} \ldots v_n)$.

Then, $h(v_i) \cdot_Q h(v_{i+1} \ldots v_n) = h(v_i \ldots v_n) = h(v_i \ldots v_j) \cdot_Q h(v_{j+1} \ldots v_n)$. Since Q does not have proper left ideals, we can apply Lemma 4.7 and get $h(v_i) = h(v_i \ldots v_j)$.

By $t_i \in T$, we have $\binom{g}{h}\binom{u_i}{v_i} \in F$. Because of $h(v_i) = h(v_i \ldots v_j)$, we get $\binom{g}{h}\binom{u_i}{v_i \ldots v_j} \in F$, and thus, $\binom{u_i}{v_i \ldots v_j} \in T$. Similarly, $t_{i+1} \ldots t_n \in T$ implies $\binom{g}{h}\binom{u_{i+1} \ldots u_n}{v_{i+1} \ldots v_n} \in F$. By $h(v_{i+1} \ldots v_n) = h(v_{j+1} \ldots v_n)$, we have $\binom{g}{h}\binom{u_{i+1} \ldots u_n}{v_{j+1} \ldots v_n} \in F$, and hence, $\binom{u_{i+1} \ldots u_n}{v_{j+1} \ldots v_n} \in T$.

Therefore, $\binom{u_i}{v_i \ldots v_j}$ and $\binom{u_{i+1} \ldots u_n}{v_{j+1} \ldots v_n}$ are a factorization of $t_i \ldots t_n$ into two traces from T. Since $(t_i, t_{i+1} \ldots t_n)$ is a most oblique cut of t and $\Pi_1\binom{u_i}{v_i \ldots v_j} = \Pi_1(t_i)$, we obtain $|\Pi_2\binom{u_i}{v_i \ldots v_j}| \leq |\Pi_2(t_i)|$. Hence, $|v_i \ldots v_j| \leq |v_i|$. Because v_i is a prefix of $v_i \ldots v_j$, we have $|v_i \ldots v_j| = |v_i|$. Consequently, $v_{i+1} \ldots v_j = \lambda$. This is a contradiction, because every trace in T contains at least one letter from A_2. Finally, our assumption $n > |Q| + 1$ lead us to a contradiction. Hence, we have $n \leq |Q| + 1$. $\square$

The method of most oblique cuts is a very suitable method to prove Proposition 4.1 in the case that the semigroup Q does not have proper left ideals. Let us consider an example where this method fails: Let $T = \binom{a}{b} \cup \{ \binom{a^n}{b^m} \mid n \geq 2, m \geq 2 \} \subseteq a^* \times b^*$. The language T satisfies all presumptions of Proposition 4.1. However, we cannot prove that $\mathrm{Gen}(T)$ has the FPP by factorizations with most oblique cuts. For every $n \geq 1$, the application of successive most oblique cuts factorizes the trace $\binom{a^n}{b^n} \in T$ into $\binom{a}{b} \ldots \binom{a}{b}$, i.e., we obtain n generators. Hence, the number of generators which we obtain by successive most oblique cuts is unlimited.

4.4 The Remaining Cases of Proposition 4.1

We prove the remaining cases of Proposition 4.1 by an induction on the number of elements in Q. In the case that Q is a singleton, we already know by Proposition 4.6 that Proposition 4.1 is true for T, because the singleton semigroup does not have proper left ideals. We show:

Proposition 4.9 *Let* $n > 1$. *Assume that Proposition 4.1 holds for every non-empty, concatenation closed language* $T' \subseteq \mathbb{M}_1^+ \times \mathbb{M}_2^+$ *which is recognized by a product automaton* $[P', R', g', h', F']$ *with* $|\Pi_2(F')| < n$. *Let* $[P, R, g, h, F]$ *be a product automaton for a language* T *such that*

- T *is a non-empty, concatenation closed language in* $\mathbb{M}_1^+ \times \mathbb{M}_2^+$,
- $|\Pi_2(F)| = n$, *and,*
- $\Pi_2(F)$ *satisfies one of the assertions (B), (C), or (D) in Proposition 4.3.*

Then, $\mathrm{Gen}(T)$ *has the FPP.*

Proof. We denote $Q = \Pi_2(F)$. If Q satisfies assertion (B), then we denote $J = U \cap V$. Hence, there is an ideal J of Q regardless of which assertion of (B), (C), or (D) Q satisfies.

We examine the language $T_J = \{ t \in T \mid h \circ \Pi_2(t) \in J \}$. If $J = \emptyset$, then $T_J = \emptyset$. Now, assume that $J \neq \emptyset$. Since the ideal J is a subsemigroup of Q, by Proposition 4.5, T_J is concatenation closed. Further T_J is an ideal of T. Also by Proposition 4.5, T_J is recognizable. More precisely, the product automaton $\mathcal{A}_J = [P, R, g, h, F \cap (P \times J)]$ defines T_J. Clearly, $\Pi_2(F \cap (P \times J))$ yields J. We have $|\Pi_2(F \cap (P \times J))| < |Q| = n$. By the inductive hypothesis, there is some integer l_J such that $T_J = \mathrm{Gen}(T_J)^{1,\ldots,l_J}$.

We show in two steps that $\mathrm{Gen}(T)$ has the FPP: At first, we show that there is some $l > 0$ such that $T \backslash T_J \subseteq \mathrm{Gen}(T)^{1,\ldots,l}$. Then, we show $T_J \subseteq \mathrm{Gen}(T)^{1,\ldots,3ll_J}$.

Fact 4.10 *There is an integer* $l > 0$ *such that* $T \backslash T_J \subseteq \mathrm{Gen}(T)^{1,\ldots,l}$.

Note that if we factorize any trace $t \in T \backslash T_J$ into some traces of T, then not any factor does belong to the ideal T_J. Otherwise, t would belong to T_J. To prove Fact 4.10, we branch into three cases depending on which assertion Q satisfies.

At first, assume that Q satisfies assertion (C). Then, we set $l = 1$. We show $T \setminus T_J \subseteq \mathrm{Gen}(T)^1$ by a contradiction. Assume some $t \in T \backslash T_J$ with $t \notin \mathrm{Gen}(T)$, i.e., there are $t_1, t_2 \in T$ with $t = t_1 t_2$. As mentioned above, we have $t_1, t_2 \notin T_J$. Thus, $h \circ \Pi_2(t_1) = h \circ \Pi_2(t_2) = r$ and $h \circ \Pi_2(t_1 t_2) = r^2 \in J$, i.e., $t \in T_J$. This is a contradiction.

Assume that Q satisfies assertion (D). By Proposition 4.5 (as for T_J), T_H is a non-empty, recognizable, and concatenation closed subset of T. Moreover $T_J \cap T_H = \emptyset$, i.e., $T \backslash T_J = T_H$. By the inductive hypothesis, since $|H| = |Q| - |J| < n$, there is an $l_H > 0$ such that $T_H = \mathrm{Gen}(T_H)^{1,\ldots,l_H}$.

We have $\mathrm{Gen}(T_H) \subseteq \mathrm{Gen}(T)$. Indeed, assume some $t \in \mathrm{Gen}(T_H)$ with $t \notin \mathrm{Gen}(T)$. Then, there are $t_1, t_2 \in T$ with $t = t_1 t_2$. As above, $t_1, t_2 \notin T_J$, i.e., $t_1, t_2 \in T_H$. This contradicts $t \in \mathrm{Gen}(T_H)$. Thus, we have $T_H = \mathrm{Gen}(T_H)^{1,\ldots,l_H} \subseteq \mathrm{Gen}(T)^{1,\ldots,l_H}$ and Fact 4.10 is true for $l = l_H$.

At last, assume that Q satisfies assertion (B). As in the previous cases, T_U and T_V are non-empty, recognizable, and concatenation closed subsets of T. Further, T_U and T_V are left ideals of T. By the inductive hypothesis, since $|U| < |Q|$ and $|V| < |Q|$, we have two integers $l_U, l_V > 0$ such that $T_U = \mathrm{Gen}(T_U)^{1,\ldots,l_U}$ and $T_V = \mathrm{Gen}(T_V)^{1,\ldots,l_V}$.

We have $U \cup V = Q$ and $U \cap V = J$. For every $t \in T$, we have $h \circ \Pi_2(t) \in U$ or $h \circ \Pi_2(t) \in V$. Thus, $T_U \cup T_V = T$. Further, for every $t \in T$, we have $h \circ \Pi_2(t) \in J$ if and only if $h \circ \Pi_2(t) \in U$ and $h \circ \Pi_2(t) \in V$. Hence, we have $T_U \cap T_V = T_J$.

To show Fact 4.10, it suffices to show $T_U \backslash T_J \subseteq \mathrm{Gen}(T)^{1,\ldots,l_U l_V + l_U}$. Then, we accordingly obtain $T_V \backslash T_J \subseteq \mathrm{Gen}(T)^{1,\ldots,l_U l_V + l_V}$ such that Fact 4.10 is true for $l = l_U l_V + \max(l_U, l_V)$.

We show $T_U \backslash T_J \subseteq \mathrm{Gen}(T)^{1,\ldots,l_U l_V + l_U}$. Assume some $t \in T_U \backslash T_J$.

Case 1: $t \in \mathrm{Gen}(T_U)\backslash T_J$

Clearly, $t \notin T_V$. If we factorize t into some traces in T, no factor belongs to the ideal T_J, i.e., no factor belongs to both T_U and T_V.

The trace t is not necessarily a generator of T. If $t \in \mathrm{Gen}(T)$, then we are done. So assume that $t \notin \mathrm{Gen}(T)$. There are some $x \in T$ and some $y \in \mathrm{Gen}(T)$ with $xy = t$. Assume y belongs to the left ideal T_V. Then, $xy \in T_V$. This is a contradiction. Thus, $y \in T_U$. Assume $x \in T_U$. Then, $xy = t$ contradicts $t \in \mathrm{Gen}(T_U)$. Hence, $x \in T_V$ and $y \in T_U$.

