Clausal Relations and C -clones.

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Introduction

Systems of operations are considered in the Clone theory. *Clones* are sets of operations on a fixed set that are closed under composition and contain all projections. The clones on a finite set D are the Galois closed sets of operations with respect to the well-known *Galois connection* $\text{Pol}_D - \text{Inv}_D$ induced by the preservation relation “an operation f *preserves* a relation ϱ ” (see [PK79]). The latter will be used frequently in the subsequent chapters.

In this dissertation a special set of relations on a finite set D , called *clausal relations* (see Definition 2.1.2), is introduced. The definition of a clausal relation is based on the notion of a *clausal constraint*, which is a disjunction of inequalities of the form $x \geq d$ and $x \leq d$, where $d \in D = \{0, 1, \dots, n-1\}$ and x belongs to a set X of variables. Clausal constraints were studied by N. Creignou, M. Hermann, A. Krokhin and G. Salzer (see [CHKS08]) to classify the complexity of constraints.

The correspondence between various sets of relations and clones has been proven to be quite fruitful in computer sciences; in particular, to identify polynomial-time solvable subclasses of the *Constraint Satisfaction Problem* (CSP) (see [Jea98], and [BKJ00]). In this thesis it is shown that clausal relations meet a sufficient condition that is known to ensure polynomial time solvability of the corresponding CSP, namely that their clone of polymorphisms contains a constant operation (see [JCG97]). Therefore, for clausal relations the CSP is solvable in polynomial time.

The set of all clones on D forms a complete lattice with respect to the set-theoretic inclusion. There are countably many clones on the two-element set, and their lattice is completely known (see [Pos41]). Furthermore, since there are continuum-many clones on a set containing at least three elements (see [PK79]), it seems to be hopeless to find a full description of this lattice. Therefore, a restriction to special relations makes sense.

Our aim is to describe clones which are determined by sets of clausal relations. These are called *C-clones* (see Definition 3.1.2). The restriction of the Galois connection $\text{Pol}_D - \text{Inv}_D$ to clausal relations gives rise to the Galois connection $\text{Pol}_D - C\text{Inv}_D$. The latter yields a smaller number of Galois closed sets of operations (*C-clones*), a fact that motivated us to find all *C-clones* on D , and to describe them.

In particular, this work can be seen as a contribution to reveal the structure of the clone lattice. This dissertation is organised as follows.

First, Chapter 1 recalls basic concepts such as clone; relational, weak Krasner, and Krasner clone. Furthermore, it provides the Galois connection $\text{Pol}_D - \text{Inv}_D$ and the definition of the Constraint Satisfaction Problem over sets of relations.

In Chapter 2 the set of clausal relations is defined. Some operations on clausal relations are studied. Moreover, in Section 2.2 we prove that this set can be partitioned into *trivial* and *non-trivial clausal relations*.

In Chapter 3 the lattice of C -clones is analysed. First, we prove that every C -clone is determined by a set of non-trivial clausal relations. Second, we describe the lattice of C -clones on the two-element set. The resulting lattice is a 5-element sublattice of the lattice described by E., L. Post (see [Pos41]). In Section 3.4 it is shown that clausal relations meet a sufficient condition that is known to ensure polynomial time solvability of the corresponding CSP. Finally, for $|D| \geq 3$ the existence of infinitely many C -clones is shown (see also [Var10]). Therefore, we analyse the lattice of C -clones up to equality of their unary bijective parts. Due to the Galois connection $\text{Pol}_D - \text{Inv}_D$ this is equivalent to the study of automorphism groups of clausal relations, the so-called *C -automorphism groups*.

In Chapter 4 we describe the lattice of C -automorphism groups. It turns out that the precondition of bijectivity imposes such strong constraints on the C -clone lattice that the resulting lattice is Boolean with $|D| - 1$ atoms. Furthermore, it is shown that the *Krasner clones* (see Section 1.3) corresponding to the C -automorphism groups are determined by some of their unary relations, which are intervals w.r.t. the natural order on D .

In Chapter 5 we study the C -clone lattice up to equality of the corresponding endomorphism monoids, which is the lattice of *C -monoids*. The structure of this lattice is considerably harder to understand, since unary relations do not suffice to describe C -monoids. However, in Section 5.1, it is shown that finite sets of clausal relations suffice to describe C -monoids. Then, in Section 5.2 the lattice of C -monoids for $|D| = 3$ is determined. After that, in Sections 5.3 and 5.4, all elements neighbouring the top (*co-atoms*) and the bottom (*atoms*) in the C -monoid lattice are characterised (see also [Var12]).

In Chapter 6 all co-atoms in the C -clone lattice are described. These co-atoms can be described by non-trivial binary clausal relations and it turns out that they coincide with the relations which characterise co-atoms in the lattice of C -monoids.

In Chapter 7 some open problems are presented that emerged while working on this dissertation.

Finally, in Chapter 8 some concluding remarks can be found.

1 Preliminaries

This chapter provides definitions, notations and conventions used in the subsequent chapters. Operations, relations, clones of operations defined on a fixed set will be studied. Hence, some notational issues associated with these concepts are addressed in Section 1.1. In Section 1.2 we discuss relational clones, while in Section 1.3 the connection between sets of relations and sets of operations is explained. Finally, in Section 1.4 the definition of the Constraint Satisfaction Problem over sets of relations is given.

Throughout the text $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ denotes the set of natural numbers, $\mathbb{N}_+ = \{1, 2, \dots\}$ denotes the set of positive natural numbers, and D (also called domain) represents the set $D = \{0, 1, \dots, n-1\}$ for a fixed natural number $n \geq 2$.

1.1 Clones

Let k be a positive natural number. A k -ary operation on the domain D is a function $f : D^k \rightarrow D$. We denote by $O_D^{(k)} := \{f \mid f : D^k \rightarrow D\}$ the set of all k -ary operations on D and by

$$O_D := \bigcup_{k=1}^{\infty} O_D^{(k)}$$

the set of all finitary operations on D . Moreover, for each $j \in \{1, \dots, k\}$ let denote by $e_j^k(d_1, \dots, d_k) := d_j$ the j -th projection of arity k and by

$$J_D := \bigcup_{k=1}^{\infty} \{e_j^k \mid 1 \leq j \leq k\}$$

the set of all projections on D .

The composition or superposition of a k -ary operation f with operations $g_1, \dots, g_k$ of arity m is the m -ary operation $f(g_1, \dots, g_k)$ defined by

$$f(g_1, \dots, g_k)(x_1, \dots, x_m) := f(g_1(x_1, \dots, x_m), \dots, g_k(x_1, \dots, x_m))$$

for all $x_1, \dots, x_m \in D$.

Now, we will define a clone.

1.1.1 Definition ([Pös04]). Any set of (finitary) operations (on a fixed base set D) that is closed under composition and contains all projections is called a *clone* on D . If $F \subseteq O_D$ is a clone, we will denote this by $F \leq O_D$.

Some basic examples of clones are the following:

- The set O_D is called the *full clone* D .
- The set J_D is called the *trivial clone* on D .
- The set of all idempotent operations on D , i.e.

$$I = \{f \in O_D^{(k)} \mid f(x, \dots, x) = x\}.$$

- All continuous operations on a topological space.

The following handy remark permits the definition of the *clone generated by a subset* F of O_D , denoted by $\langle F \rangle_{O_D}$, as the intersection of all clones that contain F , i.e.

$$\langle F \rangle_{O_D} := \bigcap \{C \mid C \leq O_D \wedge F \subseteq C\}.$$

1.1.2 Remark ([PK79]). The intersection of a set of clones on D is a clone.

Proof. Let C_i be clones on D where $i \in I$, and let $\mathcal{C} := \bigcap_{i \in I} C_i$. We show that $\mathcal{C}$ is a clone. Clearly, $J_D \subseteq \mathcal{C}$. Let k, m be positive natural numbers, $f \in O_D^{(k)}$ and let $g_1, \dots, g_k \in O_D^{(m)}$. If $\{f, g_1, \dots, g_k\} \subseteq \bigcap_{i \in I} C_i$, then we get $\{f, g_1, \dots, g_k\} \subseteq C_i$ for all $i \in I$. Since C_i are clones on D , we obtain $f(g_1, \dots, g_k) \in C_i$ for all $i \in I$, and hence $f(g_1, \dots, g_k) \in \mathcal{C}$. $\square$

It is not difficult to see that $\langle F \rangle_{O_D}$ is the smallest clone containing F .

The set $L_D = \{F \subseteq O_D \mid F \leq O_D\}$ of all clones on D forms a complete lattice¹ with respect to the set-theoretic inclusion. This lattice will be denoted by $\mathcal{L}_D := (L_D, \subseteq)$. The smallest element of this lattice is J_D while the greatest element is O_D .

Below, we freely use $\leq$ to denote also the usual order on integers. Nevertheless, we hope not to confuse the reader and be clear.

The *k-ary constant operation* (briefly, *constant*) $c_a^k : D^k \longrightarrow D$ is given by

$$c_a^k(x_1, \dots, x_k) := a$$

for all $x_1, \dots, x_k \in D$, where a is a fixed element of D . Moreover, a k -ary operation f on D is called *monotone*, if for all $x_1, \dots, x_k, y_1, \dots, y_k \in D$, $x_1 \leq y_1, \dots, x_k \leq y_k$ implies

$$f(x_1, \dots, x_k) \leq f(y_1, \dots, y_k).$$

Another example of a clone is the set

$$M := \{f : \{0, 1\}^k \longrightarrow \{0, 1\} \mid f \text{ is monotone}, k \in \mathbb{N}_+\},$$

¹In this work some notions as lattice, complete lattice, and ordered set will be not defined. More information about these concepts can be found in [DP02].

and is called the *clone of monotone Boolean functions*.

Let m be a positive natural number. An m -ary relation ϱ on D is a subset of the m -fold Cartesian product D^m . If $\varrho = \{r_1, \dots, r_k\}$ then it is often convenient to represent ϱ as a matrix $(r_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq k}} \in D^{m \times k}$, whose columns are the tuples in the relation, i.e. $r_j = (r_{1j}, r_{2j}, \dots, r_{mj})$ for all $j \in \{1, \dots, k\}$. We define by

$$R_D^{(m)} := \{\varrho \mid \varrho \subseteq D^m\}$$

the set of all m -ary relations on D and by

$$R_D := \bigcup_{m=1}^{\infty} R_D^{(m)}$$

the set of all finitary relations on D .

For $m \in \mathbb{N}_+$ and an equivalence relation θ on the set $\{0, \dots, m-1\}$, we define an m -ary relation d_θ on D to be the relation

$$d_\theta := \{(x_0, \dots, x_{m-1}) \in D^m \mid \forall (i, j) \in \theta : x_i = x_j\}$$

and call it *diagonal relation*. The special case $d_\theta = D^m$ is also called *trivial relation*. The set of all diagonal relations together with the empty relation $\emptyset$ is denoted by $\text{diag}(D)$.

There are only two *binary diagonal relations*, namely

$$\begin{aligned} \Delta_D^{(2)} &:= \{(x, x) \mid x \in D\} \quad \text{and} \\ \nabla_D^{(2)} &:= \{(x, y) \mid x, y \in D\} = D^2 \end{aligned}$$

the *identity relation* and the full Cartesian product of D with itself.

1.2 Relational clones

The definition to come will serve as an analogy of clones of operations with respect to the Galois connection $\text{Pol}_D - \text{Inv}_D$. The latter is defined in the next section.

A set Q of finitary relations on D containing all diagonal relations and being closed under composition using primitive positive formulas of first order logic with equality, is called a *relational clone* on D . The last statement is equivalently characterised by the fact [Beh09] that the set of finitary relations is closed under the following (infinite number of) partial operations on R_D : First, the two constants $\emptyset$ and $\Delta_D^{(2)}$; then for all arities $m, n \in \mathbb{N}_+$ and mappings $\alpha : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ the partial operations $\cap^{(m)}$, W_α and V_α , where

$$\begin{aligned} \cap^{(m)} : R_D^{(m)} \times R_D^{(m)} &\longrightarrow R_D^{(m)} \\ (\varrho, \sigma) &\mapsto \varrho \cap \sigma \end{aligned}$$

is the theoretic *intersection* of relations of the same arity,

$$\begin{aligned} W_\alpha : R_D^{(n)} &\longrightarrow R_D^{(m)} \\ \varrho &\mapsto \{(a_1, \dots, a_m) \in D^m \mid (a_{\alpha(1)}, \dots, a_{\alpha(n)}) \in \varrho\} \end{aligned}$$

is the *covariant substitution functor*² (applied to α), and

$$\begin{aligned} V_\alpha : R_D^{(m)} &\longrightarrow R_D^{(n)} \\ \sigma &\mapsto \{(a_{\alpha(1)}, \dots, a_{\alpha(n)}) \in D^n \mid (a_1, \dots, a_m) \in \sigma\}. \end{aligned}$$

is the *contravariant substitution functor*³ (applied to α).

In [Pös80] has been shown that by a special choice of α we obtain many well-known operations on R_D , below are given some of these operations, which will be used in the subsequent chapters.

1.2.1 Examples. Let $m \in \mathbb{N}_+$, $\varrho \in R_D^{(m)}$, $\mathbf{m} := \{1, \dots, m\}$ and let $S_{\mathbf{m}}$ be the *symmetric group* (of all permutations) on $\mathbf{m}$. By a special choice of a mapping α we obtain the following operations on relations:

- 1) *Cyclic permutation of coordinates:*

For $\alpha \in S_{\mathbf{m}}$ defined by $\alpha(i) := (i \bmod m) + 1$, we get

$$W_\alpha(\varrho) = \{(x_1, x_2, \dots, x_m) \in D^m \mid (x_2, x_3, \dots, x_m, x_1) \in \varrho\} =: \zeta(\varrho).$$

- 2) *Exchange of the first two coordinates:*

$$\text{For } \alpha \in S_{\mathbf{m}} \text{ such that } \alpha(i) := \begin{cases} 1 & \text{if } i = 2, \\ 2 & \text{if } i = 1, \\ i & \text{otherwise,} \end{cases}$$

we get $W_\alpha(\varrho) = \{(x_1, x_2, \dots, x_m) \in D^m \mid (x_2, x_1, x_3, \dots, x_m) \in \varrho\} =: \tau(\varrho)$.

- 3) *Permutation of coordinates:*

For $\pi \in S_{\mathbf{m}}$, we have $W_{\pi^{-1}} = V_\pi$.

- 4) *Identification of coordinates:*

$$\text{For } \alpha : \mathbf{m} \longrightarrow \{1, \dots, m-1\} \text{ such that } \alpha(i) := \begin{cases} m-1 & \text{if } i = m, \\ i & \text{otherwise,} \end{cases}$$

we get

$$W_\alpha(\varrho) = \{(x_1, x_2, \dots, x_{m-1}) \in D^{m-1} \mid (x_1, x_2, \dots, x_{m-1}, x_{m-1}) \in \varrho\} =: \Delta(\varrho).$$

²The name is due to $W_{\alpha_1} \circ W_{\alpha_2} = W_{\alpha_1 \circ \alpha_2}$

³Likewise, it holds $V_{\alpha_1} \circ V_{\alpha_2} = V_{\alpha_2 \circ \alpha_1}$

5) *Projections onto coordinates:*

For injective $\alpha : \{1, \dots, q\} \longrightarrow \mathbf{m}$ defined by $\alpha(k) \mapsto i_k$, we get

$$\begin{aligned} V_\alpha(\varrho) &= pr_{i_1, \dots, i_q}(\varrho) \\ &:= \{(x_{i_1}, \dots, x_{i_q}) \mid \exists x_j (j \in \mathbf{m} \setminus \{i_1, \dots, i_q\}) : (x_1, \dots, x_m) \in \varrho\}. \end{aligned}$$

Since the intersection of an arbitrary family of relational clones on D is also a relational clone, we can define the *relational clone generated by a subset Q* of R_D , in analogy to Remark 1.1.2, as the intersection of all relational clones that contain Q . This relational clone will be denoted by $[Q]_{R_D}$, and it is the least relational clone containing Q . If a subset Q of R_D is a relational clone, it will be denoted by $Q \leq R_D$.

1.2.2 Remark ([PK79]). The set $\text{diag}(D)$ of all diagonal relations on D is a relational clone. Furthermore, it is the smallest relational clone, which is contained in every relational clone $Q \leq R_D$. By definition the set R_D is the greatest relational clone.

The complete *lattice of relational clones* will be denoted by $\mathcal{L}_D^* := (L_D^*, \subseteq)$, where

$$L_D^* = \{Q \subseteq R_D \mid Q \leq R_D\}.$$

The greatest and smallest element of $\mathcal{L}_D^*$ are R_D and $\text{diag}(D)$, respectively. For a finite set D the lattice of all relational clones and of all clones are dually isomorphic. Figure 1.1 illustrates this interconnection via the Galois connection $\text{Pol}_D - \text{Inv}_D$.

In [Pos41], E., L. Post investigated and completely described all elements of the clone lattice $\mathcal{L}_{\{0,1\}}$. Figure 3.1 after Lemma 3.2.10, shows the structure of this lattice, this figure also describes the structure of the lattice of all relational clones on D if we turn it around.

1.3 The Galois connection Pol – Inv

Galois connections provide a very useful tool for the description of closure operators. In addition to this, on a finite set clones of operations and relational clones are in one-to-one correspondence via the Galois connection of polymorphisms and invariant relations.

We will start with the definition of a closure operator. As usual for a set M , the set of all subsets of M will be denoted by $\mathcal{P}(M)$.

1.3.1 Definition ([DW02]). A mapping $C : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$ is called a *closure operator* on M , if for all subsets $X, Y \subseteq M$ the following properties are satisfied:

- (i) $X \subseteq C(X)$ (extensivity),
- (ii) $X \subseteq Y \Rightarrow C(X) \subseteq C(Y)$ (monotonicity),
- (iii) $C(X) = C(C(X))$ (idempotency).

Subsets of M of the form $C(X)$ are called *closed* (with respect to the operator C) and $C(X)$ is said to be the closed set generated by X .

Next we provide the canonical definition of a Galois connection between sets M_1 and M_2 .

1.3.2 Definition ([Pös04]). Let M_1 and M_2 be two sets and let $\varrho \subseteq M_1 \times M_2$ be a binary relation. The pair (φ, ψ) of mappings, defined by

$$\begin{aligned} \varphi : \mathcal{P}(M_1) &\longrightarrow \mathcal{P}(M_2) \\ X &\mapsto \varphi(X) := \{y \in M_2 \mid \forall x \in X : (x, y) \in \varrho\}, \end{aligned}$$

and

$$\begin{aligned} \psi : \mathcal{P}(M_2) &\longrightarrow \mathcal{P}(M_1) \\ Y &\mapsto \psi(Y) := \{x \in M_1 \mid \forall y \in Y : (x, y) \in \varrho\}, \end{aligned}$$

is called the *Galois connection* induced by ϱ .

It is well-known that such pairs (φ, ψ) of mappings can be characterised as follows.

For all $X, X' \in \mathcal{P}(M_1)$ and all $Y, Y' \in \mathcal{P}(M_2)$ the following conditions are satisfied:

- (i) $X \subseteq X' \Rightarrow \varphi(X) \supseteq \varphi(X')$ and $Y \subseteq Y' \Rightarrow \psi(Y) \supseteq \psi(Y')$;
- (ii) $X \subseteq \psi\varphi(X)$ and $Y \subseteq \varphi\psi(Y)$.

These conditions imply that φ and ψ are anti-monotone and their compositions $\psi\varphi$ and $\varphi\psi$ are closure operators. Therefore, elements $X \in \mathcal{P}(M_1)$ and $Y \in \mathcal{P}(M_2)$ that are closed under $\varphi\psi$ or $\psi\varphi$, respectively, i.e.

$$X = \psi\varphi(X) \text{ and } Y = \varphi\psi(Y),$$

are called *Galois closed* with respect to the Galois connection (φ, ψ) . The sets of Galois closed elements are

$$\{\psi(Y) \mid Y \subseteq M_2\} \subseteq \mathcal{P}(M_1) \text{ and } \{\varphi(X) \mid X \subseteq M_1\} \subseteq \mathcal{P}(M_2),$$

and they form dually isomorphic complete lattices (w.r.t. inclusion).

Next, we will consider Galois connections between sets of operations and relations. They all are based on the preservation relation, which is the most important and basic notion of our investigations.

1.3.3 Definition ([Pös04]). We say that a k -ary operation f of $\mathcal{O}_D^{(k)}$ *preserves* an m -ary relation ϱ of $\mathcal{R}_D^{(m)}$, or ϱ is *invariant* for f , or f is a *polymorphism* of ϱ , denoted by $f \triangleright \varrho$, if whenever

$$r_1 = (a_{11}, \dots, a_{m1}) \in \varrho, \dots, r_k = (a_{1k}, \dots, a_{mk}) \in \varrho,$$

it follows that the following tuple also belongs to ϱ :

$$f[r_1, \dots, r_k] := (f(a_{11}, \dots, a_{1k}), \dots, f(a_{m1}, \dots, a_{mk})) \in \varrho.$$

To illustrate this definition, we represent $\varrho = \{r_1, \dots, r_k\}$ as an $m \times k$ -matrix. If we apply f to the rows of the matrix and interpret the m resulting values as a m -tuple, this has to be in ϱ as well,

$$\begin{array}{ccccccc} f(& a_{11} & a_{12} & \dots & a_{1k} &) = & \circ \\ f(& a_{21} & a_{22} & \dots & a_{2k} &) = & \circ \\ & \vdots & \vdots & \ddots & \vdots & & \\ f(& a_{m1} & a_{m2} & \dots & a_{mk} &) = & \circ \\ & \cap & \cap & \dots & \cap & & \cap \\ & \varrho & \varrho & \dots & \varrho & \Rightarrow & \varrho. \end{array}$$

This preservation relation $\triangleright$ induces a Galois connection between sets of operations and sets of relations on D .

1.3.4 Definition ([Pös04]). For a set $F \subseteq O_D$ of operations, we define $\text{Inv}_D F$ as the set of all relations that are *invariant* for all $f \in F$:

$$\text{Inv}_D F := \{\varrho \in R_D \mid \forall f \in F : f \triangleright \varrho\}.$$

Similarly, for a set $Q \subseteq R_D$ of relations, we define

$$\text{Pol}_D Q := \{f \in O_D \mid \forall \varrho \in Q : f \triangleright \varrho\}$$

as the *set of polymorphisms* of Q . Furthermore, for $k \in \mathbb{N}_+$ we abbreviate

$$\text{Pol}_D^{(k)} Q := O_D^{(k)} \cap \text{Pol}_D Q.$$

If D is known from the context we write Pol instead of Pol_D , and Inv instead of Inv_D . The operators Pol and Inv define the Galois connection $\text{Pol} - \text{Inv}$, see Theorem 1.3.5 below.

We can restrict this Galois connection on the operational side, by setting

$$\text{End } Q := O_D^{(1)} \cap \text{Pol } Q \quad \text{and} \quad \text{Aut } Q := S_D \cap \text{Pol } Q,$$

and call these sets of operations *endomorphisms* of Q and *automorphisms*⁴ of Q , respectively. In this context S_D denotes the symmetric group on D .

⁴In general such relation preserving permutations are called *weak automorphisms* of Q . However, Lemma 4.1.4 below states that for finite carrier sets relation preserving permutations are also relation reflecting, i.e. $f, f^{-1} \in \text{Pol } Q$. Hence in our case the notions of weak automorphism and automorphism of Q coincide.

Figure 1.1: Interconnection between clones and relational clones. Here, $\mathcal{D}$ denotes the relational clone $\text{diag}(D)$.

The following theorem gives essential properties of the operators Pol and Inv . They will be used many times in the subsequent chapters.

1.3.5 Theorem ([PK79]). *The operators Pol and Inv define a Galois connection between sets of operations and sets of relations on D , i.e., it follows that*

$$\begin{aligned}
 Q &\subseteq \text{Inv Pol } Q, & Q_1 &\subseteq Q_2 \Rightarrow \text{Pol } Q_1 \supseteq \text{Pol } Q_2, \\
 F &\subseteq \text{Pol Inv } F, & F_1 &\subseteq F_2 \Rightarrow \text{Inv } F_1 \supseteq \text{Inv } F_2, \\
 \text{Pol Inv Pol } Q &= \text{Pol } Q, & \text{Inv Pol Inv } F &= \text{Inv } F
 \end{aligned}$$

for $Q, Q_1, Q_2 \subseteq \mathbf{R}_D$ and $F, F_1, F_2 \subseteq \mathbf{O}_D$.

Furthermore, it follows that

$$\begin{aligned}
 Q &\subseteq \text{Inv } F \iff F \subseteq \text{Pol } Q, \\
 \text{Pol } Q_1 \cap \text{Pol } Q_2 &= \text{Pol}(Q_1 \cup Q_2), & \text{Pol } Q &= \bigcap_{Q' \subseteq Q} \text{Pol } Q', \\
 \text{Inv } F_1 \cap \text{Inv } F_2 &= \text{Inv}(F_1 \cup F_2), & \text{Inv } F &= \bigcap_{f \in F} \text{Inv } f.
 \end{aligned}$$

The lemma to come shows that for a set of relations Q , the set $\text{Pol } Q$ is always a clone.

1.3.6 Lemma ([PK79]). *Let $m \in \mathbb{N}_+$, and let $Q \subseteq R_D^{(m)}$ be a set of m -ary relations on D . Then the set $\text{Pol } Q$ is a clone.*

Proof. First, we prove $J_D \subseteq \text{Pol } Q$. We show that $e_j^k \in \text{Pol } Q$ for each $j \in \{1, \dots, k\}$, where $k \in \mathbb{N}_+$. Consider any tuples $r_1, \dots, r_k$ belonging to $\varrho \in Q$. For each $j \in \{1, \dots, k\}$ we have that

$$e_j^k[r_1, \dots, r_k] = r_j \in \varrho.$$

Thus, $e_j^k \triangleright \varrho$.

Let $k, m \in \mathbb{N}_+$, $f \in \text{Pol}^{(k)} Q$, and $g_1, \dots, g_k \in \text{Pol}^{(m)} Q$, we have to prove that $f(g_1, \dots, g_k) \in \text{Pol}^{(m)} Q$. Consider any tuples $r_1, \dots, r_m$ belonging to $\varrho \in Q$. Then

$$(f(g_1, \dots, g_k))[r_1, \dots, r_m] = f(g_1[r_1, \dots, r_m], \dots, g_k[r_1, \dots, r_m])$$

Because $g_1 \triangleright \varrho, \dots, g_k \triangleright \varrho$, we obtain $g_1[r_1, \dots, r_m] \in \varrho, \dots, g_k[r_1, \dots, r_m] \in \varrho$. Since $f \triangleright \varrho$, we get $f(g_1[r_1, \dots, r_m], \dots, g_k[r_1, \dots, r_m]) \in \varrho$. Hence, $f(g_1, \dots, g_k) \triangleright \varrho$. $\square$

Let us finish this section explaining the relationship between clone generation and the Galois connection Pol – Inv.

As for $F \subseteq O_D$ and $Q \subseteq R_D$, $\text{Pol } Q$ is a clone and $\text{Inv } F$ a relational clone, one obtains

$$\begin{aligned} \langle F \rangle_{O_D} &\subseteq \text{Pol } \text{Inv } F, \\ [Q]_{R_D} &\subseteq \text{Inv } \text{Pol } Q. \end{aligned}$$

For a finite set D it is well-known (cf. [PK79]) that the Galois closed sets with respect to the Galois connection Pol – Inv are precisely the clones and relational clones, i.e.

1.3.7 Theorem ([PK79]). *For $F \subseteq O_D$ and $Q \subseteq R_D$.*

$$\begin{aligned} \langle F \rangle_{O_D} &= \text{Pol } \text{Inv } F, \\ [Q]_{R_D} &= \text{Inv } \text{Pol } Q. \end{aligned}$$

The Galois closed sets of relations of the Galois connection Aut – Inv are special relational clones on D , so-called *Krasner clones*. They are exactly those relational clones that are additionally closed under finite unions of relations of the same arity and under complements of relations. Moreover, the Galois closed sets of relations of the Galois connection End – Inv are relational clones on D , so-called *weak-Krasner clones*. They are those relational clones that are additionally closed under finite unions of relations of the same arity.

1.4 The Constraint Satisfaction Problem

In the current dissertation we will investigate if the Constraint Satisfaction Problem over a special set of relations, which is defined in the next chapter, is tractable. Hence, some basic concepts and results associated to Constraint Satisfaction Problems are addressed in this section.

The aim in a Constraint Satisfaction Problem is to find an assignment of values to a given set of variables, subject to constraints on the values which can be assigned simultaneously to certain specified subsets of the variables [BKJ00]. Many combinatorial problems appearing in computer science and artificial intelligence can be expressed as particular subclasses of the CSP(cf. [Bul02]).

The general CSP is NP-complete, but certain restrictions on the form of the involved constraints may allow to solve the problem in polynomial time.

1.4.1 Definition ([BKJ00]). The *constraint satisfaction problem* (CSP) over an arbitrary subset Q of R_D , denoted $CSP(Q)$, is defined to be the decision problem with instance $(V, D, \mathcal{C})$, where

- **Instance:** V is a set of finite *variables*; D is a set of finite *values* (sometimes called a domain); and $\mathcal{C}$ is a set of *constraints*, $\{C_1, \dots, C_q\}$.

Each constraint $C_i \in \mathcal{C}$ is a pair $\langle s_i, \varrho_i \rangle$, where s_i is a tuple of variables of length m_i , called the *constraint scope*, and $\varrho_i \in Q$ an m_i -ary relation on D , called the *constraint relation*.

- **Question:** Does there exists a *solution* to $(V, D, \mathcal{C})$, that is, a function f from V to D , such that for each constraint $\langle s_i, \varrho_i \rangle \in \mathcal{C}$, with $s_i = (x_{i_1}, \dots, x_{i_m})$, the tuple $(f(x_{i_1}), \dots, f(x_{i_m}))$ belongs to ϱ_i ?

1.4.2 Definition ([JCG97]). If there exists an algorithm that solves every problem instance in $CSP(Q)$ in polynomial time, then we shall say that $CSP(Q)$ is a *tractable* problem, and Q is a *tractable* set of relations.

P. Jeavons has proved a strong theorem connecting the complexity of $CSP(Q)$ to the clone of operations $\text{Pol } Q$.

1.4.3 Theorem ([Jea98]). Let D be a finite set, and $Q, Q_0 \subseteq R_D$. If Q_0 is finite and $\text{Pol } Q \subseteq \text{Pol } Q_0$, then $CSP(Q_0)$ is reducible in polynomial time to $CSP(Q)$.

Informally speaking, Theorem 1.4.3 says that the complexity of $CSP(Q)$ is determined by the clone $\text{Pol } Q$.

Let us finish this chapter by giving the following sufficient condition to ensure tractability.

1.4.4 Proposition ([JCG97]). Let c_a be any constant operation with value a on D . For any subset Q of R_D , if $c_a \in \text{Pol } Q$ then $CSP(Q)$ is solvable in polynomial time.

2 Clausal relations

In this chapter we study a special set of relations so-called clausal relations. The name of this type of relations is justified by the fact that clausal relations on a 2-element set are exactly those representable by clauses, that is, disjunctions of variables and their negations. In Section 2.2 we discuss some operations under which clausal relations are closed.

2.1 Introduction

Clones that are determined by sets of clausal relations will be thoroughly discussed in the following chapters. Next, these relations will be defined.

2.1.1 Definition. Let $p, q \in \mathbb{N}_+$. For given parameters $\mathbf{a} = (a_1, \dots, a_p) \in D^p$ and $\mathbf{b} = (b_1, \dots, b_q) \in D^q$, the *clausal relation* $R_{\mathbf{b}}^{\mathbf{a}}$ of type (p, q) is the set of all tuples $(x_1, \dots, x_p, y_1, \dots, y_q) \in D^{p+q}$ satisfying

$$(x_1 \geq a_1) \vee \dots \vee (x_p \geq a_p) \vee (y_1 \leq b_1) \vee \dots \vee (y_q \leq b_q). \quad (2.1)$$

Note that whenever $a_i = 0$ for some $i \in \{1, \dots, p\}$ or $b_j = n - 1$ for some $j \in \{1, \dots, q\}$, then the relation $R_{\mathbf{b}}^{\mathbf{a}}$ is the full Cartesian power of D , i.e. $R_{\mathbf{b}}^{\mathbf{a}} = D^{p+q}$, because (2.1) is satisfied for any $(x_1, \dots, x_p, y_1, \dots, y_q) \in D^{p+q}$.

2.1.2 Definition. Let $p, q \in \mathbb{N}_+$. We use

$$\mathcal{R}_q^p := \{R_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in D^p, \mathbf{b} \in D^q\}$$

to denote the set of all *clausal relations of arity*¹ $p + q$ and

$$CR_D := \bigcup_{(p,q) \in \mathbb{N}_+^2} \mathcal{R}_q^p$$

for the set of all *finitary clausal relations* on D .

We write $R_{\mathbf{b}}^a$ for $R_{\mathbf{b}}^{(a)}$ in the case $p = 1$ and likewise $R_b^{\mathbf{a}}$ for $R_{(b)}^{\mathbf{a}}$ in the case $q = 1$. Furthermore, we avoid the parentheses of the expression (2.1).

Bellow we give two examples of clausal relations.

¹If we speak of a clausal relation of arity $p + q$, we implicitly mean also that the clausal relation is of type (p, q) .

2.1.3 Examples.

a) Let $D = \{0, 1\}$, then

$$R_1^0 = \{(x_1, y_1) \in D^2 \mid x_1 \geq 0 \vee y_1 \leq 1\} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} = D^2.$$

b) Let $D = \{0, 1, 2\}$, then

$$\begin{aligned} R_0^{(2,2)} &= \{(x_1, x_2, y_1) \in D^3 \mid x_1 \geq 2 \vee x_2 \geq 2 \vee y_1 \leq 0\} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 0 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}. \end{aligned}$$

2.2 Operations on clausal relations

In this section we investigate under which operations (namely $\bigcap$, $\bigcup$, W_α , V_α) clausal relations are closed. Then, we prove that a union of two clausal relations is also a clausal relation.

2.2.1 Lemma. *Let $p, q \in \mathbb{N}_+$. If $R_{\mathbf{b}}^{\mathbf{a}}$ and $R_{\mathbf{b}'}^{\mathbf{a}'}$ are clausal relations of arity $p + q$, where the parameters $\mathbf{a} = (a_1, \dots, a_p)$, $\mathbf{a}' = (a'_1, \dots, a'_p) \in D^p$ and $\mathbf{b} = (b_1, \dots, b_q)$, $\mathbf{b}' = (b'_1, \dots, b'_q) \in D^q$. Then it follows that*

$$R_{\mathbf{b}}^{\mathbf{a}} \cup R_{\mathbf{b}'}^{\mathbf{a}'} = R_{\mathbf{d}}^{\mathbf{c}},$$

where $\mathbf{c} = (\min\{a_1, a'_1\}, \dots, \min\{a_p, a'_p\})$ and $\mathbf{d} = (\max\{b_1, b'_1\}, \dots, \max\{b_q, b'_q\})$.

Proof. Let $\mathbf{z} = (x_1, \dots, x_p, y_1, \dots, y_q) \in R_{\mathbf{b}}^{\mathbf{a}} \cup R_{\mathbf{b}'}^{\mathbf{a}'}$,

$$\begin{aligned} &\iff x_1 \geq a_1 \vee \dots \vee x_p \geq a_p \vee y_1 \leq b_1 \vee \dots \vee y_q \leq b_q \vee \\ &\quad x_1 \geq a'_1 \vee \dots \vee x_p \geq a'_p \vee y_1 \leq b'_1 \vee \dots \vee y_q \leq b'_q \\ &\iff \bigvee_{1 \leq i \leq p} (x_i \geq a_i \vee x_i \geq a'_i) \vee \bigvee_{1 \leq j \leq q} (y_j \leq b_j \vee y_j \leq b'_j) \\ &\iff \bigvee_{1 \leq i \leq p} (x_i \geq \min\{a_i, a'_i\}) \vee \bigvee_{1 \leq j \leq q} (y_j \leq \max\{b_j, b'_j\}). \end{aligned}$$

This is equivalent to $\mathbf{z} \in R_{\mathbf{d}}^{\mathbf{c}}$. □

2.2.2 Lemma. *Let $p, q \in \mathbb{N}_+$. If $R_{\mathbf{b}}^{\mathbf{a}}$ and $R_{\mathbf{b}'}^{\mathbf{a}'}$ are clausal relations of arity $p + q$, where the parameters $\mathbf{a} = (a_1, \dots, a_p)$, $\mathbf{a}' = (a'_1, \dots, a'_p) \in D^p$ and $\mathbf{b} = (b_1, \dots, b_q)$, $\mathbf{b}' = (b'_1, \dots, b'_q) \in D^q$. Then it follows that*

$$R_{\mathbf{b}}^{\mathbf{a}} \cap R_{\mathbf{b}'}^{\mathbf{a}'} \supseteq R_{\mathbf{d}}^{\mathbf{c}},$$

where $\mathbf{c} = (\max\{a_1, a'_1\}, \dots, \max\{a_p, a'_p\})$ and $\mathbf{d} = (\min\{b_1, b'_1\}, \dots, \min\{b_q, b'_q\})$.

Proof. Let $\mathbf{z} = (x_1, \dots, x_p, y_1, \dots, y_q) \in R_{\mathbf{b}}^{\mathbf{a}} \cap R_{\mathbf{b}'}^{\mathbf{a}'}$

$$\begin{aligned}
 &\iff [x_1 \geq a_1 \vee \dots \vee x_p \geq a_p \vee y_1 \leq b_1 \vee \dots \vee y_q \leq b_q] \bigwedge \\
 &\quad [x_1 \geq a'_1 \vee \dots \vee x_p \geq a'_p \vee y_1 \leq b'_1 \vee \dots \vee y_q \leq b'_q] \\
 &\iff [(x_1 \geq a_1 \vee \dots \vee x_p \geq a_p) \wedge (x_1 \geq a'_1 \vee \dots \vee x_p \geq a'_p)] \vee \\
 &\quad [(y_1 \leq b_1 \vee \dots \vee y_q \leq b_q) \wedge (x_1 \geq a'_1 \vee \dots \vee x_p \geq a'_p)] \vee \\
 &\quad [(x_1 \geq a_1 \vee \dots \vee x_p \geq a_p) \wedge (y_1 \leq b'_1 \vee \dots \vee y_q \leq b'_q)] \vee \\
 &\quad [(y_1 \leq b_1 \vee \dots \vee y_q \leq b_q) \wedge (y_1 \leq b'_1 \vee \dots \vee y_q \leq b'_q)] \\
 &\iff \bigvee_{1 \leq i \leq p} (x_i \geq a_i \wedge x_i \geq a'_i) \vee \bigvee_{1 \leq j \leq q} (y_j \leq b_j \wedge y_j \leq b'_j) \\
 &\iff \bigvee_{1 \leq i \leq p} (x_i \geq \max\{a_i, a'_i\}) \vee \bigvee_{1 \leq j \leq q} (y_j \leq \min\{b_j, b'_j\}).
 \end{aligned}$$

Thus, $R_{\mathbf{b}}^{\mathbf{a}} \cap R_{\mathbf{b}'}^{\mathbf{a}'} \supseteq R_{\mathbf{d}}^{\mathbf{c}}$. □

In general $R_{\mathbf{b}}^{\mathbf{a}} \cap R_{\mathbf{b}'}^{\mathbf{a}'} = R_{\mathbf{d}}^{\mathbf{c}}$ does not hold, as is shown in the following example.