We deal with x. There are some $k \leq l_V$ and $x_1, \ldots, x_k \in \mathrm{Gen}(T_V)$ such that $x_1 \ldots x_k = x$.

We show by a contradiction that $x_1, \ldots, x_k \in \mathrm{Gen}(T)$. Just assume some $i \in \{1, \ldots, k\}$ such that x_i can be factorized into two traces $x_i', x_i'' \in T$. Assume that $x_i'' \in T_U$. Then, $x_i \in T_U$, which is a contradiction. Hence, $x_i'' \in T_V$. Now, assume that $x_i' \in T_V$. Then, x_i is not a generator of T_V. Thus, we have $x_i' \in T_U$ and $x_i'' \in T_V$. However, this yields a contradiction: We factorize t into $x_1 \ldots x_{i-1}x_i'$ and $x_i''x_{i+1} \ldots x_k y$. Both factors belong to T_U, because x_i' and y belong to the left ideal T_U. Hence, $t \notin \mathrm{Gen}(T_U)$ which is a contradiction.

The assumption that some trace among $x_1, \ldots, x_k$ is not a generator of T yields a contradiction. Thus, we have by $x_1, \ldots, x_k, y$ a factorization of t into generators of T. Hence, $t \in \mathrm{Gen}(T)^{1,\ldots,l_V+1}$.

Case 2: $t \in T_U \backslash T_J$

There are a $k \leq l_U$ and $t_1, \ldots, t_k \in \mathrm{Gen}(T_U)$ such that $t_1 \ldots t_k = t$. The generators $t_1, \ldots, t_k$ cannot belong to T_J. By case 1, we have $t_1, \ldots, t_k \in \mathrm{Gen}(T)^{1,\ldots,l_V+1}$. Because $k \leq l_U$, we have $t \in \mathrm{Gen}(T)^{1,\ldots,l_U l_V + l_U}$.

This completes the proof of Fact 4.10. If $J = \emptyset$, then $T = T \setminus T_J$ and Fact 4.10 just proves that $\mathrm{Gen}(T)$ has the FPP. If $J \neq \emptyset$, then it remains to prove the following fact:

Fact 4.11 *If $T_J \neq \emptyset$, then we have $\mathrm{Gen}(T_J) \subseteq \mathrm{Gen}(T)^{1,\ldots,3l}$, and thus, $T_J \subseteq \mathrm{Gen}(T)^{1,\ldots,3l l_J}$.*

For the proof of this fact, assume some $t \in \mathrm{Gen}(T_J)$. Assume traces $t_1, t_2, t_3 \in T \cup \lambda$ such that

$$t_1 t_2 t_3 = t, \qquad t_1, t_3 \in (T\backslash T_J) \cup \lambda, \qquad \text{and} \qquad t_2 \in T_J \cup \lambda.$$

There are traces t_1, t_2, t_3 which fulfill these conditions: $t_1 = \lambda$, $t_2 = t$, $t_3 = \lambda$. However, we choose a triple t_1, t_2, t_3 such that $|t_2|$ is minimal.

We have $t_1, t_3 \in \mathrm{Gen}(T)^{0,\ldots,l}$. If $t_2 \in \mathrm{Gen}(T) \cup \lambda$, then $t \in \mathrm{Gen}(T)^{1,\ldots,2l+1} \subseteq \mathrm{Gen}(T)^{1,\ldots,3l}$. If $t_2 \notin \mathrm{Gen}(T)$ and $t_2 \neq \lambda$, we can factorize t_2 into $t_2' t_2''$ with $t_2', t_2'' \in T$. Observe that we cannot have $t_1 t_2' \in T_J$ and $t_2'' t_3 \in T_J$, because this contradicts $t \in \mathrm{Gen}(T_J)$. If both $t_1 t_2'$ and $t_2'' t_3$ belong to $T\backslash T_J$, then $t \in \mathrm{Gen}(T)^{1,\ldots,2l}$. If $t_1 t_2' \in T\backslash T_J$ and $t_2'' t_3 \in T_J$, then $t_2'' \notin T_J$, otherwise $t_1 t_2'$, t_2'', and t_3 contradict the choice of t_1, t_2, t_3, because $|t_2''| < |t_2|$. Thus, $t_1 t_2'$, t_2'' and t_3 belong to $T \setminus T_J$ and $t \in \mathrm{Gen}(T)^{1,\ldots,3l}$. Similarly, if $t_1 t_2' \in T_J$ and $t_2'' t_3 \in T \setminus T_J$, we also have $t \in \mathrm{Gen}(T)^{1,\ldots,3l}$.

Therefore, we have $\mathrm{Gen}(T_J) \subseteq \mathrm{Gen}(T)^{1,\ldots,3l}$, and thus, $T_J \subseteq \mathrm{Gen}(T)^{1,\ldots,3l l_J}$. Finally, Fact 4.10 and 4.11 together show that $T \subseteq \mathrm{Gen}(T)^{1,\ldots,3l l_J}$, i.e., $\mathrm{Gen}(T)$ has the FPP. □

4.5 Completion of the Proof

Proof of Proposition 4.1. The proposition is obviously true if T is the empty set. As a conclusion of Proposition 4.6, Proposition 4.1 holds for every concatenation closed language $T \subseteq \mathbb{M}_1^+ \times \mathbb{M}_2^+$, if there is a product automaton $[P, R, g, h, F]$ for T such that $\Pi_2(F)$ is a singleton.

Assume some integer $n > 1$. Assume that Proposition 4.1 is true for every concatenation closed language $T' \subseteq \mathbb{M}_1^+ \times \mathbb{M}_2^+$, if there is a product automaton $[P', R', g', h', F']$ for T' with $|\Pi_2(F')| < n$.

Now, let T be a concatenation closed language in $\mathbb{M}_1^+ \times \mathbb{M}_2^+$ recognized by a product automaton $[P, R, g, h, F]$ with $\Pi_2(F) = n$. Then, by Proposition 4.3, the semigroup $\Pi_2(F)$ satisfies one of the assertions (A), (B), (C), or (D) such that we can apply one of the Propositions 4.6 and 4.9, respectively. $\square$

5 Inductions on Independence Alphabets

5.1 Connections in Non-Connected Monoids

This section is devoted to the proofs of Proposition 3.3 and 3.4. Assume two disjoint trace monoids $\mathbb{M}(A_1, I_1)$ and $\mathbb{M}(A_1, I_2)$. We abbreviate them by $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively. Further, we denote $\mathbb{M}(A_1, I_1)\setminus\lambda$ and $\mathbb{M}(A_2, I_2)\setminus\lambda$ by $\mathbb{M}_1^+$ and $\mathbb{M}_2^+$, respectively. Assume a recognizable language $T \subseteq \mathbb{M}_1 \times \mathbb{M}_2$. We need a particular construction. We denote:

$$W_T = T^*_{\subseteq A_1} T^*_{\subseteq A_2} \Big(T \cap (\mathbb{M}_1^+ \times \mathbb{M}_2^+) \Big) T^*_{\subseteq A_1} T^*_{\subseteq A_2} \ \cup \ (T\setminus\lambda)^+_{\subseteq A_1} (T\setminus\lambda)^+_{\subseteq A_2}.$$

We have $W_T \subseteq \mathbb{M}_1^+ \times \mathbb{M}_2^+$ and it is easy to verify that $W_T^+ = T^* \cap (\mathbb{M}_1^+ \times \mathbb{M}_2^+)$. Hence, we have $T^* = T^*_{\subseteq A_1} \cup T^*_{\subseteq A_2} \cup W_T^*$. Now, we state two facts that give characterizations for the recognizability of T^* and the finite power property of T.

Fact 5.1 *The language T^* is recognizable if and only if $T^*_{\subseteq A_1}$, $T^*_{\subseteq A_2}$, and W_T^* are recognizable.*

Proof. Recognizability of $T^*_{\subseteq A_1}$, $T^*_{\subseteq A_2}$, and W_T^* clearly implies recognizability of their union T^*. Conversely, assume T^* is recognizable. Then, $W_T^+ = T^* \cap (\mathbb{M}_1^+ \times \mathbb{M}_2^+)$, and thus, W_T^* are recognizable. Further, $T^*_{\subseteq A_1}$ and $T^*_{\subseteq A_2}$ are recognizable as we have seen in Part 2.4. $\square$

Fact 5.2 *The language T has the FPP if and only if $T_{\subseteq A_1}$, $T_{\subseteq A_2}$, and W_T have the FPP.*

Proof. Assume that T has the FPP, i.e., assume some integer n such that $T^* = T^{0,\ldots,n}$. As seen in Part 2.4, $T_{\subseteq A_1}$ and $T_{\subseteq A_2}$ have the FPP. Now, let $t \in W_T^+ \subseteq T^+$. There exists an integer m with $1 \le m \le n$ and traces $t_1, \ldots, t_m \in (T\setminus\lambda)$ such that $t = t_1 \ldots t_m$. Every t_i belongs either to $T_{\subseteq A_1} \cup T_{\subseteq A_2}$ or to $T \cap (\mathbb{M}_1^+ \times \mathbb{M}_2^+)$. Let k be the number of traces among $t_1, \ldots, t_n$ which belong to $\mathbb{M}_1^+ \times \mathbb{M}_2^+$. If $k = 0$, then $t = t_1 \ldots t_m \in T^+_{\subseteq A_1} T^+_{\subseteq A_2} \subseteq W_T$. Otherwise, $t \in W_T^k$. Consequently, $W_T^+ = W_T^{1,\ldots,n}$, i.e., W_T has the FPP.