2.2.3 Example. Let $D = \{0, 1\}$, consider $\mathbf{a} = (0, 1)$, $\mathbf{a}' = (1, 0)$, $b_1 = 0$ and $b'_1 = 1$. Let also $\mathbf{z} = (x_1, x_2, y_1) = (0, 0, 1)$. The tuple $\mathbf{z}$ belongs to $R_{b_1}^{\mathbf{a}} \cap R_{b'_1}^{\mathbf{a}'}$, because $R_{b_1}^{\mathbf{a}} = D^3 = R_{b'_1}^{\mathbf{a}'}$, but $\mathbf{z} \notin R_0^{(1,1)}$.

A special case of the covariant and the contravariant substitution functor is the permutation of the entries of the tuples in a relation. For clausal relations $R_{\mathbf{b}}^{\mathbf{a}}$ this works as follows.

Let $p, q \in \mathbb{N}_+$, $\pi_1 \in S_p$, $\pi_2 \in S_q$ and let $\pi_3 : \{1, \dots, p+q\} \longrightarrow \{1, \dots, p+q\}$ such that,

$$\pi_3(i) := \begin{cases} \pi_1(i) & \text{if } i \leq p \\ \pi_2(i-p) & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
 W_{\pi_2}^{\pi_1}(R_{\mathbf{b}}^{\mathbf{a}}) &:= W_{\pi_3}(R_{\mathbf{b}}^{\mathbf{a}}) = \{(x_1, \dots, x_p, y_1, \dots, y_q) \in D^{p+q} \mid \\
 &\quad (x_{\pi_1(1)}, \dots, x_{\pi_1(p)}, y_{\pi_2(1)}, \dots, y_{\pi_2(q)}) \in R_{\mathbf{b}}^{\mathbf{a}}\}.
 \end{aligned}$$

$$\begin{aligned}
 V_{\pi_2}^{\pi_1}(R_{\mathbf{b}}^{\mathbf{a}}) &:= V_{\pi_3}(R_{\mathbf{b}}^{\mathbf{a}}) = \{(x_{\pi_1(1)}, \dots, x_{\pi_1(p)}, y_{\pi_2(1)}, \dots, y_{\pi_2(q)}) \in D^{p+q} \mid \\
 &\quad (x_1, \dots, x_p, y_1, \dots, y_q) \in R_{\mathbf{b}}^{\mathbf{a}}\}.
 \end{aligned}$$

2.2.4 Lemma. Let $p, q \in \mathbb{N}_+$, $\pi_1 \in S_p$ and $\pi_2 \in S_q$. If $R_{\mathbf{b}}^{\mathbf{a}}$ is a clausal relation of arity $p+q$, then

$$R_{\mathbf{b} \circ \pi_2}^{\mathbf{a} \circ \pi_1} = V_{\pi_2}^{\pi_1}(R_{\mathbf{b}}^{\mathbf{a}})$$

holds, where $\mathbf{a} \circ \pi_1 = (a_{\pi_1(1)}, \dots, a_{\pi_1(p)})$, $\mathbf{b} \circ \pi_2 = (b_{\pi_2(1)}, \dots, b_{\pi_2(q)})$.

Proof. Let $(\mathbf{x}, \mathbf{y}) = (x_1, \dots, x_p, y_1, \dots, y_q) \in D^{p+q}$. Then it follows that

$$\begin{aligned}
 (\mathbf{x}, \mathbf{y}) \in R_{\mathbf{b} \circ \pi_2}^{\mathbf{a} \circ \pi_1} &\Leftrightarrow (x_1 \geq a_{\pi_1(1)} \vee \dots \vee x_p \geq a_{\pi_1(p)} \vee y_1 \leq b_{\pi_2(1)} \vee \dots \vee y_q \leq b_{\pi_2(q)}) \\
 &\Leftrightarrow (x_{\pi_1^{-1}(1)} \geq a_1 \vee \dots \vee x_{\pi_1^{-1}(p)} \geq a_p \vee y_{\pi_2^{-1}(1)} \leq b_1 \vee \dots \vee y_{\pi_2^{-1}(q)} \leq b_q) \\
 &\Leftrightarrow (x_{\pi_1^{-1}(1)}, \dots, x_{\pi_1^{-1}(p)}, y_{\pi_2^{-1}(1)}, \dots, y_{\pi_2^{-1}(q)}) \in R_{\mathbf{b}}^{\mathbf{a}} \\
 &\Leftrightarrow (\mathbf{x}, \mathbf{y}) \in W_{\pi_2}^{\pi_1^{-1}}(R_{\mathbf{b}}^{\mathbf{a}}) \stackrel{1,2,1}{\Leftrightarrow} (\mathbf{x}, \mathbf{y}) \in V_{\pi_2}^{\pi_1}(R_{\mathbf{b}}^{\mathbf{a}}). \quad \square
 \end{aligned}$$

Subsets of clausal relations are not relational clones, because they are not closed under intersections. Below, we show how the identification of the first two coordinates works in clausal relations $R_{\mathbf{b}}^{\mathbf{a}}$.

Let p, q be positive natural numbers, and let $R_{\mathbf{b}}^{\mathbf{a}}$ be a clausal relation of arity $p+q$, we denote by

$$\begin{aligned}
 \Delta^p(R_{\mathbf{b}}^{\mathbf{a}}) &:= \{(x_1, \dots, x_{p-1}, y_1, \dots, y_q) \in D^{p-1+q} \mid \\
 &\quad (x_1, x_1, x_2, \dots, x_{p-1}, y_1, \dots, y_q) \in R_{\mathbf{b}}^{\mathbf{a}}\}, \\
 \Delta_q(R_{\mathbf{b}}^{\mathbf{a}}) &:= \{(x_1, \dots, x_p, y_1, \dots, y_{q-1}) \in D^{p+q-1} \mid \\
 &\quad (x_1, \dots, x_p, y_1, y_1, y_2, \dots, y_{q-1}) \in R_{\mathbf{b}}^{\mathbf{a}}\}.
 \end{aligned}$$

2.2.5 Lemma. *Let $p, q \in \mathbb{N}_+$ and let $R_{\mathbf{b}}^{\mathbf{a}}$ be a clausal relation of arity $p+q$. Then it follows that*

$$\Delta^p(R_{\mathbf{b}}^{\mathbf{a}}) = R_{\mathbf{b}}^{(\min\{a_1, a_2\}, a_3, \dots, a_p)}.$$

Proof. Let $\mathbf{z} := (x_1, x_2, \dots, x_{p-1}, y_1, \dots, y_q) \in D^{p-1+q}$. Then it follows that

$$\begin{aligned}
 \mathbf{z} \in \Delta^p(R_{\mathbf{b}}^{\mathbf{a}}) &\Leftrightarrow (x_1, x_1, x_2, \dots, x_{p-1}, y_1, \dots, y_q) \in R_{\mathbf{b}}^{\mathbf{a}} \\
 &\Leftrightarrow (x_1 \geq a_1 \vee x_1 \geq a_2 \vee x_2 \geq a_3 \vee \dots \vee x_{p-1} \geq a_p) \vee \bigvee_{1 \leq j \leq q} (y_j \leq b_j) \\
 &\Leftrightarrow (x_1 \geq \min\{a_1, a_2\} \vee x_2 \geq a_3 \vee \dots \vee x_{p-1} \geq a_p) \vee \bigvee_{1 \leq j \leq q} (y_j \leq b_j) \\
 &\Leftrightarrow \mathbf{z} \in R_{\mathbf{b}}^{(\min\{a_1, a_2\}, a_3, \dots, a_p)}.
 \end{aligned}$$

□

Similarly, one can prove the following lemma.

2.2.6 Lemma. *Let $p, q \in \mathbb{N}_+$ and let $R_{\mathbf{b}}^{\mathbf{a}}$ be a clausal relation of arity $p+q$. Then it follows that*

$$\Delta_q(R_{\mathbf{b}}^{\mathbf{a}}) = R_{\mathbf{b}}^{(\max\{b_1, b_2\}, b_3, \dots, b_q)}.$$

From 2.2.5 and 2.2.6, we deduce the corollary below.

2.2.7 Corollary. *Let $p, q \in \mathbb{N}_+$ and let $R_{\mathbf{b}}^{\mathbf{a}}$ be a clausal relation of arity $p+q$. Then it follows that*

$$\Delta_q[\Delta^p(R_{\mathbf{b}}^{\mathbf{a}})] = R_{\mathbf{b}}^{(\min\{a_1, a_2\}, a_3, \dots, a_p, \max\{b_1, b_2\}, b_3, \dots, b_q)}.$$

The following handy lemma shows that the only trivial clausal relations are those we already noticed after Definition 2.1.1. Moreover, the non-trivial ones can be easily identified by their parameters $\mathbf{a}$ and $\mathbf{b}$.

2.2.8 Lemma. *The set CR_D can be partitioned as*

$$CR_D = \{D^{p+q} \mid p, q \in \mathbb{N}_+\} \dot{\cup} CR_D^*,$$

where

$$\{D^{p+q} \mid p, q \in \mathbb{N}_+\} = CR_D \cap \text{diag}(D)$$

are the trivial clausal relations and

$$CR_D^* := \{R_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in (D \setminus \{0\})^p, \mathbf{b} \in (D \setminus \{n-1\})^q; p, q \in \mathbb{N}_+\}$$

are the non-trivial clausal relations.

Proof. Let $p, q \in \mathbb{N}_+$, $\mathbf{a} = (a_1, \dots, a_p) \in D^p$, $\mathbf{b} = (b_1, \dots, b_q) \in D^q$. We have observed above that if one of the $a_1, \dots, a_p$ equals 0, or one of the $b_1, \dots, b_q$ equals $n-1$, then $R_{\mathbf{b}}^{\mathbf{a}}$ is a total relation, i.e. $R_{\mathbf{b}}^{\mathbf{a}} = D^{p+q}$.

We have to show $CR_D^* \cap \text{diag}(D) = \emptyset$. Let us assume the existence of a relation $\varrho \in CR_D^* \cap \text{diag}(D)$. Then there exist $\mathbf{a} \in (D \setminus \{0\})^p$ and $\mathbf{b} \in (D \setminus \{n-1\})^q$ such that

$$\varrho = R_{\mathbf{b}}^{\mathbf{a}}$$

and there exists an equivalence relation θ on the set $\mathbf{m} = \{0, \dots, m-1\}$ (where $m = p+q$) such that $\varrho = d_\theta$. We show

$$\theta = \{(x, x) \mid x \in \mathbf{m}\} = \Delta_{\mathbf{m}},$$

thus

$$\varrho = d_\theta = D^m.$$

This is a contradiction to $R_{\mathbf{b}}^{\mathbf{a}} = \varrho = D^m$, because $(\underbrace{0, \dots, 0}_q, \underbrace{n-1, \dots, n-1}_q) \notin R_{\mathbf{b}}^{\mathbf{a}}$.

Let $i, j \in \{1, \dots, p\}$ with $i \neq j$. Then

$$(\underbrace{0, \dots, 0, \overset{i}{0}, 0, \dots, 0, \overset{j}{n-1}, 0, \dots, 0}_p, \underbrace{0, \dots, 0}_q) \in R_{\mathbf{b}}^{\mathbf{a}} = d_\theta.$$

Thus $(i, j) \notin \theta$. Let $i, j \in \{1, \dots, q\}$ with $i \neq j$. Then

$$(\underbrace{n-1, \dots, n-1}_p, \underbrace{0, \dots, 0, \overset{i+p}{0}, 0, \dots, 0, \overset{j+p}{n-1}, 0, \dots, 0}_q) \in R_{\mathbf{b}}^{\mathbf{a}} = d_\theta.$$

Thus $(i+p, j+p) \notin \theta$. Let $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$. Then

$$(\underbrace{n-1, \dots, n-1, \overset{i}{n-1}, n-1, \dots, n-1}_p, \underbrace{0, \dots, 0, \overset{j+p}{0}, 0, \dots, 0}_q) \in R_{\mathbf{b}}^{\mathbf{a}} = d_\theta.$$

Thus $(i, j+p) \notin \theta$.

Hence, if $i, j \in \{1, \dots, p+q\}$ with $(i, j) \in \theta$ then $i = j$. This shows $\theta = \Delta_{\mathbf{m}}$. $\square$

An example of a non-trivial clausal relation can be found in 2.1.3 b).

Let us end this section with a lemma that shows under which conditions a non-trivial clausal relation is contained in other non-trivial clausal relation.

2.2.9 Lemma. *Let $p, q \in \mathbb{N}_+$, $\mathbf{a} = (a_1, \dots, a_p)$, $\mathbf{a}' = (a'_1, \dots, a'_p) \in (D \setminus \{0\})^p$, and $\mathbf{b} = (b_1, \dots, b_q)$, $\mathbf{b}' = (b'_1, \dots, b'_q) \in (D \setminus \{n-1\})^q$. Then*

$$R_{\mathbf{b}'}^{\mathbf{a}'} \subseteq R_{\mathbf{b}}^{\mathbf{a}}$$

holds, if and only if $a'_i \geq a_i$ for all $i \in \{1, \dots, p\}$ and $b'_j \leq b_j$ for all $j \in \{1, \dots, q\}$.

Proof. $\Rightarrow$: Let $R_{\mathbf{b}'}^{\mathbf{a}'} \subseteq R_{\mathbf{b}}^{\mathbf{a}}$ and let $\mathbf{r} := (0, \dots, 0, a'_i, 0, \dots, 0, n-1, \dots, n-1)$ with $i \in \{1, \dots, p\}$. Obviously, $\mathbf{r} \in R_{\mathbf{b}'}^{\mathbf{a}'}$. Since $R_{\mathbf{b}'}^{\mathbf{a}'} \subseteq R_{\mathbf{b}}^{\mathbf{a}}$ we get

$$0 \geq a_1 \vee \dots \vee a'_i \geq a_i \vee \dots \vee 0 \geq a_p \vee n-1 \leq b_1 \vee \dots \vee n-1 \leq b_q.$$

Hence, $a'_i \geq a_i$ because $0 \not\geq a_i$ for $i \in \{1, \dots, p\}$, and $n-1 \not\leq b_j$ for $j \in \{1, \dots, q\}$. Similarly, we obtain $b'_j \leq b_j$ for $j \in \{1, \dots, q\}$ if we consider the tuple

$$(0, \dots, 0, n-1, \dots, n-1, b_j, n-1, \dots, n-1).$$

Conversely, let $(\mathbf{x}, \mathbf{y}) = (x_1, \dots, x_p, y_1, \dots, y_q) \in R_{\mathbf{b}'}^{\mathbf{a}'}$, i.e.

$$x_1 \geq a'_1 \vee \dots \vee x_p \geq a'_p \vee y_1 \leq b'_1 \vee \dots \vee y_q \leq b'_q.$$

If $x_i \geq a'_i$ for some $i \in \{1, \dots, p\}$, then because $a'_i \geq a_i$ for all $i \in \{1, \dots, p\}$, we obtain $x_i \geq a_i$. Otherwise, because of the expression above $y_j \leq b'_j$ for some $j \in \{1, \dots, q\}$ and since $b'_j \leq b_j$ for all $j \in \{1, \dots, q\}$, we get $y_j \leq b_j$. Thus, $(\mathbf{x}, \mathbf{y}) \in R_{\mathbf{b}}^{\mathbf{a}}$. $\square$

3 C -clones

In this chapter we study a restricted version of the Galois connection $\text{Pol} - \text{Inv}$; where the relations are clausal relations. This restriction provide us with a smaller number of Galois closed sets, a fact that motivated us to describe them.

3.1 Galois connection $\text{Pol} - C\text{Inv}$

Next we present a restriction of the well-known Galois connection $\text{Pol} - \text{Inv}$ where the relations are confined to be clausal relations.

3.1.1 Definition. For $F \subseteq O_D$ we define $C\text{Inv } F := \text{Inv } F \cap CR_D$. The operators

$$C\text{Inv} : \mathcal{P}(O_D) \longrightarrow \mathcal{P}(CR_D) : \quad F \mapsto C\text{Inv } F$$

and

$$\text{Pol} : \mathcal{P}(CR_D) \longrightarrow \mathcal{P}(O_D) : \quad Q \mapsto \text{Pol } Q$$

define a *Galois connection* $\text{Pol} - C\text{Inv}$, induced by the relation $\triangleright$, between sets of operations and sets of clausal relations.

We call the Galois closed sets of operations and relations of this Galois connection C -clones and relational C -clones, respectively. They are defined more formally as follows.

3.1.2 Definition. A set $F \subseteq O_D$ of operations is called a *C -clone* if $F = \text{Pol } Q$ for some set $Q \subseteq CR_D$ of clausal relations, and a set $Q \subseteq CR_D$ is called *relational C -clone* if $Q = C\text{Inv } F$ for a set F of operations.

Every Galois connection naturally gives rise to a pair of closure operators, for the ones belonging to Definition 3.1.1, we introduce a special notation.

3.1.3 Definition. For any $F \subseteq O_D$ and any $Q \subseteq R_D$ we set

$$\begin{aligned} \langle F \rangle_c &:= \text{Pol } C\text{Inv } F, \\ [Q]_c &:= C\text{Inv } \text{Pol } Q. \end{aligned}$$

The following easy and useful observation shows that every C -clone is determined by a set of non-trivial clausal relations.

3.1.4 Remark. For $Q \subseteq CR_D$, it follows that $\text{Pol}(Q) = \text{Pol}(Q \cap CR_D^*)$.

Proof. Let $Q \subseteq CR_D$. We have $Q \cap CR_D^* \subseteq Q$, hence $\text{Pol } Q \subseteq \text{Pol}(Q \cap CR_D^*)$. The other inclusion is true because $Q \setminus CR_D^* \subseteq \{D^{p+q} \mid p, q \in \mathbb{N}_+\}$ contains only trivial relations, and $f \triangleright D^{p+q}$ for any $f \in O_D$. $\square$

Let us finish this section with a lemma explaining the relationship between the closure operator $\langle F \rangle_C$ and the clone generation, i.e. the corresponding closure operator of the Galois connection $\text{Pol} - \text{Inv}$.

3.1.5 Lemma. *For any $F \subseteq O_D$ it follows that:*

$$(a) \ \langle F \rangle_{O_D} \subseteq \langle F \rangle_C.$$

$$(b) \ \langle \langle F \rangle_C \rangle_{O_D} = \langle F \rangle_C, \text{ in particular every } C\text{-clone is a clone.}$$

Proof. The first statement follows from $C\text{Inv } F \subseteq \text{Inv } F$ for any $F \subseteq O_D$, hence $\langle F \rangle_{O_D} = \text{Pol } \text{Inv } F \subseteq \text{Pol } C\text{Inv } F = \langle F \rangle_C$. For the second statement observe that $\text{Pol } \text{Inv } \langle F \rangle_C = \langle F \rangle_C$. $\square$

Let $CL_D = \{\text{Pol } Q \mid Q \subseteq CR_D\}$ be the set of all C -clones on D . We denote by $\mathcal{CL}_D := (CL_D, \subseteq)$ the lattice of all C -clones on D . The greatest element of this lattice is O_D and smallest element is $\text{Pol } CR_D$.

If the lattice $\mathcal{CL}_D = (CL_D, \subseteq)$ is finite, then it can be presented in a Hasse diagram.¹ In such a drawing, vertices correspond to C -clones and two vertices are joined by a line segment exactly if the element corresponding to the vertex drawn higher on the page covers the element corresponding to the vertex drawn lower on the page. Here a covers b means that $a \supset b$ but $a \supset c \supset b$ holds for no c .

Our main goal is to analyse the lattice of all C -clones on D , the first step, is to describe the lattice of C -clones for $D = \{0, 1\}$.

3.2 Boolean C -clones

In this section we describe the lattice of all Boolean C -clones, i.e. when D is the set $\{0, 1\}$ that we also denote by $\mathbf{2}$ for short.

From Lemma 2.2.8, the set $CR_{\mathbf{2}}$ can be partitioned as in the following corollary.

3.2.1 Corollary. *The set $CR_{\mathbf{2}}$ can be partitioned as*

$$CR_{\mathbf{2}} = \{2^{p+q} \mid p, q \in \mathbb{N}_+\} \dot{\cup} CR_{\mathbf{2}}^*,$$

where

$$CR_{\mathbf{2}}^* = \{R_{0,q}^{1,p} \mid p, q \in \mathbb{N}_+\}$$

¹More information about Hasse diagrams can be found in [DP02] and [GW99].

and

$$R_{\mathbf{0},q}^{1,p} := R_{\overbrace{(0,\dots,0)}^{q \text{ times}}}^{\overbrace{(1,\dots,1)}^{p \text{ times}}}.$$

Observe that

$$\begin{aligned} R_{\mathbf{0},q}^{1,p} &= \{(\mathbf{x}, \mathbf{y}) \in \mathbf{2}^p \times \mathbf{2}^q \mid \exists i \in \{1, \dots, p\} : x_i = 1 \vee \exists j \in \{1, \dots, q\} : y_j = 0\} \\ &= \mathbf{2}^{p+q} \setminus \{(0, \dots, 0, 1, \dots, 1)\}. \end{aligned}$$

The remark to come, which follows from 3.1.4, shows that every Boolean C -clone can be determined by sets of non-trivial clausal relations on $\mathbf{2}$.

3.2.2 Remark. For $Q \subseteq CR_{\mathbf{2}}$, it follows that $\text{Pol}(Q) = \text{Pol}(Q \cap CR_{\mathbf{2}}^*)$.

We shall describe $\{\text{Pol } Q \mid Q \subseteq CR_{\mathbf{2}}\}$. Since $\text{Pol} - C\text{Inv}$ is a Galois connection, this set is dually isomorphic to $\{C\text{Inv } F \mid F \subseteq O_{\mathbf{2}}\}$. Furthermore, we have

$$C\text{Inv } F = \bigcap_{f \in F} C\text{Inv } f$$

for $F \subseteq O_{\mathbf{2}}$. Consequently, it suffices to regard the closed relational sets $C\text{Inv } f$ for $f \in O_{\mathbf{2}}$. Since there is a one to one correspondence between $C\text{Inv } f$ and $\langle f \rangle_C$ via the operators Pol and $C\text{Inv}$, we first consider one-generated C -clones. By Remark 3.2.2 and Definition 3.1.3,

$$\langle f \rangle_C = \text{Pol } C\text{Inv } f = \text{Pol}(C\text{Inv } f \cap CR_{\mathbf{2}}^*),$$

i.e. $\langle f \rangle_C$ is the set of all the functions that preserve all the non-trivial invariant clausal relations of f .

For the rest of this section we are going to characterise, one-generated C -clones for some special functions $f \in O_{\mathbf{2}}$, (namely $f \in \{\neg, h, \vee, \wedge, g\}$). We also use the notation from Fig. 3.1 after Lemma 3.2.10, without further explanation.

3.2.3 Lemma. *Let $\neg : \mathbf{2} \rightarrow \mathbf{2}$ be the negation operation, i.e. $\neg(0) = 1$ and $\neg(1) = 0$. Then we have*

$$\langle \neg \rangle_C = O_{\mathbf{2}}.$$

Proof. By Definition 3.1.3 and Remark 3.2.2, we have

$$\langle \neg \rangle_C = \{f \in O_{\mathbf{2}} \mid \forall p, q \in \mathbb{N}_+ : \neg \triangleright R_{\mathbf{0},q}^{1,p} \implies f \triangleright R_{\mathbf{0},q}^{1,p}\}. \quad (3.1)$$

Because of (3.1) it is enough to show $\neg \not\triangleright R_{\mathbf{0},q}^{1,p}$ for all $p, q \in \mathbb{N}_+$. Indeed, the tuple $\mathbf{r} = (1, \dots, 1, 0, \dots, 0)$ belongs to $R_{\mathbf{0},q}^{1,p} = \mathbf{2}^{p+q} \setminus \{(0, \dots, 0, 1, \dots, 1)\}$ for all $p, q \in \mathbb{N}_+$, but $\neg[\mathbf{r}] = (\neg(1), \dots, \neg(1), \neg(0), \dots, \neg(0)) = (0, \dots, 0, 1, \dots, 1) \notin R_{\mathbf{0},q}^{1,p}$. $\square$

Note that for $p = q = 1$ it holds

$$R_{0,q}^{1,p} = R_0^1 = \{(0,0), (1,0), (1,1)\} = \geq,$$

and hence

$$\text{Pol}(R_0^1) = \text{Pol}(\geq) = \text{Pol}(\leq) = M,$$

where M is the clone of all monotone Boolean functions.

3.2.4 Lemma. *Let $h \in O_2^{(3)}$ be the ternary majority function on $\mathbf{2}$ (median), i.e. $h(x, y, z) := (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ for $x, y, z \in \mathbf{2}$. Then we have*

$$\langle h \rangle_c = \text{Pol}(R_0^1).$$

Proof. We show that $h \not\preceq R_{0,q}^{1,p}$, unless $p = 1, q = 1$. We consider several cases:

- $p \geq 2, q \geq 1$: The scheme

$$\begin{array}{rcl} h(1 & 0 & 0) = 0 \\ h(0 & 1 & 0) = 0 \\ h(0 & 0 & 0) = 0 \\ & \vdots & \\ h(0 & 0 & 0) = 0 \quad (\text{row } p) \\ h(1 & 1 & 0) = 1 \\ h(1 & 1 & 1) = 1 \\ & \vdots & \\ h(1 & 1 & 1) = 1 \quad (\text{row } p + q) \end{array}$$

shows that $h \not\preceq R_{0,q}^{1,p}$, because the tuples (columns of the arguments of h) all belong to $R_{0,q}^{1,p}$, but after applying h to the tuples, one obtains a tuple (column) that does not belong to $R_{0,q}^{1,p}$.

- $p \geq 1, q \geq 2$: Likewise, the scheme

$$\begin{array}{rcl} h(1 & 0 & 0) = 0 \\ h(0 & 0 & 0) = 0 \\ & \vdots & \\ h(0 & 0 & 0) = 0 \quad (\text{row } p) \\ h(1 & 0 & 1) = 1 \\ h(1 & 1 & 0) = 1 \\ h(1 & 1 & 1) = 1 \\ & \vdots & \\ h(1 & 1 & 1) = 1 \quad (\text{row } p + q) \end{array}$$

shows that $h \not\preceq R_{0,q}^{1,p}$.

If $p = q = 1$, then $h \triangleright R_0^1$, because h is a monotone operation and $M = \text{Pol}(R_0^1)$. $\square$

3.2.5 Lemma. *Let $\wedge$ and $\vee$ be the binary operators, logical conjunction and logical disjunction, respectively. Then we have*

$$\begin{aligned}\langle \wedge \rangle_C &= \text{Pol}\{R_{\mathbf{0},q}^{1,p} \mid p = 1, q \in \mathbb{N}_+\}, \\ \langle \vee \rangle_C &= \text{Pol}\{R_{\mathbf{0},q}^{1,p} \mid q = 1, p \in \mathbb{N}_+\}.\end{aligned}$$

Proof. At first we have a look at $\wedge$:

We show that $\wedge \not\triangleright R_{\mathbf{0},q}^{1,p}$, unless $p = 1$ and $q \in \mathbb{N}_+$. We consider two cases:

- $p \geq 2, q \in \mathbb{N}_+$: The scheme

$$\begin{array}{ccccccc} 1 & \wedge & 0 & = & 0 & & \\ 0 & \wedge & 1 & = & 0 & & \\ 0 & \wedge & 0 & = & 0 & & \\ & & \vdots & & & & \\ 0 & \wedge & 0 & = & 0 & (\text{row } p) & \\ 1 & \wedge & 1 & = & 1 & & \\ & & \vdots & & & & \\ 1 & \wedge & 1 & = & 1 & (\text{row } p + q) & \end{array}$$

shows that $\wedge \not\triangleright R_{\mathbf{0},q}^{1,p}$.

- $p = 1, q \in \mathbb{N}_+$: We show

$$\wedge \triangleright R_{\mathbf{0},q}^{1,p}.$$

We assume the existence of tuples

$$(x_1, y_1, \dots, y_q), (x_2, z_1, \dots, z_q) \in R_{\mathbf{0},q}^{1,p}$$

such that

$$x_1 \wedge x_2 = 0$$

and for all $j \in \{1, \dots, q\}$

$$y_j \wedge z_j = 1.$$

Because of $x_1 \wedge x_2 = 0$ w.l.o.g. $x_1 = 0$. Then our assumption

$$(x_1, y_1, \dots, y_q) \in R_{\mathbf{0},q}^{1,p}$$

implies that there is one $j \in \{1, \dots, q\}$ such that $y_j = 0$. Thus, $y_j \wedge z_j = 0$, a contradiction.

Similarly, the result for $\vee$ can be proved. $\square$

In 3.2.1 we saw that clausal relations are either total or total without one tuple, and none of the diagonals except total relations are clausal relations. Furthermore, $C\text{Inv } O_2 = \text{Inv } O_2 \cap CR_2 = \text{diag}(\mathbf{2}) \cap CR_2$, hence we obtain the following lemma.

3.2.6 Lemma. *The following equality holds*

$$C\text{Inv } \mathbf{O}_2 = \{\mathbf{2}^{(p+q)} \mid p, q \in \mathbb{N}_+\}.$$

Observe that the unary constant operations c_0, c_1 on $\mathbf{2}$, preserve all non-trivial clausal relations on $\mathbf{2}$. That is, because $c_0(y_1) = 0$ and $c_1(x_1) = 1$ for any tuple $(x_1, \dots, x_p, y_1, \dots, y_q)$ of $\mathbf{R}_{\mathbf{0},q}^{1,p}$, i.e. For any $p, q \in \mathbb{N}_+$ we have

$$c_0 \triangleright \mathbf{R}_{\mathbf{0},q}^{1,p} \text{ and } c_1 \triangleright \mathbf{R}_{\mathbf{0},q}^{1,p}.$$

In the rest of the section we will freely use $\vee$ to denote supremum of two clones in Post's Lattice (see Fig. 3.1). Nevertheless, we hope not to confuse the reader and be clear.

3.2.7 Lemma. *The smallest Boolean C -clone is*

$$\langle \emptyset \rangle_C = \text{Pol}(C\mathbf{R}_2) = \{f \in \mathbf{O}_2 \mid \forall p, q \in \mathbb{N}_+ : f \triangleright \mathbf{R}_{\mathbf{0},q}^{1,p}\}.$$

It holds

$$\langle \emptyset \rangle_C \supseteq \langle c_0 \rangle_{\mathbf{O}_2} \vee \langle c_1 \rangle_{\mathbf{O}_2} \stackrel{\text{Fig. 3.1}}{=} \langle c_0, c_1 \rangle_{\mathbf{O}_2},$$

and

$$\begin{aligned} \langle \emptyset \rangle_C &\subsetneq \langle \wedge \rangle_C \subsetneq M, \\ \langle \emptyset \rangle_C &\subsetneq \langle \vee \rangle_C \subsetneq M. \end{aligned}$$

Furthermore, $\langle \wedge \rangle_C$ and $\langle \vee \rangle_C$ are incomparable C -clones.

Proof. From the previous observation we obtain $c_0, c_1 \in \langle \emptyset \rangle_C$. Because $\langle \emptyset \rangle_C$ is a Boolean clone, we have

$$\langle c_0 \rangle_{\mathbf{O}_2}, \langle c_1 \rangle_{\mathbf{O}_2} \subseteq \langle \langle \emptyset \rangle_C \rangle_{\mathbf{O}_2} \stackrel{3.1.5}{=} \langle \emptyset \rangle_C.$$

Thus, $\langle \emptyset \rangle_C$ is an upper bound for $\langle c_0 \rangle_{\mathbf{O}_2}$ and $\langle c_1 \rangle_{\mathbf{O}_2}$. Hence,

$$\langle c_0 \rangle_{\mathbf{O}_2} \vee \langle c_1 \rangle_{\mathbf{O}_2} \subseteq \langle \emptyset \rangle_C.$$

Because $\langle \emptyset \rangle_C = \text{Pol}(C\mathbf{R}_2)$, we have that neither $\wedge \in \langle \emptyset \rangle_C$ nor $\vee \in \langle \emptyset \rangle_C$, because $\wedge \not\triangleright \mathbf{R}_{\mathbf{0},q}^{1,p}$ for $p \geq 2$ and $\vee \not\triangleright \mathbf{R}_{\mathbf{0},q}^{1,p}$ for $q \geq 2$. Thus,

$$\wedge \in \langle \wedge \rangle_C \setminus \langle \emptyset \rangle_C,$$

$$\vee \in \langle \vee \rangle_C \setminus \langle \emptyset \rangle_C.$$

Lemma 3.2.4 implies $M = \text{Pol}(\mathbf{R}_0^1)$. Thence, (cf. 3.2.5)

$$\langle \wedge \rangle_C, \langle \vee \rangle_C \subseteq M \stackrel{3.2.4}{=} \langle h \rangle_C.$$

This inclusion is proper since

$$h \in (M \setminus \langle \wedge \rangle_C) \cap (M \setminus \langle \vee \rangle_C).$$

This holds because h is a monotone operation and $h \not\vdash R_{0,q}^{1,p}$ for $p = 1, q > 1$ and for $q = 1, p > 1$.

Because of $\wedge \not\vdash R_{0,q}^{1,p}$ for $p \geq 2$ and $q = 1$, we have $\wedge \in \langle \wedge \rangle_C \setminus \langle \vee \rangle_C$, and because of $\vee \not\vdash R_{0,q}^{1,p}$ for $p = 1$ and $q \geq 2$ we have $\vee \in \langle \vee \rangle_C \setminus \langle \wedge \rangle_C$. Consequently, the two C -clones are incomparable. $\square$

3.2.8 Lemma. *For any subset $F \subseteq O_2$ it follows that*

$$\langle c_0, c_1 \rangle_{O_2} \vee \langle F \rangle_{O_2} \subseteq \langle F \rangle_C.$$

Proof. From 3.2.7 we infer

$$\langle c_0, c_1 \rangle_{O_2} \subseteq \langle \emptyset \rangle_C \subseteq \langle F \rangle_C.$$

Because of $\langle F \rangle_C \in \mathcal{L}_2$ and $F \subseteq \langle F \rangle_C$ we have

$$\langle F \rangle_{O_2} \subseteq \langle \langle F \rangle_C \rangle_{O_2} \stackrel{3.1.5}{=} \langle F \rangle_C.$$

Consequently, we obtain $\langle c_0, c_1 \rangle_{O_2} \vee \langle F \rangle_{O_2} \subseteq \langle F \rangle_C$. $\square$

3.2.9 Lemma. *The following equalities hold*

$$\langle c_0, c_1, \wedge \rangle_{O_2} = \langle \wedge \rangle_C \quad \text{and} \quad \langle c_0, c_1, \vee \rangle_{O_2} = \langle \vee \rangle_C.$$

Proof. From 3.2.8 and Fig. 3.1 we obtain

$$\langle c_0, c_1, \wedge \rangle_{O_2} \stackrel{\text{Fig. 3.1}}{=} \langle c_0, c_1 \rangle_{O_2} \vee \langle \wedge \rangle_{O_2} \subseteq \langle \wedge \rangle_C.$$

Let us assume

$$\langle c_0, c_1, \wedge \rangle_{O_2} \subsetneq \langle \wedge \rangle_C.$$

Then, $\langle \wedge \rangle_C$ has to be a clone in $\mathcal{L}_2$ being above $\langle c_0, c_1, \wedge \rangle_{O_2}$. Because the upper cover of $\langle c_0, c_1, \wedge \rangle_{O_2}$ in $\mathcal{L}_2$ is M (see Post's Lattice, Fig. 3.1), it follows

$$M \subseteq \langle \wedge \rangle_C,$$

which is a contradiction to $M \supset \langle \wedge \rangle_C$ (cf. Lemma 3.2.7). Similarly, the claim for $\langle \vee \rangle_C$ can be proved. $\square$

The lemma to come is valid not only for $D = 2$, but also for $D = \{0, 1, \dots, n-1\}$.

3.2.10 Lemma. *For any two subsets $F, G \subseteq O_D$ the following implication holds*

$$F \subseteq G \subseteq \langle F \rangle_C \implies \langle G \rangle_C = \langle F \rangle_C.$$

Proof. Since $\langle \cdot \rangle_C$ is a closure operator, we obtain

$$\begin{aligned} F \subseteq G &\implies \langle F \rangle_C \subseteq \langle G \rangle_C \quad \text{and} \\ G \subseteq \langle F \rangle_C &\implies \langle G \rangle_C \subseteq \langle \langle F \rangle_C \rangle_C = \langle F \rangle_C. \end{aligned}$$

$\square$

Figure 3.1: Post's Lattice

3.2.11 Lemma. Let c_0, c_1 be the unary constant operations on $\mathbf{2}$ with values 0 and 1, respectively. Then it follows that

$$\langle \emptyset \rangle_c = \langle c_0, c_1 \rangle_{O_2}.$$

Proof. It holds,

$$\langle \emptyset \rangle_c \subseteq \langle \vee \rangle_c \cap \langle \wedge \rangle_c \stackrel{3.2.9}{=} \langle c_0, c_1, \wedge \rangle_{O_2} \cap \langle c_0, c_1, \vee \rangle_{O_2} \stackrel{\text{Fig. 3.1}}{=} \langle c_0, c_1 \rangle_{O_2} \stackrel{3.2.7}{\subseteq} \langle \emptyset \rangle_c.$$

□

3.2.12 Lemma. Let g be the ternary minority operation, i.e.

$$g(x, x, y) = g(x, y, x) = g(y, x, x) = y.$$

Then we have the following equalities

$$\langle g \rangle_c = \langle L \rangle_c = O_2.$$

Proof. Because of $\neg \in L$, it follows that

$$O_2 \stackrel{3.2.3}{=} \langle \neg \rangle_c \subseteq \langle L \rangle_c \subseteq O_2,$$

hence $\langle L \rangle_c = O_2$. Applying Lemma 3.2.8 to $\{g\}$ leads to

$$\{g\} \subseteq L \stackrel{\text{Fig. 3.1}}{=} \langle c_0, c_1 \rangle_{O_2} \vee \langle g \rangle_{O_2} \subseteq \langle g \rangle_c.$$

Together with 3.2.10 we infer

$$\langle L \rangle_C = \langle g \rangle_C.$$

□

3.2.13 Remark. Let $F \leq \mathbf{O}_2$ be a clone in Post's Lattice. Then we have

$$\mathcal{CL}_2 = \{\text{Pol}(C\text{Inv}(F)) \mid F \subseteq \mathbf{O}_2\} = \{\text{Pol}(C\text{Inv}(F)) \mid F \leq \mathbf{O}_2\}.$$

Proof. It is obvious that

$$\{\text{Pol}(C\text{Inv}(F)) \mid F \leq \mathbf{O}_2\} \subseteq \{\text{Pol}(C\text{Inv}(F)) \mid F \subseteq \mathbf{O}_2\}.$$

To show the other inclusion we regard $F \subseteq \mathbf{O}_2$. It holds for $G := \text{Pol Inv}(F) \leq \mathbf{O}_2$

$$\begin{aligned} \text{Pol } C\text{Inv}(G) &= \text{Pol}((\text{Inv}(G)) \cap CR_D) = \text{Pol}((\text{Inv}(\text{Pol Inv}(F))) \cap CR_D) \\ &= \text{Pol}(\text{Inv}(F) \cap CR_D) = \text{Pol } C\text{Inv}(F). \end{aligned}$$

□

The theorem below, given also in [Var10], states that there are no more Boolean C -clones than the ones already described in the previous Lemmata 3.2.4, 3.2.9, 3.2.11, 3.2.12.