Conversely, assume that $T_{\subseteq A_1}$, $T_{\subseteq A_2}$, and W_T have the FPP and let $n \ge 1$ be an integer such that $T^*_{\subseteq A_1} = T^{0,\ldots,n}_{\subseteq A_1}$, $T^*_{\subseteq A_2} = T^{0,\ldots,n}_{\subseteq A_2}$, and $W_T^* = W_T^{0,\ldots,n}$. We have

$$W_T \ = \ T^{0,\ldots,n}_{\subseteq A_1} T^{0,\ldots,n}_{\subseteq A_2} \Big(T \cap (\mathbb{M}_1^+ \times \mathbb{M}_2^+) \Big) T^{0,\ldots,n}_{\subseteq A_1} T^{0,\ldots,n}_{\subseteq A_2} \ \cup \ (T\setminus\lambda)^{1,\ldots,n}_{\subseteq A_1} (T\setminus\lambda)^{1,\ldots,n}_{\subseteq A_2} \quad \subseteq \quad T^{1,\ldots,4n+1}.$$

Then, $W_T^* = W_T^{0,\ldots,n} \subseteq T^{0,\ldots,(4n+1)n}$. Hence, we have $T^* \subseteq T^{0,\ldots,(4n+1)n}$, i.e., $T^* = T^{0,\ldots,(4n+1)n}$. $\square$

Proving Proposition 3.3 means to show the equivalence of the following four assertions, provided that both the star problem and the FPP are decidable in both $\mathbb{M}_1$ and $\mathbb{M}_2$:

1. *The star problem is decidable in $\mathbb{M}_1 \times \mathbb{M}_2$.*

2. *The star problem is decidable for recognizable subsets of $\mathbb{M}_1^+ \times \mathbb{M}_2^+$.*

3. *The FPP is decidable for recognizable subsets of $\mathbb{M}_1^+ \times \mathbb{M}_2^+$.*

4. *The FPP is decidable in $\mathbb{M}_1 \times \mathbb{M}_2$.*

Proof of Proposition 3.3. We have (2)$\Leftrightarrow$(3) by Proposition 3.2. Further, we have (1)$\Rightarrow$(2) and (4)$\Rightarrow$(3), because (2) and (3) are special cases of (1) and (4), respectively.

To show (2)$\Rightarrow$(1), assume some recognizable language $T \subseteq \mathbb{M}_1 \times \mathbb{M}_2$. We apply Fact 5.1. We determine whether $T^*_{\subseteq A_1}$ and $T^*_{\subseteq A_2}$ are recognizable. If one of these sets is not recognizable, then we are done. If both $T^*_{\subseteq A_1}$ and $T^*_{\subseteq A_2}$ are recognizable, then W_T is also recognizable. Then, we can decide whether W^*_T is recognizable, because we presume (2).

We can show (3)$\Rightarrow$(4) in the same way by Fact 5.2. $\square$

Now, we prove Proposition 3.4. Assume some trace monoid $\mathbb{M}(A, I)$ with a decidable star problem and a decidable FPP. Further, assume some letter $b \notin A$. We denote $\mathbb{M}(A, I)$ and $\mathbb{M}(A, I)\setminus\lambda$ by $\mathbb{M}$ and $\mathbb{M}^+$, respectively. To show Proposition 3.4, we have to show that both the star problem and the FPP are decidable in $\mathbb{M} \times b^*$.

In the special case that $\mathbb{M}$ is a free monoid, Proposition 3.4 was already obtained by P. Gastin, E. Ochmański, A. Petit, and B. Rozoy [11]. G. Richomme adapted it to arbitrary trace monoids $\mathbb{M}$ with a decidable star problem and a decidable FPP [29]. We follow [29], but we simplify the proof by applying Proposition 3.3. Indeed, to show Proposition 3.4, we just need to show that the FPP is decidable for recognizable languages $T \subseteq \mathbb{M}^+ \times b^+$.

For any language $T \subseteq \mathbb{M}^+ \times \{b\}^+$, we denote by $\mathrm{Inf}(T)$ the set $\{u \in \Pi_A(T) \mid \binom{u}{b^m} \in T$ for infinitely many integers $m\}$. Observe that $\mathrm{Inf}(T)$ is recognizable if T is recognizable. Indeed, in this case, we can apply Mezei's Theorem and find some recognizable subsets $L_1, \ldots, L_n \subseteq \mathbb{M}^+$ and some recognizable sets $L'_1, \ldots, L'_n \subseteq b^+$ such that T is the union of $L_i \times L'_i$ for $i \in \{1, \ldots, n\}$: $\mathrm{Inf}(T)$ is then the union of the L_i for i such that L'_i is infinite, i.e., $\mathrm{Inf}(T)$ is recognizable. We adapt Proposition 4.3 in [11]:

Lemma 5.3 *Let $T \subseteq \mathbb{M}^+ \times \{b\}^+$ be a recognizable language. The set T has the FPP if and only if $\Pi_A(T)$ has the FPP and there exists an integer s such that $\Pi_A(T)^s \subseteq \Pi_A(T)^*\mathrm{Inf}(T)\Pi_A(T)^*$.*

Proof. At first, we consider some preliminary facts. The lemma is true for $T = \emptyset$ such that we can assume that $T \neq \emptyset$. Consider an automaton $[Q, h, F]$ recognizing T. Consider the sequence $h\binom{\lambda}{\lambda}$, $h\binom{\lambda}{b}$, $h\binom{\lambda}{b^2}$,... By pumping arguments, there is some integer $1 \leq m \leq |Q|$ such that for every integer $k \geq |Q|$, we have $h\binom{\lambda}{b^{k-m}} = h\binom{\lambda}{b^k} = h\binom{\lambda}{b^{k+m}}$. Consequently, for every $u \in \mathbb{M}$ and every $k \geq |Q|$, $\binom{u}{b^k} \in T$ implies $\binom{u}{b^{k-m}} \in T$ and $\binom{u}{b^k}\binom{\lambda}{b^m}^* \subseteq T$. Then, $u \in \mathrm{Inf}(t)$.

Assume some $u \in \Pi_A(T) \setminus \mathrm{Inf}(T)$. There is some $k > 0$ with $\binom{u}{b^k} \in T$. We also have $k < |Q|$. Otherwise, we could conclude by pumping that $u \in \mathrm{Inf}(t)$. Now, the following fact is immediate:

> **Fact 5.4** *For every $u \in \Pi_A(T)$, there exists an integer $1 \leq k < |Q|$ such that $\binom{u}{b^k} \in T$. Moreover, if $u \in \mathrm{Inf}(T)$, then $\binom{u}{b^k}\binom{\lambda}{b^m}^* \subseteq T$.*

Assume that T has the FPP, i.e., there is some integer n such that $T^* = T^{0,\ldots,n}$. Then, $\Pi_A(T)$ also has the FPP. We show that $\Pi_A(T)^s \subseteq \Pi_A(T)^*\mathrm{Inf}(T)\Pi_A(T)^*$ for $s = n|Q| + 1$. Assume some $u \in \Pi_A(T)^s$. There are traces $t_1, \ldots, t_s \in T$ and some $v \in b^+$ such that $t_1 \ldots t_s = \binom{u}{v}$. We have $|v| \geq s > n|Q|$. Because T has the FPP, we can factorize t into traces $t'_1, \ldots, t'_{n'} \in T$ for some $n' \leq n$. Because $|v| > n|Q|$, there is some trace t'_i among $t'_1, \ldots, t'_{n'}$ with $|\Pi_b(t'_i)| > |Q|$. Because of the pumping arguments mentioned above, we have $\Pi_A(t'_i) \in \mathrm{Inf}(T)$. Then, we have $u = \Pi_A(t'_1 \ldots t'_{i-1})\Pi_A(t'_i)\Pi_A(t'_{i+1} \ldots t'_{n'}) \in \Pi_A(T)^*\mathrm{Inf}(T)\Pi_A(T)^*$.

Conversely, assume that $\Pi_A(T)^* \subseteq \Pi_A(T)^{0,\ldots,n}$ and $\Pi_A(T)^s \subseteq \Pi_A(T)^*\mathrm{Inf}(T)\Pi_A(T)^*$ for some integers $n \geq 1, s \geq 1$. We have $\Pi_A(T)^{s,\ldots,\infty} \subseteq \Pi_A(T)^*\mathrm{Inf}(T)\Pi_A(T)^*$. Hence, we can freely assume that $s > (2n + 1)|Q|$. We show another fact:

Fact 5.5 *For every $t \in T^s$, there is some trace $t' \in T^{1,\ldots,s-1}$ such that $\Pi_A(t) = \Pi_A(t')$, $|t|_b < |t'|_b$, and $t'\binom{\lambda}{b^m}^* \subseteq T^{1,\ldots,s-1}$.*

Let write $t = \binom{u}{v}$ for some $u \in \mathbb{M}^+$ and $v \in b^+$. Note that $|v| \geq s$. We can factorize u into u_1, u_2, u_3 with $u_1, u_3 \in \Pi_A(T)^*$ and $u_2 \in \mathrm{Inf}(T)$. Because $\Pi_A(T)$ has the FPP, we have $u_1 \in \Pi_A(T)^{0,\ldots,n}$. Consequently, there is some trace $t_1 \in T^{0,\ldots,n}$ with $\Pi_A(t_1) = u_1$. By Fact 5.4, we can choose t_1 such that $|\Pi_b(t_1)| < n|Q|$. Accordingly, there is some $t_3 \in T^{0,\ldots,n}$ with $\Pi_A(t_1) = u_1$ and $|\Pi_b(t_3)| < n|Q|$. Further, there is some trace $t_2 \in T$ such that $\Pi_A(t_2) = u_2$, $|\Pi_b(t_2)| < |Q|$, and $t_2\binom{\lambda}{b^m}^* \subseteq T$. Then, $t_1t_2t_3$ is the desired trace t', above.