3.2.14 Theorem. *The lattice of all Boolean C -clones is*

$$\mathcal{CL}_2 = \{\perp, \langle \wedge \rangle_C, \langle \vee \rangle_C, \langle h \rangle_C, \mathbf{O}_2\},$$

where

$$\perp := \langle c_0, c_1 \rangle_{\mathbf{O}_2}, \quad \langle \wedge \rangle_C = \langle c_0, c_1, \wedge \rangle_{\mathbf{O}_2}, \quad \langle \vee \rangle_C = \langle c_0, c_1, \vee \rangle_{\mathbf{O}_2}, \quad \text{and} \quad \langle h \rangle_C = M.$$

Proof. The next six equalities are consequences of Lemma 3.2.10 and previous lemmata.

$$\begin{aligned} \left\{ \langle C \rangle_C \mid C \in [J_2, \langle c_0, c_1 \rangle_{\mathbf{O}_2}]_{\mathcal{L}_2} \right\} &= \{ \langle c_0, c_1 \rangle_{\mathbf{O}_2} \} \\ \left\{ \langle C \rangle_C \mid C \in [\langle \wedge \rangle_{\mathbf{O}_2}, \langle \wedge, c_0, c_1 \rangle_{\mathbf{O}_2}]_{\mathcal{L}_2} \right\} &= \{ \langle \wedge, c_0, c_1 \rangle_{\mathbf{O}_2} \} \\ \left\{ \langle C \rangle_C \mid C \in [\langle \vee \rangle_{\mathbf{O}_2}, \langle \vee, c_0, c_1 \rangle_{\mathbf{O}_2}]_{\mathcal{L}_2} \right\} &= \{ \langle \vee, c_0, c_1 \rangle_{\mathbf{O}_2} \} \\ \left\{ \langle C \rangle_C \mid C \in [\langle \neg \rangle_{\mathbf{O}_2}, \mathbf{O}_2]_{\mathcal{L}_2} \right\} &= \{ \mathbf{O}_2 \} \\ \left\{ \langle C \rangle_C \mid C \in [\langle g \rangle_{\mathbf{O}_2}, \mathbf{O}_2]_{\mathcal{L}_2} \right\} &= \{ \mathbf{O}_2 \} \\ \left\{ \langle C \rangle_C \mid C \in [\langle h \rangle_{\mathbf{O}_2}, M]_{\mathcal{L}_2} \right\} &= \{ M \} \end{aligned}$$

The next four equalities will be shown below.

$$\begin{aligned} \left\{ \langle C \rangle_c \mid C \in [\langle \wedge \rangle_{O_2}, M]_{\mathcal{L}_2} \setminus [\langle \wedge \rangle_{O_2}, \langle \wedge, c_0, c_1 \rangle_{O_2}]_{\mathcal{L}_2} \right\} &= \{M\} \\ \left\{ \langle C \rangle_c \mid C \in [\langle \wedge \rangle_{O_2}, O_2]_{\mathcal{L}_2} \setminus [\langle \wedge \rangle_{O_2}, \langle \wedge, c_0, c_1 \rangle_{O_2}]_{\mathcal{L}_2}, C \not\subseteq M \right\} &= \{O_2\} \\ \left\{ \langle C \rangle_c \mid C \in [\langle \vee \rangle_{O_2}, M]_{\mathcal{L}_2} \setminus [\langle \vee \rangle_{O_2}, \langle \vee, c_0, c_1 \rangle_{O_2}]_{\mathcal{L}_2} \right\} &= \{M\} \\ \left\{ \langle C \rangle_c \mid C \in [\langle \vee \rangle_{O_2}, O_2]_{\mathcal{L}_2} \setminus [\langle \vee \rangle_{O_2}, \langle \vee, c_0, c_1 \rangle_{O_2}]_{\mathcal{L}_2}, C \not\subseteq M \right\} &= \{O_2\} \end{aligned}$$

Regarding a clone $C \leq O_2$ with $\langle \wedge \rangle_{O_2} \subseteq C$, but $C \notin [\langle \wedge \rangle_{O_2}, \langle \wedge, c_0, c_1 \rangle_{O_2}]_{\mathcal{L}_2}$ yields (using Lemma 3.2.8)

$$C_1 := (\langle c_0, c_1 \rangle_{O_2} \vee C) \subseteq \langle C \rangle_c.$$

If $C \subseteq M$, then one can read off Post's Lattice that $M = C_1$. Furthermore, because of monotonicity of $\langle \cdot \rangle_c$ yields $\langle C \rangle_c \subseteq \langle M \rangle_c = M$. Thus,

$$\langle C \rangle_c = M.$$

Otherwise, (i. e. $C \not\subseteq M$) leads to

$$O_2 = C_1 \subseteq \langle C \rangle_c \subseteq O_2.$$

The proof for $\vee$ instead of $\wedge$ is similar. Knowing all these 10 equalities and applying Remark 3.2.13, one obtains

$$\mathcal{CL}_2 = \{\perp, \langle \wedge \rangle_c, \langle \vee \rangle_c, M, O_2\},$$

the clones which are described in the previous lemmata. $\square$

The previous theorem does not only describe the set of all the Boolean C -clones but also the operations that these contain. For example, $\langle \vee \rangle_c$ contains the operations $c_0, c_1, \vee$, all the projections and compositions of these functions.

As we mentioned, when we describe C -clones at the same time we are describing relational C -clones. The lattices of Boolean C -clones and of Boolean relational C -clones are dually isomorphic as is shown in the next figure.

For the remainder of this section, we restrict ourselves to the following: Given a C -clone $\text{Pol}(Q)$ with $Q \subseteq CR_2^*$, find a minimal subset $Q_1 \subseteq Q$, such that

$$\text{Pol}(Q) = \text{Pol}(Q_1).$$

The motivation for the restriction is to establish that all Boolean C -clones can be described by a finite number of clausal relations.

3.2.15 Lemma. *The following equalities hold*

$$\begin{aligned} O_2 &= \text{Pol}(\emptyset), \\ M &= \text{Pol}(R_0^1), \\ \langle \wedge \rangle_C &= \text{Pol}\{R_{0,q}^{1,p} \mid p = 1, q \in \mathbb{N}_+\} = \text{Pol}(R_{(0,0)}^1), \\ \langle \vee \rangle_C &= \text{Pol}\{R_{0,q}^{1,p} \mid q = 1, p \in \mathbb{N}_+\} = \text{Pol}(R_0^{(1,1)}), \\ \langle c_0, c_1 \rangle_{O_2} &= \text{Pol}\{R_{0,q}^{1,p} \mid p, q \in \mathbb{N}_+\} = \text{Pol}(R_{(0,0)}^{(1,1)}). \end{aligned}$$

Proof. The characterization of O_2 is trivial. The second statement follows from Lemma 3.2.4. The arguments for the rest of the equalities are very similar, so w.l.o.g. we will only deal with the characterization of $\langle \wedge \rangle_C$.

“ $\subseteq$ ”: We have

$$\{R_{(0,0)}^1\} \subseteq \{R_{0,q}^{1,p} \mid p = 1, q \in \mathbb{N}_+\}$$

hence

$$W := \text{Pol}\{R_{(0,0)}^1\} \supseteq \text{Pol}\{R_{0,q}^{1,p} \mid p = 1, q \in \mathbb{N}_+\} = \langle \wedge \rangle_C.$$

“ $\supseteq$ ”: From the proof of Lemma 3.2.4 we know $M = \langle h \rangle_C \not\subseteq W$, applying the above-established facts that

$$\langle c_0, c_1, \wedge \rangle_{O_2} \stackrel{3.2.9}{=} \langle \wedge \rangle_C \subseteq W$$

and C -clones are clones, one can read off of Post’s Lattice that

$$W = \langle c_0, c_1, \wedge \rangle_{O_2} = \langle \wedge \rangle_C.$$

□

To finish this section, let us summarize the results obtained so far in two tables. Below, we show clausal relations that are preserved by functions $f \in O_2$, (namely $f \in \{\wedge, \vee, \neg, c_0, c_1, h, g\}$). These functions generate atoms in the Post’s Lattice.

TABLE 1

f	$f \triangleright R_{\mathbf{0},q}^{1,p}$	$f \not\triangleright R_{\mathbf{0},q}^{1,p}$	$\langle f \rangle_{\mathcal{C}}$
$\wedge$	$R_{\mathbf{0},q}^{1,p}$ for $p = 1, q \in \mathbb{N}_+$	$R_{\mathbf{0},q}^{1,p}$ for $p \geq 2, q \in \mathbb{N}_+$	$\langle \wedge \rangle_{\mathcal{C}} = \langle c_0, c_1, \wedge \rangle_{\mathbf{O}_2}$
$\vee$	$R_{\mathbf{0},q}^{1,p}$ for $p \in \mathbb{N}_+, q = 1$	$R_{\mathbf{0},q}^{1,p}$ for $p \in \mathbb{N}_+, q \geq 2$	$\langle \vee \rangle_{\mathcal{C}} = \langle c_0, c_1, \vee \rangle_{\mathbf{O}_2}$
$\neg$		$R_{\mathbf{0},q}^{1,p}$ for $p, q \in \mathbb{N}_+$	$\langle \neg \rangle_{\mathcal{C}} = \mathbf{O}_2$
c_0	$R_{\mathbf{0},q}^{1,p}$ for $p, q \in \mathbb{N}_+$		$c_0 \in \langle \emptyset \rangle_{\mathcal{C}}$
c_1	$R_{\mathbf{0},q}^{1,p}$ for $p, q \in \mathbb{N}_+$		$c_1 \in \langle \emptyset \rangle_{\mathcal{C}}$
h	$R_{\mathbf{0},q}^{1,p}$ for $p = 1, q = 1$	$R_{\mathbf{0},q}^{1,p}$ for $(p, q) \neq (1, 1)$	$\langle h \rangle_{\mathcal{C}} = M$
g		$R_{\mathbf{0},q}^{1,p}$ for $p, q \in \mathbb{N}_+$	$\langle g \rangle_{\mathcal{C}} = \mathbf{O}_2$

In the following table, we give minimal subsets Q_i of clausal relations to determine every Boolean C -clone.

TABLE 2

C -clones	$C\text{Inv}(C_i) \setminus \{D^{p+q} \mid p, q \in \mathbb{N}_+\}$ where $1 \leq i \leq 6$	$C_i = \text{Pol } Q_i$ where $Q_i \in CR_D^*$
$C_1 = \langle \emptyset \rangle_{\mathcal{C}} = \langle c_0, c_1 \rangle_{\mathbf{O}_2}$	$C\text{Inv}(C_1) = \{R_{\mathbf{0},q}^{1,p} \mid p = q = 2\}$	$\text{Pol } R_{(0,0)}^{(1,1)}$
$C_2 = \langle \wedge \rangle_{\mathcal{C}} = \langle c_0, c_1, \wedge \rangle_{\mathbf{O}_2}$	$C\text{Inv}(C_2) = \{R_{\mathbf{0},q}^{1,p} \mid p = 1, q = 2\}$	$\text{Pol } R_{(0,0)}^1$
$C_3 = \langle \vee \rangle_{\mathcal{C}} = \langle c_0, c_1, \vee \rangle_{\mathbf{O}_2}$	$C\text{Inv}(C_3) = \{R_{\mathbf{0},q}^{1,p} \mid p = 2, q = 1\}$	$\text{Pol } R_0^{(1,1)}$
$C_4 = \langle h \rangle_{\mathcal{C}} = M$	$C\text{Inv}(C_4) = \{R_{\mathbf{0},q}^{1,p} \mid p = 1, q = 1\}$	$\text{Pol } R_0^1$
$C_5 = \langle \neg \rangle_{\mathcal{C}} = \mathbf{O}_2$	$C\text{Inv}(C_5) = \emptyset$	$\text{Pol } \emptyset$
$C_6 = \langle g \rangle_{\mathcal{C}} = \mathbf{O}_2$	$C\text{Inv}(C_6) = \emptyset$	$\text{Pol } \emptyset$

3.3 The duality principle for C -clones

In previous sections properties (statements) of the lower parameters $\mathbf{b}$ of clausal relations $R_{\mathbf{b}}^{\mathbf{a}}$, could be handled analogously to properties of the upper parameters $\mathbf{a}$. In this section we explain how a statement for the upper parameters $\mathbf{a}$ of clausal relations $R_{\mathbf{b}}^{\mathbf{a}}$ will correspond to a dual statement concerning the lower parameters $\mathbf{b}$.

Let $\alpha : D \rightarrow D$ be the permutation defined by

$$\alpha(x) := n - 1 - x.$$

We define the *dual element* of $x \in D$ by $x^\partial := x^\alpha := \alpha(x) = n - 1 - x$. For every u, v belonging to D , it holds

$$u \leq v \iff u^\alpha \geq v^\alpha. \quad (3.2)$$

Furthermore, $(v^\alpha)^\alpha = v$, i.e. $\alpha^{-1} = \alpha$. Since (3.2) and because α is an isomorphism, we have that α is an order-isomorphism² from $(D, \leq)$ to $(D, \geq)$.

Let $m \in \mathbb{N}_+$, for an m -ary tuple $\mathbf{x} = (x_1, \dots, x_m)$ on D^m , we denote by

$$\mathbf{x}^\partial := \mathbf{x}^\alpha := (x_1^\alpha, x_2^\alpha, \dots, x_m^\alpha) = (n - 1 - x_1, n - 1 - x_2, \dots, n - 1 - x_m),$$

the *dual m -ary tuple* on D^m . For an m -ary relation ϱ on D , we associate a relation

$$\varrho^\alpha := \{\mathbf{x}^\alpha \mid \mathbf{x} \in \varrho\}, \quad (3.3)$$

and for a set Q of relations on D , we let $Q^\alpha := \{\varrho^\alpha \mid \varrho \in Q\}$.

Let $k \in \mathbb{N}_+$. The *dual* of a k -ary operation f on D is the k -ary operation $f^\partial := f^\alpha$, where

$$\begin{aligned} f^\alpha(\mathbf{x}) &:= (f(\mathbf{x}^{\alpha^{-1}}))^\alpha = (f(\mathbf{x}^\alpha))^\alpha \\ &= n - 1 - f(n - 1 - x_1, \dots, n - 1 - x_k). \end{aligned}$$

It is not difficult to verify that

$$f(\mathbf{x}^\alpha) = ((f(\mathbf{x}^\alpha))^\alpha)^\alpha = (f^\alpha(\mathbf{x}))^\alpha, \quad (3.4)$$

$$(f^\alpha)^\alpha(\mathbf{x}) = f(\mathbf{x}) = (f^\alpha(\mathbf{x}^\alpha))^\alpha. \quad (3.5)$$

To every $F \subseteq O_D$, we assign $F^\partial = F^\alpha := \{f^\alpha \mid f \in F\}$ as the *set of dual operations* on D . In what follows we need the following lemma and remark.

3.3.1 Lemma. *Let $f \in O_D$ and let $\varrho \in R_D$. Then it follows that*

$$f \triangleright \varrho \iff f^\alpha \triangleright \varrho^\alpha.$$

²In this work some notions such as order-isomorphism or linearly ordered set will be not defined explicitly. A nice introduction to these concepts can be found in [DP02].

Proof. Let k, m be positive natural numbers, $f \in O_D^{(k)}$ and let $\varrho \in R_D^{(m)}$. First, we show $f \triangleright \varrho$ implies $f^\alpha \triangleright \varrho^\alpha$. Consider any m -ary tuples $\mathbf{r}_1^\alpha, \dots, \mathbf{r}_k^\alpha$ belonging to ϱ^α , where $\mathbf{r}_l^\alpha = (x_{1l}^\alpha, x_{2l}^\alpha, \dots, x_{ml}^\alpha)$ for $l \in \{1, \dots, k\}$. Since $\mathbf{r}_1^\alpha, \dots, \mathbf{r}_k^\alpha \in \varrho^\alpha$, we obtain that $\mathbf{r}_1, \dots, \mathbf{r}_k \in \varrho$. Because $f \triangleright \varrho$, we get $f[\mathbf{r}_1, \dots, \mathbf{r}_k] \in \varrho$. Hence,

$$\begin{aligned} f[\mathbf{r}_1, \dots, \mathbf{r}_k] &= (f(x_{11}, x_{12}, \dots, x_{1k}), \dots, f(x_{m1}, x_{m2}, \dots, x_{mk})) \\ &\stackrel{(3.5)}{=} ((f^\alpha(x_{11}^\alpha, x_{12}^\alpha, \dots, x_{1k}^\alpha))^\alpha, \dots, (f^\alpha(x_{m1}^\alpha, x_{m2}^\alpha, \dots, x_{mk}^\alpha))^\alpha) \in \varrho \end{aligned}$$

The latter implies $(f^\alpha(x_{11}^\alpha, x_{12}^\alpha, \dots, x_{1k}^\alpha), \dots, f^\alpha(x_{m1}^\alpha, x_{m2}^\alpha, \dots, x_{mk}^\alpha)) \in \varrho^\alpha$ and hence $f^\alpha[\mathbf{r}_1^\alpha, \dots, \mathbf{r}_k^\alpha] \in \varrho^\alpha$. Conversely, let $f^\alpha \triangleright \varrho^\alpha$, using the first part of this proof, we get

$$f = (f^\alpha)^\alpha \triangleright (\varrho^\alpha)^\alpha = \varrho. \quad \square$$

Let F be a set of operations on D and let Q be a set of relations on D . We define

$$\text{Pol}^\alpha(Q) := (\text{Pol } Q^\alpha)^\alpha \quad \text{and} \quad \text{Inv}^\alpha(F) := (\text{Inv } F^\alpha)^\alpha.$$

3.3.2 Remark. Let $F \subseteq O_D$ and $Q \subseteq R_D$. Then

$$\text{Pol}^\alpha(Q) = \text{Pol } Q \quad \text{and} \quad \text{Inv}^\alpha(F) = \text{Inv } F.$$

Proof. By definition $\text{Pol}^\alpha(Q) = (\text{Pol } Q^\alpha)^\alpha$, and hence

$$\begin{aligned} \text{Pol}^\alpha(Q) &= (\text{Pol } Q^\alpha)^\alpha = (\{f \in O_D \mid \forall \varrho^\alpha \in Q^\alpha : f \triangleright \varrho^\alpha\})^\alpha \\ &= (\{f \in O_D \mid \forall \varrho \in Q : f \triangleright \varrho^\alpha\})^\alpha \stackrel{3.3.1}{=} (\{f \in O_D \mid \forall \varrho \in Q : f^\alpha \triangleright \varrho\})^\alpha \\ &= \{f^\alpha \in O_D \mid \forall \varrho \in Q : f^\alpha \triangleright \varrho\} = \{g \in O_D \mid \forall \varrho \in Q : g \triangleright \varrho\} = \text{Pol}(Q). \end{aligned}$$

Similarly, one can prove $\text{Inv}^\alpha(F) = \text{Inv } F$. $\square$

Note that for every tuple $\mathbf{a}$ that belongs to $(D \setminus \{0\})^p$, its dual $\mathbf{a}^\alpha$ belongs to $(D \setminus \{n-1\})^p$. Similarly, for $\mathbf{b} \in (D \setminus \{n-1\})^q$, we get $\mathbf{b}^\alpha \in (D \setminus \{0\})^q$.

Let $\mathbf{R}_\mathbf{b}^\mathbf{a} \in CR_{(D, \leq)}$ be a clausal relation of arity $p+q$, with respect to the linearly ordered set $(D, \leq)$. That is

$$\mathbf{R}_\mathbf{b}^\mathbf{a} = \{(\mathbf{x}, \mathbf{y}) \in D^{p+q} \mid x_1 \geq a_1 \vee \dots \vee x_p \geq a_p \vee y_1 \leq b_1 \vee \dots \vee y_q \leq b_q\}.$$

Using (3.3) above, we see that $(\mathbf{R}_\mathbf{b}^\mathbf{a})^\alpha$ is not a clausal relation with respect to $(D, \leq)$, but w.r.t. $(D, \geq)$. i.e.

$$\begin{aligned} (\mathbf{R}_\mathbf{b}^\mathbf{a})^\alpha &= \{(\mathbf{u}, \mathbf{v}) \in D^{p+q} \mid u_1 \leq a_1^\alpha \vee \dots \vee u_p \leq a_p^\alpha \vee v_1 \geq b_1^\alpha \vee \dots \vee v_q \geq b_q^\alpha\} \\ &= \mathbf{R}_{\mathbf{b}^\alpha}^{\mathbf{a}^\alpha} \text{ w.r.t. } (D, \geq). \end{aligned} \quad (3.6)$$

Next we define an operator $P_{p+q} : R_D \longrightarrow R_D$, which swaps the first p coordinates of every tuple belonging to the relation ϱ with the last q coordinates. That is,

$$P_{p+q}(\varrho) := \{(\mathbf{y}, \mathbf{x}) \in D^{q+p} \mid (\mathbf{x}, \mathbf{y}) \in \varrho\}.$$

Note that for $\pi \in S_{p+q}$ defined by

$$\pi(k) := \begin{cases} k + q & \text{if } k \leq p, \\ k - p & \text{if } k > p, \end{cases}$$

we have $P_{p+q}(\varrho) = V_\pi(\varrho)$.

The operator operates on $(R_{\mathbf{b}}^{\mathbf{a}})^\alpha$ as follows

$$\begin{aligned} P_{p+q}((R_{\mathbf{b}}^{\mathbf{a}})^\alpha) &= \{(\mathbf{v}, \mathbf{u}) \mid (\mathbf{u}, \mathbf{v}) \in (R_{\mathbf{b}}^{\mathbf{a}})^\alpha\} \\ &= \{(\mathbf{v}, \mathbf{u}) \mid u_1 \leq a_1^\alpha \vee \dots \vee u_p \leq a_p^\alpha \vee v_1 \geq b_1^\alpha \vee \dots \vee v_q \geq b_q^\alpha\} \\ &= \{(\mathbf{v}, \mathbf{u}) \mid v_1 \geq b_1^\alpha \vee \dots \vee v_q \geq b_q^\alpha \vee u_1 \leq a_1^\alpha \vee \dots \vee u_p \leq a_p^\alpha\} \\ &= R_{\mathbf{a}^\alpha}^{\mathbf{b}^\alpha} \text{ w.r.t. } (D, \leq). \end{aligned} \tag{3.7}$$

3.3.3 Definition. Let $R_{\mathbf{b}}^{\mathbf{a}} \in CR_{(D, \leq)}$ be a clausal relation of arity $p + q$, we define the *dual clausal relation* of arity $q + p$ by

$$(R_{\mathbf{b}}^{\mathbf{a}})^\partial := P_{p+q}((R_{\mathbf{b}}^{\mathbf{a}})^\alpha) \stackrel{(3.7)}{=} R_{\mathbf{a}^\alpha}^{\mathbf{b}^\alpha}.$$

The latter implies $P_{p+q}^{-1}((R_{\mathbf{b}}^{\mathbf{a}})^\partial) = (R_{\mathbf{b}}^{\mathbf{a}})^\alpha$. Note also that $(R_{\mathbf{b}}^{\mathbf{a}})^\partial = R_{\mathbf{a}^\alpha}^{\mathbf{b}^\alpha} \in CR_{(D, \leq)}$.

The set of clausal relations is defined over a linearly ordered set $(D, \leq)$. From (3.6) we obtain $(CR_{(D, \leq)})^\alpha = CR_{(D, \geq)}$. It follows that the set of duals clausal relations equals

$$(CR_{(D, \leq)})^\partial := P[(CR_{(D, \leq)})^\alpha] \stackrel{(3.6)}{=} P[CR_{(D, \geq)}] \stackrel{(3.7)}{=} CR_{(D, \leq)}.$$

Here the notation $P[Q]$ means application of the corresponding operator P_{p+q} to every relation in Q .

3.3.4 Lemma. Let $f \in O_D$ and $R_{\mathbf{b}}^{\mathbf{a}} \in CR_D$. Then it holds

$$f \triangleright R_{\mathbf{b}}^{\mathbf{a}} \iff f^\partial = f^\alpha \triangleright R_{\mathbf{a}^\alpha}^{\mathbf{b}^\alpha} = (R_{\mathbf{b}}^{\mathbf{a}})^\partial.$$

Proof. Let k be a positive natural number and let $f \in O_D^{(k)}$, such that $f \triangleright R_{\mathbf{b}}^{\mathbf{a}}$. We show $f^\alpha \triangleright R_{\mathbf{a}^\alpha}^{\mathbf{b}^\alpha}$. Let $\mathbf{r}_1, \dots, \mathbf{r}_k \in R_{\mathbf{a}^\alpha}^{\mathbf{b}^\alpha}$, where

$$\mathbf{r}_l = (\mathbf{y}_l, \mathbf{x}_l) = (y_{1l}, \dots, y_{ql}, x_{1l}, \dots, x_{pl})$$

for $l \in \{1, \dots, k\}$. Then

$$\begin{aligned} y_{1l} &\geq b_1^\alpha \vee \dots \vee y_{ql} \geq b_q^\alpha \vee x_{1l} \leq a_1^\alpha \vee \dots \vee x_{pl} \leq a_p^\alpha \\ &\stackrel{(3.2)}{\iff} x_{1l}^\alpha \geq a_1 \vee \dots \vee x_{pl}^\alpha \geq a_p \vee y_{1l}^\alpha \leq b_1 \vee \dots \vee y_{ql}^\alpha \leq b_q \\ &\iff (x_{1l}^\alpha, \dots, x_{pl}^\alpha, y_{1l}^\alpha, \dots, y_{ql}^\alpha) \in R_{\mathbf{b}}^{\mathbf{a}} \end{aligned}$$

Since $f \triangleright R_{\mathbf{b}}^{\mathbf{a}}$, we get

$$\begin{aligned}
 f(x_{11}^\alpha, \dots, x_{1k}^\alpha) &\geq a_1 \vee \dots \vee f(x_{p1}^\alpha, \dots, x_{pk}^\alpha) \geq a_p \vee \\
 f(y_{11}^\alpha, \dots, y_{1k}^\alpha) &\leq b_1 \vee \dots \vee f(y_{q1}^\alpha, \dots, y_{qk}^\alpha) \leq b_q \\
 \stackrel{(3.4)(3.2)}{\iff} f^\alpha(y_{11}, \dots, y_{1k}) &\geq b_1^\alpha \vee \dots \vee f^\alpha(y_{q1}, \dots, y_{qk}) \geq b_q^\alpha \\
 f^\alpha(x_{11}, \dots, x_{1k}) &\leq a_1^\alpha \vee \dots \vee f^\alpha(x_{p1}, \dots, x_{pk}) \leq a_p^\alpha \\
 \text{Thus } f^\alpha[\mathbf{r}_1, \dots, \mathbf{r}_k] &\in R_{\mathbf{a}^\alpha}^{\mathbf{b}^\alpha}.
 \end{aligned}$$

Let $f^\alpha \triangleright R_{\mathbf{a}^\alpha}^{\mathbf{b}^\alpha}$, using the above direction, we obtain $f = (f^\alpha)^\alpha \triangleright R_{(\mathbf{b}^\alpha)^\alpha}^{(\mathbf{a}^\alpha)^\alpha} = R_{\mathbf{b}}^{\mathbf{a}}$. $\square$

We define $C\text{Inv}^\alpha F := (C\text{Inv}(F^\alpha))^\alpha$. Below we show that $C\text{Inv}^\alpha F$ is also a set of invariant clausal relations but w.r.t. $(D, \geq)$.

$$\begin{aligned}
 C\text{Inv}^\alpha F &= (C\text{Inv}(F^\alpha))^\alpha = (\{\varrho \in CR_D \mid \forall f^\alpha \in F^\alpha : f^\alpha \triangleright \varrho\})^\alpha \\
 &= (\{\varrho \in CR_D \mid \forall f \in F : f^\alpha \triangleright \varrho\})^\alpha \stackrel{3.3.4}{=} (\{\varrho \in CR_D \mid \forall f \in F : f \triangleright \varrho^\alpha\})^\alpha \\
 &= \{\varrho^\alpha \mid \varrho \in CR_D \wedge \forall f \in F : f \triangleright \varrho^\alpha\} = \{\varrho^\alpha \mid \varrho \in CR_D \wedge \varrho^\alpha \in \text{Inv } F\} \\
 &= \{\varrho^\alpha \in (CR_{(D, \leq)})^\alpha \mid \varrho^\alpha \in \text{Inv } F\} = \{\sigma \in (CR_{(D, \leq)})^\alpha \mid \sigma \in \text{Inv } F\} \\
 &= (CR_{(D, \leq)})^\alpha \cap \text{Inv } F = CR_{(D, \geq)} \cap \text{Inv } F.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 P[C\text{Inv}^\alpha F] &= P[(CR_{(D, \leq)})^\alpha \cap \text{Inv } F] = P[(CR_{(D, \leq)})^\alpha] \cap P[\text{Inv } F] \\
 &= CR_D \cap P[\text{Inv } F] = CR_D \cap \text{Inv } F = C\text{Inv } F,
 \end{aligned} \tag{3.8}$$

$$(C\text{Inv } F)^\alpha = (C\text{Inv}(F^\alpha))^\alpha = C\text{Inv}^\alpha F^\alpha \stackrel{(3.8)}{=} P^{-1}[C\text{Inv } F^\alpha] \tag{3.9}$$

$$(\text{Pol } Q)^\partial = (\text{Pol } Q)^\alpha \stackrel{3.3.2}{=} (\text{Pol}^\alpha Q)^\alpha = ((\text{Pol } Q)^\alpha)^\alpha = \text{Pol } Q^\alpha \tag{3.10}$$

We define

$$(C\text{Inv } F)^\partial := P[(C\text{Inv } F)^\alpha] \stackrel{(3.9)}{=} P[P^{-1}[C\text{Inv } F^\alpha]] = C\text{Inv } F^\alpha = C\text{Inv } F^\partial.$$

$$\begin{aligned}
 (\langle F \rangle_c)^\partial &= (\text{Pol } C\text{Inv } F)^\partial = (\text{Pol } C\text{Inv } F)^\alpha \stackrel{(3.10)}{=} \text{Pol } (C\text{Inv } F)^\alpha \\
 &\stackrel{(3.9)}{=} \text{Pol } P^{-1}[C\text{Inv } F^\alpha] = \text{Pol } C\text{Inv } F^\alpha = \text{Pol } C\text{Inv } F^\partial = \langle F^\partial \rangle_c.
 \end{aligned} \tag{3.11}$$

3.3.5 Definition. If S is a statement about clausal relations with respect to the linearly ordered set $(D, \leq)$, then we obtain another statement S^∂ about clausal relations with respect to $(D, \geq)$ as follows. Taking $(D, \leq)$ as the canonical representative of an n -element chain, we obtain from S a statement $S(L, \sqsubseteq)$ about clausal relations with respect to any linearly ordered set $(L, \sqsubseteq)$, making use of the unique order isomorphism from $(D, \leq)$ onto $(L, \sqsubseteq)$. In particular, for $(L, \sqsubseteq) = (D, \geq)$ we can deduce an intermediate statement $S^\alpha \stackrel{\alpha}{=} S(D, \geq)$ from S . Interpreting S^α again as a statement about clausal relations with respect to $(D, \leq)$ we obtain the dual statement S^∂ .

For example from 3.2.15 we infer

$$\{\wedge, c_0, c_1\} = \langle \wedge \rangle_C = \text{Pol } R_{(0,0)}^1.$$

We translate this statement to a dual statement

$$\begin{aligned} (\langle \wedge \rangle_C)^\partial &\stackrel{(3.11)}{=} \langle \wedge^\partial \rangle_C = \langle \vee \rangle_C \quad \text{and} \\ (\text{Pol } R_{(0,0)}^1)^\partial &\stackrel{(3.10)}{=} \text{Pol } (R_{(0,0)}^1)^\alpha = \text{Pol } P_{1+2}^{-1}(R_{(0,0)}^1)^\partial = \text{Pol } (R_{(0,0)}^1)^\partial = \text{Pol } R_0^{(1,1)}. \end{aligned}$$

Thus, we obtain the dual statement $S^\partial : \langle \vee \rangle_C = \text{Pol } R_0^{(1,1)}$.

3.3.6 Remark. The last step in dualising a statement S from the intermediate statement S^α to the dual statement S^∂ is not completely straightforward, since clausal relations with respect to $(D, \geq)$ are generally not clausal relations with respect to $(D, \leq)$, only up to permutation of variables. However, via the mentioned variable permutation they are in one-to-one correspondence to each other. As the relation $\triangleright$ is not affected by variable permutations. This is, for instance, the reason for the second part of Lemma 3.3.4.

3.3.7 Theorem (Duality principle). *Let S be a statement about clausal relations with respect to $(D, \leq)$. Then,*

$$S \text{ is true} \iff S^\partial \text{ is true}.$$

Proof. It is inherent to the applied method of dualisation that S is true iff S^α is true iff S^∂ is true. $\square$

This duality principle is used to simplify proofs hereafter. If a lemma asserts two statements that are dual to each other, we usually prove only one of them.

3.4 Some C -clones on a finite set.

In this section we show that clausal relations meet a sufficient condition that is known to ensure polynomial time solvability of the corresponding CSP. Furthermore, in order to analyse the lattice of all C -clones we collect some statements that generalise Lemmata 3.2.4, 3.2.5, 3.2.6 and 3.2.8 to the case $|D| \geq 3$. Then, we describe the smallest C -clone on a finite set D .

3.4.1 Lemma. *Let $l \in \mathbb{N}_+$, c_0^l, c_{n-1}^l be l -ary constant operations with values 0 and $n-1$, respectively. Then it follows that*

$$\text{Pol } CR_D = \langle \{c_0^l, c_{n-1}^l\} \rangle_{O_D}.$$

Proof. First, we show that $\langle \{c_0^l, c_{n-1}^l\} \rangle_{O_D} \subseteq \text{Pol } CR_D$. The set J_D of all projections on D belongs to $\text{Pol } R_D \subseteq \text{Pol } CR_D$. Let $R_{\mathbf{b}}^{\mathbf{a}} \in CR_D$ with $\mathbf{a} \in D^p$ and $\mathbf{b} \in D^q$ and $r_1, \dots, r_l \in R_{\mathbf{b}}^{\mathbf{a}}$. Since $0 \leq b_1$ and $n-1 \geq a_1$, we get

$$c_0^l[r_1, \dots, r_l] = (0, \dots, 0, 0, \dots, 0) \in R_{\mathbf{b}}^{\mathbf{a}},$$

$$\text{and } c_{n-1}^l[r_1, \dots, r_l] = (n-1, \dots, n-1, n-1, \dots, n-1) \in R_{\mathbf{b}}^{\mathbf{a}}.$$

Consequently, $c_0^l \triangleright R_{\mathbf{b}}^{\mathbf{a}}$ and $c_{n-1}^l \triangleright R_{\mathbf{b}}^{\mathbf{a}}$. Conversely, we assume that for some $k \in \mathbb{N}_+$ there is a function

$$f \in \left(\text{Pol}^{(k)} CR_D \right) \setminus \{ \{c_0^k, c_{n-1}^k\} \cup J_D \}.$$

- Since $f \neq c_0^k$, there is some $(y'_1, \dots, y'_k) \in D^k$ such that $f(y'_1, \dots, y'_k) > 0$.
- As $f \neq c_{n-1}^k$, there is some $(x'_1, \dots, x'_k) \in D^k$ such that $f(x'_1, \dots, x'_k) < n-1$.

Since $f \notin J_D$, we get $f \neq e_j^k$ for all $j \in \{1, \dots, k\}$. Hence, for all j belonging to $\{1, \dots, k\}$ there exists a tuple $(d_1^j, \dots, d_j^j, \dots, d_k^j)$ in D^k depending on j such that

$$f(d_1^j, \dots, d_j^j, \dots, d_k^j) \neq d_j^j, \quad (\star)$$

We write these tuples as rows in the following matrix

$$\begin{pmatrix} d_1^1 & d_2^1 & \dots & d_k^1 \\ d_1^2 & d_2^2 & \dots & d_k^2 \\ & & \vdots & \\ d_1^k & d_2^k & \dots & d_k^k \end{pmatrix}.$$

We can reorder the rows of this matrix in such way that the rows where $(\star)$ is satisfied because of “ $<$ ” are listed first, followed by the rows where $(\star)$ is satisfied because of “ $>$ ”. That is, for some permutation $\pi \in S_{\{1, \dots, k\}}$, there is $j \in \{1, \dots, k+1\}$ such that for every $i \in \{1, \dots, j-1\}$ we get

$$f(d_1^{\pi(i)}, \dots, d_k^{\pi(i)}) < d_{\pi(i)}^{\pi(i)},$$

and for every $i \in \{j, \dots, k\}$ we get $f(d_1^{\pi(i)}, \dots, d_k^{\pi(i)}) > d_{\pi(i)}^{\pi(i)}$. Consider the tuples

$$r_1 := (x'_1, d_1^{\pi(1)}, \dots, d_1^{\pi(k)}, y'_1), \dots, r_k := (x'_k, d_k^{\pi(1)}, \dots, d_k^{\pi(k)}, y'_k).$$

The scheme

$$\begin{array}{ccccccc}
 & r_1 & r_2 & \dots & r_k & & \\
 & x'_1 & x'_2 & \dots & x'_k &) & < n-1 \\
 1: & f(& d_1^{\pi(1)} & d_2^{\pi(1)} & \dots & d_k^{\pi(1)} &) < d_{\pi(1)}^{\pi(1)} \\
 & & & & \vdots & & \vdots \\
 j-1: & f(& d_1^{\pi(j-1)} & d_2^{\pi(j-1)} & \dots & d_k^{\pi(j-1)} &) < d_{\pi(j-1)}^{\pi(j-1)} \\
 j: & f(& d_1^{\pi(j)} & d_2^{\pi(j)} & \dots & d_k^{\pi(j)} &) > d_{\pi(j)}^{\pi(j)} \\
 & & & & \vdots & & \vdots \\
 & f(& d_1^{\pi(k)} & d_2^{\pi(k)} & \dots & d_k^{\pi(k)} &) > d_{\pi(k)}^{\pi(k)} \\
 & f(& y'_1 & y'_2 & \dots & y'_k &) > 0
 \end{array}$$

shows that $f \not\models R_{d_{\pi(j)}^{\pi(j)}, \dots, d_{\pi(k)}^{\pi(k)}, 0}^{n-1, d_{\pi(1)}^{\pi(1)}, \dots, d_{\pi(j-1)}^{\pi(j-1)}}$. This is because the tuples (columns of the arguments of f) all belong to $R_{d_{\pi(j)}^{\pi(j)}, \dots, d_{\pi(k)}^{\pi(k)}, 0}^{n-1, d_{\pi(1)}^{\pi(1)}, \dots, d_{\pi(j-1)}^{\pi(j-1)}}$, since in the i -th column the entry in the $(\pi^{-1}(i) + 1)$ -st row is d_i^i , satisfying the clausal expression with equality. However, after applying f to the tuples, one obtains a tuple (column) that does not belong to $R_{d_{\pi(j)}^{\pi(j)}, \dots, d_{\pi(k)}^{\pi(k)}, 0}^{n-1, d_{\pi(1)}^{\pi(1)}, \dots, d_{\pi(j-1)}^{\pi(j-1)}}$. This is impossible, since $f \in \text{Pol}^{(k)} CR_D$.

Consequently, $(\text{Pol}^{(k)} CR_D) \setminus \{\{c_0^k, c_{n-1}^k\} \cup J_D\} = \emptyset$ holds for every $k \in \mathbb{N}_+$, which proves the claim. $\square$

At this point, we know that for every $k \in \mathbb{N}_+$ the k -ary constant operation c_0^k belongs to $\text{Pol}(CR_D)$. The latter together with Proposition 1.4.4, make the constraint satisfaction problem over clausal relations, denoted by $\text{CSP}(CR_D)$, solvable in polynomial time.

Analogously to Lemma 3.2.8, and using the previous lemma, the lemma to come can be proved.

3.4.2 Lemma. *For any subset $F \subseteq O_D$ and all $l \in \mathbb{N}_+$ it follows that*

$$\langle \{c_0^l, c_{n-1}^l\} \rangle_{O_D} \vee \langle F \rangle_{O_D} \subseteq \langle F \rangle_C.$$

In 2.2.8 we saw that clausal relations are either trivial or non-trivial, and none of the diagonals except for the full Cartesian powers are clausal relations. Furthermore, $C\text{Inv } O_D = \text{Inv } O_D \cap CR_D = \text{diag}(D) \cap CR_D$ holds. Hence, we obtain the following lemma.

3.4.3 Lemma. *The following equality holds*

$$C\text{Inv } O_D = \{D^{p+q} \mid p, q \in \mathbb{N}_+\}.$$

In the following proofs, we use that the tuple $(n-1, \dots, n-1, 0, \dots, 0)$ belongs to all non-trivial clausal relations $R_{\mathbf{b}}^{\mathbf{a}}$, but $(0, \dots, 0, n-1, \dots, n-1)$ does not belong to $R_{\mathbf{b}}^{\mathbf{a}}$.

3.4.4 Lemma. *Let $h \in \mathcal{O}_D^{(3)}$ be the ternary majority operation on D defined by $h(x, y, z) := (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$ where $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$. Then we have*

$$\langle h \rangle_C = \text{Pol} \{R_b^a \mid a \in D \setminus \{0\}, b \in D \setminus \{n-1\}\}.$$

Proof. Let $R_b^a \in CR_D^*$, i.e. $\mathbf{a} \in (D \setminus \{0\})^p$ and $\mathbf{b} \in (D \setminus \{n-1\})^q$. We show that $h \not\vdash R_b^a$ unless $p = 1, q = 1$.

- $p \geq 2, q \geq 1$: The scheme

$$\begin{array}{ccccccc} h(\begin{array}{ccc} n-1 & 0 & 0 \end{array}) & = & 0 \\ h(\begin{array}{ccc} 0 & n-1 & 0 \end{array}) & = & 0 \\ h(\begin{array}{ccc} 0 & 0 & 0 \end{array}) & = & 0 \\ & & \vdots & & \vdots \\ h(\begin{array}{ccc} 0 & 0 & 0 \end{array}) & = & 0 & (\text{row } p) \\ h(\begin{array}{ccc} n-1 & n-1 & 0 \end{array}) & = & n-1 \\ h(\begin{array}{ccc} n-1 & n-1 & n-1 \end{array}) & = & n-1 \\ & & \vdots & & \vdots \\ h(\begin{array}{ccc} n-1 & n-1 & n-1 \end{array}) & = & n-1 & (\text{row } p+q) \end{array}$$

shows that $h \not\vdash R_b^a$, because the tuples (columns of the arguments of h) all belong to R_b^a , but after applying h to the tuples, one obtains a tuple (column) that does not belong to R_b^a .

- $p \geq 1, q \geq 2$: Likewise, the scheme

$$\begin{array}{ccccccc} h(\begin{array}{ccc} n-1 & 0 & 0 \end{array}) & = & 0 \\ h(\begin{array}{ccc} 0 & 0 & 0 \end{array}) & = & 0 \\ & & \vdots & & \vdots \\ h(\begin{array}{ccc} 0 & 0 & 0 \end{array}) & = & 0 & (\text{row } p) \\ h(\begin{array}{ccc} n-1 & 0 & n-1 \end{array}) & = & n-1 \\ h(\begin{array}{ccc} n-1 & n-1 & 0 \end{array}) & = & n-1 \\ h(\begin{array}{ccc} n-1 & n-1 & n-1 \end{array}) & = & n-1 \\ & & \vdots & & \vdots \\ h(\begin{array}{ccc} n-1 & n-1 & n-1 \end{array}) & = & n-1 & (\text{row } p+q) \end{array}$$

- $p = q = 1$: Let R_b^a be any non-trivial clausal relation of arity 2, and let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in R_b^a$. We consider two cases:

- Suppose that there exist $i, j \in \{1, 2, 3\}$ with $i \neq j : x_i \geq a, x_j \geq a$. Since $h(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$, h is totally symmetric, i.e. $h(x_1, x_2, x_3) = h(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$ for all $\pi \in S_3$. Thus, w.l.o.g. we consider $i = 1, j = 2$, i.e.

$$x_1 \geq a, x_2 \geq a, x_3 \geq x_3$$

because h is monotone, we obtain $h(x_1, x_2, x_3) \geq h(a, a, x_3) = a$, and hence

$$(h(x_1, x_2, x_3), h(y_1, y_2, y_3)) \in R_b^a$$

- Otherwise, there exist $i, j \in \{1, 2, 3\}$ with $i \neq j : y_i \leq b, y_j \leq b$, because $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in R_b^a$. Furthermore, since h is monotone we get $h(y_1, y_2, y_3) \leq b$. Thus,

$$(h(x_1, x_2, x_3), h(y_1, y_2, y_3)) \in R_b^a.$$

The last two cases show $h \triangleright R_b^a$. $\square$

We finish this section with the next lemma, which characterise C -clones generated by the binary operations $\min$ and $\max$.

3.4.5 Lemma. *Let $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$ for $x, y \in D$. Then it holds*

$$\begin{aligned} \langle \wedge \rangle_C &= \text{Pol}\{R_b^a \mid a \in D \setminus \{0\}, \mathbf{b} \in (D \setminus \{n-1\})^q\}, \\ \langle \vee \rangle_C &= \text{Pol}\{R_b^a \mid \mathbf{a} \in (D \setminus \{0\})^p, b \in D \setminus \{n-1\}\}. \end{aligned}$$

Proof. At first we look at $\wedge$: Let $R_b^a \in CR_D^*$. We show that $\wedge \not\triangleright R_b^a$, unless when $p = 1$ and $q \in \mathbb{N}_+$. We consider two cases:

- $p \geq 2$: The scheme

$$\begin{array}{ccccccc} n-1 & \wedge & 0 & = & 0 & & \\ 0 & \wedge & n-1 & = & 0 & & \\ 0 & \wedge & 0 & = & 0 & & \\ & \vdots & & & & & \\ 0 & \wedge & 0 & = & 0 & & \text{(row } p) \\ n-1 & \wedge & n-1 & = & n-1 & & \\ & \vdots & & & & & \\ n-1 & \wedge & n-1 & = & n-1 & & \text{(row } p+q) \end{array}$$

shows that $\wedge \not\triangleright R_b^a$.

- $p = 1, q \in \mathbb{N}_+$: We show $\wedge \triangleright R_b^a$ for all $R_b^a \in CR_D^*$. We assume the existence of tuples $(x_1, y_1, \dots, y_q), (z_1, y'_1, \dots, y'_q) \in R_b^a$ such that

$$x_1 \wedge z_1 = 0,$$

and for all $j \in \{1, \dots, q\}$

$$y_j \wedge y'_j = n-1.$$

Hence, $y_j = y'_j = n-1$ for all $j \in \{1, \dots, q\}$. Because of $x_1 \wedge z_1 = 0$ w.l.o.g. $x_1 = 0$. Thus,

$$(x_1, y_1, \dots, y_q) = (0, n-1, \dots, n-1) \in R_b^a,$$

which is a contradiction.

We obtain the second statement, if we apply the duality principle to the first statement. $\square$

3.5 Infinite chains of C -clones.

In contrast to Boolean C -clones, we show that for $|D| \geq 3$ there are infinitely many C -clones by constructing infinite descending chains of such clones.

Let $\{0, 1, 2\} \subseteq D$ and $m \in \mathbb{N}_+$. Consider the following clausal relation:

$$\varrho_m := R_{\underbrace{(1, \dots, 1)}_{m \text{ times}} \underbrace{(1, \dots, 1)}_{m \text{ times}}} =$$

$$\{(x_1, \dots, x_m, y_1, \dots, y_m) \in D^{2m} \mid x_1 \geq 1 \vee \dots \vee x_m \geq 1 \vee y_1 \leq 1 \vee \dots \vee y_m \leq 1\}.$$

Observe that the tuple $(0, \dots, 0, 2, \dots, 2)$ does not belong to ϱ_m .

3.5.1 Proposition. *If ϱ_m is the $2m$ -ary relation defined above, then*

$$\text{Pol}(\varrho_{m-1}) \supsetneq \text{Pol}(\varrho_m)$$

holds for any $m \in \mathbb{N}_+$.

Proof. Let $n \in \mathbb{N}_+$ and $f \in \text{Pol}^{(n)}(\varrho_m)$. We have to show $f \in \text{Pol}^{(n)}(\varrho_{m-1})$. Let $r_1, \dots, r_n \in \varrho_{m-1}$, where $r_k =: (x_{1k}, \dots, x_{m-1k}, y_{1k}, \dots, y_{m-1k})$ for $k \in \{1, \dots, n\}$. Then

$$\begin{aligned} & x_{1k} \geq 1 \vee \dots \vee x_{m-1k} \geq 1 \vee y_{1k} \leq 1 \vee \dots \vee y_{m-1k} \leq 1 \\ \Leftrightarrow & x_{1k} \geq 1 \vee \dots \vee x_{m-1k} \geq 1 \vee x_{m-1k} \geq 1 \vee y_{1k} \leq 1 \vee \dots \vee y_{m-1k} \leq 1 \vee y_{m-1k} \leq 1. \end{aligned}$$

We define for $k \in \{1, \dots, n\}$

$$\begin{aligned} r'_k &= (\bar{x}_{1k}, \dots, \bar{x}_{m-1k}, \bar{x}_{mk}, \bar{y}_{1k}, \dots, \bar{y}_{m-1k}, \bar{y}_{mk}) \\ &:= (x_{1k}, \dots, x_{m-1k}, x_{m-1k}, y_{1k}, \dots, y_{m-1k}, y_{m-1k}). \end{aligned}$$

The new tuples $r'_1, \dots, r'_n$ belong to ϱ_m because of the above expression.

Because $f \triangleright \varrho_m$, we have $(c_1, \dots, c_{m-1}, c_m, d_1, \dots, d_{m-1}, d_m) := f[r'_1, \dots, r'_n] \in \varrho_m$. Then, by construction, we have $c_{m-1} = c_m$, $d_{m-1} = d_m$, and

$$\begin{aligned} c_{m-1} \geq 1 \vee c_m \geq 1 & \iff c_{m-1} \geq 1, \\ d_{m-1} \leq 1 \vee d_m \leq 1 & \iff d_{m-1} \leq 1. \end{aligned}$$

Therefore, $(c_1, \dots, c_{m-1}, d_1, \dots, d_{m-1}) = f[r_1, \dots, r_n] \in \varrho_{m-1}$, i.e. $f \triangleright \varrho_{m-1}$.

Now we show that the inclusion is proper. Let f be a $2m$ -ary operation such that

$$f(x_1, \dots, x_m, \dots, x_{2m}) := \begin{cases} 0 & \text{if there is only one 1 among the first } m \text{ entries} \\ & \text{and 0 in the other entries,} \\ 2 & \text{if there is only one 1 between the } m+1, \dots, 2m \\ & \text{entries and 2 in the other entries,} \\ 1 & \text{otherwise.} \end{cases}$$

That is

$$f(1, 0, \dots, \overset{m}{0}, 0, \dots, \overset{2m}{0}) = \dots = f(0, \dots, 0, \overset{m}{1}, 0, \dots, 0) = 0$$

$$f(2, 2, \dots, \overset{m+1}{1}, 2, \dots, 2) = \dots = f(2, \dots, 2, \overset{m+1}{2}, \dots, 2, 1) = 2.$$

We show $f \notin \text{Pol}^{(2m)}(\varrho_m)$.

Consider the tuples $r_1, \dots, r_{2m}$ belonging to ϱ_m , such that

$$r_1 = (1, 0, \dots, \overset{m}{0}, 2, \dots, \overset{2m}{2}), \dots, r_{2m} = (0, \dots, 0, 2, \dots, 2, \overset{2m}{1}).$$

$f[r_1, \dots, r_{2m}] \notin \varrho_m$ because,

	1	2	...	m	m+1	m+2	...	2m			
1	$f($	1	0	...	0	0	0	...	0	)	=0
2	$f($	0	1	...	0	0	0	...	0	)	=0
				$\ddots$				$\vdots$			$\vdots$
m	$f($	0	0	...	1	0	0	...	0	)	=0
m+1	$f($	2	2	...	2	1	2	...	2	)	=2
m+2	$f($	2	2	...	2	2	1	...	2	)	=2
				$\vdots$				$\ddots$			$\vdots$
2m	$f($	2	2	...	2	2	2	...	1	)	=2
											$\notin \mathcal{Q}_m$

after we apply f to the tuples in ϱ_m , we obtain a tuple that does not belong to ϱ_m . However, $f \in \text{Pol}^{(2m)}(\varrho_{m-1})$ as the following argument shows. Consider any the tuples $r_1^*, \dots, r_{2m}^*$ belonging to ϱ_{m-1} , where

$$r_k^* = (x_{1k}, \dots, x_{m-1k}, y_{1k}, \dots, y_{m-1k})$$

for $k \in \{1, \dots, 2m\}$. Note that, by definition, every tuple r_k^* satisfies the following expression:

$$x_{1k} \geq 1 \vee \dots \vee x_{m-1k} \geq 1 \vee y_{1k} \leq 1 \vee \dots \vee y_{m-1k} \leq 1.$$

Let us regard the tuples $r_k^* \in \varrho_{m-1}$ as the columns of a matrix $A \in D^{2(m-1) \times (2m)}$. We construct a matrix $(b_{ij})_{\substack{1 \leq i \leq 2(m-1) \\ 1 \leq j \leq 2m}} = B \in \{0, 1\}^{2(m-1) \times (2m)}$ in the following way:

$$b_{ij} := \begin{cases} 1 & \text{if } a_{ij} \geq 1 \text{ and } 1 \leq i \leq m-1, \\ 1 & \text{if } a_{ij} \leq 1 \text{ and } m \leq i \leq 2(m-1), \\ 0 & \text{otherwise.} \end{cases}$$

In every column r_k^* there is at least one element $d \in D$ such that $d \geq 1$ or $d \leq 1$, hence any column of B contains at least one entry $b_{ij} = 1$. Since there are only $2(m-1)$ rows but $2m$ columns, there is at least one row of B , say the l -st row, containing two entries 1. Consequently,

$$f(r_{l1}^*, \dots, r_{l2m}^*) = 1$$

from the definition of f . Then $f[r_1^*, \dots, r_{2m}^*] \in \varrho_{m-1}$, hence $f \triangleright \varrho_{m-1}$. $\square$

Similarly to 3.5.1, if we consider

- a fixed number c belonging $\{1, \dots, n-2\}$, $m \in \mathbb{N}_+$,
- the clausal relation ϱ'_m defined by

$$\begin{aligned} \varrho'_m &:= R_{\underbrace{(c, \dots, c)}_{m \text{ times}} \underbrace{(c, \dots, c)}_{m \text{ times}}} \\ &= \{(\mathbf{x}, \mathbf{y}) \in D^{2m} \mid x_1 \geq c \vee \dots \vee x_m \geq c \vee y_1 \leq c \vee \dots \vee y_m \leq c\}, \end{aligned}$$

- and $g \in O_D^{(2m)}$ defined by

$$g(x_1, \dots, x_m, \dots, x_{2m}) := \begin{cases} 0 & \text{if there is only one } c \text{ among the first} \\ & m \text{ entries and } 0 \text{ in the other entries,} \\ n-1 & \text{if there is only one } c \text{ among} \\ & \text{the entries } m+1, \dots, 2m \text{ and } n-1 \text{ in} \\ & \text{the other entries,} \\ c & \text{otherwise,} \end{cases}$$

then the proposition below can be proved.

3.5.2 Proposition. *Let $c \in \{1, \dots, n-2\}$ be fixed, and let ϱ'_m be the $2m$ -ary relation defined above. Then*

$$\text{Pol}(\varrho'_{m-1}) \supsetneq \text{Pol}(\varrho'_m)$$

holds for any $m \in \mathbb{N}_+$.

Next we show that

$$\text{Pol} R_{\underbrace{(c, \dots, c)}_{m \text{ times}} \underbrace{(c, \dots, c)}_{m \text{ times}}} \quad \text{and} \quad \text{Pol} R_{\underbrace{(d, \dots, d)}_{m \text{ times}} \underbrace{(d, \dots, d)}_{m \text{ times}}},$$

where $m \in \mathbb{N}_+$, $c, d \in D \setminus \{0, n-1\}$ are fixed and $c \neq d$, are incomparable.

Let $m \in \mathbb{N}_+$, we define σ_m by

$$\sigma_m := R_{\underbrace{(d, \dots, d)}_{m \text{ times}} \underbrace{(d, \dots, d)}_{m \text{ times}}} = \{(\mathbf{x}, \mathbf{y}) \in D^{2m} \mid x_1 \geq d \vee \dots \vee x_m \geq d \vee y_1 \leq d \vee \dots \vee y_m \leq d\}.$$

3.5.3 Lemma. *Let $c, d \in D \setminus \{0, n-1\}$ be fixed and let ϱ'_m and σ_m be the $2m$ -ary relations defined above. If $c \neq d$, then*

$$\text{Pol}(\varrho'_m) \not\subseteq \text{Pol}(\sigma_m),$$

holds for any $m \in \mathbb{N}_+$.

Proof. Fix $c, d \in \{1, \dots, n-2\}$, consider $\text{Pol } \varrho'_m$ and $\text{Pol } \sigma_m$, where $c \neq d$. As $c \neq d$, we consider two cases:

- 1) Let $c < d$ and f be a $2m$ -ary operation such that

$$f(x_1, \dots, x_m, \dots, x_{2m}) := \begin{cases} c & \text{if there is exactly one } d \text{ in the first} \\ & m \text{ entries and } 0 \text{ in the other entries,} \\ n-1 & \text{otherwise.} \end{cases}$$

It will be shown that $f \triangleright \varrho'_m$ but $f \not\triangleright \sigma_m$.

Let $\mathbf{r}_k := (x_{1k}, \dots, x_{mk}, y_{1k}, \dots, y_{mk})$ be any tuples, where $k \in \{1, \dots, 2m\}$, belonging to ϱ'_m . We define

$$f[\mathbf{r}_1, \dots, \mathbf{r}_{2m}] := (e_1, \dots, e_m, o_1, \dots, o_m).$$

As $\text{Im}(f) \in \{c, n-1\}$, we get $f(x_{11}, \dots, x_{12m}) = e_1 \geq c$, and hence

$$f[\mathbf{r}_1, \dots, \mathbf{r}_{2m}] \in \varrho'_m.$$

Thus, $f \triangleright \varrho'_m$.

Below we show that $f \not\triangleright \sigma_m$. The schema

		1	2	...	m	m+1	m+2	...	2m		
1	f	d	0	...	0	0	0	...	0	)	= c
2	f	0	d	...	0	0	0	...	0	)	= c
				$\ddots$				$\vdots$			$\vdots$
m	f	0	0	...	d	0	0	...	0	)	= c
m+1	f	n-1	n-1	...	n-1	d	n-1	...	n-1	)	= n-1
m+2	f	n-1	n-1	...	n-1	n-1	d	...	n-1	)	= n-1
				$\vdots$				$\ddots$			$\vdots$
2m	f	n-1	n-1	...	n-1	n-1	n-1	...	d	)	= n-1

shows that $f \not\triangleright \sigma_m$, because the tuples (columns of the arguments of f) all belong to σ_m . However, after applying f to the tuples, one obtains a tuple (column) that does not belong to σ_m , (because $c \not\leq d$ and $n-1 \not\leq d$).

- 2) If $d < c$, then, similarly to case 1) one can prove that $\text{Pol } \varrho'_m$ is not contained in $\text{Pol } \sigma_m$, by using $g \in \mathcal{O}_D^{(2m)}$ such that

$$g(x_1, \dots, x_m, \dots, x_{2m}) := \begin{cases} c & \text{if there is exactly one } d \text{ between the} \\ & m+1, \dots, 2m \text{ entries and } n-1 \\ & \text{in the other entries,} \\ 0 & \text{otherwise.} \end{cases}$$

□

Note that if $c \neq d$ implies $\text{Pol } \varrho'_m \not\subseteq \text{Pol } \sigma_m$, then automatically (changing the role of c and d , which remain unequal) $\text{Pol } \sigma_m \not\subseteq \text{Pol } \varrho'_m$ also holds.

From Proposition 3.5.2 and the previous lemma, we get $n - 2$ different infinite descending chains of C -clones. They are depicted in Figure 3.2 below.

Figure 3.2: Infinite descending chains in the lattice of C -clones.

At this point it has been proved that there are infinitely many C -clones. However, we cannot answer the following question:

Is the set of all C -clones defined on a finite set with at least three elements, countable or uncountable? We leave this question open, which is also presented in Chapter 7.

Below we prove that if we permute the components of $\mathbf{a}$ or $\mathbf{b}$ of the clausal relation $R_{\mathbf{b}}^{\mathbf{a}}$, the C -clone $\text{Pol } R_{\mathbf{b}}^{\mathbf{a}}$ stays the same. To this end we need an auxiliary statement. For a finite set D we have $[Q]_{R_D} = \text{Inv Pol } Q$, so $\text{Pol } [Q]_{R_D} = \text{Pol Inv Pol } Q$, whence 3.5.4 follows by setting $Q = \{R_{\mathbf{b}}^{\mathbf{a}}\}$.

3.5.4 Remark. Let $R_{\mathbf{b}}^{\mathbf{a}}$ be a clausal relation. Then, $\text{Pol } [R_{\mathbf{b}}^{\mathbf{a}}]_{R_D} = \text{Pol } R_{\mathbf{b}}^{\mathbf{a}}$ holds.

3.5.5 Lemma. Let $p, q \in \mathbb{N}_+$, $\pi_1 \in S_p$ and $\pi_2 \in S_q$. Then

$$\text{Pol } R_{\mathbf{b}}^{\mathbf{a}} = \text{Pol } R_{\mathbf{b} \circ \pi_2}^{\mathbf{a} \circ \pi_1}$$

holds, where $\mathbf{a} = (a_1, \dots, a_p) \in (D \setminus \{0\})^p$, $\mathbf{b} = (b_1, \dots, b_q) \in (D \setminus \{n - 1\})^q$, $\mathbf{a} \circ \pi_1 = (a_{\pi_1(1)}, \dots, a_{\pi_1(p)})$, and $\mathbf{b} \circ \pi_2 = (b_{\pi_2(1)}, \dots, b_{\pi_2(q)})$.

Proof. From Lemma 2.2.4, we infer $R_{\mathbf{b} \circ \pi_2}^{\mathbf{a} \circ \pi_1} = V_{\pi_2}^{\pi_1}(R_{\mathbf{b}}^{\mathbf{a}})$. First, we show that the relational clone $[R_{\mathbf{b}}^{\mathbf{a}}]_{R_D}$ is equal to $[V_{\pi_2}^{\pi_1}(R_{\mathbf{b}}^{\mathbf{a}})]_{R_D}$.

Observe that $V_{\pi_2}^{\pi_1}(R_{\mathbf{b}}^{\mathbf{a}}) \in [R_{\mathbf{b}}^{\mathbf{a}}]_{R_D}$. Moreover,

$$R_{\mathbf{b}}^{\mathbf{a}} = V_{id}^{\pi_1}(R_{\mathbf{b}}^{\mathbf{a}}) = V_{\pi_2 \circ \pi_2^{-1}}^{\pi_1 \circ \pi_2^{-1}}(R_{\mathbf{b}}^{\mathbf{a}}) = V_{\pi_2^{-1}}^{\pi_1^{-1}}(V_{\pi_2}^{\pi_1}(R_{\mathbf{b}}^{\mathbf{a}})) \in [V_{\pi_2}^{\pi_1}(R_{\mathbf{b}}^{\mathbf{a}})]_{R_D},$$

$$\text{Pol } R_{\mathbf{b}}^{\mathbf{a}} \stackrel{3.5.4}{=} \text{Pol } [R_{\mathbf{b}}^{\mathbf{a}}]_{R_D} = \text{Pol } [V_{\pi_2}^{\pi_1}(R_{\mathbf{b}}^{\mathbf{a}})]_{R_D} = \text{Pol } [R_{\mathbf{b} \circ \pi_2}^{\mathbf{a} \circ \pi_1}]_{R_D} \stackrel{3.5.4}{=} \text{Pol } R_{\mathbf{b} \circ \pi_2}^{\mathbf{a} \circ \pi_1}.$$

□

For what is left of this section, we will address the following task.

Given a C -clone $\text{Pol}(R_{\mathbf{b}}^{\mathbf{a}, a'})$, where

$$R_{\mathbf{b}}^{\mathbf{a}, a'} = \left\{ (x_1, \dots, x_p, x_{p+1}, y_1, \dots, y_q) \in D^{p+q+1} \mid \right. \\ \left. x_1 \geq a_1 \vee \dots \vee x_p \geq a_p \vee x_{p+1} \geq a' \vee y_1 \leq b_1 \vee \dots \vee y_q \leq b_q \right\},$$

find a relation $R_{\mathbf{b}}^{\mathbf{a}}$ of lesser arity, such that

$$\text{Pol}(R_{\mathbf{b}}^{\mathbf{a}, a'}) \subseteq \text{Pol}(R_{\mathbf{b}}^{\mathbf{a}}) \quad \text{or} \quad \text{Pol}(R_{\mathbf{b}}^{\mathbf{a}}) \subseteq \text{Pol}(R_{\mathbf{b}}^{\mathbf{a}, a'}).$$

3.5.6 Proposition. *Let $p, q \in \mathbb{N}_+$, $\mathbf{a} = (a_1, \dots, a_p) \in (D \setminus \{0\})^p$, $a' \in D \setminus \{0\}$ and $\mathbf{b} = (b_1, \dots, b_q) \in (D \setminus \{n-1\})^q$. Then*

$$\text{Pol}(R_{\mathbf{b}}^{\mathbf{a}, a'}) \subseteq \text{Pol}(R_{\mathbf{b}}^{\mathbf{a}})$$

holds, if and only if there exists some $i \in \{1, \dots, p\}$, such that $a' \geq a_i$.

Proof. $\Rightarrow$ Let $k \in \mathbb{N}_+$ and let $f \in O_D^{(k)}$ be the k -ary operation on D defined by

$$f(\mathbf{x}) := \begin{cases} a' & \text{if } \mathbf{x} = (a_1, \dots, a_1) \in (D \setminus \{0\})^k \\ n-1 & \text{otherwise.} \end{cases}$$

We show $f \in \text{Pol}^{(k)}(R_{\mathbf{b}}^{\mathbf{a}, a'})$. Let $r_1, r_2, \dots, r_k$ be any tuples belonging to $R_{\mathbf{b}}^{\mathbf{a}, a'}$, where

$$r_l = (x_{1l}, \dots, x_{pl}, x_{p+1l}, y_{1l}, \dots, y_{ql})$$

for $l \in \{1, \dots, k\}$. Then $(c_1, \dots, c_{p+1}, d_1, \dots, d_q) := f[r_1, \dots, r_k] \in R_{\mathbf{b}}^{\mathbf{a}, a'}$ because $c_{p+1} = f(x_{p+11}, \dots, x_{p+1k}) \geq a'$. Hence, $f \in \text{Pol}^{(k)}(R_{\mathbf{b}}^{\mathbf{a}, a'})$ and $f \in \text{Pol}^{(k)}(R_{\mathbf{b}}^{\mathbf{a}})$ by supposition. Let $r'_1 = \dots = r'_k = (x_1, \dots, x_p, y_1, \dots, y_q) := (a_1, \dots, a_1, 0, \dots, 0)$. Obviously, r'_l belongs to $R_{\mathbf{b}}^{\mathbf{a}}$ for $l \in \{1, \dots, k\}$, thus,

$$f[r'_1, \dots, r'_k] = (a', \dots, a', n-1, \dots, n-1) \in R_{\mathbf{b}}^{\mathbf{a}}.$$

Because $n-1 \not\geq b_j$ for all $j \in \{1, \dots, q\}$, there must be one $i \in \{1, \dots, p\}$ such that $a' \geq a_i$.

$\Leftarrow$ Thanks to 3.5.5, we assume $i = p$, i.e. $a' \geq a_p$, so $\min\{a_p, a'\} = a_p$.

Let f be any k -ary operation with $f \in \text{Pol}^{(k)}(\mathbf{R}_{\mathbf{b}}^{\mathbf{a}, a'})$ we have to show $f \in \text{Pol}^{(k)}(\mathbf{R}_{\mathbf{b}}^{\mathbf{a}})$. Let $r_1, \dots, r_k \in \mathbf{R}_{\mathbf{b}}^{\mathbf{a}}$, where $r_l = (x_{1l}, \dots, x_{pl}, y_{1l}, \dots, y_{ql})$ for $l \in \{1, \dots, k\}$. Then

$$\begin{aligned} & x_{1l} \geq a_1 \vee \dots \vee x_{pl} \geq a_p \vee y_{1l} \leq b_1 \vee \dots \vee y_{ql} \leq b_q \\ \iff & x_{1l} \geq a_1 \vee \dots \vee x_{pl} \geq \min\{a_p, a'\} \vee y_{1l} \leq b_1 \vee \dots \vee y_{ql} \leq b_q \\ \iff & x_{1l} \geq a_1 \vee \dots \vee x_{pl} \geq a_p \vee x_{pl} \geq a' \vee y_{1l} \leq b_1 \vee \dots \vee y_{ql} \leq b_q \quad (3.12) \end{aligned}$$

We define for $l \in \{1, \dots, k\}$

$$r'_l = (\bar{x}_{1l}, \dots, \bar{x}_{pl}, \bar{x}_{p+1l}, \bar{y}_{1l}, \dots, \bar{y}_{ql}) := (x_{1l}, \dots, x_{pl}, x_{pl}, y_{1l}, \dots, y_{ql}).$$

The tuples $r'_1, \dots, r'_k$ belong to $\mathbf{R}_{\mathbf{b}}^{\mathbf{a}, a'}$ because of (3.12).

Because $f \triangleright \mathbf{R}_{\mathbf{b}}^{\mathbf{a}, a'}$ we have $(c_1, \dots, c_p, c_{p+1}, d_1, \dots, d_q) := f[r'_1, \dots, r'_k] \in \mathbf{R}_{\mathbf{b}}^{\mathbf{a}, a'}$. Then, by construction, we have $c_p = c_{p+1}$, and

$$c_p \geq a_p \vee c_{p+1} \geq a' \geq a_p \iff c_p \geq a_p.$$

Therefore, $(c_1, \dots, c_p, d_1, \dots, d_q) \in \mathbf{R}_{\mathbf{b}}^{\mathbf{a}}$, i.e. $f \triangleright \mathbf{R}_{\mathbf{b}}^{\mathbf{a}}$. $\square$

Using the duality principle for C -clones, the dual statement of 3.5.6 is the following.

3.5.7 Proposition. *Let $p, q \in \mathbb{N}_+$, $\mathbf{b} = (b_1, \dots, b_q) \in (D \setminus \{n-1\})^q$, $b' \in D \setminus \{n-1\}$ and $\mathbf{a} = (a_1, \dots, a_p) \in (D \setminus \{0\})^p$. Then*

$$\text{Pol}(\mathbf{R}_{\mathbf{b}, b'}^{\mathbf{a}}) \subseteq \text{Pol}(\mathbf{R}_{\mathbf{b}}^{\mathbf{a}})$$

holds, if and only if there exists some $j \in \{1, \dots, q\}$ such that $b' \leq b_j$.

The following lemma shows a special case where the inclusion, presented in Proposition 3.5.6 above, is proper.

3.5.8 Lemma. *Let $k \in \mathbb{N}_{\geq 2}$ and let $|D| \geq 2$. Then*

$$\text{Pol}^{(k)}(\mathbf{R}_0^{n-1, n-1}) \subsetneq \text{Pol}^{(k)}(\mathbf{R}_0^{n-1}).$$

Proof. First, we show $\text{Pol}^{(2)}(\mathbf{R}_0^{n-1, n-1}) \subsetneq \text{Pol}^{(2)}(\mathbf{R}_0^{n-1})$. To prove the last statement, it is enough, thanks to 3.5.6, to show that $f \triangleright \mathbf{R}_0^{n-1}$, but $f \not\triangleright \mathbf{R}_0^{n-1, n-1}$ for some $f \in \mathcal{O}_D^{(2)}$. Let f be the binary operation on D defined by

$$f(x_1, x_2) := \begin{cases} n-1 & \text{if } x_1 = x_2 = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

We show that f does not preserve $R_0^{n-1, n-1}$. Consider, $r_1 = (n-1, 0, n-1)$ and $r_2 = (0, n-1, n-1)$. Clearly, the tuples r_1, r_2 belong to $R_0^{n-1, n-1}$, but $f[r_1, r_2]$, which is equal to $(f(n-1, 0), f(0, n-1), f(n-1, n-1)) = (0, 0, n-1)$, does not belong to $R_0^{n-1, n-1}$.

To prove that f preserves R_0^{n-1} , we consider any two tuples $r'_1 = (x_1, y_1)$ and $r'_2 = (x_2, y_2)$ belonging to R_0^{n-1} , i.e. $x_l = n-1 \vee y_l = 0$ for $l \in \{1, 2\}$. Now we consider two cases:

- If $(x_1, x_2) = (n-1, n-1)$, then

$$f[r'_1, r'_2] = (f(x_1, x_2), f(y_1, y_2)) = (n-1, f(y_1, y_2)) \in R_0^{n-1}.$$

- If $(x_1, x_2) \neq (n-1, n-1)$ because $r'_1, r'_2 \in R_0^{n-1}$, then we get $y_l = 0$ for some $l \in \{1, 2\}$. Hence, $(f(x_1, x_2), f(y_1, y_2)) = (0, 0) \in R_0^{n-1}$.

To show the lemma for $k \in \mathbb{N}_{\geq 2}$, we define $g(x_1, x_2, \dots, x_k) = f(x_1, x_2)$. Evidently, $g \in \langle f \rangle_{O_D}$, furthermore $f(x_1, x_2) = g(x_1, x_2, x_2, \dots, x_2)$, so $f \in \langle g \rangle_{O_D}$. Hence, $\langle f \rangle_{O_D} = \langle g \rangle_{O_D}$, and $\text{Inv } f = \text{Inv } \langle f \rangle_{O_D} = \text{Inv } \langle g \rangle_{O_D} = \text{Inv } g$. So, because of the first part of the proof, we have $g \triangleright R_0^{n-1}$, but $g \not\triangleright R_0^{n-1, n-1}$. $\square$

4 C -automorphism groups and Krasner C -clones

As we already mentioned our aim is the analysis of the lattice of all C -clones. This seems to be fairly hard as we exhibited already infinitely many C -clones for $|D| \geq 3$. A first modest step towards our goal is to describe all C -clones correlating with Krasner clones on the relational side. Since the latter correspond exactly to all permutation groups on D (cf. [PK79]), there are only finitely many such clones for finite D .

4.1 Characterisation

In this section we characterise all groups and Krasner clones that are determined by sets of clausal relations. To this end we present a restriction of the Galois connection $\text{Aut} - \text{Inv}$, where the relations are clausal relations.

4.1.1 Definition. For $G \subseteq S_D$ we define $C\text{Inv } G := \text{Inv } G \cap CR_D$. The operators

$$\begin{aligned} C\text{Inv} : \mathcal{P}(S_D) &\longrightarrow \mathcal{P}(CR_D) : & G &\mapsto C\text{Inv } G \\ \text{Aut} : \mathcal{P}(CR_D) &\longrightarrow \mathcal{P}(S_D) : & Q &\mapsto \text{Aut } Q \end{aligned}$$

define a *Galois connection* $\text{Aut} - C\text{Inv}$ between symmetric groups and clausal relations.

4.1.2 Definition. A permutation group $G \subseteq S_D$ is called *C -automorphism group* if $G = \text{Aut } Q$ for some set $Q \subseteq CR_D$ of clausal relations, and a set $Q \subseteq CR_D$ is called *Krasner C -clone* if $Q = C\text{Inv } G$ for a permutation group G .

In this sense the C -automorphism groups allow to analyse the lattice of all C -clones up to equality of their unary bijective part. From Remark 3.1.4 we deduce that every C -automorphism group can be determined by sets of non-trivial clausal relations, i.e.

4.1.3 Remark. For $Q \subseteq CR_D$, it follows that $\text{Aut}(Q) = \text{Aut}(Q \cap CR_D^*)$.

We shall describe $\{\text{Aut } Q \mid Q \subseteq CR_D\} \stackrel{4.1.3}{=} \{\text{Aut } Q' \mid Q' \subseteq CR_D^*\}$. Since

$$\text{Aut } Q' = \bigcap_{R_b^a \in Q'} \text{Aut } R_b^a$$

for $Q' \subseteq CR_D^*$, it suffices to regard C -automorphism groups¹ $\text{Aut } R_{\mathbf{b}}^{\mathbf{a}}$ with parameters $\mathbf{a} = (a_1, \dots, a_p) \in (D \setminus \{0\})^p$ and $\mathbf{b} = (b_1, \dots, b_q) \in (D \setminus \{n-1\})^q$. Observe that, by definition, a clausal relation $R_{\mathbf{b}}^{\mathbf{a}}$ can be written² as

$$R_{\mathbf{b}}^{\mathbf{a}} = \bigcup_{i=1}^p (D^{i-1} \times [a_i, n-1] \times D^{p+q-i}) \cup \bigcup_{j=1}^q (D^{p+j-1} \times [0, b_j] \times D^{q-j}) \quad (4.1)$$

For the rest of this section, we are going to study C -automorphism groups determined by only one non-trivial clausal relation. In order to characterise them, we use the following lemma, which shows that, on a finite set D , permutations preserving some relations automatically preserve its complement.

4.1.4 Lemma ([PK79]). *Let $m \in \mathbb{N}_+$, $f \in S_D$, $\varrho \in R_D^{(m)}$ and let $\varrho^C := D^m \setminus \varrho$. Then*

$$f \triangleright \varrho \iff f^{-1} \triangleright \varrho \iff f \triangleright \varrho^C \iff f^{-1} \triangleright \varrho^C.$$

Using this lemma, for a permutation $f \in S_D$ and a relation $R_{\mathbf{b}}^{\mathbf{a}} \in CR_D^*$, the proposition $f \triangleright R_{\mathbf{b}}^{\mathbf{a}}$ can always be replaced by the preservation of the *complement* $(R_{\mathbf{b}}^{\mathbf{a}})^C = \prod_{i=1}^p [0, a_i) \times \prod_{j=1}^q (b_j, n-1]$. Hence,

$$\text{Aut } R_{\mathbf{b}}^{\mathbf{a}} = \text{Aut } ((R_{\mathbf{b}}^{\mathbf{a}})^C) = \text{Aut } \left(\prod_{i=1}^p [0, a_i) \times \prod_{j=1}^q (b_j, n-1] \right).$$

4.1.5 Lemma. *Let $R_{\mathbf{b}}^{\mathbf{a}} \in CR_D^*$ be a non-trivial clausal relation, that is to say $\mathbf{a} = (a_1, \dots, a_p) \in (D \setminus \{0\})^p$ and $\mathbf{b} = (b_1, \dots, b_q) \in (D \setminus \{n-1\})^q$. Then it holds*

$$\begin{aligned} \text{Pol } \{(R_{\mathbf{b}}^{\mathbf{a}})^C\} &= \text{Pol } \left(\prod_{i=1}^p [0, a_i) \times \prod_{j=1}^q (b_j, n-1] \right) \\ &= \text{Pol } (\{[0, a_i) \mid i \in \{1, \dots, p\}\} \cup \{(b_j, n-1] \mid j \in \{1, \dots, q\}\}). \end{aligned}$$

Proof. Let $W := \text{Pol } (\{[0, a_i) \mid i \in \{1, \dots, p\}\} \cup \{(b_j, n-1] \mid j \in \{1, \dots, q\}\})$, and let $f \in W$, we show $f \in \text{Pol } \{(R_{\mathbf{b}}^{\mathbf{a}})^C\}$. As $f \in W$, it is equivalent to

$$\{[0, a_i) \mid i \in \{1, \dots, p\}\} \cup \{(b_j, n-1] \mid j \in \{1, \dots, q\}\} \subseteq \text{Inv } f.$$

Since $\text{Inv } f$ is a relational clone of R_D , it is closed under $\times$, it follows that also $\prod_{i=1}^p [0, a_i) \times \prod_{j=1}^q (b_j, n-1] \in \text{Inv } f$, which is equivalent to

$$f \in \text{Pol } \left(\prod_{i=1}^p [0, a_i) \times \prod_{j=1}^q (b_j, n-1] \right) = \text{Pol } \{(R_{\mathbf{b}}^{\mathbf{a}})^C\}.$$

¹For convenience of notation we occasionally also write $\text{Aut } \varrho$ instead of $\text{Aut}\{\varrho\}$ for just one relation $\varrho \in R_D$.