Now, we show that T has the FPP. We show that every trace in $T^{(m+1)s}$ belongs to $T^{1,\ldots,(m+1)s-1}$. Just assume some trace $t \in T^{(m+1)s}$. We can factorize t into $t_0, \ldots, t_m \in T^s$. Let $t'_0, \ldots, t'_m$ be the traces which we obtain by applying Fact 5.5 on $t_0, \ldots, t_m$. For $0 \leq i \leq m$, we define $n_i = |t_i|_b - |t'_i|_b$. Consider the integers $n_0, n_0+n_1, \ldots, n_0+\ldots+n_m$. There are two integers $0 \leq i < j \leq m$ such that $n_0+\ldots+n_i$ and $n_0+\ldots+n_j$ are equal modulo m. Hence, $n_{i+1}+\ldots+n_j$ is a multiple of m. Thus, we have $t_{i+1}\ldots t_j \in t'_{i+1}\ldots t'_j\binom{\lambda}{b^m}^* \subseteq T^{1,\ldots,(j-i)(s-1)}$. Consequently, $t = t_1\ldots t_n \in T^{1,\ldots,(m+1)s-1}$. □

Based on this characterization, we can prove Proposition 3.4 (following some ideas of [11]).

Proof of Proposition 3.4. By Proposition 3.3, it suffices to show that we can decide the FPP for recognizable languages $T \subseteq \mathbb{M}^+ \times b^+$. It suffices to show that the characterization in Lemma 5.3 is decidable. Assume some recognizable language $T \subseteq \mathbb{M}^+ \times b^+$. At first, we determine whether $\Pi_A(T)$ has the FPP. If this is not the case, we are done. Otherwise, we know that $\Pi_A(T)^*$ is recognizable, and we still have to show how to decide whether there is some integer s with $\Pi_A(T)^s \subseteq \Pi_A(T)^*\mathrm{Inf}(T)\Pi_A(T)^*$. As already mentioned, the set $\mathrm{Inf}(T)$ is recognizable. Hence, $\Pi_A(T)^*\mathrm{Inf}(T)\Pi_A(T)^*$ is recognizable. Assume some automaton $[Q, h, F]$ for $K = \mathbb{M} \setminus \Pi_A(T)^*\mathrm{Inf}(T)\Pi_A(T)^*$. If we have $\Pi_A(T)^{|Q|+1} \subseteq \Pi_A(T)^*\mathrm{Inf}(T)\Pi_A(T)^*$, then $|Q|+1$ is the desired integer s. Conversely, assume $\Pi_A(T)^{|Q|+1} \not\subseteq \Pi_A(T)^*\mathrm{Inf}(T)\Pi_A(T)^*$. Then, there are traces $u_1, \ldots, u_{|Q|+1} \in \Pi_A(T)$ such that $u_1\ldots u_{|Q|+1} \in K$. There are two integers $1 \leq i < j \leq |Q|+1$ such that $h(u_1\ldots u_i) = h(u_1\ldots u_j)$. Then, h yields the same value on every trace in $u_1\ldots u_i(u_{i+1}\ldots u_j)^*$. We have $u_1\ldots u_i(u_{i+1}\ldots u_j)^*(u_{j+1}\ldots u_{|Q|+1}) \subseteq K$, i.e., none of these traces belongs to $\Pi_A(T)^*\mathrm{Inf}(T)\Pi_A(T)^*$. On the other hand, we have $u_1\ldots u_i(u_{i+1}\ldots u_j)^n u_{j+1}\ldots u_{|Q|+1} \subseteq \Pi_A(T)^{|Q|+1+(n-1)(j-i)}$ for any integer n. Consequently, the desired integer s exists if and only if $\Pi_A(T)^{|Q|+1} \subseteq \Pi_A(T)^*\mathrm{Inf}(T)\Pi_A(T)^*$. We can decide this condition by standard techniques of automata theory. □

5.2 Connections in Connected Monoids

This section is entirely and uniquely devoted to the proof of Proposition 3.5: *Assume a connected independence alphabet (A, I). The star problem (resp. the FPP) is decidable in $\mathbb{M}(A, I)$ if and only if it is decidable in $\mathbb{M}(B, I)$ for every strict subset $B \subset A$.*

Obviously, the decidability of the star problem (resp. the FPP) in $\mathbb{M}(A, I)$ implies its decidability in $\mathbb{M}(B, I)$ for every subset $B \subseteq A$. Now, consider the other direction. Assume that the star problem (resp. the FPP) is decidable in $\mathbb{M}(B, I)$ for every strict subset $B \subset A$. Further, assume a recognizable language $T \subseteq \mathbb{M}(A, I)$. We can decide the star problem (resp. the FPP) in two special cases. Firstly, if there is some letter in A which does not occur in any trace in T, then we can decide whether T^* is recognizable (resp. T has the FPP), because $T \subseteq \mathbb{M}(B, I)$ for some $B \subset A$. Secondly, if every trace in T contains every letter of A, i.e., if $T_{=A} = T$, then T^* is recognizable by Proposition 2.2, and we can decide whether T has the FPP by Proposition 2.3.

The idea is to use the decidability in these special cases to show decidability for arbitrary recognizable language $T \subseteq \mathbb{M}(A, I)$. To achieve this, we recall a construction[1] by G. Pighizzini [28]. We show by Lemma 5.6 and Fact 5.7 two technical results. Then, we state Lemma 5.8 and 5.9 which give characterizations for recognizability of T^* and the finite power property of T. At last, we show that these characterizations are decidable.

G. Pighizzini called a *composition* of A a sequence $\alpha_1, \dots, \alpha_s$, $s \geq 1$ of non-empty, mutually disjoint subsets of A whose union yields A. Clearly, we have $s \leq |A|$. Let $\mathrm{Comp}(A)$ denote the set of all compositions of A. For $T \subseteq \mathbb{M}(A, I)$, let $X = (T^*)_{\subset A}$ and

$$Z_T = \cup_{(\alpha_1,\dots,\alpha_s)\in \mathrm{Comp}(A)} XY_{\alpha_1}XY_{\alpha_2}\dots XY_{\alpha_s}X,$$

where $Y_{\alpha_i} = \cup_{\alpha_i \subseteq B \subseteq A} T_{=B}$ for $1 \leq i \leq s$. Note that $alph(t) = A$ for every trace $t \in Z_T$. Hence, Z_T is a connected language, since (A, I) is connected. G. Pighizzini proved that $(T^*)_{=A} = Z_T^+$ [28]. Here, we need a slightly stronger result, because we are not only interested in the star problem, but also in the FPP.

Lemma 5.6 *For any $T \subseteq \mathbb{M}(A, I)$ and $n \geq 1$, $(T^n)_{=A} \subseteq Z_T^{1,\dots,n}$.*

We prove this lemma by G. Pighizzini's proof for $(T^*)_{=A} = Z_T^+$.

Proof. Let $Z = Z_T$ and $t \in (T^n)_{=A}$: $t = t_1t_2\dots t_n$ with $t_i \in T$ and $alph(t_1t_2\dots t_n) = A$. If $n = 1$ then $t \in Z$. Assume $n > 1$ and for every integer m, $1 \leq m < n$, $(T^m)_{=A} \subseteq Z^{1,\dots,m}$. Let denote by $j_1, j_2, \dots, j_k$ the integers i such that $alph(t_1 \dots t_i) \neq alph(t_1 \dots t_{i-1})$ and further, let $\alpha_r = alph(t_1 \dots t_{j_r}) \setminus alph(t_1 \dots t_{j_r - 1})$ for $1 \leq r \leq k$. By construction, if $r \neq r'$, then $\alpha_r \cap \alpha_{r'} = \emptyset$ and, since $t_1 \dots t_{j_1 - 1} = \lambda$, $\cup_{r=1}^k \alpha_r = alph(t_1 \dots t_{j_k}) = A$. So $(\alpha_1, \dots, \alpha_k)$ is a composition of A. Observe, $\lambda \in X$ and (since for all $r \in \{2, \dots, k\}$, $alph(t_{j_{r-1}+1} \dots t_{j_r - 1}) \subset A$) $t_{j_{r-1}+1} \dots t_{j_r - 1} \in X$. Moreover, since for every $r \in \{1, \dots, k\}$, $\alpha_r \subseteq alph(t_{j_r}) \subseteq A$, we have $t_{j_r} \in Y_{\alpha_r}$. Therefore, $t_1 \dots t_{j_k} \in XY_{\alpha_1}XY_{\alpha_2} \dots XY_{\alpha_k} \subseteq Z$. If $alph(t_{j_k+1} \dots t_n) \subset A$, then $t_{j_k+1} \dots t_n \in X$ and $t \in Z$. On the other hand, if $alph(t_{j_k+1} \dots t_n) = A$, then since $1 \leq j_k \leq n - 1$, by inductive hypothesis, $t_{j_k+1} \dots t_n \in Z^{1,\dots,n-j_k}$ and thus, $t \in ZZ^{1,\dots,n-j_k} \subseteq Z^{1,\dots,n}$. □

Now, using the following fact, we can prove two characterizations.

Fact 5.7 *If T is a recognizable subset of a connected trace monoid $\mathbb{M}(A, I)$ and if for every strict subset $B \subset A$, $T^*_{\subseteq B}$ is recognizable, then Z_T is recognizable.*

Proof. Indeed, firstly $X = (T^*)_{\subset A} = \cup_{B \subset A} T^*_{\subseteq B}$ is recognizable. Further, for every subset $B \subseteq A$, $T_{=B}$ is recognizable, and thus, for every subset $C \subseteq A$, $Y_C = \cup_{C \subseteq B \subseteq A} T_{=B}$ is also recognizable. Since $\mathrm{Comp}(A)$ is finite, it follows that Z_T is recognizable. □

Lemma 5.8 *Let $\mathbb{M}(A, I)$ be a connected trace monoid. Let $T \subseteq \mathbb{M}(A, I)$ be a recognizable set. The set T^* is recognizable if and only if for every strict subset $B \subset A$, $T^*_{\subseteq B}$ is recognizable.*

Proof. If T^* is recognizable, then we have already seen in Part 2.4 that for every strict subset $B \subset A$, $T^*_{\subseteq B}$ is recognizable.