²Here $[i, j]$ denotes a closed interval of D w.r.t. the natural order on D .

Conversely, relational clones are also closed under projections to coordinates, and it holds

$$\text{pr}_i(\mathbf{R}_{\mathbf{b}}^{\mathbf{a}})^C = [0, a_i) \quad \text{and} \quad \text{pr}_{p+j}(\mathbf{R}_{\mathbf{b}}^{\mathbf{a}})^C = (b_j, n-1]$$

for $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$. So if $f \in \text{Pol} \prod_{i=1}^p [0, a_i) \times \prod_{j=1}^q (b_j, n-1]$, equivalently $\prod_{i=1}^p [0, a_i) \times \prod_{j=1}^q (b_j, n-1] \in \text{Inv } f$, then also

$$\text{pr}_i(\mathbf{R}_{\mathbf{b}}^{\mathbf{a}})^C = [0, a_i) \in \text{Inv } f \quad \text{and} \quad \text{pr}_{p+j}(\mathbf{R}_{\mathbf{b}}^{\mathbf{a}})^C = (b_j, n-1] \in \text{Inv } f$$

for all $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$. This implies

$$\{[0, a_i) \mid i \in \{1, \dots, p\}\} \cup \{(b_j, n-1] \mid j \in \{1, \dots, q\}\} \subseteq \text{Inv } f,$$

equivalently to $f \in \text{Pol} (\{[0, a_i) \mid i \in \{1, \dots, p\}\} \cup \{(b_j, n-1] \mid j \in \{1, \dots, q\}\})$. $\square$

By Lemma 4.1.4 it holds for $f \in S_D$ and $b < n-1$

$$f \triangleright (b, n-1] \iff f \triangleright (b, n-1]^C = [0, b] = [0, b+1),$$

that is $\text{Aut}(b, n-1] = \text{Aut}[0, b+1)$. Combining this observation with the previous lemma, we obtain for all $\mathbf{R}_{\mathbf{b}}^{\mathbf{a}} \in CR_D^*$

$$\begin{aligned} \text{Aut} \{\mathbf{R}_{\mathbf{b}}^{\mathbf{a}}\} &= \text{Aut} \{(\mathbf{R}_{\mathbf{b}}^{\mathbf{a}})^C\} \\ &\stackrel{4.1.5}{=} \text{Aut} (\{[0, a_i) \mid i \in \{1, \dots, p\}\} \cup \{(b_j, n-1] \mid j \in \{1, \dots, q\}\}) \\ &= \text{Aut} (\{[0, a_i) \mid i \in \{1, \dots, p\}\} \cup \{[0, b_j+1) \mid j \in \{1, \dots, q\}\}) \\ &= \text{Aut} (\text{Int } \mathbf{R}_{\mathbf{b}}^{\mathbf{a}}), \end{aligned}$$

where $\text{Int } \mathbf{R}_{\mathbf{b}}^{\mathbf{a}} := \{[0, a_i) \mid i \in \{1, \dots, p\}\} \cup \{[0, b_j+1) \mid j \in \{1, \dots, q\}\}$ are the intervals belonging to $\mathbf{R}_{\mathbf{b}}^{\mathbf{a}}$. This enables us to simplify our Galois connection once more:

4.1.6 Corollary ([BV10]). *The lattice of C -automorphism groups is*

$$\{\text{Aut } Q \mid Q \subseteq CR_D\} = \{\text{Aut } Q \mid Q \subseteq \mathcal{I}\},$$

where $\mathcal{I} := \{[0, a) \mid a \in D \setminus \{0\}\}$.

Proof. For $Q \subseteq CR_D$, Remark 4.1.3 yields

$$\begin{aligned} \text{Aut } Q &= \text{Aut} (Q \cap CR_D^*) = \bigcap \{\text{Aut} \{\varrho\} \mid \varrho \in Q \cap CR_D^*\} \\ &= \bigcap \{\text{Aut } \text{Int } \varrho \mid \varrho \in Q \cap CR_D^*\} = \text{Aut} \left(\bigcup \{\text{Int } \varrho \mid \varrho \in Q \cap CR_D^*\} \right), \end{aligned}$$

where $\bigcup \{\text{Int } \varrho \mid \varrho \in Q \cap CR_D^*\} \subseteq \mathcal{I}$.

Conversely, for $u \in D \setminus \{0\}$ one has $\text{Int } \mathbf{R}_{(u-1)}^{(u)} = \{[0, u)\}$, and so

$$\text{Aut} \left\{ \mathbf{R}_{(u-1)}^{(u)} \right\} = \text{Aut } \text{Int } \mathbf{R}_{(u-1)}^{(u)} = \text{Aut} \{[0, u)\}.$$

Hence, for all $U \subseteq D \setminus \{0\}$ holds

$$\text{Aut} \{[0, u) \mid u \in U\} = \text{Aut} \left\{ \mathbf{R}_{(u-1)}^{(u)} \mid u \in U \right\}. \quad \square$$

4.1.7 Corollary. For $Q \subseteq CR_D$ the corresponding Krasner clone is generated by its unary part:

$$\text{Inv Aut } Q = \text{Inv Aut Inv}^{(1)} \text{ Aut } Q.$$

Proof. Let $Q^* := Q \cap CR_D^*$ denote the non-trivial part of Q and abbreviate

$$\text{Inv Aut} =: [\cdot]'. \quad \square$$

Then as demonstrated in Corollary 4.1.6, one has $\text{Aut}(Q) = \text{Aut}(Q^*) = \text{Aut } Q'$, where $Q' := \bigcup \{ \text{Int } \rho \mid \rho \in Q^* \}$. We will show $Q' \subseteq [Q]'^{(1)} \subseteq [Q]'$, then

$$[Q]' = \text{Inv Aut}(Q) = \text{Inv Aut}(Q') = [Q]' \subseteq [[Q]'^{(1)}]' \subseteq [[Q]']' = [Q]'$$

yields the claim. To prove $Q' \subseteq [Q]'^{(1)}$, it suffices to see $\text{Int } \rho \subseteq [Q]'^{(1)}$ for any $\rho \in Q^* \subseteq Q$. Obviously, $\text{Int } \rho$ is a set of unary relations. By definition, for $\rho = R_{\mathbf{b}}^{\mathbf{a}}$ holds

$$[0, a_i) = pr_i(R_{\mathbf{b}}^{\mathbf{a}})^C \in [\rho]' \subseteq [Q]' \quad \text{and} \quad [0, b_j + 1) = (pr_{p+j}(R_{\mathbf{b}}^{\mathbf{a}})^C)^C \in [\rho]' \subseteq [Q]'$$

for $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$. Thus, $\text{Int } \rho \subseteq [Q]'$ concluding the argument. $\square$

Now the lattice of C -automorphism groups can be treated by computer. The following example shows it for $|D| = 2$, $|D| = 3$, and $|D| = 4$.

4.1.8 Example. In the next cross tables, is shown which $f \in S_D$ preserve which intervals in $\mathcal{I}$. The rows are labelled by functions f_k followed by their tuple z of values. Note that the index k corresponds to these tuples as it is just the interpretation of $z = f_k(0)f_k(1) \dots f_k(n-1)$ as a n -adic number: $k = \sum_{i=0}^{n-1} f_k(i)|D|^{n-1-i}$.

$\triangleright$	$[0, 1)$
$f_1 : 01$	$\times$
$f_2 : 10$

The lattice of all C -automorphism groups on the set $D = \{0, 1\}$.

$\triangleright$	$[0, 1)$	$[0, 2)$
$f_5 : 012$	$\times$	$\times$
$f_7 : 021$	$\times$	
$f_{11} : 102$		$\times$
$f_{15} : 120$		
$f_{19} : 201$		
$f_{21} : 210$

The lattice of all C -automorphism groups on the set $D = \{0, 1, 2\}$.

$\triangleright$	$[0, 1)$	$[0, 2)$	$[0, 3)$
$f_{27} : 0123$	$\times$	$\times$	$\times$
$f_{30} : 0132$	$\times$	$\times$	
$f_{39} : 0213$	$\times$		$\times$
$f_{45} : 0231$	$\times$		
$f_{54} : 0312$	$\times$		
$f_{57} : 0321$	$\times$		
$f_{75} : 1023$		$\times$	$\times$
$f_{78} : 1032$		$\times$	
$f_{99} : 1203$			$\times$
$f_{108} : 1230$			
$f_{114} : 1302$			
$f_{120} : 1320$			
$f_{135} : 2013$			$\times$
$f_{141} : 2031$			
$f_{147} : 2103$			$\times$
$f_{156} : 2130$			
$f_{177} : 2301$			
$f_{180} : 2310$			
$f_{198} : 3012$			
$f_{201} : 3021$			
$f_{210} : 3102$			
$f_{216} : 3120$			
$f_{225} : 3201$			
$f_{228} : 3210$

The lattice of all C -automorphism groups on the set $D = \{0, 1, 2, 3\}$

In this example it could be seen that the lattice is Boolean with $n - 1$ atoms. In fact, this is true in general:

4.1.9 Theorem ([BV10]). *The lattice of all C -automorphism groups is dually isomorphic to $(\mathcal{P}(D \setminus \{0\}), \subseteq)$ via the following isomorphism*

$$\begin{aligned} \phi : (\mathcal{P}(D \setminus \{0\}), \subseteq) &\longrightarrow (\{\text{Aut } Q \mid Q \subseteq CR_D\}, \supseteq) \\ U &\longmapsto \text{Aut } \{[0, u) \mid u \in U\}. \end{aligned}$$

Proof. Obviously, $(\mathcal{P}(\mathcal{I}), \subseteq) \cong (\mathcal{P}(D \setminus \{0\}), \subseteq)$, mapping each $U \subseteq D \setminus \{0\}$ to $\{[0, u) \mid u \in U\}$. So from Corollary 4.1.6 one obtains that ϕ is a surjective order preserving map. Now it will be shown that it is also order reflecting. This will automatically yield injectivity. To this end we consider $U, U' \subseteq D \setminus \{0\}$ such that $\phi(U) \supseteq \phi(U')$. Take any $u \in U$. Finiteness of D yields that $U' = \{u_1, \dots, u_k\}$ such that w.l.o.g. $0 =: u_0 < u_1 < u_2 < \dots < u_\nu \leq u < u_{\nu+1} < \dots < u_k$ for some $0 \leq \nu \leq k$, where $1 \leq k \leq |D|$ or $U' = \emptyset$. For $U' \neq \emptyset$ assume, to obtain

a contradiction, $u_\nu < u$, and let $f \in S_D$ be the transposition swapping u and u_ν . Evidently, $f \in \phi(U') \subseteq \phi(U)$, so $f \triangleright [0, u]$. The assumption $u_\nu < u$ yields $u_\nu \in [0, u]$, implying $u = f(u_\nu) < u$, a contradiction. Hence, $u = u_\nu \in U'$, which shows $U \subseteq U'$. If $U' = \emptyset$ the same argument with $\nu = 0$ yields a contradiction as $\phi(U') = S_D$. Consequently, this case does not occur, concluding the proof. $\square$

We finish this section characterising the sort of permutation groups occurring in the lattice of all C -automorphism groups. To do this we need an additional auxiliary statement.

4.1.10 Lemma. *Let $a_1, a_2 \in D$ with $a_1 < a_2$. Then it holds*

$$\text{Aut}\{[0, a_1), [a_1, a_2)\} = \text{Aut}\{[0, a_1), [0, a_2)\}.$$

Proof. The claim is equivalent to the following equality of Krasner clones

$$\text{Inv Aut}\{[0, a_1), [a_1, a_2)\} = \text{Inv Aut}\{[0, a_1), [0, a_2)\}.$$

As Krasner clones are closed under intersection, union and complements, one obtains

$$\begin{aligned} [a_1, a_2) &= [0, a_2) \setminus [0, a_1) = [0, a_2) \cap [0, a_1)^C \in \text{Inv Aut}\{[0, a_1), [0, a_2)\}, \\ [0, a_2) &= [0, a_1) \cup [a_1, a_2) \in \text{Inv Aut}\{[0, a_1), [a_1, a_2)\}, \end{aligned}$$

showing that the Krasner clones are mutually contained in each other. $\square$

4.1.11 Theorem ([BV10]). *Let a natural number $m \in \mathbb{N}$ and a finite sequence of elements $0 =: a_0 < a_1 < \dots < a_m < a_{m+1} := n$ be given. Then it holds,*

$$\text{Aut}\{[0, a_1), [0, a_2), \dots, [0, a_m)\} \cong S_{[a_0, a_1)} \times S_{[a_1, a_2)} \times \dots \times S_{[a_{m-1}, a_m)} \times S_{[a_m, n)}.$$

Proof. First, we use induction on $m \in \mathbb{N}$ to prove

$$\text{Aut}\{[0, a_1), [0, a_2), \dots, [0, a_m)\} = \text{Aut}\{[0, a_1), [a_1, a_2), \dots, [a_{m-1}, a_m)\}.$$

The case $m = 0$ is trivial since both sides of the equation yield $\text{Aut} \emptyset$. Assuming the claim is true for $m \in \mathbb{N}$, we show its truth for $m + 1$:

$$\begin{aligned} \text{Aut}\{[0, a_1), \dots, [0, a_{m+1})\} &= \\ &= \text{Aut}\{[0, a_1), \dots, [0, a_{m-1})\} \cap \text{Aut}\{[0, a_m), [0, a_{m+1})\} \\ &\stackrel{4.1.10}{=} \text{Aut}\{[0, a_1), \dots, [0, a_{m-1})\} \cap \text{Aut}\{[0, a_m), [a_m, a_{m+1})\} \\ &= \text{Aut}\{[0, a_1), \dots, [0, a_{m-1}), [0, a_m)\} \cap \text{Aut}\{[a_m, a_{m+1})\} \\ &= \text{Aut}\{[0, a_1), [a_1, a_2), \dots, [a_{m-1}, a_m)\} \cap \text{Aut}\{[a_m, a_{m+1})\} \\ &= \text{Aut}\{[0, a_1), [a_1, a_2), \dots, [a_{m-1}, a_m), [a_m, a_{m+1})\}. \end{aligned}$$

Next we prove

$$\text{Aut}\{[a_0, a_1), \dots, [a_{m-1}, a_m)\} = \text{Aut}\{[a_0, a_1), \dots, [a_{m-1}, a_m), [a_m, n)\}.$$

Choose $f \in \text{Aut}\{[a_0, a_1), \dots, [a_{m-1}, a_m)\} = \text{Aut}\{[0, a_1), [0, a_2), \dots, [0, a_m)\}$ arbitrarily. Especially, $f \triangleright [0, a_m)$, which is, by Lemma 4.1.4, equivalent to f preserves $[0, a_m)^C = [a_m, n)$. This shows the inclusion of $\text{Aut}\{[a_i, a_{i+1}) \mid 0 \leq i < m\}$ in $\text{Aut}\{[a_i, a_{i+1}) \mid 0 \leq i \leq m\}$, and the converse is not difficult. Finally, we show that

$$\text{Aut}\{[a_0, a_1), \dots, [a_{m-1}, a_m), [a_m, n)\} \cong S_{[a_0, a_1)} \times \dots \times S_{[a_{m-1}, a_m)} \times S_{[a_m, n)}.$$

We define

$$\begin{aligned} \varphi : S_{[a_0, a_1)} \times \dots \times S_{[a_m, n)} &\longrightarrow \text{Aut}\{[a_0, a_1), \dots, [a_{m-1}, a_m), [a_m, n)\} \\ (g_0, \dots, g_m) &\mapsto \varphi(g_0, \dots, g_m) \end{aligned}$$

by

$$\varphi(g_0, \dots, g_m)(x) := g_i(x) \text{ iff } x \in [a_i, a_{i+1})$$

for all $i \in \{0, \dots, m-1\}$. As the intervals $\{[a_i, a_{i+1}) \mid 0 \leq i \leq m\}$ form a partition of D , $\varphi(g_0, \dots, g_m)$ is well defined. In order to prove that φ is an isomorphism, we have to show that φ is a bijective homomorphism. To verify the homomorphism property, take $(g_0, \dots, g_m), (h_0, \dots, h_m) \in \prod_{i=0}^m S_{[a_i, a_{i+1})}$ and let $x \in D$. Then, apparently, $x \in [a_i, a_{i+1})$ for some $0 \leq i \leq m$, and

$$\begin{aligned} \varphi(g_0 \circ h_0, \dots, g_m \circ h_m)(x) &= (g_i \circ h_i)(x) = g_i(x) \circ h_i(x) \\ &= \varphi(g_0, \dots, g_m)(x) \circ \varphi(h_0, \dots, h_m)(x). \end{aligned}$$

It is clear by definition that φ is injective. Finally, we show that φ is surjective. Let $h \in \text{Aut}\{[a_0, a_1), \dots, [a_{m-1}, a_m), [a_m, n)\}$, then we define $g_i = h|_{[a_i, a_{i+1})}$, yielding $\varphi(g_1, \dots, g_m) = h$. $\square$

Figure 4.1: The lattice of all C -automorphism groups on the set $D = \{0, 1, 2, 3\}$.

Applying the last theorem to all C -automorphism groups for $|D| = 4$, the case shown in the Example 4.1.8, one obtains the lattice depicted in Figure 4.1.

4.2 C -automorphism groups and the partition lattice

The remark in Figure 4.1 hints at another way to understand the structure of this lattice: It is isomorphic (see Proposition 4.2.1 below) to a sublattice $\mathbb{P}$ of the partition lattice on D , consisting of those partitions all blocks of which are intervals w.r.t. the natural order on D . That is, $P \in \text{Part}(D)$ is such a partition, iff for every block $B \in P$ holds

$$[\min B, \max B] \subseteq B.$$

Combining this observation with the arguments used in the proof of Theorem 4.1.11, one can see that the C -automorphism group belonging to such a partition P is given as a direct product of the symmetric groups corresponding to the blocks of P (cf. also Figure 4.1).

4.2.1 Proposition ([BV10]). *The lattice of C -automorphism groups can be embedded into the lattice $(\text{Part}(D), \leq)$ of all partitions on D . The image of this embedding is the lattice $\mathbb{P} := \{P \in \text{Part}(D) \mid \forall B \in P : [\min B, \max B] \subseteq B\}$.*

Proof. By Theorem 4.1.9 it is sufficient to embed $(\mathcal{P}(D \setminus \{0\}), \supseteq)$ in $(\text{Part}(D), \leq)$. We define an order-reversing mapping $\psi : \mathcal{P}(D \setminus \{0\}) \rightarrow \text{Part}(D)$ in the following way: For every $U \subseteq D \setminus \{0\}$ there is some $m \in \mathbb{N}$ such that $U = \{a_1, \dots, a_m\}$ and $0 =: a_0 < a_1 < a_2 < \dots < a_m < a_{m+1} := n$. Using this we set

$$\psi(U) := \{[a_i, a_{i+1}) \mid 0 \leq i \leq m\}.$$

Clearly, $\psi(U)$ is a partition in $\mathbb{P}$. Conversely, every partition $P \in \mathbb{P}$ arises in the form $P = \psi(U)$ for the set $U := D \cap \{1 + \max B \mid B \in P\}$.

It remains to be shown that ψ is an injective lattice homomorphism. To this end regard $m, k \in \mathbb{N}$ and arbitrary finite sequences

$$\begin{aligned} 0 =: a_0 < a_1 < a_2 < \dots < a_m < a_{m+1} &:= n \text{ and} \\ 0 =: b_0 < b_1 < b_2 < \dots < b_k < b_{k+1} &:= n, \end{aligned}$$

and let $U := \{a_1, \dots, a_m\}$ and $V := \{b_1, \dots, b_k\}$.

To show injectivity assume $\psi(U) = \psi(V)$. Now an inductive argument shows $U = V$: the block $[0, a_1) \in \psi(U) = \psi(V)$, and the only block in $\psi(V)$ containing 0 is $[0, b_1)$, so $[0, a_1) = [0, b_1)$. This implies $a_1 = b_1$. Then

$$[b_1, a_2) = [a_1, a_2) \in \psi(U) = \psi(V),$$

and the only block of $\psi(V)$ containing b_1 is $[b_1, b_2)$. So $[a_1, a_2) = [b_1, b_2)$, implying $a_2 = b_2$ etc. This continues until $n = a_{m+1} = b_{k+1}$ has been shown, and this means that $b_{m+1} = n = b_{k+1}$. So $k = m$, and $U = V$.

To show $\psi(U \cup V) = \psi(U) \wedge \psi(V)$, take any $v \in U \cup V$ and regard the block $[v, w) \in \psi(U \cup V)$ given by $w := \min \{y \in U \cup V \mid v < y\}$. Clearly, $a_i \leq v < a_{i+1}$ for some $0 \leq i \leq m$, and by definition of w , one has $w \leq a_{i+1}$. This shows that the block

$[v, w] \subseteq [a_i, a_{i+1}] \in \psi(U)$ and, as $v \in U \cup V$ was arbitrary, that $\psi(U \cup V) \leq \psi(U)$. An analogous argument yields $\psi(U \cup V) \leq \psi(V)$, so

$$\psi(U \cup V) \leq \psi(U) \wedge \psi(V).$$

Conversely, take any block $B \in \psi(U) \wedge \psi(V)$ and regard some $x \in B$. As

$$\psi(U) \wedge \psi(V) \leq \psi(U),$$

there is some $0 \leq i \leq m$ such that $B \subseteq [a_i, a_{i+1})$. Likewise, there is some $0 \leq j \leq k$ such that $B \subseteq [b_j, b_{j+1})$. Then also

$$c := \max \{a_i, b_j\} \leq x < \min \{a_{i+1}, b_{j+1}\} =: e.$$

As $B \neq \emptyset$, $c < e$. Let $d := \min \{y \in U \cup V \mid c < y\}$, then by minimality $d \leq e$. Assuming $d < e$ and $d \in U$ would contradict $a_i \leq c < d < e \leq a_{i+1}$, the assumption on the order in U , and similarly, $d \in V$ would contradict $b_j \leq c < d < e \leq b_{j+1}$. Hence, $d = e$ and $x \in [c, e) = [c, d)$, i.e. $B \subseteq [c, d) \in \psi(U \cup V)$. As $B \in \psi(U) \wedge \psi(V)$ was arbitrary, this shows $\psi(U) \wedge \psi(V) \leq \psi(U \cup V)$.

Using similar but not necessarily more complicated arguments, below we prove that $\psi(U) \vee \psi(V) = \psi(U \cap V)$.

First we show $\psi(U) \leq \psi(U \cap V)$. Choose any $0 \leq i \leq m$ and define elements

$$c := \max \{y \in U \cap V \mid y \leq a_i\} \text{ and } d := \min \{y \in U \cap V \mid c < y\}.$$

By definition we have $c \leq a_i$ and $c < d$. Assuming $d \leq a_i$, implies by maximality of c the contradiction $d \leq c < d$. Thus $a_i < d$; as $d \in U$ we obtain $a_{i+1} \leq d$. This means $[a_i, a_{i+1}) \subseteq [c, d)$, and as $a_i \in U$ was arbitrary, every block in $\psi(U)$ is contained in some block of $\psi(U \cap V)$. This shows

$$\psi(U) \leq \psi(U \cap V),$$

and $\psi(V) \leq \psi(U \cap V)$ can be proven with analogous arguments. Hence,

$$\psi(U) \vee \psi(V) \leq \psi(U \cap V).$$

To verify $\psi(U) \vee \psi(V) \geq \psi(U \cap V)$, we regard any block $[u, v) \in \psi(U \cap V)$. Note that $v = \min \{y \in U \mid y > u \text{ and } y \in V\}$. As $u, v \in U$, there is a subsequence

$$u = a'_0 < a'_1 < a'_2 < \cdots < a'_t = v$$

of $a_0, a_1, a_2, \dots, a_m$, such that $[u, v) = \bigcup_{0 \leq s < t} [a'_s, a'_{s+1})$.

We show by induction on $1 \leq r \leq t$ that there is some block $B \in \psi(U) \vee \psi(V)$ with $[a'_0, a'_r) \subseteq B$. The base $r = 1$ is simple: $[a'_0, a'_1) \in \psi(U) \leq \psi(U) \vee \psi(V)$, so there is such a block B . We now proceed with the inductive step from $1 \leq r$ to $r+1 \leq t$. By the induction hypothesis, we know of the existence of a block $B \in \psi(U) \vee \psi(V)$ with $[a'_0, a'_r) \subseteq B$. Let $b := \max \{y \in V \mid y \leq a'_r\}$, especially $b \leq a'_r$. Since $0 < r < t$,

we have $u = a'_0 < a'_r < a'_t = v$, so by minimality of v , $a'_r \notin V$. This shows $a'_r \neq b$, i.e. $a'_r < b$. Define $b^+ := \min \{y \in V \mid y > b\}$, then $b < b^+$. The maximality of b implies $b^+ > a'_r$, i.e. $b < a'_r < b^+$. Thus, $[b, b^+) \in \psi(V) \leq \psi(U) \vee \psi(V)$. Hence, there is some block $B' \in \psi(U) \vee \psi(V)$ with $a'_r \in [b, b^+) \subseteq B'$. Furthermore, as $V \ni u = a'_0 < a'_r$, the maximality of b implies $u = a'_0 \leq b < a'_r$, so $b \in [a'_0, a'_r) \subseteq B$. Thus $b \in B \cap B'$, i.e. $B = B'$ as they are blocks of a partition. Putting this together with our previous observations, we obtain $a'_r \in [b, b^+) \subseteq B' = B$. Since $[a'_r, a'_{r+1}) \in \psi(U) \leq \psi(U) \vee \psi(V)$, there is a block $B'' \in \psi(U) \vee \psi(V)$ such that $a'_r \in [a'_r, a'_{r+1}) \subseteq B''$. Hence $a'_r \in B \cap B''$, so $B = B''$ and $[a'_r, a'_{r+1}) \subseteq B'' = B$. This shows

$$[a'_0, a'_{r+1}) = [a'_0, a'_r) \cup [a'_r, a'_{r+1}) \subseteq B,$$

finishing the induction.

Using the result just proven for $r = t$, one obtains $[u, v) = [a'_0, a'_t) \in B$ for some block $B \in \psi(U) \vee \psi(V)$. For the block $[u, v) \in \psi(U \cap V)$ was arbitrarily chosen, this yields $\psi(U \cap V) \leq \psi(U) \vee \psi(V)$, concluding the proof. $\square$

5 C -monoids and weak Krasner C -clones

In this chapter we study the C -clone lattice on a finite set up to equality of the corresponding endomorphism monoids. Here, unary relations are not any more enough to determine the lattice of all relational clones given by C -monoids, as given by C -automorphism groups (See Chapter 4), and hence its structure is considerably harder to understand.

5.1 Describing C -monoids

In this section we describe some monoids and weak Krasner clones that are determined by sets of clausal relations. The restriction of the Galois connection $\text{End} - \text{Inv}$ where the relations are clausal relations define the following Galois connection.

5.1.1 Definition. For $F \subseteq \mathcal{O}_D^{(1)}$ we define $C\text{Inv } F := \text{Inv } F \cap CR_D$. The operators

$$\begin{aligned} C\text{Inv} : \mathcal{P}(\mathcal{O}_D^{(1)}) &\longrightarrow \mathcal{P}(CR_D) : & F &\mapsto C\text{Inv } F, \\ \text{End} : \mathcal{P}(CR_D) &\longrightarrow \mathcal{P}(\mathcal{O}_D^{(1)}) : & Q &\mapsto \text{End } Q \end{aligned}$$

define a *Galois connection* $\text{End} - C\text{Inv}$ between unary operations and clausal relations.

5.1.2 Definition. A set $F \subseteq \mathcal{O}_D^{(1)}$ is called a *clausal monoid*, or *C -monoid* for short if $F = \text{End } Q$ for some set $Q \subseteq CR_D$ of clausal relations. A set $Q \subseteq CR_D$ is called a *weak Krasner C -clone* if $Q = C\text{Inv } F$ for some set F of unary operations.

Every Galois connection gives rise to a pair of closure operators. For the ones belonging to Definition 5.1.1 we introduce a special notation.

5.1.3 Definition. For any $F \subseteq \mathcal{O}_D^{(1)}$ and any $Q \subseteq CR_D$ we set

$$\begin{aligned} \langle F \rangle_{\mathcal{CO}_D^{(1)}} &:= \text{End } C\text{Inv } F, \\ [Q]_{w\mathcal{C}} &:= C\text{Inv } \text{End } Q. \end{aligned}$$

Our aim is to describe $\{\text{End } Q \mid Q \subseteq CR_D\}$ all C -monoids on D . According to Remark 3.1.4 every C -monoid can be determined by sets of non-trivial clausal relations, i.e.

$$\{\text{End } Q \mid Q \subseteq CR_D\} \stackrel{3.1.4}{=} \{\text{End } Q' \mid Q' \subseteq CR_D^*\}.$$

Since

$$\text{End } Q' = \bigcap_{R_{\mathbf{b}}^{\mathbf{a}} \in Q'} \text{End } R_{\mathbf{b}}^{\mathbf{a}}$$

for $Q' \subseteq CR_D^*$, it suffices to regard C -monoids of the form $\text{End } R_{\mathbf{b}}^{\mathbf{a}}$ with parameters $\mathbf{a} = (a_1, \dots, a_p) \in (D \setminus \{0\})^p$ and $\mathbf{b} = (b_1, \dots, b_q) \in (D \setminus \{n-1\})^q$.

With the help of Propositions 5.1.4, 5.1.5, and Corollary 5.1.6 below we can identify finite subsets of clausal relations, which are sufficient to describe the lattice of all C -monoids.

5.1.4 Proposition. *Let $\mathbf{a} = (a_1, \dots, a_p) \in (D \setminus \{0\})^p$ and $b_1, \dots, b_q \in D \setminus \{n-1\}$, where $p, q \in \mathbb{N}_+$. Then, it follows that*

$$\text{End } R_{b_1, \dots, b_q, b_q}^{\mathbf{a}} = \text{End } R_{b_1, \dots, b_q}^{\mathbf{a}}.$$

Proof. “ $\subseteq$ ”: Let $f \in \text{End } R_{b_1, \dots, b_q, b_q}^{\mathbf{a}}$. We have to show $f \in \text{End } R_{b_1, \dots, b_q}^{\mathbf{a}}$. Let $\mathbf{r} := (x_1, \dots, x_p, y_1, \dots, y_q) \in R_{b_1, \dots, b_q}^{\mathbf{a}}$, i.e.

$$\begin{aligned} x_1 \geq a_1 \vee \dots \vee x_p \geq a_p \vee y_1 \leq b_1 \vee \dots \vee y_q \leq b_q \\ \iff x_1 \geq a_1 \vee \dots \vee x_p \geq a_p \vee y_1 \leq b_1 \vee \dots \vee y_q \leq b_q \vee y_q \leq b_q. \end{aligned}$$

Because of the above expression we obtain $(x_1, \dots, x_p, y_1, \dots, y_q, y_q) \in R_{b_1, \dots, b_q, b_q}^{\mathbf{a}}$. Since $f \triangleright R_{b_1, \dots, b_q, b_q}^{\mathbf{a}}$, we have

$$\begin{aligned} f(x_1) \geq a_1 \vee \dots \vee f(x_p) \geq a_p \vee f(y_1) \leq b_1 \vee \dots \vee f(y_q) \leq b_q \vee f(y_q) \leq b_q \\ \iff f(x_1) \geq a_1 \vee \dots \vee f(x_p) \geq a_p \vee f(y_1) \leq b_1 \vee \dots \vee f(y_q) \leq b_q \\ \iff f[\mathbf{r}] \in R_{b_1, \dots, b_q}^{\mathbf{a}}. \end{aligned}$$

“ $\supseteq$ ”: Let $f \in \text{End } R_{b_1, \dots, b_q}^{\mathbf{a}}$. Assume that $f \notin \text{End } R_{b_1, \dots, b_q, b_q}^{\mathbf{a}}$, hence there is one $\mathbf{r}' := (x'_1, \dots, x'_p, y'_1, \dots, y'_q, y'_{q+1}) \in R_{b_1, \dots, b_q, b_q}^{\mathbf{a}}$ such that $f[\mathbf{r}'] \notin R_{b_1, \dots, b_q, b_q}^{\mathbf{a}}$, i.e.

$$x'_1 \geq a_1 \vee \dots \vee x'_p \geq a_p \vee y'_1 \leq b_1 \vee \dots \vee y'_q \leq b_q \vee y'_{q+1} \leq b_q \quad (5.1)$$

$$\text{and } f(x'_1) < a_1 \wedge \dots \wedge f(x'_p) < a_p \wedge f(y'_1) > b_1 \wedge \dots \wedge f(y'_q) > b_q \wedge f(y'_{q+1}) > b_q.$$

We consider two cases:

Case 1: If $\mathbf{r}^* := (x'_1, \dots, x'_p, y'_1, \dots, y'_q) \in R_{b_1, \dots, b_q}^{\mathbf{a}}$, then it follows that $f(x'_i) < a_i$ for all i belonging to $\{1, \dots, p\}$ and $f(y'_j) > b_j$ for all $j \in \{1, \dots, q\}$. Consequently, $f[\mathbf{r}^*]$ does not belong to $R_{b_1, \dots, b_q}^{\mathbf{a}}$, which is a contradiction to $f \triangleright R_{b_1, \dots, b_q}^{\mathbf{a}}$.

Case 2: If $\mathbf{r}^* \notin R_{b_1, \dots, b_q}^{\mathbf{a}}$, i.e.,

$$x'_1 < a_1 \wedge \dots \wedge x'_p < a_p \wedge y'_1 > b_1 \wedge \dots \wedge y'_q > b_q.$$

Because of (5.1) we have $y'_{q+1} \leq b_q$. Furthermore, the tuple

$$\mathbf{r}^{**} := (x'_1, \dots, x'_p, y'_1, \dots, y'_{q-1}, y'_{q+1})$$

belongs to $R_{b_1, \dots, b_q}^{\mathbf{a}}$. However,

$$f(x'_1) < a_1 \wedge \dots \wedge f(x'_p) < a_p \wedge f(y'_1) > b_1 \wedge \dots \wedge f(y'_{q-1}) > b_{q-1} \wedge f(y'_{q+1}) > b_q.$$

Thus, we get $f[\mathbf{r}^{**}] \notin R_{b_1, \dots, b_q}^{\mathbf{a}}$, which is also a contradiction to $f \triangleright R_{b_1, \dots, b_q}^{\mathbf{a}}$. $\square$

The dual statement of 5.1.4 is the following.

5.1.5 Proposition. *Let $a_1, \dots, a_p \in D \setminus \{0\}$ and $\mathbf{b} = (b_1, \dots, b_q) \in (D \setminus \{n-1\})^q$, where $p, q \in \mathbb{N}_+$. Then it follows that*

$$\text{End } R_{\mathbf{b}}^{a_1, \dots, a_p, a_p} = \text{End } R_{\mathbf{b}}^{a_1, \dots, a_p}.$$

From Lemma 3.5.5 we infer that if we permute the components of $\mathbf{a}$ or $\mathbf{b}$ of the clausal relation $R_{\mathbf{b}}^{\mathbf{a}}$, the C -monoid $\text{End } R_{\mathbf{b}}^{\mathbf{a}}$ stays the same. This is pointed out in the following corollary.

5.1.6 Corollary. *Let $p, q \in \mathbb{N}_+$, $\pi_1 \in S_p$ and $\pi_2 \in S_q$. Then*

$$\text{End } R_{\mathbf{b}}^{\mathbf{a}} = \text{End } R_{\mathbf{b} \circ \pi_2}^{\mathbf{a} \circ \pi_1}$$

holds, where $\mathbf{a} = (a_1, \dots, a_p) \in (D \setminus \{0\})^p$, $\mathbf{b} = (b_1, \dots, b_q) \in (D \setminus \{n-1\})^q$, $\mathbf{a} \circ \pi_1 = (a_{\pi_1(1)}, \dots, a_{\pi_1(p)})$, and $\mathbf{b} \circ \pi_2 = (b_{\pi_2(1)}, \dots, b_{\pi_2(q)})$.

Due to Propositions 5.1.4, 5.1.5 and Corollary 5.1.6, it is sufficient for the study of C -monoids to regard C -monoids $\text{End } R_{\mathbf{b}}^{\mathbf{a}}$, where $R_{\mathbf{b}}^{\mathbf{a}} \in CR_D^*$ is a non-trivial clausal relation, and all entries of the tuple $\mathbf{a}$ and of the tuple $\mathbf{b}$ are pairwise distinct, respectively. This implies that $1 \leq q, p < n$.

Reordering the entries in the tuples of a relation does not change its set of polymorphisms. This, implies that reordering the components of $\mathbf{a}$ or $\mathbf{b}$ of a clausal relation $R_{\mathbf{b}}^{\mathbf{a}}$, does not change the set of endomorphisms that preserve it. That is $\text{End } R_{\mathbf{b}}^{\mathbf{a}} = \text{End } R_{\tilde{\mathbf{b}}}^{\tilde{\mathbf{a}}}$, where $\tilde{\mathbf{a}}$ has the same entries as $\mathbf{a}$ but in ascending order and $\tilde{\mathbf{b}}$ has the same entries as $\mathbf{b}$ but in descending order. In the following sections we will only consider such clausal relations.

From Propositions 3.5.6 and 3.5.7 we infer the following corollaries.

5.1.7 Corollary. *Let $p, q \in \{1, \dots, n-1\}$, $\mathbf{b} = (b_1, \dots, b_q) \in (D \setminus \{n-1\})^q$, and $a_1, \dots, a_p, a' \in D \setminus \{0\}$. Then the inclusion*

$$\text{End } R_{\mathbf{b}}^{a_1, \dots, a_p, a'} \subseteq \text{End } R_{\mathbf{b}}^{a_1, \dots, a_p}$$

holds if and only if there exists some $i \in \{1, \dots, p\}$ with $a' \geq a_i$.

5.1.8 Corollary. *Let $p, q \in \{1, \dots, n-1\}$, $\mathbf{a} = (a_1, \dots, a_p) \in (D \setminus \{0\})^p$, and $b_1, \dots, b_q, b' \in D \setminus \{n-1\}$. Then the inclusion*

$$\text{End } R_{b_1, \dots, b_q, b'}^{\mathbf{a}} \subseteq \text{End } R_{b_1, \dots, b_q}^{\mathbf{a}}$$

holds if and only if there exists some $j \in \{1, \dots, q\}$ with $b' \leq b_j$.