Conversely, assume that for every strict subset $B \subset A$, $T^*_{\subseteq B}$ is recognizable. By Fact 5.7, Z_T is recognizable. Moreover, Z_T is connected such that by Proposition 2.2 Z_T^* and $Z_T^+ = Z_T^* \setminus \lambda$ are recognizable. It is easy to verify that $Z_T^+ \subseteq (T^+)_{=A}$, and by Lemma 5.6 it holds $(T^+)_{=A} \subseteq Z_T^+$. Hence, $Z_T^+ = (T^+)_{=A}$, and thus, $T^* = (T^*)_{\subset A} \cup Z_T^+$. Consequently, T^* is recognizable. □

[1] For the same purpose, we can also consider a similar construction introduced in [29]

Lemma 5.9 *Let $\mathbb{M}(A,I)$ be a connected trace monoid. Let $T \subseteq \mathbb{M}(A,I)$ be a recognizable set. The set T has the FPP if and only if for every strict subset $B \subset A$, $T_{\subseteq B}$ has the FPP and Z_T has the FPP.*

Proof. Assume T has the FPP, i.e., we have $T^* = T^{0,\ldots,n}$ for some integer n. We have seen in Part 2.4, for every subset $B \subset A$, $T_{\subseteq B}$ has the FPP. Moreover, using Lemma 5.6, we have $Z_T^+ \subseteq (T^*)_{=A} \subseteq (\cup_{i=1}^n T^i)_{=A} = \cup_{i=1}^n (T^i)_{=A} \subseteq \cup_{i=1}^n Z_T^{1,\ldots,i} \subseteq Z_T^{1,\ldots,n}$. Hence, $Z_T^* = Z_T^{0,\ldots,n}$, i.e., Z_T has the FPP.

Conversely, let m be an integer such that $Z_T^* = Z_T^{0,\ldots,m}$ and for every strict subset $B \subset A$, $T_{\subseteq B}^* = T_{\subseteq B}^{0,\ldots,m}$. Let $X = (T^*)_{\subset A}$. Observe $X \subseteq T^{0,\ldots,m}$. Moreover, for every subset $\alpha \subseteq A$, $Y_\alpha \subseteq T$ and then $Z_T = \cup_{(\alpha_1,\ldots,\alpha_s)\in \mathrm{Comp}(A)} XY_{\alpha_1}XY_{\alpha_2}\ldots XY_{\alpha_s}X \subseteq \cup_{(\alpha_1,\ldots,\alpha_s)\in\mathrm{Comp}(A)} T^{1,\ldots,(s+1)m+s}$. Since for $(\alpha_1,\ldots,\alpha_s) \in \mathrm{Comp}(A)$, we have $1 \leq s \leq |A|$. If we note $k = (|A|+1)m + |A|$, then we have $Z_T \subseteq T^{1,\ldots,k}$. Now, let $t \in T^+$. If $alph(t) \subset A$, $t \in X \subseteq T^{0,\ldots,m}$. If $alph(t) = A$, from $Z_T^+ = (T^*)_{=A}$, $t \in Z_T^+ = Z_T^{1,\ldots,m} \subseteq T^{1,\ldots,mk}$. Thus, $T^* \subseteq T^{0,\ldots,mk}$, i.e., T has the FPP. □

Finally, we are able to prove Proposition 3.5.

Proof of Proposition 3.5. Let $\mathbb{M}(A,I)$ be a connected trace monoid. If the star problem (resp. FPP) is decidable in $\mathbb{M}(A,I)$, then it is decidable in $\mathbb{M}(B,I)$ for every strict subset $B \subset A$, because every recognizable language in $\mathbb{M}(B,I)$ is also recognizable in $\mathbb{M}(A,I)$.

Now, assume that the star problem is decidable in $\mathbb{M}(B,I)$ for every strict subset $B \subset A$. Assume a recognizable language $T \subseteq \mathbb{M}(A,I)$. By Lemma 5.8, we can decide whether T^* is recognizable by deciding whether $T_{\subseteq B}^* = (T_{\subseteq B})^*$ for $B \subset A$ is recognizable.

Assume that the FPP is decidable in $\mathbb{M}(B,I)$ for every strict subset $B \subset A$. We apply Lemma 5.9. We check for every strict subset $B \subset A$ whether $T_{\subseteq B}$ has the FPP. If one of the languages $T_{\subseteq B}$ for $B \subset A$ does not have the FPP, then T cannot have the FPP. Otherwise, we still have to check whether Z_T has the FPP. Because $T_{\subseteq B}^*$ is recognizable for $B \subset A$, the language Z_T is also recognizable by Fact 5.7. Then, we can decide whether Z_T has the FPP by Proposition 2.3. □

5.3 Decidability Equivalence

In this section, we prove Theorem 3.1: *The star problem is decidable in some trace monoid* $\mathbb{M}(A,I)$ *if and only if the FPP is decidable in* $\mathbb{M}(A,I)$.

Proof of Theorem 3.1. We prove the theorem by an induction on (A,I). Assume some trace monoid $\mathbb{M}(A,I)$ with $|A| = 1$. Then, $\mathbb{M}(A,I)$ is a free monoid in which the star problem is obviously decidable and the FPP is decidable due to [15, 30].

Now, assume a trace monoid $\mathbb{M}(A,I)$ such that for every strict subset $B \subset A$, either both problems are undecidable in $\mathbb{M}(B,I)$ or both problems are decidable in $\mathbb{M}(B,I)$.

If there is some strict subset $B \subset A$ such that both problems are undecidable in $\mathbb{M}(B,I)$, then both problems are undecidable, and thus, equivalent in $\mathbb{M}(A,I)$. Hence, we only consider the case that both problems are decidable in $\mathbb{M}(B,I)$ for every $B \subset A$.

If (A,I) is connected, both problems are decidable, and thus, equivalent in $\mathbb{M}(A,I)$ by Proposition 3.5. If (A,I) is non-connected, we can split A into two disjoint subsets A_1 and A_2 such that $A_1 \times A_2 \subseteq I$. Then, we can regard $\mathbb{M}(A,I)$ as $\mathbb{M}(A_1,I) \times \mathbb{M}(A_2,I)$. We have by Proposition 3.3 the equivalence of both problems in $\mathbb{M}(A,I)$. □

5.4 Decidability in Trace Monoids without C4

In this section, we want to prove Theorem 3.6: *both the star problem and the FPP are decidable in trace monoids without C4.*

Let us recall that a trace monoid $\mathbb{M}(A,I)$ is without C4 if whatever are 4 letters a,b,c,d in A, $\mathbb{M}(\{a,b,c,d\},I) \neq \{a,b\}^* \times \{c,d\}^*$. In particular, given any subset $B \subset A$, $\mathbb{M}(B,I)$ is also a trace monoid without C4.

Proof of Theorem 3.6. We prove by an induction on A that the star problem and the FPP are decidable in trace monoids without C4. For singletons A, $\mathbb{M}(A,I)$ is a free monoid in which the star problem is obvious and the FPP is decidable due to [15, 30].

Now, assume a trace monoid $\mathbb{M}(A,I)$ without C4. Assume further that both the star problem and the FPP are decidable in $\mathbb{M}(B,I)$ for every $B \subset A$. If $\mathbb{M}(A,I)$ is connected, the result an immediate conclusion from Proposition 3.5. If $\mathbb{M}(A,I)$ is non-connected, since (A,I) is without C4, we can write $A = B \cup \{b\}$ with $b \notin B$ and $B \times \{b\} \subseteq I$. Then, the result is an immediate consequence of Proposition 3.4. □

5.5 A General Characterization

In this section, we prove Proposition 3.7 using results from Sections 5.1 and 5.2: *for every recognizable trace language T in any trace monoid, T^* is recognizable if and only if* $\mathrm{Conn}(T^*) \cup \mathrm{NConn}(T)$ *has the FPP.*

Proof of Proposition 3.7. Let $\mathbb{M} = \mathbb{M}(A,I)$ be a trace monoid. If $|A| \leq 1$, then $\mathbb{M}$ is a free monoid and each trace of $\mathbb{M}$ is connected. For every recognizable language $T \subseteq \mathbb{M}$, T^* is recognizable and $\mathrm{Conn}(T^*) = T^*$ has the FPP.

Assume $|A| > 1$ and for any strict subset $B \subset A$, the result of Proposition 3.7 is true in $\mathbb{M}(B,I)$. Let T be a recognizable language in $\mathbb{M}$ and $S_T = \mathrm{Conn}(T^*) \cup \mathrm{NConn}(T)$. Observe $S_T^* = T^*$.

1. First consider T^* is recognizable. By inductive hypothesis, for every strict subset $B \subsetneq A$, $S_{T_{\subseteq B}} = \mathrm{Conn}(T^*_{\subseteq B}) \cup \mathrm{NConn}(T_{\subseteq B})$ has the FPP. Let n be an integer such that for every strict subset $B \subset A$, $S^*_{T_{\subseteq B}} = S^{0,\ldots,n}_{T_{\subseteq B}}$. Let $t \in T^* = S_T^*$. If $alph(t) \subset A$, $t \in S^{0,\ldots,n}_{T_{\subseteq alph(t)}} \subseteq S_T^{0,\ldots,n}$. Assume $alph(t) = A$. If (A,I) is connected, then $t \in \mathrm{Conn}(T^*) \subseteq S_T$. If (A,I) is non-connected, then we split A into two disjoint subsets A_1 and A_2 with $A_1 \times A_2 \subseteq I$. Let $M_1^+ = \mathbb{M}(A_1,I) \setminus \lambda$, $M_2^+ = \mathbb{M}(A_2,I) \setminus \lambda$, and
$$W_T = T^*_{\subseteq A_1} T^*_{\subseteq A_2} (T \cap \mathbb{M}_1^+ \times \mathbb{M}_2^+) T^*_{\subseteq A_1} T^*_{\subseteq A_2} \cup (T \setminus \lambda)^+_{\subseteq A_1} (T \setminus \lambda)^+_{\subseteq A_2}$$
as in Part 5.1. From Fact 5.1, W_T^* is recognizable. From Proposition 3.3, W_T has the FPP, i.e., there exists an integer $m \geq 1$ such that $W_T^* = W_T^{0,\ldots,m}$. Since $t \in W_T^*$, $t \in W_T^{0,\ldots,m}$. Since the subsets $S_{T_{\subseteq A_1}}$ and $S_{T_{\subseteq A_2}}$ of S_T have the FPP, we have $W_T \subseteq S_T^{0,\ldots,2n} T S_T^{0,\ldots,2n} \cup S_T^{2,\ldots,2n}$. Since $T \subseteq S_T$, $W_T \subseteq S_T^{1,\ldots,4n+1}$ and $t \in S_T^{1,\ldots,(4n+1)m}$. Since $(4n+1)m > n$, $S_T^* = S_T^{0,\ldots,(4n+1)m}$, i.e., S_T has the FPP.