From Propositions and Corollaries 5.1.4 to 5.1.8 we infer the existence of the following finite descending chains in the lattice of C -monoids on a finite set D :

$$\text{End } R_0^{1, 2, \dots, n-2, n-1} \subseteq \text{End } R_0^{1, 2, \dots, n-2} \subseteq \dots \subseteq \text{End } R_0^{1, 2} \subseteq \text{End } R_0^1,$$

and

$$\text{End } R_{n-2,n-3,\dots,1,0}^{n-1} \subseteq \text{End } R_{n-2,n-3,\dots,1}^{n-1} \subseteq \dots \subseteq \text{End } R_{n-2,n-3}^{n-1} \subseteq \text{End } R_{n-2}^{n-1}.$$

Lemma 5.1.9 shows some equalities between C -monoids, which enable us to reduce the number of non-trivial clausal relations to consider for determining C -monoids. Corollary 5.1.10 below states the numbers of relations resulting from this reduction.

5.1.9 Lemma. *Let $f \in O_D^{(1)}$, $\mathbf{a} \in (D \setminus \{0\})^p$, $\mathbf{b} \in (D \setminus \{n-1\})^q$. Then we have:*

1. *For $p \geq 0$ and $q \geq 1$ the following equivalence holds:*

$$f \triangleright R_{\mathbf{b}}^{1,\mathbf{a}} \iff f \triangleright R_{\mathbf{b},0}^{1,\mathbf{a}}.$$

Consequently, $\text{End } R_{\mathbf{b}}^{1,\mathbf{a}} = \text{End } R_{\mathbf{b},0}^{1,\mathbf{a}}$.

2. *For $p \geq 1$ and $q \geq 0$ the following equivalence holds:*

$$f \triangleright R_{n-2,\mathbf{b}}^{\mathbf{a}} \iff f \triangleright R_{n-2,\mathbf{b}}^{\mathbf{a},n-1}.$$

Consequently, $\text{End } R_{n-2,\mathbf{b}}^{\mathbf{a}} = \text{End } R_{n-2,\mathbf{b}}^{\mathbf{a},n-1}$.

Proof. We show that $\text{End } R_{\mathbf{b}}^{1,\mathbf{a}} = \text{End } R_{\mathbf{b},0}^{1,\mathbf{a}}$. The inclusion “ $\supseteq$ ” follows from Corollary 5.1.8. Conversely, let $f \in \text{End } R_{\mathbf{b}}^{1,\mathbf{a}}$, and let $\mathbf{r} := (u, \mathbf{x}, \mathbf{y}, v) \in R_{\mathbf{b},0}^{1,\mathbf{a}}$. The latter is equivalent to $(u, \mathbf{x}, \mathbf{y}) \in R_{\mathbf{b}}^{1,\mathbf{a}}$ or $v = 0$. We consider two cases:

- If $(u, \mathbf{x}, \mathbf{y}) \in R_{\mathbf{b}}^{1,\mathbf{a}}$, then from $f \triangleright R_{\mathbf{b}}^{1,\mathbf{a}}$ we obtain that $(f(u), f[\mathbf{x}], f[\mathbf{y}]) \in R_{\mathbf{b}}^{1,\mathbf{a}}$. Hence, we get that $f[\mathbf{r}] = (f(u), f[\mathbf{x}], f[\mathbf{y}], f(v)) \in R_{\mathbf{b},0}^{1,\mathbf{a}}$.
- If $(u, \mathbf{x}, \mathbf{y}) \notin R_{\mathbf{b}}^{1,\mathbf{a}}$, then we have $u = 0$ and $(\mathbf{x}, \mathbf{y}) \notin R_{\mathbf{b}}^{\mathbf{a}}$. Since $\mathbf{r} \in R_{\mathbf{b},0}^{1,\mathbf{a}}$, we obtain $v = 0$. Let us assume $f[\mathbf{r}] = (f(u), f[\mathbf{x}], f[\mathbf{y}], f(v)) \notin R_{\mathbf{b},0}^{1,\mathbf{a}}$. Hence, $0 = f(u)$ and $0 = f(u) = f(0) = f(v) > 0$, which is a contradiction. Thus, it follows that $f[\mathbf{r}] \in R_{\mathbf{b},0}^{1,\mathbf{a}}$.

The second statement follows from the duality principle for C -clones:

$$\begin{aligned} f \triangleright R_{n-2,\mathbf{b}}^{\mathbf{a}} &\stackrel{3.3.4}{\iff} f^\alpha \triangleright R_{\mathbf{a}^\alpha}^{n-2^\alpha,\mathbf{b}^\alpha} \iff f^\alpha \triangleright R_{n-1-\mathbf{a}}^{1,n-1-\mathbf{b}} \\ &\stackrel{\text{first part}}{\iff} f^\alpha \triangleright R_{n-1-\mathbf{a},0}^{1,n-1-\mathbf{b}} \stackrel{3.3.4}{\iff} f = (f^\alpha)^\alpha \triangleright R_{1^\alpha,(n-1-\mathbf{b})^\alpha}^{(n-1-\mathbf{a})^\alpha,0^\alpha} = R_{n-2,\mathbf{b}}^{\mathbf{a},n-1}. \quad \square \end{aligned}$$

Applying Lemma 5.1.9 to C -monoids on $D = \{0, 1, 2\}$ yields

$$[R_1^1]_{wC} = [R_{(1,0)}^1]_{wC} = [R_1^{(1,2)}]_{wC} = [R_{(1,0)}^{(1,2)}]_{wC}.$$

These equalities can be read from the lattice of C -monoids on $D = \{0, 1, 2\}$. (cf. Fig. 5.3)

5.1.10 Corollary. *Let $n \geq 3$. Then the number of non-trivial clausal relations to determine all C -monoids on D can be reduced from*

$$\frac{1}{4}4^n - 2^n + 1$$

to

$$\frac{9}{64}4^n - 2^{n-1} + 1.$$

Proof. The number of all non-trivial clausal relations $R_{\mathbf{b}}^{\mathbf{a}}$ where all entries of the tuple $\mathbf{a}$ and of the tuple $\mathbf{b}$ are pairwise distinct, and where all the entries in $\mathbf{a}$ are in ascending order, and in $\mathbf{b}$ are in descending order on $D = \{0, \dots, n-1\}$ is

$$\begin{aligned} |(\mathcal{P}(\{1, \dots, n-1\}) \setminus \{\emptyset\}) \times (\mathcal{P}(\{0, \dots, n-2\}) \setminus \{\emptyset\})| &= (2^{n-1} - 1)^2 \\ &= \frac{1}{4}4^n - 2^n + 1. \end{aligned}$$

From the first part of Lemma 5.1.9 we consider relations $R_{\mathbf{d}}^{\mathbf{c}}$, where the parameters $\mathbf{c}$ contain 1 but not 0, and where the parameters $\mathbf{d}$ contain 0 but not $n-1$, and where $\mathbf{d} \neq (0)$. The number of this kind of relations is $\frac{2^{n-1}}{2} \cdot (\frac{2^{n-1}}{2} - 1)$. From the second part of Lemma 5.1.9, we consider $R_{\mathbf{d}}^{\mathbf{c}}$, where the parameters $\mathbf{c}$ contain $n-1$ but not 0, and where the parameters $\mathbf{d}$ contain $n-2$ but not $n-1$ and where $\mathbf{c} \neq (n-1)$. The number of these relations is again $\frac{2^{n-1}}{2} \cdot (\frac{2^{n-1}}{2} - 1) = 2^{n-2} \cdot (2^{n-2} - 1)$.

This way some combinations of parameters have been counted twice (those being subject to the first and the second part of Lemma 5.1.9). That are precisely those tuples $\mathbf{c}$ containing 1 and $n-1$ but no 0, and those tuples $\mathbf{d}$ with 0 and $n-2$ but not with $n-1$. The number of those relations is $(2^{n-3})^2$. This means that the number of relations that are needed to determine all C -monoids on D can be reduced to

$$\begin{aligned} &\frac{1}{4}4^n - 2^n + 1 - 2 \cdot 2^{n-2}(2^{n-2} - 1) + (2^{n-3})^2 \\ &= 2^{2n} \left(\frac{2^4 - 2^3 + 1}{2^6} \right) - 2^{n-1} + 1 \\ &= 4^n \left(\frac{9}{64} \right) - 2^{n-1} + 1. \end{aligned} \quad \square$$

The following example shows the reduction of the number of non-trivial clausal relations to determine C -monoids on $D = \{0, 1, 2, 3\}$.

5.1.11 Example. Let $D = \{0, 1, 2, 3\}$. At the beginning of the section we explain that to describe all C -monoids on D , it is sufficient to regard C -monoids of the form $\text{End } R_{\mathbf{d}}^{\mathbf{c}}$, where $R_{\mathbf{d}}^{\mathbf{c}}$ is a non-trivial clausal relation without double parameters. Hence, the set of all non-trivial clausal relations on D without double parameters to regard is:

$$S := \left\{ \begin{array}{ccccccc} R_0^1, & R_1^1, & R_2^1, & R_{(1,0)}^1, & R_{(2,0)}^1, & R_{(2,1)}^1, & R_{(2,1,0)}^1, \\ R_0^2, & R_1^2, & R_2^2, & R_{(1,0)}^2, & R_{(2,0)}^2, & R_{(2,1)}^2, & R_{(2,1,0)}^2, \\ \vdots & & & \vdots & & & \vdots \\ R_0^{(1,2,3)}, & R_1^{(1,2,3)}, & R_2^{(1,2,3)}, & R_{(1,0)}^{(1,2,3)}, & R_{(2,0)}^{(1,2,3)}, & R_{(2,1)}^{(1,2,3)}, & R_{(2,1,0)}^{(1,2,3)} \end{array} \right\}.$$

and its cardinality is:

$$|S| = |(\mathcal{P}(\{1, 2, 3\}) \setminus \{\emptyset\}) \times (\mathcal{P}(\{0, 1, 2\}) \setminus \{\emptyset\})| = (2^3 - 1)^2 = \frac{1}{4}4^4 - 2^4 + 1 = 49.$$

The first part of Lemma 5.1.9 states the equality of the C -monoids $\text{End } R_{\mathbf{b}}^{1,\mathbf{a}}$ and $\text{End } R_{\mathbf{b},0}^{1,\mathbf{a}}$. Hence we look for relations $R_{\mathbf{d}}^{\mathbf{c}} = R_{\mathbf{b},0}^{1,\mathbf{a}} \in S$, where $\mathbf{c}$ contains 1 but not 0, and $\mathbf{d}$ contains 0 but not 3. Furthermore, since $q \geq 1$ we are not going to consider tuples, where $\mathbf{d} = (0)$. The set of these relations is

$$A = \left\{ \begin{array}{cccccc} R_{(1,0)}^1, & R_{(2,0)}^1, & R_{(2,1,0)}^1, & R_{(1,0)}^{(1,2)}, & R_{(2,0)}^{(1,2)}, & R_{(2,1,0)}^{(1,2)}, \\ R_{(1,0)}^{(1,3)}, & R_{(2,0)}^{(1,3)}, & R_{(2,1,0)}^{(1,3)}, & R_{(1,0)}^{(1,2,3)}, & R_{(2,0)}^{(1,2,3)}, & R_{(2,1,0)}^{(1,2,3)} \end{array} \right\}.$$

From the second part of Lemma 5.1.9 we take a look at the relations $R_{\mathbf{d}}^{\mathbf{e}} = R_{(2,\mathbf{b})}^{(\mathbf{a},3)} \in S$ where $\mathbf{e}$ contains 3 but not 0, and $\mathbf{d}$ contains 2 but not 3, and $\mathbf{e} \neq (3)$. The set of these relations is:

$$B = \left\{ \begin{array}{cccccc} R_2^{(1,3)}, & R_{(2,0)}^{(1,3)}, & R_{(2,1)}^{(1,3)}, & R_{(2,1,0)}^{(1,3)}, & R_2^{(2,3)}, & R_{(2,0)}^{(2,3)}, \\ R_{(2,1)}^{(2,3)}, & R_{(2,1,0)}^{(2,3)}, & R_2^{(1,2,3)}, & R_{(2,0)}^{(1,2,3)}, & R_{(2,1)}^{(1,2,3)}, & R_{(2,1,0)}^{(1,2,3)} \end{array} \right\}.$$

Now some relations have been considered twice, namely those relations $R_{\mathbf{d}}^{\mathbf{c}}$ where $\mathbf{c}$ contains 1 and 3 and where $\mathbf{d}$ contains 0 and 2. These are

$$C = \left\{ R_{(2,0)}^{(1,3)}, R_{(2,1,0)}^{(1,3)}, R_{(2,0)}^{(1,2,3)}, R_{(2,1,0)}^{(1,2,3)} \right\}.$$

The number of relations to describe C -monoids on $D = \{0, 1, 2, 3\}$ can be reduced from 49 to $29 = |S| - |A| - |B| + |C|$. Relations that are not crossed out in the scheme below are sufficient to determine all C -monoids for $|D| = 4$.

R_0^1	R_1^1	R_2^1	$R_{(1,0)}^1$	$R_{(2,0)}^1$	$R_{(2,1)}^1$	$R_{(2,1,0)}^1$
R_0^2	R_1^2	R_2^2	$R_{(1,0)}^2$	$R_{(2,0)}^2$	$R_{(2,1)}^2$	$R_{(2,1,0)}^2$
R_0^3	R_1^3	R_2^3	$R_{(1,0)}^3$	$R_{(2,0)}^3$	$R_{(2,1)}^3$	$R_{(2,1,0)}^3$
$R_0^{(1,2)}$	$R_1^{(1,2)}$	$R_2^{(1,2)}$	$R_{(1,0)}^{(1,2)}$	$R_{(2,0)}^{(1,2)}$	$R_{(2,1)}^{(1,2)}$	$R_{(2,1,0)}^{(1,2)}$
$R_0^{(1,3)}$	$R_1^{(1,3)}$	$R_2^{(1,3)}$	$R_{(1,0)}^{(1,3)}$	$R_{(2,0)}^{(1,3)}$	$R_{(2,1)}^{(1,3)}$	$R_{(2,1,0)}^{(1,3)}$
$R_0^{(2,3)}$	$R_1^{(2,3)}$	$R_2^{(2,3)}$	$R_{(1,0)}^{(2,3)}$	$R_{(2,0)}^{(2,3)}$	$R_{(2,1)}^{(2,3)}$	$R_{(2,1,0)}^{(2,3)}$
$R_0^{(1,2,3)}$	$R_1^{(1,2,3)}$	$R_2^{(1,2,3)}$	$R_{(1,0)}^{(1,2,3)}$	$R_{(2,0)}^{(1,2,3)}$	$R_{(2,1)}^{(1,2,3)}$	$R_{(2,1,0)}^{(1,2,3)}$

In the next section we investigate the lattice of all C -monoids for $|D| = 3$.

5.2 C -monoids for $|D| = 3$

In the following we present the Hasse diagram of the lattice $(CL_{\mathbf{3}}^{(1)}, \subseteq)$, where $CL_{\mathbf{3}}^{(1)}$ is the set of all C -monoids on $\mathbf{3} := \{0, 1, 2\}$.

Due to Propositions 5.1.4, 5.1.5 and Corollary 5.1.6 the problem of determining all C -monoids on the three element set is reduced to the task which functions $f \in O_{\mathbf{3}}^{(1)}$ (See Table 5.1) preserve which non-trivial clausal relations $R_{\mathbf{b}}^{\mathbf{a}}$, where all entries of the tuple $\mathbf{a}$ and of the tuple $\mathbf{b}$ are pairwise distinct respectively, and where all the entries in $\mathbf{a}$ are in ascending order, and in $\mathbf{b}$ are in descending order. That is, $R_{\mathbf{b}}^{\mathbf{a}}$ belongs to

$$\left\{ R_0^1, R_0^2, R_1^1, R_1^2, R_{(1,0)}^2, R_{(1,0)}^1, R_0^{(1,2)}, R_1^{(1,2)}, R_{(1,0)}^{(1,2)} \right\}.$$

In the following tables, the index k of the functions is just the interpretation of $f_k(0)f_k(1)f_k(2)$ as a 3-adic number: $k = \sum_{i=0}^2 |3|^{2-i} \cdot f_k(i)$. For example

$$17 = 3^2 \cdot f_{17}(0) + 3^1 \cdot f_{17}(1) + 3^0 \cdot f_{17}(2)$$

	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
1	0	0	0	1	1	1	2	2	2	0	0	0	1	1
2	0	1	2	0	1	2	0	1	2	0	1	2	0	1

	f_{14}	f_{15}	f_{16}	f_{17}	f_{18}	f_{19}	f_{20}	f_{21}	f_{22}	f_{23}	f_{24}	f_{25}	f_{26}
0	1	1	1	1	2	2	2	2	2	2	2	2	2
1	1	2	2	2	0	0	0	1	1	1	2	2	2
2	2	0	1	2	0	1	2	0	1	2	0	1	2

Figure 5.1: Tables of unary functions on $\mathbf{3}$.

Next example provides us with the lattice of all C -monoids on $\mathbf{3}$.

5.2.1 Example. The following cross table shows which $f \in O_{\mathbf{3}}^{(1)}$ preserve which relations ϱ belonging to $\left\{ R_0^1, R_0^2, R_1^1, R_1^2, R_{(1,0)}^2, R_{(1,0)}^1, R_0^{(1,2)}, R_1^{(1,2)}, R_{(1,0)}^{(1,2)} \right\}$. The rows are labelled by functions f_k (see Table 5.1), the columns are labeled by the relations themselves. Furthermore, in the case when $f \not\varrho$ we give a tuple $\mathbf{r}$ that belongs to ϱ but where $f[\mathbf{r}] \notin \varrho$. For example: In the third row, third column, we have that $(2, 2) \in R_0^2$. However, $(f_1(2), f_1(2)) = (1, 1) \notin R_0^2$.

$\triangleright$	R_0^1	R_0^2	R_1^1	R_1^2	$R_{(1,0)}^2$	$R_{(1,0)}^1$	$R_0^{(1,2)}$	$R_1^{(1,2)}$	$R_{(1,0)}^{(1,2)}$
f_0	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
f_1	$(1, 2)$	$(2, 2)$	$\times$	$\times$	$\times$	$\times$	$(1, 0, 2)$	$\times$	$\times$
f_2	$(1, 2)$	$\times$	$(1, 2)$	$\times$	$\times$	$(1, 2, 2)$	$(1, 0, 2)$	$(1, 1, 2)$	$(1, 1, 2, 2)$
f_3	$(2, 1)$	$(2, 1)$	$\times$	$\times$	$\times$	$\times$	$(2, 2, 1)$	$\times$	$\times$
f_4	$\times$	$(2, 2)$	$\times$	$\times$	$\times$	$\times$	$(0, 2, 1)$	$\times$	$\times$
f_5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
f_6	$(2, 1)$	$(2, 1)$	$(0, 1)$	$(0, 1)$	$(2, 2, 1)$	$(2, 1, 1)$	$(2, 2, 1)$	$(2, 2, 1)$	$(2, 2, 1, 1)$
f_7	$\times$	$(2, 2)$	$(0, 1)$	$(0, 1)$	$(0, 1, 2)$	$(0, 1, 1)$	$(0, 2, 1)$	$(0, 2, 1)$	$(0, 2, 1, 1)$
f_8	$\times$	$\times$	$(0, 1)$	$(0, 1)$	$(0, 0, 1)$	$(0, 1, 1)$	$\times$	$(0, 0, 1)$	$(0, 0, 1, 1)$
f_9	$(2, 0)$	$(1, 0)$	$\times$	$\times$	$\times$	$\times$	$(1, 2, 0)$	$\times$	$\times$
f_{10}	$(1, 0)$	$(0, 0)$	$\times$	$\times$	$\times$	$\times$	$(1, 2, 0)$	$\times$	$\times$
f_{11}	$(1, 2)$	$(1, 2)$	$(1, 2)$	$\times$	$(1, 2, 0)$	$(1, 2, 2)$	$(1, 1, 0)$	$(1, 1, 2)$	$(1, 1, 2, 2)$
f_{12}	$(2, 1)$	$(2, 1)$	$\times$	$\times$	$\times$	$\times$	$(2, 2, 0)$	$\times$	$\times$
f_{13}	$\times$	$(2, 2)$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
f_{14}	$\times$	$(0, 0)$	$\times$	$\times$	$(1, 2, 0)$	$\times$	$\times$	$\times$	$\times$
f_{15}	$(2, 1)$	$(2, 1)$	$(2, 1)$	$(0, 1)$	$(2, 1, 0)$	$(2, 1, 1)$	$(2, 2, 1)$	$(2, 2, 1)$	$(2, 2, 1, 1)$
f_{16}	$\times$	$(0, 0)$	$\times$	$(0, 1)$	$(2, 1, 0)$	$\times$	$\times$	$\times$	$\times$
f_{17}	$\times$	$(0, 0)$	$\times$	$(0, 1)$	$(0, 1, 1)$	$\times$	$\times$	$\times$	$\times$
f_{18}	$(2, 0)$	$(2, 0)$	$(2, 0)$	$(2, 0)$	$(2, 0, 0)$	$(2, 0, 0)$	$(2, 2, 0)$	$(2, 2, 0)$	$(2, 2, 0, 0)$
f_{19}	$(1, 2)$	$(2, 0)$	$(1, 0)$	$(1, 0)$	$(1, 0, 0)$	$(1, 0, 0)$	$(1, 1, 0)$	$(1, 1, 0)$	$(1, 1, 0, 0)$
f_{20}	$(1, 2)$	$(1, 0)$	$(1, 0)$	$(1, 0)$	$(1, 0, 0)$	$(1, 0, 0)$	$(1, 1, 0)$	$(1, 1, 0)$	$(1, 1, 0, 0)$
f_{21}	$(2, 0)$	$(2, 0)$	$(2, 0)$	$(2, 0)$	$(2, 0, 0)$	$(2, 0, 0)$	$(2, 2, 0)$	$(2, 2, 0)$	$(2, 2, 0, 0)$
f_{22}	$\times$	$(2, 0)$	$\times$	$(2, 0)$	$(2, 0, 0)$	$\times$	$\times$	$\times$	$\times$
f_{23}	$\times$	$(1, 0)$	$\times$	$(1, 0)$	$(1, 0, 0)$	$\times$	$\times$	$\times$	$\times$
f_{24}	$(2, 0)$	$(2, 0)$	$(2, 0)$	$(2, 0)$	$(2, 0, 0)$	$(2, 0, 0)$	$(2, 2, 0)$	$(2, 2, 0)$	$(2, 2, 0, 0)$
f_{25}	$\times$	$(2, 1)$	$\times$	$(2, 1)$	$(2, 0, 0)$	$\times$	$\times$	$\times$	$\times$
f_{26}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

Figure 5.2: Table of C -monoids on **3**.

The previous table was first determined by hand. Later we verified the cross table with a computer program that determines which unary functions on D (for $|D| \leq 5$) preserve which non-trivial clausal relations $R_{\mathbf{b}}^{\mathbf{a}}$, where all entries of the tuple $\mathbf{a}$ and of the tuple $\mathbf{b}$ are pairwise distinct, respectively.

We present only the pseudo code of the procedure, which verifies whether a given unary function f preserves a given relation of the above mentioned type.

Pseudo code to verify whether $f \triangleright R_{\mathbf{b}}^{\mathbf{a}}$.

- S1 Given: $n = |D|$, $p, q \in \{1, \dots, n-1\}$, a unary function f on D , and tuples $\mathbf{a} \in (D \setminus \{0\})^p$, $\mathbf{b} \in (D \setminus \{n-1\})^q$ where all entries of the tuple $\mathbf{a}$ and of $\mathbf{b}$ are pairwise distinct, respectively.
- S2 Determine all tuples $\mathbf{r} := (x_1, \dots, x_p, y_1, \dots, y_q) \in D^{p+q}$ that satisfy
$$x_1 \geq a_1 \vee \dots \vee x_p \geq a_p \vee y_1 \leq b_1 \vee \dots \vee y_q \leq b_q$$
and collect them in $R_{\mathbf{b}}^{\mathbf{a}}$.
- S3 For each $\mathbf{r} \in R_{\mathbf{b}}^{\mathbf{a}}$
test whether $f[\mathbf{r}] \in R_{\mathbf{b}}^{\mathbf{a}}$ holds.
In case it does not, then return $f \not\triangleright R_{\mathbf{b}}^{\mathbf{a}}$ and write the tuple $\mathbf{r}$.
If all tests were successful, then return $f \triangleright R_{\mathbf{b}}^{\mathbf{a}}$.

This pseudo code is well-defined because it is an easy application of the Definition 1.3.3. It is complete because the number of tuples in the relation is finite, and all tuples are verified.

Figure 5.3 shows the concept lattice of the formal context (cross table) from Table 5.2. A detailed introduction to concepts, concept lattices, formal contexts can be found in [GW99]. Here these concepts will be not defined.

The pairs of the form $(\langle F \rangle_{CO_D^{(1)}}, CInv F)$, where $\langle F \rangle_{CO_D^{(1)}}$ is a C -monoid and $CInv F$ is the corresponding weak Krasner C -clone, and of the form $(End Q, [Q]_{wC})$, where $End Q$ is a C -monoid and $[Q]_{wC}$ is the corresponding weak Krasner C -clone can be read from the diagram.

In the diagram we used the reduced labelling as is common in formal concept analysis (see [GW99]). This means that each function and each clausal relation appear only once in the diagram. The pair of the form $(\langle f \rangle_{CO_D^{(1)}}, CInv f)$ is labelled with f , and of the form $(End R_{\mathbf{b}}^{\mathbf{a}}, [R_{\mathbf{b}}^{\mathbf{a}}]_{wC})$ is labelled with $R_{\mathbf{b}}^{\mathbf{a}}$.

For every node in the line diagram one can read the functions belonging to the corresponding C -monoid, by collecting all functions located at this node and at nodes which can be reached by descending line paths from this node. For example: The node with label f_4 corresponds to $(\langle f_4 \rangle_{CO_D^{(1)}}, CInv f_4)$. Hence, we get

$$\langle f_4 \rangle_{CO_D^{(1)}} = \{f_0, f_4, f_5, f_{13}, f_{26}\}.$$

Dually, for every node one read the set of relations belonging to the corresponding weak Krasner C -clone by following all line paths going upwards from the node and noting down all relations assigned to these nodes. Hence, we obtain

$$CInv\{f_4\} = \{R_0^1, R_1^1, R_1^2, R_{(1,0)}^2, R_{(1,0)}^1, R_1^{(1,2)}, R_{(1,0)}^{(1,2)}\}.$$

Similarly, the circle with label $R_0^{(1,2)}$ corresponds to $(\text{End } R_0^{(1,2)}, [R_0^{(1,2)}]_{wC})$, where

$$\text{End } R_0^{(1,2)} = \{f_0, f_5, f_8, f_{13}, f_{14}, f_{16}, f_{17}, f_{22}, f_{23}, f_{25}, f_{26}\} \text{ and}$$

$$[R_0^{(1,2)}]_{wC} = \{R_0^{(1,2)}, R_0^1\}.$$

Figure 5.3: The lattice of all C -monoids for $\mathbf{3}$.

Note that in the lattice that is displayed in Figure 5.3 the co-atoms are of the form; $\langle F \rangle_{CO_D^{(1)}} = \text{End } R_b^a$, where R_b^a is a non trivial clausal relation of arity two. In the following section we show that all co-atoms in the lattice of C -monoids on an arbitrary finite set D are of this form. Moreover, we describe all atoms in the lattice of C -monoids.

5.3 Maximal C -monoids

In this section all co-atoms in the lattice of C -monoids for $|D| \geq 3$ are described. These are formally defined as follows.

5.3.1 Definition. A C -monoid F is called *maximal* if $F \neq O_D^{(1)}$ and between F and $O_D^{(1)}$ there are no more C -monoids. That is, for every C -monoid F' the condition $F \subsetneq F'$ implies $F' = O_D^{(1)}$.

Theorem 5.3.5 below, given also in [Var12], states that a C -monoid is maximal if it can be determined by only one non-trivial clausal relation of arity two. To prove this theorem we need some auxiliary statements such as Lemmata 5.3.3, 5.3.4 and Remark 5.3.2. The latter states that C -monoids of the form $\text{End } R_b^a$, where $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$, are candidates for maximal C -monoids.

5.3.2 Remark. Let $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$. Then

$$\text{End } R_b^a \subsetneq O_D^{(1)}.$$

Proof. Let f be the unary operation on D defined by $f(x) := n-1-x$. We show that $f \notin \text{End } R_b^a$. Clearly, the tuple $\mathbf{r} := (n-1, 0)$ belongs to R_b^a . Moreover, as $0 \not\geq a$ and $n-1 \not\leq b$ we obtain $(0, n-1) = f[\mathbf{r}] \notin R_b^a$. $\square$

5.3.3 Lemma. Let $f \in O_D^{(1)}$, $a_1, \dots, a_p \in D \setminus \{0\}$ and $b_1, \dots, b_q \in D \setminus \{n-1\}$. Then we have

$$f \triangleright R_{b_1, \dots, b_q}^{a_1, \dots, a_p} \Rightarrow f \triangleright R_{\max\{b_1, \dots, b_q\}}^{\min\{a_1, \dots, a_p\}}.$$

Proof. Let $f \in O_D^{(1)}$ and $f \triangleright R_{b_1, \dots, b_q}^{a_1, \dots, a_p}$. We have to show that f preserves $R_{\max\{b_1, \dots, b_q\}}^{\min\{a_1, \dots, a_p\}}$. Let $\mathbf{r} = (x_1, y_1) \in R_{\max\{b_1, \dots, b_q\}}^{\min\{a_1, \dots, a_p\}}$, i.e.

$$\begin{aligned} x_1 &\geq \min\{a_1, \dots, a_p\} \vee y_1 \leq \max\{b_1, \dots, b_q\} \\ \iff x_1 &\geq a_1 \vee \dots \vee x_1 \geq a_p \vee y_1 \leq b_1 \vee y_1 \leq b_2 \vee \dots \vee y_1 \leq b_q \end{aligned} \quad (5.2)$$

Because of (5.2), the tuple $(x_1, \dots, x_1, y_1, \dots, y_1)$ belongs to $R_{b_1, \dots, b_q}^{a_1, \dots, a_p}$. Furthermore, since $f \triangleright R_{b_1, \dots, b_q}^{a_1, \dots, a_p}$, we obtain

$$\begin{aligned} f(x_1) &\geq a_1 \vee \dots \vee f(x_1) \geq a_p \vee f(y_1) \leq b_1 \vee \dots \vee f(y_1) \leq b_q \\ \iff f(x_1) &\geq \min\{a_1, \dots, a_p\} \vee f(y_1) \leq \max\{b_1, \dots, b_q\}. \end{aligned}$$

Hence, we get $f[\mathbf{r}] = (f(x_1), f(y_1)) \in R_{\max\{b_1, \dots, b_q\}}^{\min\{a_1, \dots, a_p\}}$. $\square$

We can also prove that any two C -monoids, $\text{End } R_{b_1}^{a_1}$ and $\text{End } R_{b_2}^{a_2}$, where

$$(a_1, b_1) \neq (a_2, b_2)$$

and $(a_1, b_1), (a_2, b_2) \in (D \setminus \{0\}) \times (D \setminus \{n-1\})$, are incomparable.

5.3.4 Lemma. Let $a_1, a_2 \in D \setminus \{0\}$, $b_1, b_2 \in D \setminus \{n-1\}$. If $(a_1, b_1) \neq (a_2, b_2)$, then we have

$$\text{End } R_{b_1}^{a_1} \not\subseteq \text{End } R_{b_2}^{a_2} \quad \text{and} \quad \text{End } R_{b_2}^{a_2} \not\subseteq \text{End } R_{b_1}^{a_1}.$$

Proof. Consider $\text{End } R_{b_1}^{a_1}$ and $\text{End } R_{b_2}^{a_2}$, where $(a_1, b_1) \neq (a_2, b_2)$. We divide the proof in two cases.

Case 1 If $a_1 \neq a_2$, then w.l.o.g. $a_1 < a_2 \leq n - 1$. First, we show that $\text{End } R_{b_1}^{a_1}$ is not contained in $\text{End } R_{b_2}^{a_2}$. Let f be the unary operation on D defined by

$$f(x) := \begin{cases} a_1 & \text{if } x = a_2, \\ n - 1 & \text{otherwise.} \end{cases}$$

Let $\mathbf{r} := (x_1, y_1) \in R_{b_1}^{a_1}$. Since $f(x_1) \geq a_1$, we have $f[\mathbf{r}] = (f(x_1), f(y_1)) \in R_{b_1}^{a_1}$ and hence, $f \in \text{End } R_{b_1}^{a_1}$. Since $\mathbf{r}^* := (a_2, 0) \in R_{b_2}^{a_2}$ and $f[\mathbf{r}^*] = (a_1, n - 1) \notin R_{b_2}^{a_2}$, it follows that $f \notin \text{End } R_{b_2}^{a_2}$.

To prove $\text{End } R_{b_2}^{a_2} \not\subseteq \text{End } R_{b_1}^{a_1}$, consider $g \in O_D^{(1)}$ defined by

$$g(x) := \begin{cases} 0 & \text{if } x = a_1, \\ x & \text{otherwise.} \end{cases}$$

We show $g \triangleright R_{b_2}^{a_2}$. Let $\mathbf{r} = (x_1, y_1) \in R_{b_2}^{a_2}$, i.e. $x_1 \geq a_2 \vee y_1 \leq b_2$. We consider two sub-cases:

- If $x_1 \geq a_2 > a_1$, then $g(x_1) = x_1 \geq a_2$, and $g[\mathbf{r}] \in R_{b_2}^{a_2}$.
- If $y_1 \leq b_2$, then $g(y_1) \leq b_2$. Hence, $g[\mathbf{r}] \in R_{b_2}^{a_2}$.

Since $\mathbf{r}' := (a_1, n - 1) \in R_{b_1}^{a_1}$ and $g[\mathbf{r}'] = (0, n - 1) \notin R_{b_1}^{a_1}$, we get $g \not\triangleright R_{b_1}^{a_1}$.

Case 2 If $b_1 \neq b_2$, then w.l.o.g. $0 \leq b_1 < b_2$. First, we prove $\text{End } R_{b_2}^{a_2} \not\subseteq \text{End } R_{b_1}^{a_1}$. Consider $f^* \in O_D^{(1)}$ defined by

$$f^*(x) := \begin{cases} b_2 & \text{if } x = b_1, \\ 0 & \text{otherwise.} \end{cases}$$

Because $f^*(y_1) \leq b_2$ for any $(x_1, y_1) \in R_{b_2}^{a_2}$, we have $f^* \triangleright R_{b_2}^{a_2}$. Furthermore, $f^* \not\triangleright R_{b_1}^{a_1}$; since $(n - 1, b_1) \in R_{b_1}^{a_1}$ and $(f^*(n - 1), f^*(b_1)) = (0, b_2) \notin R_{b_1}^{a_1}$. Finally, to prove $\text{End } R_{b_1}^{a_1} \not\subseteq \text{End } R_{b_2}^{a_2}$ consider $g^* \in O_D^{(1)}$ defined by

$$g^*(x) := \begin{cases} n - 1 & \text{if } x = b_2, \\ x & \text{otherwise.} \end{cases}$$

To show $g^* \triangleright R_{b_1}^{a_1}$, let $\mathbf{r} = (x_1, y_1) \in R_{b_1}^{a_1}$, i.e. $x_1 \geq a_1 \vee y_1 \leq b_1$. We consider two sub-cases:

- If $x_1 \geq a_1$, then $g^*(x_1) \geq a_1$. Hence, $g^*[\mathbf{r}] \in R_{b_1}^{a_1}$.
- If $y_1 \leq b_1 < b_2$, then $g^*(y_1) = y_1 \leq b_1$. Hence, we get $g^*[\mathbf{r}] \in R_{b_1}^{a_1}$.

Moreover, because $(0, b_2) \in R_{b_2}^{a_2}$ and $(0, n - 1) = (g^*(0), g^*(b_2)) \notin R_{b_2}^{a_2}$, it follows that $g^* \not\triangleright R_{b_2}^{a_2}$. $\square$

Now we can characterise all maximal C -monoids.

5.3.5 Theorem. *Let $M \subseteq O_D^{(1)}$ be a C -monoid. M is maximal if and only if there are elements $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$ such that*

$$M = \text{End } R_b^a.$$

Proof. “ $\Rightarrow$ ” Let $M \subseteq O_D^{(1)}$ be a maximal C -monoid. Then there exists a relation $\varrho \in CR_D^*$ such that $M = \text{End } \varrho$. This means there are elements $a_1 \dots a_p \in D \setminus \{0\}$ and $b_1, \dots, b_q \in D \setminus \{n-1\}$ such that $M = \text{End } R_{b_1, \dots, b_q}^{a_1, \dots, a_p}$. From 5.3.3 and 5.3.2, we have

$$M = \text{End } R_{b_1, \dots, b_q}^{a_1, \dots, a_p} \stackrel{5.3.3}{\subseteq} \text{End } R_{\max\{b_1, \dots, b_q\}}^{\min\{a_1, \dots, a_p\}} \stackrel{5.3.2}{\subsetneq} O_D^{(1)}.$$

Since M is maximal, we have $M = \text{End } R_{\max\{b_1, \dots, b_q\}}^{\min\{a_1, \dots, a_p\}}$.

“ $\Leftarrow$ ” Let $\text{End } R_b^a$ be a C -monoid, where $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$. Because of 5.3.2, we have $\text{End } R_b^a \subsetneq O_D^{(1)}$. Since the lattice of the C -monoids is finite, there is a maximal C -monoid M such that

$$\text{End } R_b^a \subseteq M \subsetneq O_D^{(1)}.$$

The first part of this proof implies the existence of elements $c \in D \setminus \{0\}$ and $d \in D \setminus \{n-1\}$ such that $M = \text{End } R_d^c$. Hence, $\text{End } R_b^a \subseteq \text{End } R_d^c$. Because of 5.3.4, we have $a = c$ and $b = d$. Thus, $M = \text{End } R_b^a$ is maximal. $\square$

Finally, from Theorem 5.3.5, we deduce there are $(n-1)^2$ maximal C -monoids on an n -element set D , for $|D| \geq 3$.

5.4 Minimal C -monoids

In this section, all atoms in the lattice of C -monoids for $|D| \geq 3$ are described, these are formally defined as follows.

5.4.1 Definition. Let id_D be the identity on D . A C -monoid F is *minimal*, if $F \neq \langle id_D \rangle_{CO_D^{(1)}}$ and between F and $\langle id_D \rangle_{CO_D^{(1)}}$ there are no more C -monoids.

We begin the characterisation of all minimal C -monoids by describing the smallest C -monoid.

5.4.2 Lemma. *Let c_0, c_{n-1} be unary constants operations with values 0 and $n-1$ respectively, and let id_D be the identity on D . Then*

$$\text{End } CR_D = \{c_0, c_{n-1}, id_D\}.$$

Proof. We first show $\{c_0, c_{n-1}, id_D\} \subseteq \text{End } CR_D$. The identity id_D belongs to $\text{End } R_D \subseteq \text{End } CR_D$. Furthermore, let $R_{\mathbf{b}}^{\mathbf{a}} \in CR_D$ with $\mathbf{a} \in D^p$, $\mathbf{b} \in D^q$ and consider $\mathbf{r} \in R_{\mathbf{b}}^{\mathbf{a}}$. Since $0 \leq b_j$ and $n-1 \geq a_j$, we have $c_0[\mathbf{r}] = (0, \dots, 0) \in R_{\mathbf{b}}^{\mathbf{a}}$ and $c_{n-1}[\mathbf{r}] = (n-1, \dots, n-1) \in R_{\mathbf{b}}^{\mathbf{a}}$. Consequently,

$$c_0 \triangleright R_{\mathbf{b}}^{\mathbf{a}} \text{ and } c_{n-1} \triangleright R_{\mathbf{b}}^{\mathbf{a}}.$$

Conversely, we assume that there is a function $f \in (\text{End } CR_D) \setminus \{c_0, c_{n-1}, id_D\}$. Since $f \neq id_D$, there is some $c \in D$ such that $f(c) \neq c$. As $f \neq c_0$, there is some $y \in D$ such that $f(y) > 0$. Finally, $f \neq c_{n-1}$ implies there is some $x \in D$ such that $f(x) < n-1$. We divide the remaining proof in two cases:

- If $f(c) < c$, then since $(c, y) \in R_0^c$ and $(f(c), f(y)) \notin R_0^c$, it follows that $f \not\triangleright R_0^c$. This is impossible, since $f \in \text{End } CR_D$.
- If $f(c) > c$, then from $(x, c) \in R_c^{n-1}$ and $(f(x), f(c)) \notin R_c^{n-1}$, we receive $f \not\triangleright R_c^{n-1}$, which is also a contradiction.

Consequently, $(\text{End } CR_D) \setminus \{c_0, c_{n-1}, id_D\} = \emptyset$, which proves the claim. $\square$

In the rest of this section we shall characterise one-generated C -monoids for some functions $f \in O_D^{(1)}$ (namely $f \in \{f_a^-, f_a^+, c_a \mid 0 < a < n-1\}$ defined hereafter).

For $0 < a < n-1$ let $f_a^-, f_a^+, c_a \in O_D^{(1)}$ be the unary operations defined by

$$f_a^-(x) := \begin{cases} a-1 & \text{if } x = a \\ x & \text{otherwise,} \end{cases} \quad f_a^+(x) := \begin{cases} a+1 & \text{if } x = a \\ x & \text{otherwise,} \end{cases}$$

and $c_a(x) := a$ for all $x \in D$. These functions allow us to characterise the minimal C -monoids.

The lemma below states that clausal relations belonging to the weak Krasner C -clones of f_a^- are those clausal relations $R_{\mathbf{b}}^{\mathbf{e}}$ such that $e_1, \dots, e_p \notin \{0, a\}$.

5.4.3 Lemma. *Let $0 < a < n-1$. Then,*

$$CR_D^* \cap C\text{Inv } f_a^- = \{R_{\mathbf{b}}^{\mathbf{e}} \mid \mathbf{e} \in (D \setminus \{0, a\})^p, \mathbf{b} \in (D \setminus \{n-1\})^q\}.$$

Proof. We define

$$W := \{R_{\mathbf{b}}^{\mathbf{e}} \mid \mathbf{e} \in (D \setminus \{0, a\})^p, \mathbf{b} \in (D \setminus \{n-1\})^q\} \subseteq CR_D^*.$$

First, we show $W \subseteq C\text{Inv } f_a^-$. Let $R_{\mathbf{b}}^{\mathbf{e}} \in W$, we have to prove that $f_a^- \triangleright R_{\mathbf{b}}^{\mathbf{e}}$. By the equation (4.1) before Lemma 4.1.4, it is enough to prove $f_a^- \triangleright [e_i, n-1]$ for all $i \in \{1, \dots, p\}$ and $f_a^- \triangleright [0, b_j]$ for all $j \in \{1, \dots, q\}$. Let $y \in [0, b_j]$. Since $f_a^-(y) \leq y \leq b_j$, one obtains $f_a^- \triangleright [0, b_j]$ for all $j \in \{1, \dots, q\}$.

To prove $f_a^- \triangleright [e_i, n-1]$ for all $i \in \{1, \dots, p\}$, let us assume the opposite, i.e., there is one $x \in [e_i, n-1]$ such that $f_a^-(x) \notin [e_i, n-1]$. The latter implies $f_a^-(x) < e_i$. By definition of f_a^- there are only two cases to consider.

- 1) If $x = a$, then $a = x \geq e_i > f_a^-(x) = f_a^-(a)$ and by definition $f_a^-(a) = a-1$. Consequently, $e_i = a$. This is impossible, since $R_{\mathbf{b}}^{\mathbf{e}} \in W$.
- 2) If $x \neq a$, then $x = f_a^-(x) < e_i$, a contradiction to $x \in [e_i, n-1]$.

We show $CR_D^* \cap C\text{Inv } f_a^- \subseteq W$. Assume that there is a relation $R_{\mathbf{b}'}^{e'}$ belonging to $C\text{Inv } f_a^- \cap CR_D^*$ but not to W . Hence, $a = e'_i$ for some $i \in \{1, \dots, p\}$, i.e., $R_{\mathbf{b}'}^{e'}$ is equal to $R_{\mathbf{b}'}^{(e'_1, \dots, e'_{i-1}, a, e'_{i+1}, \dots, e'_p)}$. Then

$$\mathbf{r} := (0, \dots, 0, \overset{i}{a}, 0, \dots, 0, n-1, \dots, n-1) \in R_{\mathbf{b}'}^{(e'_1, \dots, e'_{i-1}, a, e'_{i+1}, \dots, e'_p)}.$$

Since $f_a^- \triangleright R_{\mathbf{b}'}^{e'}$, we receive $f_a^-[\mathbf{r}] \in R_{\mathbf{b}'}^{(e'_1, \dots, e'_{i-1}, a, e'_{i+1}, \dots, e'_p)}$. But this yields to a contradiction since

$$\begin{aligned} 0 < e'_1 \wedge \dots \wedge 0 < e'_{i-1} \wedge a - 1 < a \wedge 0 < e'_{i+1} \wedge \dots \wedge 0 < e'_p \\ & \wedge n-1 > b'_1 \wedge \dots \wedge n-1 > b'_q. \end{aligned}$$

□

Similarly, one can prove following statement, which states that clausal relations belonging to the weak Krasner C -clones of f_a^+ are those clausal relations $R_{\mathbf{b}}^e$ such that $b_1, \dots, b_q$ do not belong to $\{a, n-1\}$.

5.4.4 Lemma. *Let $0 < a < n-1$. Then,*

$$CR_D^* \cap C\text{Inv } f_a^+ = \{R_{\mathbf{b}}^e \mid \mathbf{e} \in (D \setminus \{0\})^p, \mathbf{b} \in (D \setminus \{a, n-1\})^q\}.$$

We see below that the one-generated C -monoids of functions of the form f_a^- or f_a^+ are atoms in the lattice of all C -monoids. We show for $g \in \{f_a^-, f_a^+\}$ that whenever f belongs to $\langle g \rangle_{CO_D^{(1)}}$ and f does not belong to the smallest C -monoid, then $f = g$, in particular $\langle f \rangle_{CO_D^{(1)}} = \langle g \rangle_{CO_D^{(1)}}$. To do this we need some additional auxiliary statements.

5.4.5 Lemma. *Let $f \in O_D^{(1)}$, $c \in D$ and let c_0 and c_{n-1} be unary constants operations with values 0 and $n-1$, respectively. Then the following holds:*

- 1) *If $f(c) < c$, then $f = c_0$ or $f \not\in R_0^c$.*
- 2) *If $f(c) > c$, then $f = c_{n-1}$ or $f \not\in R_c^{n-1}$.*

Proof. Let $f(c) < c$. If $f \neq c_0$, then there is some $y \in D$ such that $f(y) \neq 0$. Hence, we get $(c, y) \in R_0^c$, and $(f(c), f(y)) \notin R_0^c$. Thus, $f \not\in R_0^c$. Similarly, the second statement can be proved. □

5.4.6 Corollary. *Let $0 < a < n-1$, c_0 , c_{n-1} . Then we have*

$$\begin{aligned} g \in \langle f_a^- \rangle_{CO_D^{(1)}} &\Rightarrow g \in \{c_0, c_{n-1}\} \text{ or } g(c) = c \text{ for all } c \in D \setminus \{a\}, \\ g \in \langle f_a^+ \rangle_{CO_D^{(1)}} &\Rightarrow g \in \{c_0, c_{n-1}\} \text{ or } g(c) = c \text{ for all } c \in D \setminus \{a\}. \end{aligned}$$

Proof. Let $g \in \langle f_a^- \rangle_{CO_D^{(1)}}$. By Lemma 5.4.3, it follows $g \triangleright R_{\mathbf{b}}^e$ where $\mathbf{e} \in (D \setminus \{0, a\})^p$. Assume that $g \neq c_0$ and $g \neq c_{n-1}$, and there is some $c \in D \setminus \{a\}$ such that $g(c) \neq c$. Since $g(c) \neq c$, we consider two cases.

- If $g(c) < c$, then from Lemma 5.4.5 we infer $g \not\triangleright R_0^c$ as $g \neq c_0$. However, as $g \in \text{End } C\text{Inv } f_a^-$ and $R_0^c \in C\text{Inv } f_a^-$, we obtain $g \triangleright R_0^c$, which is a contradiction.
- Similarly, if $g(c) > c$, then $g \not\triangleright R_c^{n-1}$ because $g \neq c_{n-1}$. As $g \in \text{End } C\text{Inv } f_a^-$ and $R_c^{n-1} \in C\text{Inv } f_a^-$, we obtain $g \triangleright R_c^{n-1}$, which is also a contradiction.

Similarly, one can prove the second statement. $\square$

Finally, we obtain the following corollary.

5.4.7 Corollary. *Let $0 < a < n - 1$, c_0, c_{n-1} and id_D . Then we have*

- 1) *If $f \in (\text{End } C\text{Inv } f_a^-) \setminus \{c_0, c_{n-1}, id_D\}$, then $f = f_a^-$.*
- 2) *If $f \in (\text{End } C\text{Inv } f_a^+) \setminus \{c_0, c_{n-1}, id_D\}$, then $f = f_a^+$.*

Proof. Let $f \in (\text{End } C\text{Inv } f_a^-) \setminus \{c_0, c_{n-1}, id_D\}$. We show that $f(a) = a - 1$ and $f(x) = x$ for $x \in D \setminus \{a\}$. The preconditions, together with Corollary 5.4.6, imply that $f(c) = c$ for all $c \neq a$. As $f \neq id_D$, $n - 1 \geq d := f(a) \neq a$. Assume that $d > a$. Since $f \neq c_{n-1}$ and because of 5.4.5, we infer $f \not\triangleright R_a^{n-1}$. This is impossible, since $R_a^{n-1} \in C\text{Inv } f_a^-$ and $f \in \text{End } C\text{Inv } f_a^-$. Thus, $d < a$. Assume $d < a - 1$. As $f \neq c_0$, there is some $y \in D$ such that $f(y) \not\geq 0$. Obviously, $(a, y) \in R_0^{a-1}$ but $(f(a), f(y)) \notin R_0^{a-1}$. Hence, we get $f \not\triangleright R_0^{a-1}$. This is impossible, since $f \in \text{End } C\text{Inv } f_a^-$ and R_0^{a-1} belongs to $C\text{Inv } f_a^-$. Hence, $f(a) = d = a - 1$. Thus, $f = f_a^-$. Similarly, the second statement can be proved. $\square$

Proposition 5.4.8 below, given also in [Var12], states that $\langle c_a \rangle_{CO_D^{(1)}}$, $0 < a < n - 1$ are minimal and describes their weak Krasner C -clones.

5.4.8 Proposition. *Let $0 < a < n - 1$ and c_a be the unary constant operation with value a . Then we have*

$$CR_D^* \cap C\text{Inv}(c_a) = \{R_d^e \mid \exists i \in \{1, \dots, p\} : e_i \leq a \text{ or } \exists j \in \{1, \dots, q\} : d_j \geq a\},$$

and

$$\text{End } C\text{Inv}(c_a) = \{id_D, c_0, c_{n-1}, c_a\}, \text{ i.e. } \langle c_a \rangle_{CO_D^{(1)}} \text{ is minimal.}$$

Proof. We define

$$W := \{R_d^e \mid \exists i \in \{1, \dots, p\} : e_i \leq a \text{ or } \exists j \in \{1, \dots, q\} : d_j \geq a\} \subseteq CR_D^*.$$

Since $c_a[(x_1, \dots, x_p, y_1, \dots, y_q)] = (a, \dots, a) \in R_d^e$ for all $R_d^e \in W$, we have

$$C\text{Inv}(c_a) \supseteq W.$$

In order to prove $W \supseteq C\text{Inv } c_a \cap CR_D^*$, we assume that there is a relation $R_d^{e'}$ belonging to $C\text{Inv}(c_a) \cap CR_D^*$ but not to W . The latter implies $a < e'_i$ for all $i \in \{1, \dots, p\}$ and $d'_j < a$ for all $j \in \{1, \dots, q\}$. Let $\mathbf{r} \in R_d^{e'}$. We have $c_a[\mathbf{r}] = (a, \dots, a) \notin R_d^{e'}$,

which is a contradiction to $c_a \triangleright R_{\mathbf{d}'}^e$.

We show $\text{End } C\text{Inv}(c_a) = \{id_D, c_0, c_{n-1}, c_a\}$. The inclusion “ $\supseteq$ ” is trivial. Conversely, take $f \in \text{End } C\text{Inv}(c_a)$. Assume $f \neq c_0, f \neq c_{n-1}$ and $f \neq id_D$. Since f is not id_D , there is some $c \in D$ such that $f(c) \neq c$. As $f \neq c_0$, there is some $y \in D$ such that $f(y) > 0$. Finally, $f \neq c_{n-1}$ implies that there is some $x \in D$ such that $f(x) < n - 1$. For the rest of the proof we show $f = c_a$.

- 1) For all $d \in [a, n - 1]$ and for all $z \in [0, d]$ we have $f(z) \leq d$.
Let $d \in [a, n - 1]$ and choose any element $z \in [0, d]$. As $z \leq d$, it follows $(x, z) \in R_d^{n-1}$. Since $d \geq a$, we receive $R_d^{n-1} \in C\text{Inv } c_a$ and hence, $f \triangleright R_d^{n-1}$. Thus, $(f(x), f(z)) \in R_d^{n-1}$. For $f(x) < n - 1$, it follows $f(z) \leq d$.
- 2) For all $e \in [0, a]$ and for all $z \in [e, n - 1]$ we have $f(z) \geq e$.
The proof is analogous to 1).
- 3) For all $d \in [a, n - 1]$ it holds that $f(d) \leq d$, and for all $e \in [0, a]$ we have $f(e) \geq e$. In particular, $f(a) = a$.
The first statement is a special case of 1), the second one a special case of 2). A combination of the two latter results yields $f(a) = a$.
- 4) For all $z \in [0, a]$ we have $f(z) \leq a$, and for all $z \in [a, n - 1]$ we have $f(z) \geq a$.
The claims follow from 1) and 2), respectively.
- 5) For all $z \in [0, a]$ it holds that $z \leq f(z) \leq a$, and for all $z \in [a, n - 1]$ we have $a \leq f(z) \leq z$.
This is a combination of 3) and 4).
- 6) For all $z \in [0, a)$ such that $f(z) \neq z$ we have $f(z) = a$.
Let $z \in [0, a)$ and $f(z) \neq z$, then by 5) we obtain $z < f(z) \leq a$. Obviously, $(z, z) \in R_z^a$. Since $R_z^a \in C\text{Inv } c_a$, we get $f \triangleright R_z^a$. Thus, $(f(z), f(z)) \in R_z^a$. Since $f(z) > z$, it follows $f(z) \geq a$.
- 7) For all $z \in (a, n - 1]$ such that $f(z) \neq z$, we have $f(z) = a$.
The proof is analogous to 6).
- 8) For all $z \in D$, it holds that $f(z) = z$ or $f(z) = a$.
This is a combination of 3), 6) and 7).
- 9) There exists a $v \in D \setminus \{a\}$ such that $f(v) = a$.
We assume, $f \neq id_D$. Hence, so there is some $v \in D$ such that $f(v) \neq v$. Statement 3) implies that $v \neq a$, and from 8) we infer $f(v) = a$.
- 10) If $f(0) = a$, then $f(z) \leq a$ for all $z \in D$.
Let $f(0) = a$ and $z \in D$. Then it holds that $(a, 0, z) \in R_{(0,a)}^{n-1}$, and $R_{(0,a)}^{n-1}$ belongs to $C\text{Inv } c_a$. Hence, we get $(a, a, f(z)) = (f(a), f(0), f(z)) \in R_{(0,a)}^{n-1}$. Since $0 < a < n - 1$, it follows $f(z) \leq a$.

- 11) If $f(n-1) = a$, then $f(z) \geq a$ for all $z \in D$.

The proof is analogous to 10).

- 12) If there exists a $v \in [0, a)$ such that $f(v) = a$, then $f = c_a$.

Let $v \in [0, a)$ such that $f(v) = a$. Let $z \in D$, then $(z, v) \in R_v^a$. Since $R_v^a \in C\text{Inv } c_a$, we receive $f \triangleright R_v^a$. Hence $(f(z), a) = (f(z), f(v)) \in R_v^a$. As $a > v$, we infer

$$f(z) \geq a. \quad (\star)$$

Now 4) implies $f(z) \leq a$ for all $z \in [0, a]$. This yields the precondition of 10). Hence, we can infer $f(z) \leq a$ for all $z \in D$. That together with $(\star)$, implies $f = c_a$.

- 13) Now we can prove $f = c_a$.

In case that the precondition of 12) is fulfilled, we are done. Otherwise, from 9) we infer $f(v) = a$ for some $v \in (a, n-1]$. For any $z \in D$ we have that $(v, z) \in R_a^v \in C\text{Inv } c_a$. Since $f \triangleright R_a^v$, it follows $(a, f(z)) \in R_a^v$. Because of $a < v$, we get

$$f(z) \leq a. \quad (\star\star)$$

Now 5) implies $a \leq f(z)$ for all $z \in [a, n-1]$. This yields the precondition of 11). Hence, it follows $f(z) \geq a$ for all $z \in D$. That together with $(\star\star)$ yields to $f = c_a$. $\square$

We finish this section by proving that there are no other minimal C -monoids than the ones already described in the previous Lemmata 5.4.3, 5.4.4 and Proposition 5.4.8. To do this we need an additional auxiliary statement.

5.4.9 Lemma. *Let $f \in O_D^{(1)}$ be an unary operation and $f \notin \{c_0, c_{n-1}\}$. If for every $a \in \{1, \dots, n-2\}$ there exists $\varrho_a \notin C\text{Inv } c_a$ such that $f \triangleright \varrho_a$, then $f(0) = 0$ and $f(n-1) = n-1$.*

Proof. Let $f \in O_D^{(1)} \setminus \{c_0, c_{n-1}\}$. To prove $f(0) = 0$ and $f(n-1) = n-1$, let us assume $f(0) \neq 0$ or $f(n-1) \neq n-1$. First we assume $f(0) > 0$, we consider two cases:

- If $f(0) =: a \in \{1, 2, \dots, n-2\}$, then there exists a $\varrho_a \notin C\text{Inv } c_a$ and $f \triangleright \varrho_a$. Since $\varrho_a \notin C\text{Inv } c_a$, the parameters of the relation $\varrho_a = R_{d_1, \dots, d_q}^{e_1, \dots, e_p}$ fulfil $a < e_i$ for all $i \in \{1, \dots, p\}$ and $a > d_j$ for all $j \in \{1, \dots, q\}$. Since $0 \leq d_1$, the tuple $\mathbf{r} := (0, \dots, \overset{p}{0}, \dots, \overset{p+q}{0})$ belongs to ϱ_a , but $f[\mathbf{r}] = (a, \dots, \overset{p}{a}, \dots, \overset{p+q}{a}) \notin \varrho_a$. This contradicts $f \triangleright \varrho_a$.
- If $f(0) = n-1$, then $f \neq c_{n-1}$ implies that there is some $x \in D \setminus \{0\}$ such that $f(x) < n-1$. Let $f(x) =: b \in \{0, 1, \dots, n-2\}$, again we consider two subcases.
 1. If $b \in \{1, \dots, n-2\}$, then similar to the first case we obtain a contradiction to $f \triangleright \varrho_b$ for some $\varrho_b \notin C\text{Inv } c_b$.

2. Let $f(x) = b = 0$ for $x \in D \setminus \{0\}$. By assumption and since $|D| \geq 3$, there exists ϱ_1 such that $\varrho_1 \notin C\text{Inv } c_1$ and $f \triangleright \varrho_1$. Since $\varrho_1 \notin C\text{Inv } c_1$, the relation $\varrho_1 = R_{d_1, \dots, d_q}^{e_1, \dots, e_p}$ satisfies $1 < e_i$ for all $i \in \{1, \dots, p\}$ and $1 > d_j$ for all $j \in \{1, \dots, q\}$. The tuple $\mathbf{r}^* := (x, \dots, \overset{p}{x}, 0, \dots, \overset{p+q}{0})$ belongs to ϱ_1 because of $0 \leq d_1$. However, we receive

$$f[\mathbf{r}^*] = (0, \dots, \overset{p}{0}, n-1, \dots, \overset{p+q}{n-1}) \notin \varrho_1,$$

in contradiction to $f \triangleright \varrho_1$.

Thus, $f(0) = 0$. Similarly, we can prove that $f(n-1) = n-1$. $\square$

5.4.10 Theorem. *For all $f \in O_D^{(1)} \setminus \{id_D, c_0, c_{n-1}\}$ there exists $a \in \{1, \dots, n-2\}$ such that*

$$\begin{aligned} C\text{Inv } f &\subseteq C\text{Inv } c_a, \quad \text{i.e.} \quad \langle c_a \rangle_{CO_D^{(1)}} \subseteq \langle f \rangle_{CO_D^{(1)}}, \\ \text{or } C\text{Inv } f &\subseteq C\text{Inv } f_a^-, \quad \text{i.e.} \quad \langle f_a^- \rangle_{CO_D^{(1)}} \subseteq \langle f \rangle_{CO_D^{(1)}}, \\ \text{or } C\text{Inv } f &\subseteq C\text{Inv } f_a^+, \quad \text{i.e.} \quad \langle f_a^+ \rangle_{CO_D^{(1)}} \subseteq \langle f \rangle_{CO_D^{(1)}}. \end{aligned}$$

Proof. Let $f \in O_D^{(1)} \setminus \{id_D, c_0, c_{n-1}\}$. Assume that for all $a \in \{1, \dots, n-2\}$ it holds that

$$\begin{aligned} C\text{Inv } f &\not\subseteq C\text{Inv } c_a, \\ \text{and } C\text{Inv } f &\not\subseteq C\text{Inv } f_a^-, \\ \text{and } C\text{Inv } f &\not\subseteq C\text{Inv } f_a^+. \end{aligned}$$

Hence, we infer that for all $a \in \{1, \dots, n-2\}$ it holds that

$$\begin{aligned} \exists \varrho_a &\notin C\text{Inv } c_a : f \triangleright \varrho_a, \\ \text{and } \exists \varrho_a^- &\notin C\text{Inv } f_a^- : f \triangleright \varrho_a^-, \\ \text{and } \exists \varrho_a^+ &\notin C\text{Inv } f_a^+ : f \triangleright \varrho_a^+. \end{aligned}$$

As for all $a \in \{1, \dots, n-2\}$ there is some $\varrho_a \notin C\text{Inv } c_a$ such that $f \triangleright \varrho_a$, we infer from Lemma 5.4.9 that $f(n-1) = n-1$ and $f(0) = 0$.

Since for all $a \in \{1, \dots, n-2\}$ there is some $\varrho_a^- \in C\text{Inv } f$ such that $f_a^- \not\triangleright \varrho_a^-$, we obtain from Lemma 5.4.3 that

$$\varrho_a^- = R_{\mathbf{b}}^{e_1, \dots, e_{i-1}, a, e_{i+1}, \dots, e_p},$$

for some $i \in \{1, \dots, p\}$, $e_k \neq 0$ for $k \neq i$, and $\mathbf{b} \in (D \setminus \{n-1\})^q$. The tuple

$$\mathbf{r} := (0, \dots, 0, \overset{i}{a}, 0, \dots, \overset{p}{0}, n-1, \dots, n-1)$$

belongs to ϱ_a^- . Since $f \triangleright \varrho_a^-$, we receive

$$(0, \dots, 0, f(a), 0, \dots, 0, n-1, \dots, n-1) = f[\mathbf{r}] \in \varrho_a^-,$$

and hence,

$$f(a) \geq a. \quad (5.3)$$

As for all $a \in \{1, \dots, n-2\}$ there is some $\varrho_a^+ \in C\text{Inv } f$ such that $f_a^+ \not\leq \varrho_a^+$, we obtain from Lemma 5.4.4 that

$$\varrho_a^+ = R_{b_1, \dots, b_{j-1}, a, b_{j+1}, \dots, b_q}^{\mathbf{e}}$$

for some $j \in \{1, \dots, q\}$, $b_k \neq n-1$ for $k \neq j$ and $\mathbf{e} \in (D \setminus \{0\})^p$. Let

$$\mathbf{r}^* := (0, \dots, \overset{p}{0}, n-1, \dots, n-1, \overset{j}{a}, n-1, \dots, n-1) \in \varrho_a^+.$$

Since $f \triangleright \varrho_a^+$, we receive

$$f[\mathbf{r}^*] = (0, \dots, 0, n-1, \dots, n-1, f(a), n-1, \dots, n-1) \in \varrho_a^+.$$

Hence,

$$f(a) \leq a. \quad (5.4)$$

We have $f(a) = a$ for all $a \in \{1, \dots, n-2\}$, due to the Inequalities (5.3) and (5.4). Furthermore, $f(0) = 0$ and $f(n-1) = n-1$. Thus, $f = id_D$. This is impossible, since $f \notin \{c_0, c_{n-1}, id_D\}$. $\square$

To finish this section, we present all results obtained so far in the Figure 5.4. Here we see the following:

- The rectangle labeled as $\langle f_a^- \rangle_{CO_D^{(1)}}$, $0 < a < n-1$, denotes minimal C -monoids that are generated by f_a^- , where $0 < a < n-1$.
- The rectangle labeled as $\langle f_a^+ \rangle_{CO_D^{(1)}}$, $0 < a < n-1$, denotes minimal C -monoids that are generated by f_a^+ , where $0 < a < n-1$.
- The rectangle labeled as $\langle c_a^{(1)} \rangle_{CO_D^{(1)}}$, $0 < a < n-1$, denotes minimal C -monoids that are generated by the unary constant operation $c_a^{(1)}$ with value a , where $a \in \{1, \dots, n-2\}$.

There are $n-2$ different minimal C -monoids on D , with $|D| \geq 3$, of the form $\langle f_a^- \rangle_{CO_D^{(1)}}$, $n-2$ of the form $\langle f_a^+ \rangle_{CO_D^{(1)}}$, and $n-2$ of the form $\langle c_a^{(1)} \rangle_{CO_D^{(1)}}$. Furthermore, due to Theorem 5.4.10 there are no other minimal C -monoids than the ones already described. Thus, there are $3(n-2)$ minimal C -monoids on a n -element set D , where $|D| \geq 3$.

 Figure 5.4: The lattice of C -monoids for $|D| \geq 3$.

5.5 Special C -monoids for $|D| \geq 3$

In this section some C -monoids and weak Krasner C -clones on D , where $|D| \geq 3$, are described. Hereafter, the image of f is denoted by $\text{Im}(f)$.

5.5.1 Lemma. *Let $k \in \{1, \dots, n-2\}$ be fixed, $F := \{f \in O_D^{(1)} \mid \text{Im}(f) \subseteq \{0, \dots, k\}\}$. Then*

$$CR_D^* \cap C\text{Inv } F = \{R_b^a \in CR_D^* \mid \exists j \in \{1, \dots, q\} : b_j \geq k\}.$$

Proof. Fix $k \in \{1, \dots, n-2\}$. We define

$$W := \{R_b^a \in CR_D^* \mid \exists j \in \{1, \dots, q\} : b_j \geq k\} \subseteq CR_D^*.$$

First, we show $W \subseteq C\text{Inv } F$. Let $\varrho \in W$ and $f \in F$, we have to prove that $f \triangleright \varrho$. Since $\varrho \in W$, we get $\varrho = R_{\mathbf{b}}^{\mathbf{a}} \in CR_D^*$ with $b_j \geq k$ for some $j \in \{1, \dots, q\}$. Consider any tuple $\mathbf{r} := (x_1, \dots, x_p, y_1, \dots, y_q)$ belonging to ϱ . Since $\text{Im}(f) \subseteq \{0, \dots, k\}$, we obtain $f(y_j) \leq k \leq b_j$. Thus, $f[\mathbf{r}] \in \varrho$.

To prove $W \supseteq CR_D^* \cap C\text{Inv } F$, we assume there is a relation $R_{\mathbf{b}'}^{\mathbf{a}'}$ belonging to $CR_D^* \cap C\text{Inv } F$ but not to W . Hence, the parameters of the relation $R_{\mathbf{b}'}^{\mathbf{a}'}$ fulfill $b'_j < k$ for all $j \in \{1, \dots, q\}$. Consider the unary function f_k defined by

$$f_k(x) := \begin{cases} k & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, f_k belongs to F . Furthermore, the tuple $(n-1, \dots, n-1, 0, \dots, 0)$ belongs to $R_{\mathbf{b}'}^{\mathbf{a}'}$. Since $f_k \triangleright R_{\mathbf{b}'}^{\mathbf{a}'}$, we obtain

$$(f_k(n-1), \dots, f_k(n-1), f_k(0), \dots, f_k(0)) = (0, \dots, 0, k, \dots, k) \in R_{\mathbf{b}'}^{\mathbf{a}'}$$

However, this yields to a contradiction since $0 \not\geq a'_i$ for all $i \in \{1, \dots, p\}$ and $k \not\leq b'_j$ for all $j \in \{1, \dots, q\}$. $\square$

If $D = \{0, 1, 2\}$ and $k = 1$, then $F = \{f \in O_D^{(1)} \mid \text{Im}(f) \subseteq \{0, 1\}\}$ is equal to $\{f_1, f_3, f_4, f_9, f_{10}, f_{12}, f_{13}\}$ (see Table 5.1). From Lemma 5.5.1 we infer

$$\begin{aligned} CR_D^* \cap C\text{Inv } F &= \{R_{\mathbf{b}}^{\mathbf{a}} \in CR_D^* \mid \exists j \in \{1, \dots, q\} : b_j \geq 1\} \\ &= \{R_1^1, R_1^2, R_{(1,0)}^2, R_{(1,0)}^1, R_1^{(1,2)}, R_{(1,0)}^{(1,2)}\}. \end{aligned}$$

The last equality can be read from the lattice of all C -monoids on $D = \{0, 1, 2\}$ depicted in the Figure 5.3.

Similarly, the following lemma can be proved.

5.5.2 Lemma. *Let $k \in \{1, \dots, n-2\}$ be fixed, and let*

$$F := \{f \in O_D^{(1)} \mid \text{Im}(f) \subseteq \{k, \dots, n-1\}\}.$$

Then, $CR_D^ \cap C\text{Inv } F = \{R_{\mathbf{b}}^{\mathbf{a}} \in CR_D^* \mid \exists i \in \{1, \dots, p\} : a_i \leq k\}$.*

If $D = \{0, 1, 2\}$ and $k = 1$, then $F = \{f \in O_D^{(1)} \mid \text{Im}(f) \subseteq \{1, 2\}\}$ is equal to $\{f_{13}, f_{14}, f_{16}, f_{17}, f_{22}, f_{23}, f_{25}, f_{26}\}$, and from Lemma 5.5.2 we infer

$$CR_D^* \cap C\text{Inv } F = \{R_1^1, R_0^1, R_0^{(1,2)}, R_{(1,0)}^1, R_1^{(1,2)}, R_{(1,0)}^{(1,2)}\}.$$

5.5.3 Lemma. *Let $k \in \{1, \dots, n-2\}$, and $s \in \{1, \dots, k\}$ be fixed, let D be the domain with at least four elements. Let also $F := \{f \in O_D^{(1)} \mid \text{Im}(f) \subseteq \{s, \dots, k\}\}$. Then,*

$$\begin{aligned} CR_D^* \cap C\text{Inv } F &= \{R_{\mathbf{b}}^{\mathbf{a}} \in CR_D^* \mid \exists i \in \{1, \dots, p\} : a_i \leq s\} \cup \\ &\quad \{R_{\mathbf{b}}^{\mathbf{a}} \in CR_D^* \mid \exists j \in \{1, \dots, q\} : b_j \geq k\}. \end{aligned}$$

Proof. Fix $k \in \{1, \dots, n-2\}$ and $s \in \{1, \dots, k\}$. We define

$$\begin{aligned} W_1 &:= \{R_{\mathbf{b}}^{\mathbf{a}} \in CR_D^* \mid \exists i \in \{1, \dots, p\} : a_i \leq s\} \subseteq CR_D^*, \\ W_2 &:= \{R_{\mathbf{b}}^{\mathbf{a}} \in CR_D^* \mid \exists j \in \{1, \dots, q\} : b_j \geq k\} \subseteq CR_D^*. \end{aligned}$$

First, we show $W_1 \cup W_2 \subseteq C\text{Inv } F$. Let $\varrho \in W_1 \cup W_2$ and $f \in F$, we have to prove that $f \triangleright \varrho$. Since $\varrho \in W_1 \cup W_2$, we consider two cases:

- If $\varrho \in W_1$, then $\varrho = R_{\mathbf{b}}^{\mathbf{a}}$ with $a_i \leq s$ for some $i \in \{1, \dots, p\}$. Consider any tuple $\mathbf{r} := (x_1, \dots, x_p, y_1, \dots, y_q)$ belonging to ϱ . Because $\text{Im}(f) \subseteq \{s, \dots, k\}$, we get $f(x_i) \geq s \geq a_i$. Thus, $f[\mathbf{r}] \in \varrho$.
- If $\varrho \notin W_1$, then ϱ belongs to W_2 . The latter implies $\varrho = R_{\mathbf{b}}^{\mathbf{a}}$ with $b_j \geq k$ for some $j \in \{1, \dots, q\}$. For any tuple $\mathbf{r} = (x_1, \dots, x_p, y_1, \dots, y_q)$ belonging to ϱ , we have $f[\mathbf{r}] \in \varrho$, since $f(y_j) \leq k \leq b_j$.

We show $W_1 \cup W_2 \supseteq CR_D^* \cap C\text{Inv } F$. Assume there is a relation $R_{\mathbf{b}'}^{\mathbf{a}'}$ belonging to $CR_D^* \cap C\text{Inv } F$ but not to $W_1 \cup W_2$. Hence, the parameters of $R_{\mathbf{b}'}^{\mathbf{a}'}$ fulfill $a'_i > s$ for all $i \in \{1, \dots, p\}$ and $b'_j < k$ for all $j \in \{1, \dots, q\}$. Let f_k^s be the unary function defined by

$$f_k^s(x) := \begin{cases} s & \text{if } x = n-1, \\ k & \text{otherwise.} \end{cases}$$

By definition, the function f_k^s belongs to F . Since $(n-1, \dots, n-1, 0, \dots, 0) \in R_{\mathbf{b}'}^{\mathbf{a}'}$ and $f_k^s \triangleright R_{\mathbf{b}'}^{\mathbf{a}'}$, we obtain

$$(f_k^s(n-1), \dots, f_k^s(n-1), f_k^s(0), \dots, f_k^s(0)) = (s, \dots, s, k, \dots, k) \in R_{\mathbf{b}'}^{\mathbf{a}'}$$

However, this yields to a contradiction since $a'_i > s$ for all $i \in \{1, \dots, p\}$ and $b'_j < k$ for all $j \in \{1, \dots, q\}$. $\square$

For what is left of this section, we will characterise one-generated C -monoids for some functions $f \in O_D^{(1)}$, (namely $f \in \{g_k, g'_k \mid 0 < k < n-1\}$) defined hereafter. For $k \in \{1, \dots, n-2\}$, let g_k, g'_k be the unary operations on D given by

$$g_k(x) := \min(c_k(x), x) \quad \text{and} \quad g'_k(x) := \max(c_k(x), x),$$

where c_k is the unary constant operation with value k .

The C -monoids generated by these functions allow us to characterise some upper covers of the minimal C -monoids generated by c_k , where $k \in \{1, \dots, n-2\}$.

5.5.4 Lemma. *Let $k \in \{1, \dots, n-2\}$ be fixed, c_k be the unary constant operation with value k , and let g_k be the unary operation on D defined by*

$$g_k(x) := \min(c_k(x), x).$$

Then we have

$$CR_D^* \cap C\text{Inv } g_k = \{R_{\mathbf{b}}^{\mathbf{a}} \mid \forall i \in \{1, \dots, p\} : a_i \leq k \text{ or } \exists j \in \{1, \dots, q\} : b_j \geq k\}.$$

Proof. Let $k \in \{1, \dots, n-2\}$ be fixed. We define

$$W := \{R_{\mathbf{b}}^{\mathbf{a}} \mid \forall i \in \{1, \dots, p\} : a_i \leq k \text{ or } \exists j \in \{1, \dots, q\} : b_j \geq k\} \subseteq CR_D^*.$$

First, we show $W \subseteq C\text{Inv } g_k$. Let $\varrho \in W$ and let $\mathbf{r} := (x_1, \dots, x_p, y_1, \dots, y_q)$ be any tuple belonging to ϱ . Since $\varrho \in W$, we get $\varrho = R_{\mathbf{b}}^{\mathbf{a}}$, where $b_j \geq k$ for some $j \in \{1, \dots, q\}$ or $a_i \leq k$ for all $i \in \{1, \dots, p\}$. We consider the following cases:

- If there is $j \in \{1, \dots, q\}$ such that $b_j \geq k$, then, since $\text{Im}(g_k) \in \{1, \dots, k\}$, we get $g_k(y_j) \leq k \leq b_j$. Thus, $g_k[\mathbf{r}] \in \varrho$.
- Otherwise, for every $j \in \{1, \dots, q\}$ we get $b_j < k$. Then because $\varrho \in W$, we get $a_i \leq k$ for all $i \in \{1, \dots, p\}$. Since $\mathbf{r} \in \varrho$, one obtains

$$x_1 \geq a_1 \vee \dots \vee x_p \geq a_p \vee y_1 \leq b_1 \vee \dots \vee y_q \leq b_q,$$

and we consider the following subcases:

1. If $y_j \leq b_j$ for some $j \in \{1, \dots, q\}$, then $y_j \leq b_j < k$, and hence, $g_k(y_j) = y_j \leq b_j$. Consequently, $g_k[\mathbf{r}] \in \varrho$.
2. If $x_i \geq a_i \in \{1, \dots, k\}$ for some $i \in \{1, \dots, p\}$, then either the element x_i belongs to $\{1, \dots, k\}$ or $x_i > k$. If $x_i \in \{1, \dots, k\}$, then $g_k(x_i) = x_i \geq a_i$, and hence $g_k[\mathbf{r}] \in \varrho$. If $x_i > k$, then $g_k(x_i) = k \geq a_i$, and likewise $g_k[\mathbf{r}] \in \varrho$.