2. Conversely, assume that S_T has the FPP. This implies that for every strict subset $B \subset A$, the set $S_{T_{\subseteq B}}$ has the FPP, and, by inductive hypothesis, $T^*_{\subseteq B}$ is recognizable. If (A,I) is connected, T^* is recognizable from Lemma 5.8. Otherwise, $\mathrm{Conn}(T^*) \subseteq T^*_{\subset A}$. From $T^* = S_T^*$, and since there exists some $n \geq 1$ with $S_T^* \subseteq S_T^{0,\ldots,n}$, we obtain $T^* = (T^*_{\subset A} \cup T)^{0,\ldots,n}$. The language T is recognizable. Since $T^*_{\subset A} = \cup_{B \subsetneq A} T^*_{\subseteq B}$, $T^*_{\subset A}$ is also recognizable. Thus, T^* is recognizable. □

6 On Some Ideas to Solve the Star Problem

In this section, we examine some conjectures on the star problem and the FPP. First in Part 6.1, we generalize a result from E. Ochmański [26]. In this paper, the author gave two conjectures. In Part 6.2, we solve one of them showing the exact frontier of its validity. In Part 6.3, we answer partially the second conjecture. Finally, in Part 6.4, we examine an idea from M. Latteux.

6.1 Sets Containing only one Non-Connected Trace

Here, we prove Proposition 3.8: *In any trace monoid, the star problem and the FPP are decidable for languages containing at most one non-connected trace.* This result was already proved by E. Ochmański for monoids of the form $A^* \times B^*$ [26]. At first, we adapt this result and its proof to monoids of the form $\mathbb{M}(A_1, I_1) \times \mathbb{M}(A_2, I_2)$.

Lemma 6.1 *Let $\mathbb{M}_1 = \mathbb{M}(A_1, I_1)$ and $\mathbb{M}_2 = \mathbb{M}(A_2, I_2)$ be two disjoint trace monoids, $T_1 \subseteq \mathbb{M}_1$, $T_2 \subseteq \mathbb{M}_2$ be languages, and $t_1 \in \mathbb{M}_1$, $t_2 \in \mathbb{M}_2$ be non-empty traces. If both T_1^* and T_2^* are recognizable, then the following three assertions are equivalent in $\mathbb{M}_1 \times \mathbb{M}_2$:*

1. *$(T_1 \cup T_2 \cup \binom{t_1}{t_2})^*$ is recognizable.*
2. *$T_1^* \cup t_1$ and $T_2^* \cup t_2$ have the FPP.*
3. *$(T_1 \cup T_2)^* \cup \binom{t_1}{t_2}$ has the FPP.*

Proof. We denote $T = T_1 \cup T_2$ and $t = \binom{t_1}{t_2}$. We have $(T \cup t)^* = T^*(tT^*)^*$. Further, $T^* \cup t$ has the FPP if and only if there exists an integer n such that $(T^* \cup t)^* = T^*(tT^*)^{0,\ldots,n}$.

(1)⇒(2) It suffices to show that $T_1^* \cup t_1$ has the FPP. By Mezei's Theorem, there are an integer k, and some non-empty languages $K_1, \ldots, K_k \subseteq \mathbb{M}_1$, and $L_1, \ldots, L_k \subseteq \mathbb{M}_2$ such that $(T \cup t)^* = K_1 \times L_1 \cup \ldots \cup K_k \times L_k$. Let n be an integer such that for $i \in \{1, \ldots, k\}$ we have some trace $v_i \in L_i$ with $|v_i| \leq n$. Assume any trace $u \in (T_1^* \cup t_1)^*$. There is some $v \in \mathbb{M}_2$ with $|v| \leq n$ such that $\binom{u}{v} \in (T \cup t)^*$. Because $|v| \leq n$ and $t_2 \neq \lambda$, we have $\binom{u}{v} \in T^*(tT^*)^{0,\ldots,n}$. Thus, we have $u \in T_1^*(t_1T_1^*)^{0,\ldots,n}$, i.e., $u \in (T_1^* \cup t_1)^{0,\ldots,2n+1}$.

(2)⇒(3) By hypothesis, there is an integer $n \geq 1$ such that $(T_1 \cup t_1)^* = T_1^*(t_1T_1^*)^{0,\ldots,n-1}$, and $(T_2 \cup t_2)^* = T_2^*(t_2T_2^*)^{0,\ldots,n-1}$. We prove $(T \cup t)^* = T^*(tT^*)^{0,\ldots,n^3-1}$.

It is sufficient to prove: $T^*(tT^*)^{n^3} \subseteq T^*(tT^*)^{0,\ldots,n^3-1}$.

Let $s \in T^*(tT^*)^{n^3}$: $s = \Pi_{A_1}(s)\Pi_{A_2}(s)$. We factorize $\Pi_{A_1}(s)$ into $y_1 \ldots y_{n^2}$ where $y_i \in T_1^*(t_1T_1^*)^n$ for $i \in \{1, \ldots, n^2\}$. For $i \in \{1, \ldots, n^2\}$ there exists some integer $k_i \in \{1, \ldots, n\}$, such that $y_i \in T_1^*(t_1T_1^*)^{n-k_i}$. Moreover, since the integers k_i can take at most n values, there exists a value $n_1 \in \{1, \ldots, n\}$ such that there are (at least) n integers $i \in \{1, \ldots, n^2\}$ with $k_i = n_1$.

In the same way, $\Pi_{A_2}(t) = z_1 \ldots z_{n^2}$ with $z_i \in T_2^*(t_2T_2^*)^n$ and there is an integer $n_2 \in \{1, \ldots, n\}$ such that for (at least) n integers $i \in \{1, \ldots, n^2\}$, we have $z_i \in T_2^*(t_2T_2^*)^{n-n_2}$.

Considering n_2 integers i with $y_i \in T_1^*(t_1T_1^*)^{n-n_1}$ (for other i, $y_i \in T_1^*(t_1T_1^*)^n$), and n_1 integers j with $z_j \in T_2^*(t_2T_2^*)^{n-n_2}$ (for other j, $z_j \in T_2^*(t_2T_2^*)^n$), we get $y_1 \ldots y_{n^2} \in T_1^*(t_1T_1^*)^{n^3-n_1n_2}$ and $z_1 \ldots z_{n^2} \in T_2^*(t_2T_2^*)^{n^3-n_1n_2}$. Thus, $s \in T^*(tT^*)^{n^3-n_1n_2}$ and since $n_1n_2 \geq 1$, $t \in T^*(tT^*)^{0,\ldots,n^3-1}$.

(3)⇒(1) The sets T_1^* and T_2^* are recognizable. Hence, $(T_1 \cup T_2)^* = T_1^*T_2^*$ is recognizable. Because $(T_1 \cup T_2)^* \cup \binom{t_1}{t_2}$ has the FPP, its iteration is recognizable, i.e., $(T_1 \cup T_2 \cup \binom{t_1}{t_2})^*$ is recognizable. □

Now, we prove that the star problem and the FPP are decidable for recognizable languages containing at most one non-connected trace.

Proof of Proposition 3.8. Because of Proposition 2.2 and 2.3, the result is known for languages containing only connected traces, it is sufficient to prove by an induction on the independence alphabet that, for a connected recognizable trace language C and a non-connected trace t, it is both decidable whether $(C \cup t)^*$ is recognizable and whether $C \cup t$ has the FPP.

Assume some independence alphabet (A, I). If $|A| = 1$, then $\mathbb{M}(A, I)$ is a free monoid: There are only connected traces such that the previous questions are empty. Now, assume that $|A| > 1$ and for every strict subset $B \subset A$, the inductive hypothesis is true in $\mathbb{M}(B, I)$.

Assume a connected recognizable set C and a non-connected trace t. We denote $T = C \cup t$.

Assume that (A, I) is connected. We can apply the results from Part 5.2. By Lemma 5.8, T^* is recognizable if and only if for every strict subset $B \subset A$, the language $T^*_{\subseteq B}$ is recognizable. For every $B \subset A$, we can decide recognizability of $T^*_{\subseteq B}$ by the inductive hypothesis, because there is at most one non-connected trace (namely t) in $T_{\subseteq B}$. We define the language Z_T as in Part 5.2. By Lemma 5.9, T has the FPP if and only if Z_T and for every strict subset $B \subset A$, the language $T_{\subseteq B}$ has the FPP. We can decide these by Proposition 2.3 and the inductive hypothesis, respectively.

Now, assume that (A, I) is not connected: $A = A_1 \cup A_2$ with $A_1 \times A_2 \subseteq I$. At first, assume $t \notin \mathbb{M}(A_1, I_1)^+ \times \mathbb{M}(A_2, I_2)^+$. Then, we can split T into two disjoint languages $T = T_{\subseteq A_1} \cup T_{\subseteq A_2}$. We have $T^* = T^*_{\subseteq A_1} T^*_{\subseteq A_2}$. Consequently, T^* is recognizable if and only if both $T^*_{\subseteq A_1}$ and $T^*_{\subseteq A_2}$ are recognizable. Further, T has the FPP if and only if both $T_{\subseteq A_1}$ and $T_{\subseteq A_2}$ have the FPP. We can decide these conditions by the inductive hypothesis, because there is at most one non-connected trace in $T_{\subseteq A_1}$ and $T_{\subseteq A_2}$.

Assume $t \in \mathbb{M}(A_1, I_1)^+ \times \mathbb{M}(A_2, I_2)^+$. We denote $t_1 = \Pi_{A_1}(t)$ and $t_2 = \Pi_{A_2}(t)$. Then, we have $T = T_{\subseteq A_1} \cup T_{\subseteq A_2} \cup \binom{t_1}{t_2}$ and $T^* = (T^*_{\subseteq A_1} \cup T^*_{\subseteq A_2} \cup \binom{t_1}{t_2})^*$. Since T contains exactly one non-connected trace, namely $\binom{t_1}{t_2}$, the sets $T_{\subseteq A_1}$ and $T_{\subseteq A_2}$ are connected, and thus, $T^*_{\subseteq A_1}$ and $T^*_{\subseteq A_2}$ are recognizable from Proposition 2.2. We can use Lemma 6.1: T^* is recognizable if and only if both $T^*_{\subseteq A_1} \cup t_1$ and $T^*_{\subseteq A_2} \cup t_2$ have the FPP. This is decidable by the inductive hypothesis. It remains to show how to decide whether T has the FPP. By Lemma 6.1, this is the case if and only if each of the sets $T_{\subseteq A_1}$, $T_{\subseteq A_2}$, $T^*_{\subseteq A_1} \cup t_1$, and $T^*_{\subseteq A_2} \cup t_2$ has the FPP. Since $T^*_{\subseteq A_1}$ and $T^*_{\subseteq A_2}$ are recognizable, this is decidable by inductive hypothesis. $\square$

6.2 Contradicting a Conjecture by E. Ochmański

In this part, we prove Proposition 3.9, i.e., we show that for every finite language T in some trace monoid without P3, if T^* is recognizable, then there is some trace $t \in T$ such that $(T \setminus t)^*$ is recognizable. We also show that the same assertion is false in P3.

At first, we show a lemma concerning the star problem for finite languages in trace monoids without P3.

Lemma 6.2 *Let $\mathbb{M}(A, I)$ be a trace monoid without* P3. *For any finite language $T \subset \mathbb{M}$, the following two assertions are equivalent:*

1. *T^* is recognizable.*
2. *For every $a \in A$ which occurs in some non-connected trace in T, there is a trace in a^+ in T.*

Proof. For every three distinct letters $a, b, c \in A$ with aIb and bIc, we also have aIc. Otherwise, a, b, c would form a P3. Hence, we can split A into m mutually disjoint subsets $A_1, \ldots, A_m$ for some integer $m \geq 1$ such that for any two distinct letters $a, b \in A$, we have aIb if and only if there is some $i \in \{1, \ldots, m\}$ such that $a, b \in A_i$. For $i \in \{1, \ldots, m\}$, the trace monoid $\mathbb{M}(A_i, I)$ is totally commutative.

(1)$\Rightarrow$(2) First observe that this part was already proved in a more general context in [22, Corollary 4.2]. In order to be self contained, we prove it. Assume that (2) is false and consider an integer i between 1 and m. Let a be a letter in A_i which occurs in some trace in T, but not any trace from a^+ belongs to T. Since T^* is recognizable, $T^*_{\subseteq A_i}$ is recognizable. From Proposition 3.7, $\mathrm{Conn}(T^*_{\subseteq A_i}) \cup \mathrm{NConn}(T_{\subseteq A_i})$ has the FPP. This is a contradiction, because the number of occurrences of the letter a in traces of $\mathrm{Conn}(T^*_{\subseteq A_i}) \cup \mathrm{NConn}(T_{\subseteq A_i})$ is non-zero and limited by some integer (T is finite).

(2)$\Rightarrow$(1) At first, we consider the case of a totally commutative monoid ($m = 1$). Choose some integer $n > 0$ such that for every letter a which occurs in T, we have $a^n \in T^*$. Further, let $k = |\mathrm{NConn}(T)|$. Because $\mathbb{M}(A, I)$ is totally commutative, we have $T^* = \mathrm{NConn}(T)^*\mathrm{Conn}(T)^*$. We show $T^* = \mathrm{NConn}(T)^{0,\ldots,nk-1}\mathrm{Conn}(T)^*$ which implies that T^* is recognizable. It is sufficient to show that $\mathrm{NConn}(T)^{nk} \subseteq \mathrm{NConn}(T)^{0,\ldots,nk-n}\mathrm{Conn}(T)^*$. Assume some trace $t \in \mathrm{NConn}(T)^{nk}$. There is some $s \in \mathrm{NConn}(T)$ such that $t \in \mathrm{NConn}(T)^{nk-n}s^n$. We have $s^n \in \mathrm{Conn}(T)^*$, because for every $a \in alph(s)$, we have $a^n \in \mathrm{Conn}(T)^*$. Hence, we have $t \in \mathrm{NConn}(T)^{0,\ldots,nk-n}\mathrm{Conn}(T)^*$.

Now, consider the general case ($m \geq 1$). By inductively applying Lemma 5.8, we can show that T^* is recognizable by showing that $T^*_{\subseteq A_i}$ is recognizable for every $i \in \{1, \ldots, m\}$. The languages $T_{\subseteq A_i}$ are subsets of totally commutative monoids such that we can apply the case shown above. $\square$

Proof of Proposition 3.9. Assume some finite language T in a trace monoid without P3 such that T^* is recognizable. If T is connected, then $(T \setminus t)$ is connected, and from Proposition 2.2 $(T \setminus t)^*$ is recognizable for any $t \in T$. So assume some non-connected trace $t \in T$. Because T^* is recognizable, T satisfies assertion (2) in Lemma 6.2. Thus, also $T \setminus t$ satisfies assertion (2) in Lemma 6.2, and $(T \setminus t)^*$ is recognizable .

To contradict the assertion in any trace monoid with P3, it suffices to give a counter example in P3. We consider the finite language $T = \{\binom{ca}{\lambda}, \binom{c}{\lambda}, \binom{aa}{\lambda}, \binom{\lambda}{b}, \binom{ac}{b}, \binom{aca}{b}\}$. To verify that T^* is recognizable, we show

$$T^* = \mathrm{Conn}(T)^*\mathrm{NConn}(T)\mathrm{Conn}(T)^* \cup \mathrm{Conn}(T)^*.$$

Observe that $\binom{c\{a,c\}^*}{b^*} \subseteq \mathrm{Conn}(T)^* \subseteq T^*$. Assume some trace $\binom{u}{v} \in T^*$. If $u = \lambda$, $v = \lambda$, or $u \in c\{a, c\}^*$, then $\binom{u}{v} \in \mathrm{Conn}(T)^*$. Hence, it suffices to consider that $v \neq \lambda$ and $u \in a\{a, c\}^*$. Assume that $|u|_c = 0$. Then, $\binom{u}{v} \in \mathrm{Conn}(T)^*$. Assume that $|u|_c = 1$. There are two integers $i \geq 1$ and $j \geq 0$ such that $u = a^i c a^j$. Depending on whether the integers i or j are even or odd, $\binom{u}{v}$ belongs to $\binom{aa}{\lambda}^*\binom{c}{\lambda}\binom{aa}{\lambda}^*\binom{\lambda}{b}^+$, $\binom{aa}{\lambda}^*\binom{ac}{b}\binom{aa}{\lambda}^*\binom{\lambda}{b}^*$, $\binom{aa}{\lambda}^*\binom{ca}{\lambda}\binom{aa}{\lambda}^*\binom{\lambda}{b}^+$, or $\binom{aa}{\lambda}^*\binom{aca}{b}\binom{aa}{\lambda}^*\binom{\lambda}{b}^*$, i.e., $\binom{u}{v} \in \mathrm{Conn}(T)^*\mathrm{NConn}(T)\mathrm{Conn}(T)^* \cup \mathrm{Conn}(T)^*$. Finally, assume that $|u|_c > 1$. We factorize $\binom{u}{v}$ into $\binom{u'}{v'}\binom{u''}{\lambda}$ such that $|u'|_c = 1$ and $u'' \in c\{a, c\}^*$.
Then, we have $\binom{u'}{v'} \in \mathrm{Conn}(T)^*\mathrm{NConn}(T)\mathrm{Conn}(T)^* \cup \mathrm{Conn}(T)^*$ and $\binom{u''}{\lambda} \in \mathrm{Conn}(T)^*$.

On other part, whatever is the trace we delete from T, the iteration of the obtained set is not recognizable. Since the family of recognizable sets is closed by intersection, this can be observed from the following six relations:

- $\{\binom{c}{\lambda}, \binom{aa}{\lambda}, \binom{\lambda}{b}, \binom{ac}{b}, \binom{aca}{b}\}^* \cap \binom{(acc)^*}{b^*} = \binom{acc}{b}^*\binom{\lambda}{b}^*$
- $\{\binom{ca}{\lambda}, \binom{aa}{\lambda}, \binom{\lambda}{b}, \binom{ac}{b}, \binom{aca}{b}\}^* \cap \binom{(acca)^*}{b^*} = \binom{acca}{b}^*\binom{\lambda}{b}^*$

- $\{\binom{ca}{\lambda}, \binom{c}{\lambda}, \binom{\lambda}{b}, \binom{ac}{b}, \binom{aca}{b}\}^* \cap \binom{(acca)^*}{b^*} = \binom{acca}{b}^* \binom{\lambda}{b}^*$
- $\{\binom{ca}{\lambda}, \binom{c}{\lambda}, \binom{aa}{\lambda}, \binom{ac}{b}, \binom{aca}{b}\}^* \cap \binom{(ac)^*}{b^*} = \binom{\lambda}{\lambda} \cup \binom{ac}{b}^+ \binom{ac}{\lambda}^*$
- $\{\binom{ca}{\lambda}, \binom{c}{\lambda}, \binom{aa}{\lambda}, \binom{\lambda}{b}, \binom{aca}{b}\}^* \cap \binom{(aca)^*}{b^*} = \binom{aca}{b}^* \binom{\lambda}{b}^*$
- $\{\binom{ca}{\lambda}, \binom{c}{\lambda}, \binom{aa}{\lambda}, \binom{\lambda}{b}, \binom{ac}{b}\}^* \cap \binom{(ac)^*}{b^*} = \binom{ac}{b}^* \binom{\lambda}{b}^*$

We can easily verify by Mezei's Theorem that the languages which we obtained by the intersections are not recognizable. Hence, for any $t \in T$, the language $(T \setminus t)^*$ is not recognizable. $\square$

6.3 On the Second Conjecture by E. Ochmański

Here, we consider a trace monoid $\mathbb{M}(A, I)$ without C4 and a finite subset $T \subset \mathbb{M}$ which contains at least two traces. We prove Proposition 3.10: *if T^* is not recognizable, then there exists a trace $t \in T$ such that $(T \setminus t)^*$ is not recognizable.* Let recall that this is a partial answer for a conjecture by E. Ochmański [26].

Proof of Proposition 3.10. Assume that for every $t \in T$, $(T \setminus t)^*$ is recognizable. We show that T^* is recognizable. Let A be the set of the letters occurring in traces of T. We have $T \subseteq \mathbb{M}(A, I)$.

At first, assume that (A, I) is connected. Assume some strict subset $B \subset A$. There exists a trace $t \in T$ such that $alph(t) \not\subseteq B$. Then, since $(T \setminus t)^*$ is recognizable, $T^*_{\subseteq B} = (T \setminus t)^*_{\subseteq B}$ is also recognizable. Hence, T^* is recognizable by Lemma 5.8.

Now, assume that (A, I) is not connected, i.e., $A = A_1 \cup \{b\}$ with $A_1 \times \{b\} \subseteq I$. Assume that some trace $t \in b^+$ belongs to T. Then, we have $T^* = (T \setminus t)^* t^*$. Hence, T^* is recognizable.

Conversely, assume $T \cap \binom{\lambda}{b^+}$ is empty. We show that this yields a contradiction. Some trace in $\mathbb{M}(A_1, I)^+ \times b^+$ belongs to T. Because $|T| \geq 2$, we can choose some $t \in T$ such that some trace in $\mathbb{M}(A_1, I)^+ \times b^+$ belongs to $T \setminus t$. We denote $X = T \setminus t$. The iteration X^* is recognizable. By Proposition 3.7, $\mathrm{Conn}(X^*) \cup \mathrm{NConn}(X)$ have the FPP. This is a contradiction. The letter b does not occur in the traces in $\mathrm{Conn}(X^*)$, otherwise some trace in $\binom{\lambda}{b^+}$ would belong to X and T. The set $\mathrm{NConn}(X)$ is finite because X and T are finite. Thus, the number of occurrences of the letter b in traces in $\mathrm{Conn}(X^*) \cup \mathrm{NConn}(X)$ is limited by some integer. Hence, this set cannot have the FPP. $\square$

6.4 On M. Latteux' Conjecture

In this part, we prove Proposition 3.11: *Assume some trace monoid $\mathbb{M}$ which is not a free monoid. For every integer $n_0 > 0$, there is some recognizable language $T \subseteq \mathbb{M}$ such that T has the FPP, but $[T^{0,\ldots,n}]^{-1}$ does not have the FPP for any integer $n \in \{1, \ldots, n_0\}$.*

Proof of Proposition 3.11. It suffices to show the claim for the trace monoid $a^* \times b^*$. Assume some integer $n_0 > 0$. Let $k = 2n_0 + 1$. We define a recognizable language $T \subseteq a^* \times b^*$.

$$T = \binom{a^k}{b^k} \cup \binom{a}{(b^k)^+} \cup \binom{(a^k)^+}{b}$$

We show by Lemma 5.3 that T has the FPP. We have $\Pi_a(T) = a \cup (a^k)^+$ and $\mathrm{Inf}(\Pi_a(T)) = a$. Further, we have $\Pi_a(T)^* = a^*$. The language $\Pi_a(T)$ has the FPP, because any word in a^* can be factorized into (at most) one word of the form $(a^k)^+$ followed by at most $k - 1$ times the word a. We also have $\Pi_a(T)^1 \subseteq \Pi_a(T)^* \mathrm{Inf}(T) \Pi_a(T)^* = a^+$. Consequently, T has the FPP by Lemma 5.3.

We examine the iteration $T^{0,\dots,n}$ for $n \in \{1,\dots,n_0\}$. We have

$$T^{0,\dots,n} = \underbrace{\bigcup_{x \in \{0,\dots,n\}} \binom{a^{xk}}{b^{xk}}}_{=T_1} \cup \underbrace{\bigcup_{\substack{x,y \in \{1,\dots,n-1\},\\ x+y \le n}} \binom{a^x a^{yk} (a^k)^*}{b^y b^{xk} (b^k)^*}}_{=T_2} \cup \bigcup_{\substack{x \in \{0,\dots,n-1\},\\ y \in \{1,\dots,n\},\\ x+y \le n}} \Bigg(\underbrace{\binom{a^{xk} a^y}{b^{xk} b^{yk} (b^k)^*}}_{=T_3} \cup \underbrace{\binom{a^{xk} a^{yk} (a^k)^*}{b^{xk} b^y}}_{=T_4} \Bigg)$$

The set T_1 covers all traces which we obtain by the concatenation of the trace $\binom{a^k}{b^k}$ at most n times. The language T_2 covers all the traces in $T^{0,\dots,n}$ which we obtain by the concatenation of at least one trace from $\binom{a}{(b^k)^+}$, at least one trace from $\binom{(a^k)^+}{b}$, and possibly some traces $\binom{a^k}{b^k}$. The sets T_3 and T_4 cover the remaining concatenations.

We show that $[T^{0,\dots,n}]^{-1}$ does not have the FPP. We define the language $L = [T^{0,\dots,n}]^{-1}$. We have $a^{nk}b^{nk}, b^{nk}a^{nk} \in [T_1]^{-1} \subseteq L$. We examine words of the form $a^{nk}(b^{2nk}a^{2nk})^+b^{nk} \in L^*$.

Assume that L has the FPP. Then, every word of the form $a^{nk}(b^{2nk}a^{2nk})^+b^{nk}$ can be factorized into a limited number of words from L. By choosing a word from $a^{nk}(b^{2nk}a^{2nk})^+b^{nk}$ of sufficient length and factorizing it into a limited number of words from L, we obtain a factorization which includes some word in L with more than $4nk$ letters. Consequently, there is some $l > 0$ such that $a^{nk}(b^{2nk}a^{2nk})^l b^{nk}$ can be factorized into words from L, and there is one word $w \in L$ in the factorization with $|w| > 4nk$.

However, we show that this yields a contradiction. Note that $|w|_a \ge 2nk$ and $|w|_b \ge 2nk$. Hence, we have $w \notin [T_1 \cup T_3 \cup T_4]^{-1}$, i.e., $w \in [T_2]^{-1}$.

Assume that the first and the last letter of w are a. Then, $|w|_b$ is multiple of $2nk$ such that $w \notin [T_2]^{-1}$, because there are not any traces $t \in T_2$ such that $|t|_b$ is a multiple of $2nk$. If the first and the last letter of w is b, then $|w|_a$ is multiple of $2nk$, and we obtain a contradiction, accordingly.

Consequently, the first letter of w is the letter a and the last one is b, or vice versa. Assume that the first letter of w is a. There is an integer $1 \le i < 2nk$, such that $w \in a^i(b^{2nk}a^{2nk})^*b^+$. Note that in the division of i by k, we get some remainder between 1 and n (x in the expression for T_2).

The word w cannot be the first factor in the factorization of $a^{nk}(b^{2nk}a^{2nk})^l b^{nk}$ because i is not a multiple of k. We examine the predecessor w' of w in the factorization. Depending on whether the first letter of w' is a or b, w' satisfies some property: Either $|w'|_b$ is a multiple of $2nk$ or $|w'|_a + i$ yields a multiple of $2nk$. Assume $w' \in [T_1]^{-1}$. Then, $|w'|_b$ is not a multiple of $2nk$, but $|w'|_a + i$ cannot yield a multiple of $2nk$, because k divides $|w'|_a$ but k does not divide i. Assume $w' \in [T_2]^{-1}$. Then, $|w'|_b$ is not a multiple of $2nk$. Further, similar to the division of i by k, we obtain in the division of $|w'|_a$ by k some remainder between 1 and n. Hence, we obtain in the division of $|w'|_a + i$ some remainder between 2 and $2n$ (let recall $k > 2n$), i.e., $|w'|_a + i$ is not a multiple of $2nk$. If $w' \in [T_4]^{-1}$, we obtain a contradiction accordingly to the cases $w' \in [T_1]^{-1}$ and $w' \in [T_2]^{-1}$. Consequently, $w' \in [T_3]^{-1}$ which implies $w' = a^j b^{2nk} a^{2nk-i}$ for some integer j. Then, we have $|w'|_a < nk$, i.e., $j + 2nk - i < nk$. Together with $i < 2nk$, we obtain $j < nk$. Further, the division of j by k yields some remainder between 2 and $2n$.

Thus, w' cannot be the first factor in the factorization of $a^{nk}(b^{2nk}a^{2nk})^l b^{nk}$. We examine its predecessor w''. Similarly to w', the word w'' has to satisfy one property: Either $|w''|_b$ is a multiple of $2nk$ or $|w'|_a + j$ yields a multiple of $2nk$. As above, we conclude $w'' \in [T_3]^{-1}$, i.e., we have $w'' \in a^* b^{2nk} a^{2nk-j}$. Because $j < nk$, we have $|w''|_a > nk$. Such words do not belong to $[T_3]^{-1}$.

Consequently, the desired word w'' does not exist. From the assumption that the first letter of w is a we concluded a contradiction. If we assume that the first letter of w is b and the last one is a, we accordingly obtain a contradiction. Hence, the desired word w cannot exist, i.e., the assumption that L has the FPP yields a contradiction. □

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