Second, we show $CR_D^* \cap C\text{Inv } g_k \subseteq W$. Assume there is a relation ϱ belonging to $CR_D^* \cap C\text{Inv } g_k$ but not to W . Hence, $\varrho = R_{\mathbf{b}'}^{\mathbf{e}'}$, where $e'_i > k$ for some $i \in \{1, \dots, p\}$ and $b'_j < k$ for all $j \in \{1, \dots, q\}$. The tuple $\mathbf{r} := (0, \dots, 0, \overset{i}{n-1}, 0, \dots, \overset{p}{0}, \overset{1}{k}, \dots, \overset{q}{k})$ belongs to ϱ . Since $g_k \triangleright R_{\mathbf{b}'}^{\mathbf{e}'}$, we have $g_k[\mathbf{r}] \in R_{\mathbf{b}'}^{\mathbf{e}'}$. However, this yields a contradiction since $g_k[\mathbf{r}] = (0, \dots, 0, \overset{i}{k}, 0, \dots, \overset{p}{0}, \overset{1}{k}, \dots, \overset{q}{k})$ and

$$0 < e'_1 \wedge \dots \wedge 0 < e'_{i-1} \wedge k < e'_i \wedge 0 < e'_{i+1} \wedge \dots \wedge 0 < e'_p \wedge k > b'_1 \wedge \dots \wedge k > b'_q.$$

□

The dual statement of 5.5.4 is the following.

5.5.5 Lemma. *Let $k \in \{1, \dots, n-2\}$ be fixed, c_k be the unary constant operation with value k , and let g'_k be the unary operation on D , defined by*

$$g'_k(x) := \max(c_k(x), x).$$

Then we have

$$CR_D^* \cap C\text{Inv } g'_k = \{R_{\mathbf{b}}^{\mathbf{a}} \mid \exists i \in \{1, \dots, p\} : a_i \leq k \text{ or } \forall j \in \{1, \dots, q\} : b_j \geq k\}.$$

In following propositions we establish that the one-generated C -monoids of functions g_k or g'_k for fixed $k \in \{1, \dots, n-2\}$, are upper covers of $\langle c_k \rangle_{CO_D^{(1)}}$ in the lattice of all C -monoids.

5.5.6 Proposition. *Let $k \in \{1, \dots, n-2\}$ be fixed, and let g_k be the unary operation given by $g_k(x) = \min(c_k(x), x)$ for $x \in D$. Then*

$$\langle g_k \rangle_{CO_D^{(1)}} = \{g_k, c_k, c_0, c_{n-1}, id_D\} = \langle c_0, c_{n-1}, g_k \rangle_{O_D^{(1)}}.$$

Proof. First, we show $\{g_k, c_k, c_0, c_{n-1}, id_D\} = \langle c_0, c_{n-1}, g_k \rangle_{O_D^{(1)}}$. Clearly,

$$\{g_k, c_0, c_{n-1}, id_D\} \subseteq \langle c_0, c_{n-1}, g_k \rangle_{O_D^{(1)}}.$$

Furthermore, as $c_k(x) = \min(c_k(c_{n-1}(x)), c_{n-1}(x)) = g_k \circ c_{n-1}(x)$, we have that c_k belongs to $\langle c_0, c_{n-1}, g_k \rangle_{O_D^{(1)}}$. Thus, $\{g_k, c_0, c_{n-1}, id_D, c_k\} \subseteq \langle c_0, c_{n-1}, g_k \rangle_{O_D^{(1)}}$.

Let $F := \{id_D, c_0, c_k, c_{n-1}, g_k\}$. Because F belongs to $\langle c_0, c_{n-1}, g_k \rangle_{O_D^{(1)}}$, in order to calculate all functions that belong to $\langle c_0, c_{n-1}, g_k \rangle_{O_D^{(1)}}$, we first calculate all compositions $x \circ y$ between functions x, y of F .

$x \setminus y$	id_D	c_0	c_k	c_{n-1}	g_k
id_D	id_D	c_0	c_k	c_{n-1}	g_k
c_0	c_0	c_0	c_k	c_{n-1}	c_0
c_k	c_k	c_0	c_k	c_{n-1}	c_k
c_{n-1}	c_{n-1}	c_0	c_k	c_{n-1}	c_k
g_k	g_k	c_0	c_k	c_{n-1}	g_k

From the previous table we obtain that F is closed under composition of functions, i.e., there are no more functions in $\langle c_0, c_{n-1}, g_k \rangle_{O_D^{(1)}}$ than those belonging to F .

Consequently, $\langle c_0, c_{n-1}, g_k \rangle_{O_D^{(1)}} = F = \{id_D, c_0, c_k, c_{n-1}, g_k\}$.

Second, we show $\langle g_k \rangle_{CO_D^{(1)}} = \langle c_0, c_{n-1}, g_k \rangle_{O_D^{(1)}}$. We have $\langle c_0, c_{n-1}, g_k \rangle_{O_D^{(1)}} \subseteq \langle g_k \rangle_{CO_D^{(1)}}$. In order to show $\langle g_k \rangle_{CO_D^{(1)}} \subseteq \langle c_0, c_{n-1}, g_k \rangle_{O_D^{(1)}}$, we will prove that

$$\text{if } g \in \langle g_k \rangle_{CO_D^{(1)}} \setminus \{c_0, c_{n-1}, c_k, id_D\}, \text{ then } g = g_k.$$

Take $g \in \text{End } C\text{Inv}(g_k) = \langle g_k \rangle_{CO_D^{(1)}}$. Suppose $g \neq c_0, g \neq c_k, g \neq c_{n-1}$ and $g \neq id_D$. Since g is not id_D , there is some $c \in D$ such that $g(c) \neq c$. As $g \neq c_0$, there is some $y \in D$ such that $g(y) > 0$, and $g \neq c_{n-1}$ implies that there is some $x \in D$ such that $g(x) < n-1$. Finally, as $g \neq c_k$ there is some $d \in D$ such that $g(d) \neq k$. For the rest of the proof we show $g = g_k$.

1) For all $z \in [0, k]$ we have $g(z) \leq k$.

Let $z \in [0, k]$. As $z \leq k$, it follows $(x, z) \in R_k^{n-1}$. From Lemma 5.5.4, we infer that the relation R_k^{n-1} belongs to $C\text{Inv } g_k$, and hence $g \triangleright R_k^{n-1}$. Thus, $(g(x), g(z)) \in R_k^{n-1}$. For $g(x) < n-1$, it follows $g(z) \leq k$.

- 2) For $z \in [k, n-1]$ we have $g(z) \geq k$.
 Let $z \in [k, n-1]$. As $z \geq k$, we have $(z, y) \in R_0^k$. Furthermore, $R_0^k \in C\text{Inv } g_k$, so $g \triangleright R_0^k$. Thus, $(g(z), g(y)) \in R_0^k$. As $g(y) > 0$, it follows $g(z) \geq k$.
- 3) For all $z \in [0, k]$ we have $g(z) \geq z$.
 Let $z \in [0, k]$. As $z \leq k$, the relation R_0^z belongs to $C\text{Inv } g_k$. Clearly, the tuple (z, y) belongs to R_0^z and hence $g \triangleright R_0^z$. Thus, $(g(z), g(y)) \in R_0^z$. For $g(y) > 0$, it follows $g(z) \geq z$.
- 4) For all $z \in [k, n-1]$ we have $g(z) \leq z$.
 Let $z \in [k, n-1]$. As $z \geq k$, the relation R_z^{n-1} belongs to $C\text{Inv } g_k$. Clearly, the tuple (x, z) belongs to R_z^{n-1} and hence $g \triangleright R_z^{n-1}$. Thus, $(g(x), g(z)) \in R_z^{n-1}$. For $g(x) < n-1$, it follows $g(z) \leq z$.
- 5) For all $z \in [0, k]$ it holds that $z \leq g(z) \leq k$, and for all $z \in [k, n-1]$ we have $k \leq g(z) \leq z$. In particular, $g(k) = k$.
 This is a combination of 1), 2), 3), and 4).
- 6) For all $z \in [0, k)$ such that $g(z) \neq k$ we have $g(z) = z$.
 Let $z \in [0, k)$ and $g(z) \neq k$, then by 5) we obtain $z \leq g(z) < k$. Obviously, $(z, z) \in R_z^k$. Since $R_z^k \in C\text{Inv } g_k$, we get $g \triangleright R_z^k$. Thus, $(g(z), g(z)) \in R_z^k$. Since $g(z) < k$, it follows $g(z) \leq z$.
- 7) If $g(0) = k$, then $g(z) \leq k$ for all $z \in D$.
 Let $g(0) = k$ and $z \in D$. Then it holds that $(k, 0, z) \in R_{(0,k)}^{n-1}$, and $R_{(0,k)}^{n-1}$ belongs to $C\text{Inv } g_k$. Hence, we get $(k, k, g(z)) = (g(k), g(0), g(z)) \in R_{(0,k)}^{n-1}$. Since $0 < k < n-1$, it follows $g(z) \leq k$.
- 8) If there exists a $v \in [0, k)$ such that $g(v) = k$, then $g = c_k$.
 Let $v \in [0, k)$ such that $g(v) = k$. Let $z \in D$, then $(z, v) \in R_v^k$. Since $R_v^k \in C\text{Inv } g_k$, we receive $g \triangleright R_v^k$, and hence $(g(z), k) = (g(z), g(v)) \in R_v^k$. As $k > v$, we infer

$$g(z) \geq k. \tag{5.5}$$
 Now 5) implies $g(z) \leq k$ for all $z \in [0, k]$. This yields the precondition of 7). Hence, we can infer $g(z) \leq k$ for all $z \in D$. That together with (5.5), implies $g = c_k$.
- 9) For all $v \in [0, k]$ we have $g(v) = v$.
 From 5) we infer $g(k) = k$. Since $g \neq c_k$, by 8) we infer that for all $v \in [0, k)$, $g(v) \neq k$. This yields the precondition of 6). Hence, $g(v) = v$ for all $v \in [0, k]$.
- 10) For all $z \in (k, n-1]$ we have $g(z) = k$.
 Since g is not id_D , there is some $c \in D$ such that $g(c) \neq c$. By 9), we have $c \in (k, n-1]$ and by 5) we obtain $g(c) < c$. Let $z \in D$. The relation R_k^c belongs to $C\text{Inv } g_k$, hence $g \triangleright R_k^c$. Clearly, the tuple (c, z) belongs to R_k^c .

Thus, $(g(c), g(z)) \in R_k^c$. For $g(c) < c$, it follows $g(z) \leq k$ and by 5) $g(z) \geq k$ for all $z \in (k, n-1]$. From the last two inequalities, we get $g(z) = k$ for all $z \in (k, n-1]$. $\square$

Similarly, one can prove

5.5.7 Proposition. *Let $k \in \{1, \dots, n-2\}$ be fixed, and let g_k be the unary operation given by $g'_k(x) = \max(c_k(x), x)$ for $x \in D$. Then*

$$\langle g'_k \rangle_{CO_D^{(1)}} = \{g'_k, c_k, c_0, c_{n-1}, id_D\} = \langle c_0, c_{n-1}, g'_k \rangle_{O_D^{(1)}}.$$

From the restriction of the Galois connection $\text{End} - \text{Inv}$ to the Galois connection $\text{End} - C\text{Inv}$, we get $\text{End Inv}(F) = \langle F \rangle_{O_D^{(1)}} \subseteq \langle F \rangle_{CO_D^{(1)}}$, for any set F of operations on D . Hence, some essentially unary C -clones (C -monoids) generated by some F may contain more functions, which are not obtained in the usual form, that is, by composition of functions from F . One example is $\langle g_k \rangle_{CO_D^{(1)}}$, see also Proposition 5.5.6. The question is to determine the new functions in

$$\langle F \rangle_{CO_D^{(1)}} \setminus \langle F \rangle_{O_D^{(1)}}$$

in a constructive way that is essentially different from applying $\text{End } C\text{Inv}$.

To end this chapter, let us summarize the results obtained in Proposition 5.5.6 and 5.5.7 in the following figure.

6 Maximal C -clones

6.1 Description

In this section we describe all co-atoms in the lattice of all C -clones for $|D| \geq 3$, these are formally defined as follows.

6.1.1 Definition. A C -clone F is called *maximal* if $F \neq O_D$ and between F and O_D there are no more C -clones. That is, for every C -clone F' the condition $F \subsetneq F'$ implies $F' = O_D$.

Theorem 6.1.5 below states that a C -clone is maximal if it can be determined by only one non-trivial clausal relation of arity two. To prove this theorem we need some auxiliary statements such as Lemmata 6.1.3, 6.1.4 and Remark 6.1.2. The latter states that C -clones of the form $\text{Pol } R_b^a$, where $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$, are candidates for maximal C -clones.

6.1.2 Remark. Let $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$. Then

$$\text{Pol } R_b^a \subsetneq O_D.$$

Proof. Assume $\text{Pol } R_b^a = O_D$. Hence, we get $O_D^{(1)} \cap \text{Pol } R_b^a = O_D^{(1)}$, i.e. $\text{End } R_b^a = O_D^{(1)}$. The latter is a contradiction to $\text{End } R_b^a \subsetneq O_D^{(1)}$ (cf. Remark 5.3.2) $\square$

6.1.3 Lemma. Let $f \in O_D$, $a_1, \dots, a_p \in D \setminus \{0\}$ and $b_1, \dots, b_q \in D \setminus \{n-1\}$. Then we have

$$f \triangleright R_{b_1, \dots, b_q}^{a_1, \dots, a_p} \Rightarrow f \triangleright R_{\max\{b_1, \dots, b_q\}}^{\min\{a_1, \dots, a_p\}}.$$

Proof. Let $k \in \mathbb{N}_+$ and f be a k -ary function on D such that $f \triangleright R_{b_1, \dots, b_q}^{a_1, \dots, a_p}$. We have to show that f preserves $R_{\max\{b_1, \dots, b_q\}}^{\min\{a_1, \dots, a_p\}}$. For $l \in \{1, \dots, k\}$ we consider $\mathbf{r}_l := (x_l, y_l)$ to be any tuples that belong to $R_{\max\{b_1, \dots, b_q\}}^{\min\{a_1, \dots, a_p\}}$. The latter implies

$$\begin{aligned} x_l &\geq \min\{a_1, \dots, a_p\} \vee y_l \leq \max\{b_1, \dots, b_q\} \\ \iff x_l &\geq a_1 \vee \dots \vee x_l \geq a_p \vee y_l \leq b_1 \vee \dots \vee y_l \leq b_q \end{aligned}$$

Because of the expression above, the tuples $(x_l, \dots, x_l, y_l, \dots, y_l)$ belong to $R_{b_1, \dots, b_q}^{a_1, \dots, a_p}$, where $l \in \{1, \dots, k\}$. Furthermore, since $f \triangleright R_{b_1, \dots, b_q}^{a_1, \dots, a_p}$, we obtain

$$\begin{aligned} f(x_1, \dots, x_k) &\geq a_1 \vee f(x_1, \dots, x_k) \geq a_2 \vee \dots \vee f(x_1, \dots, x_k) \geq a_p \vee \\ f(y_1, \dots, y_k) &\leq b_1 \vee f(y_1, \dots, y_k) \leq b_2 \vee \dots \vee f(y_1, \dots, y_k) \leq b_q \\ \iff f(x_1, \dots, x_k) &\geq \min\{a_1, \dots, a_p\} \vee f(y_1, \dots, y_k) \leq \max\{b_1, \dots, b_q\}. \end{aligned}$$

Thus, $f[\mathbf{r}_1, \dots, \mathbf{r}_k] = (f(x_1, \dots, x_k), f(y_1, \dots, y_k)) \in R_{\max\{b_1, \dots, b_q\}}^{\min\{a_1, \dots, a_p\}}$. $\square$

We can also prove that any two C -monoids, $\text{Pol } R_{b_1}^{a_1}$ and $\text{Pol } R_{b_2}^{a_2}$, where

$$(a_1, b_1) \neq (a_2, b_2)$$

and $(a_1, b_1), (a_2, b_2) \in (D \setminus \{0\}) \times (D \setminus \{n-1\})$, are incomparable.

6.1.4 Lemma. *Let $a_1, a_2 \in D \setminus \{0\}$, $b_1, b_2 \in D \setminus \{n-1\}$. If $(a_1, b_1) \neq (a_2, b_2)$, then we have*

$$\text{Pol } R_{b_1}^{a_1} \not\subseteq \text{Pol } R_{b_2}^{a_2} \quad \text{and} \quad \text{Pol } R_{b_2}^{a_2} \not\subseteq \text{Pol } R_{b_1}^{a_1}.$$

Proof. Let $\text{Pol } R_{b_1}^{a_1}$ and $\text{Pol } R_{b_2}^{a_2}$, where $(a_1, b_1) \neq (a_2, b_2)$. Let us assume

$$\text{Pol } R_{b_1}^{a_1} \subseteq \text{Pol } R_{b_2}^{a_2} \quad \text{or} \quad \text{Pol } R_{b_2}^{a_2} \subseteq \text{Pol } R_{b_1}^{a_1}.$$

If $\text{Pol } R_{b_1}^{a_1} \subseteq \text{Pol } R_{b_2}^{a_2}$, then $\text{End } R_{b_1}^{a_1} = O_D^{(1)} \cap \text{Pol } R_{b_1}^{a_1} \subseteq O_D^{(1)} \cap \text{Pol } R_{b_2}^{a_2} = \text{End } R_{b_2}^{a_2}$. The latter is a contradiction to $\text{End } R_{b_1}^{a_1} \not\subseteq \text{End } R_{b_2}^{a_2}$. (cf. Lemma 5.3.4)

If $\text{Pol } R_{b_2}^{a_2} \subseteq \text{Pol } R_{b_1}^{a_1}$, then $\text{End } R_{b_2}^{a_2} = O_D^{(1)} \cap \text{Pol } R_{b_2}^{a_2} \subseteq O_D^{(1)} \cap \text{Pol } R_{b_1}^{a_1} = \text{End } R_{b_1}^{a_1}$, which is a contradiction to $\text{End } R_{b_2}^{a_2} \not\subseteq \text{End } R_{b_1}^{a_1}$. $\square$

Below we characterise all maximal C -clones.

6.1.5 Theorem. *Let $M \subseteq O_D$ be a C -clone. M is maximal if and only if there are elements $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$ such that*

$$M = \text{Pol } R_b^a.$$

Proof. “ $\Rightarrow$ ” Let $M \subseteq O_D$ be a maximal C -clone. Then there exists a relation $\varrho \in CR_D^*$ such that $M = \text{Pol } \varrho$. This means there are elements $a_1 \dots a_p \in D \setminus \{0\}$ and $b_1, \dots, b_q \in D \setminus \{n-1\}$ such that $M = \text{Pol } R_{b_1, \dots, b_q}^{a_1, \dots, a_p}$. From 6.1.3 and 6.1.2, we have

$$M = \text{Pol } R_{b_1, \dots, b_q}^{a_1, \dots, a_p} \stackrel{6.1.3}{\subseteq} \text{Pol } R_{\max\{b_1, \dots, b_q\}}^{\min\{a_1, \dots, a_p\}} \stackrel{6.1.2}{\subsetneq} O_D.$$

Since M is maximal, we have $M = \text{Pol } R_{\max\{b_1, \dots, b_q\}}^{\min\{a_1, \dots, a_p\}}$.

“ $\Leftarrow$ ” Let $\text{Pol } R_b^a$ be a C -clone, where $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$. To prove that $\text{Pol } R_b^a$ is maximal we consider a C -clone C such that $\text{Pol } R_b^a \subsetneq C$ and we have to show that $C = O_D$.

Since $\text{Pol } R_b^a \subsetneq C$, there is some $f \in C$ and $f \notin \text{Pol } R_b^a$. As $f \in C$ and $\text{Pol } R_b^a \subsetneq C$, we get $\{f\} \cup \text{Pol } R_b^a \subseteq C$. Hence,

$$\text{Pol } C\text{Inv}(\{f\} \cup \text{Pol } R_b^a) = \langle \{f\} \cup \text{Pol } R_b^a \rangle_C \subseteq \langle C \rangle_C = C \subseteq O_D. \quad (6.1)$$

We consider two cases:

- If $\text{Pol } C\text{Inv}(\{f\} \cup \text{Pol } R_b^a) = O_D$, then from (6.1) we infer $C = O_D$. Thus, $\text{Pol } R_b^a$ is maximal.

- If $\text{Pol } C\text{Inv}(\{f\} \cup \text{Pol } R_b^a) \subsetneq O_D$, then

$$C\text{Inv}(\{f\} \cup \text{Pol } R_b^a) \supsetneq C\text{Inv } O_D \stackrel{3.4.3}{=} \{D^{p+q} \mid p, q \in \mathbb{N}_+\},$$

and hence there is some relation ϱ belonging to $C\text{Inv}(\{f\} \cup \text{Pol } R_b^a)$, which is non-trivial. Thus, $\varrho = R_{\mathbf{b}'}^{\mathbf{a}'}$, where $\mathbf{a}' \in (D \setminus \{0\})^p$ and $\mathbf{b}' \in (D \setminus \{n-1\})^q$.

Furthermore,

$$\text{Pol } R_{\max\{b'_1, \dots, b'_q\}}^{\min\{a'_1, \dots, a'_p\}} \stackrel{6.1.3}{\supseteq} \text{Pol } \varrho \supseteq \text{Pol } C\text{Inv}(\{f\} \cup \text{Pol } R_b^a) \supseteq \text{Pol } C\text{Inv}(\text{Pol } R_b^a) = \text{Pol } R_b^a.$$

Hence, $f \in \text{Pol } R_{\max\{b'_1, \dots, b'_q\}}^{\min\{a'_1, \dots, a'_p\}} \supseteq \text{Pol } R_b^a$. Because of 6.1.4, we have $a = \min\{a'_1, \dots, a'_p\}$ and $b = \max\{b'_1, \dots, b'_q\}$. Thus, $\text{Pol } R_{\max\{b'_1, \dots, b'_q\}}^{\min\{a'_1, \dots, a'_p\}} = \text{Pol } R_b^a$ and hence $f \in \text{Pol } R_b^a$, which is a contradiction to $f \notin \text{Pol } R_b^a$. $\square$

Due to 6.1.5 and 3.4.4, we get

$$\begin{aligned} \langle h \rangle_C &= \text{Pol } \{R_b^a \mid a \in D \setminus \{0\}, b \in D \setminus \{n-1\}\} \\ &\stackrel{6.1.5}{=} \bigcap \{M \mid M \text{ maximal } C\text{-clone}\}. \end{aligned}$$

Note that one non-trivial clausal relation of arity two can be used to describe both a co-atom in the lattice of C -clones and a co-atom in the lattice of C -monoids, in contrast to the normal maximal clones where this does not happen. In fact, there are also $(n-1)^2$ maximal C -clones on an n -element set D .

6.2 C -complete operations

We return now to unary operations, referring back to Figure 5.3, we can see that there is a considerable number of unary operations, located at the top in this lattice. In this section these sets of operations are characterised.

6.2.1 Definition. A set $F \subseteq O_D^{(1)}$ of unary operations on D is C -complete if

$$\langle F \rangle_{CO_D^{(1)}} = O_D^{(1)}.$$

Theorem 6.2.3 below states that the set containing only one unary operation is C -complete if it does not preserve all non-trivial clausal relations of arity two. To prove this theorem we need the following statement.

6.2.2 Lemma. Let $F \subseteq O_D^{(1)}$. Then F is C -complete if and only if F is not contained in any maximal C -monoid.

Proof. “ $\Rightarrow$ ” Let $F \subseteq O_D^{(1)}$ be a C -complete set of unary operations. Assume F is contained in M for some maximal C -monoid M . The latter implies

$$O_D^{(1)} = \langle F \rangle_{CO_D^{(1)}} \subseteq \langle M \rangle_{CO_D^{(1)}} = M \subsetneq O_D^{(1)},$$

which is a contradiction.

“ $\Leftarrow$ ” In order to show $\langle F \rangle_{CO_D^{(1)}} = O_D^{(1)}$, we assume $\langle F \rangle_{CO_D^{(1)}} \subsetneq O_D^{(1)}$. Since $O_D^{(1)}$ is finite, the lattice of all C -monoids is also finite, thus there exists a maximal C -monoid M which contains $\langle F \rangle_{CO_D^{(1)}}$, hence $F \subseteq \langle F \rangle_{CO_D^{(1)}} \subseteq M$, which is also a contradiction. $\square$

6.2.3 Theorem. *Let $g \in O_D^{(1)}$ be a unary operation, $\{g\}$ is C -complete if and only if $g \not\vdash R_b^a$ for all $(a, b) \in D \setminus \{0\} \times D \setminus \{n-1\}$.*

Proof. Let $\{g\} \in O_D^{(1)}$ be C -complete. By Theorem 5.3.5 each maximal C -monoid M is of the form $M = \text{End } R_b^a$ where $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$. Thus, $\{g\} \not\subseteq M \iff g \not\vdash R_b^a$ for all $(a, b) \in D \setminus \{0\} \times D \setminus \{n-1\}$. Therefore, the theorem follows from 6.2.2. $\square$

In the following if we speak of a C -complete unary operation, we implicitly mean that the singleton set containing the unary operation is C -complete.

Due to Theorem 6.2.3 for $g \in O_D^{(1)}$, deciding if $\{g\}$ is C -complete is equivalent to the task of finding for all $(a, b) \in D \setminus \{0\} \times D \setminus \{n-1\}$ a tuple $\mathbf{r}$ belonging to R_b^a such that $g[\mathbf{r}] \notin R_b^a$.

Observe that the tuples $\mathbf{t}_d := (n-1, d)$ and $\mathbf{t}'_d := (d, 0)$, where $d \in D$, belong to R_b^a for all $(a, b) \in D \setminus \{0\} \times D \setminus \{n-1\}$. However, the tuple $(0, n-1)$ does not belong to R_b^a .

A form to find conditions under which some unary operations are C -complete, is to look for unary operations g_d on D such that $g_d[\mathbf{t}_d] = (0, n-1)$, for some $d \in \{0, 1, \dots, n-1\}$. The latter implies

$$g_d(n-1) = 0 \text{ and } g_d(d) = n-1 \text{ and } d \neq n-1.$$

From the previous observation and Theorem 6.2.3, we infer that for fixed d belonging to $\{0, \dots, n-2\}$, the sets $\{g_d\}$, where g_d satisfies

$$g_d(x) := \begin{cases} n-1 & \text{if } x = d, \\ 0 & \text{if } x = n-1, \end{cases} \quad (I_d)$$

are C -complete.

Similarly, if we consider unary operations $\hat{g}_d$ on D such that $\hat{g}_d[\mathbf{t}'_d] = (0, n-1)$, i.e. such that $\hat{g}_d(d) = 0$ and $\hat{g}_d(0) = n-1$ and $d \neq 0$. For fixed $d \in \{1, \dots, n-1\}$ the sets $\{\hat{g}_d\}$, where $\hat{g}_d$ satisfies

$$\hat{g}_d(x) := \begin{cases} 0 & \text{if } x = d, \\ n-1 & \text{if } x = 0, \end{cases} \quad (I'_d)$$

are C -complete.

Below the number of unary operations that satisfy (I_d) or (I'_d) are counted. Since (I_d) for $d = 0$ is identical to (I'_d) for $d = n - 1$, we count unary operations that satisfy (I'_d) for $0 < d < n - 1$, or (I_d) for $0 \leq d < n - 1$.

- For $D = \{0, 1\}$ there is only one operation g_0 satisfying (I_d) . It is equal to $\neg$ the negation operation.
- For $D = \{0, 1, 2\}$, there are seven unary operations that are C -complete (see Fig.5.3), all of them are listed in the following table, its last row is labelled by the number of operations that satisfy (I_d) for $d \in \{0, 1\}$ or (I'_d) for $d = 1$. For example in the last row, second column, $3 = \left| \left\{ f \in O_D^{(1)} \mid f \text{ satisfy } I_0 \right\} \right|$.

x	$g_0(x)$	$g_1(x)$	$\widehat{g}_1(x)$	Σ
0	2	01	2	
1	012	2	0	
2	0	0	12	
	3	2	2	$3 + 2 + 2 = 7$

- For $D = \{0, 1, 2, 3\}$, there are 58 operations of the form (I_d) for $d \in \{0, 1, 2\}$ or (I'_d) for $d \in \{1, 2\}$, calculated as follows.

x	$g_0(x)$	$g_1(x)$	$g_2(x)$	$\widehat{g}_1(x)$	$\widehat{g}_2(x)$	Σ
0	3	012	012	3	3	
1	0123	3	012	0	123	
2	0123	0123	3	0123	0	
3	0	0	0	123	123	
	4^2	3×4	3^2	4×3	3^2	$= 58$

- For $|D| \geq 3$, the number of operations of the form (I_d) for $0 \leq d < n - 1$ or of the form (I'_d) for $0 < d < n - 1$ is:

$$\left| \left\{ f \in O_D^{(1)} \mid f \text{ satisfy } (I_d); 0 \leq d < n - 1 \right\} \right| = \sum_{i=0}^{n-2} |D|^{(|D|-2)-i} * (|D| - 1)^i,$$

$$\left| \left\{ f \in O_D^{(1)} \mid f \text{ satisfy } (I'_d); 0 < d < n - 1 \right\} \right| = \sum_{i=0}^{n-3} |D|^{(|D|-3)-i} * (|D| - 1)^{i+1}.$$

Contrary to $D = \{0, 1, 2\}$, on $D = \{0, 1, 2, 3\}$, there are more unary operations that are C -complete, but they do neither satisfy (I_d) for $0 \leq d < 3$ nor (I'_d) for $0 < d < 3$. For example, the unary operation g^* such that

$$g^*(0) = 1, \quad g^*(1) = 3, \quad g^*(2) = 0, \quad g^*(3) = 2.$$

does neither satisfy (I_d) where $0 \leq d < 3$ nor (I'_d) where $0 < d < 3$. However, it is C -complete, as it is proved in the following remark.

6.2.4 Remark. Let $D = \{0, 1, 2, 3\}$ and g^* be defined as above. Then $\{g^*\}$ is C -complete.

Proof. Let $M := \{R_b^a \mid a \in \{1, 2, 3\}, b \in \{0, 1, 2\}\}$. Due to Theorem 6.2.3, it is enough to prove that g^* does not preserve ϱ for any $\varrho \in M$. Consider $M = M_1 \dot{\cup} M_2$, where

$$M_1 = \{R_0^a \mid a \in \{1, 2, 3\}\} \text{ and } M_2 = \{R_b^a \mid a \in \{1, 2, 3\}, b \in \{1, 2\}\},$$

and let $\varrho \in M$. We consider two cases:

If $\varrho \in M_1$, then $\varrho = R_0^a$, where $a \in \{1, 2, 3\}$. As the tuple $(2, 0)$ belongs to ϱ , and $(g^*(2), g^*(0)) = (0, 1)$ does not, we get $g^* \not\models \varrho$.

If $\varrho \in M_2$, then $\varrho = R_b^a$, where $a \in \{1, 2, 3\}$ and $b \in \{1, 2\}$. Since the tuple $(2, 1)$ belongs to ϱ , and $(g^*(2), g^*(1)) = (0, 3)$ does not belong to ϱ , we obtain $g^* \not\models \varrho$. $\square$

More conditions under which unary operations on D are C -complete, for $|D| \geq 4$, are established below.

Let $M := \{R_b^a \mid a \in D \setminus \{0\}, b \in D \setminus \{n-1\}\}$ be the set of all non-trivial clausal relations of arity two on D . For fixed $e_1 \in \{1, \dots, n-2\}$ we divide M in two disjoint sets $M = M_1 \dot{\cup} M_2$, where

$$M_1 = \{R_b^a \mid b < e_1, a \in D \setminus \{0\}\} \text{ and } M_2 = \{R_b^a \mid n-1 > b \geq e_1, a \in D \setminus \{0\}\}.$$

Due to Theorem 6.2.3 a set $\{f\}$ is C -complete for $f \in O_D^{(1)}$, if one finds a tuple $\mathbf{r}$ belonging to ϱ for all $\varrho \in M = M_1 \dot{\cup} M_2$ such that $f[\mathbf{r}] \notin \varrho$.

Let $d \in D$ be fixed. The tuple $(d, 0)$ belongs to ϱ for all $\varrho \in M_1$, and the tuple (d, e_1) belongs to ϱ for all $\varrho \in M_2$.

The equality $f[(d, e_1)] = (0, n-1)$ is equivalent to

$$f(d) = 0 \text{ and } f(e_1) = n-1 \text{ and } d \neq e_1.$$

If this condition is fulfilled, then $f \not\models \varrho$ for all $\varrho \in M_2$. Furthermore, the fact $f[(d, 0)] \in \{0\} \times \{e_1, e_1 + 1, \dots, n-1\}$ is equivalent to

$$f(d) = 0 \text{ and } f(0) \in [e_1, n-1] \text{ and } d \neq 0.$$

In this case f preserves no $\varrho \in M_1$. From the latter and Theorem 6.2.3, we infer.

If there are elements $e_1 \in \{1, \dots, n-2\}$ and $d \in D$, such that $0 \neq d \neq e_1$, then the sets $\{f\}$ on D , where f satisfies

- 1) $f(d) = 0$,
- 2) $f(e_1) = n-1$,
- 3) $f(0) \geq e_1$,

are C -complete.

Note that the unary operation g^* (see 6.2.4) satisfies the latter conditions, for $e_1 = 1$ and $d = 2$.

Similarly, for fixed $e_1 \in \{1, \dots, n-2\}$ we divide M as $M = M_1 \dot{\cup} M_2$, where

$$M_1 = \{R_b^a \mid a \geq e_1, b \in D \setminus \{n-1\}\} \text{ and } M_2 = \{R_b^a \mid 0 < a < e_1, b \in D \setminus \{n-1\}\}.$$

Let $d \in D$ be fixed. The tuple $(n-1, d)$ belongs to ϱ for all $\varrho \in M_1$, and the tuple (e_1, d) belongs to ϱ for all $\varrho \in M_2$.

The fact $f[(n-1, d)] \in \{e_1-1, e_1-2, \dots, 0\} \times \{n-1\}$ is equivalent to

$$f(n-1) \in [0, e_1-1] \text{ and } f(d) = n-1 \text{ and } d \neq n-1.$$

In this case f preserves no $\varrho \in M_1$. Furthermore, the equality $f[(e_1, d)] = (0, n-1)$, is equivalent to

$$f(e_1) = 0 \text{ and } f(d) = n-1 \text{ and } d \neq e_1.$$

If this condition is fulfilled, then $f \not\vdash \varrho$ for all $\varrho \in M_2$. Thus, we get the following assertion:

If there are elements $e_1 \in \{1, \dots, n-2\}$ and $d \in D$ such that $n-1 \neq d \neq e_1$, then the sets of unary operations $\{f\}$, where f satisfies

$$1') \quad f(d) = n-1,$$

$$2') \quad f(e_1) = 0,$$

$$3') \quad f(n-1) \leq e_1-1,$$

are C -complete.

On $D = \{0, 1, 2, 3, 4\}$ there are still more unary operations that are C -complete but they do neither satisfy the conditions 1), 2), 3) nor the conditions 1'), 2'), 3'). Furthermore, they are neither of the form (I_d) nor of the form (I'_d) . For example¹ the function f_{1104} given by

$$f_{1104}(0) = 1, \quad f_{1104}(1) = 3, \quad f_{1104}(2) = 4, \quad f_{1104}(3) = 0, \quad f_{1104}(4) = 4.$$

is one of them.

More conditions under which unary operations are C -complete can be found as follows. We divide the set M in three disjoint sets $M = M_1 \dot{\cup} M_2 \dot{\cup} M_3$, where

$$M_1 = \{R_b^a \mid b < e_1, a \in D \setminus \{0\}\},$$

$$M_2 = \{R_b^a \mid e_1 \leq b < e_2, a \in D \setminus \{0\}\},$$

$$M_3 = \{R_b^a \mid e_2 \leq b < n-1, a \in D \setminus \{0\}\},$$

¹The index k corresponding to this function is just the interpretation of $f_k(0)f_k(1)\dots f_k(4)$ as a 5-adic number: $k = \sum_{i=0}^4 f_k(i)5^{4-i}$.

for certain $e_1, e_2 \in \{1, \dots, n-2\}$ satisfying $e_1 \neq e_2$.

We look for tuples belonging to ϱ for all $\varrho \in M = M_1 \dot{\cup} M_2 \dot{\cup} M_3$. Let $d \in D$ be fixed. The tuple $(d, 0)$ belongs to ϱ for all $\varrho \in M_1$, the tuple (d, e_1) belongs to ϱ for all $\varrho \in M_2$, and the tuple (d, e_2) belongs to ϱ for all $\varrho \in M_3$. Finally, we look for conditions, under which unary operations f do not preserve any for $\varrho \in M$.

- The equality $f[(d, e_2)] = (0, n-1)$ is equivalent to

$$f(d) = 0 \quad \text{and} \quad f(e_2) = n-1 \quad \text{and} \quad d \neq e_2.$$

If this condition is fulfilled, then $f \not\vdash \varrho$ for all $\varrho \in M_3$.

- The fact $f[(d, e_1)] \in \{0\} \times \{e_2, e_2 + 1, \dots, n-1\}$, is equivalent to

$$f(d) = 0 \quad \text{and} \quad f(e_1) \in [e_2, n-1] \quad \text{and} \quad d \neq e_1.$$

In this case $f \not\vdash \varrho$ for all $\varrho \in M_2$,

- and $f[(d, 0)] \in \{0\} \times \{e_1, e_1 + 1, \dots, n-1\}$, is equivalent to

$$f(d) = 0 \quad \text{and} \quad f(0) \in [e_1, n-1] \quad \text{and} \quad d \neq 0.$$

In this case f preserves no $\varrho \in M_1$.

From the previous conditions and Theorem 6.2.3 we infer:

If there are elements $e_1, e_2 \in \{1, \dots, n-2\}$ and $d \in D$ such that

$$0 \neq d \neq e_1 \neq e_2 \neq d,$$

then sets of unary operations $\{f\}$, where f satisfies

- i) $f(d) = 0$,
- ii) $f(0) \geq e_1$,
- iii) $f(e_1) \geq e_2$,
- iv) $f(e_2) = n-1$,

are C -complete.

Note that the unary operation f_{1104} defined above satisfies the latter conditions for $e_1 = 1, e_2 = 2$ and $d = 3$.

Similarly, for $e_1, e_2 \in \{1, \dots, n-2\}$, where $e_1 \neq e_2$, we divide $M = M_1 \dot{\cup} M_2 \dot{\cup} M_3$, where

$$\begin{aligned} M_1 &= \{R_b^a \mid a \geq e_1, b \in D \setminus \{n-1\}\}, \\ M_2 &= \{R_b^a \mid e_2 \leq a < e_1, b \in D \setminus \{n-1\}\}, \\ M_3 &= \{R_b^a \mid 0 < a < e_2, b \in D \setminus \{n-1\}\}. \end{aligned}$$

Finally, we get: If there are elements $e_1, e_2 \in \{1, \dots, n-2\}$ and $d \in D$ such that $0 \neq d \neq e_1 \neq e_2 \neq d$, then sets of unary operations $\{f\}$, where f satisfies

$$\text{i}') \quad f(d) = n - 1$$

$$\text{ii}') \quad f(n - 1) < e_1,$$

$$\text{iii}') \quad f(e_1) < e_2,$$

$$\text{iv}') \quad f(e_2) = 0,$$

are C -complete.

Further conditions under which the sets $\{f\}$ for $f \in O_D^{(1)}$ are C -complete, can be found if we divide M in more disjoint subsets. Possibly the conditions obtained in this way are enough to determine when a set containing only one unary operation on D is C -complete. We leave it as an open problem, to solve the following question.

Given an arbitrary unary operation f on a finite set D , decide if the set $\{f\}$ is C -complete or not.

7 Problems for future research

In this chapter some open problems are presented that emerged while working on this dissertation, but, unfortunately, could not be solved so far.

The first questions are closely related to cardinalities of the C -clone lattice $\mathcal{CL}_D$ for a finite set $|D| \geq 3$.

- P1. In Section 3.5, we showed that for $|D| \geq 3$ there exist infinitely many C -clones. However, how many C -clones exactly do exist? Are there only countably many C -clones?

To show that there are only countably many C -clones, one could try to show that for all $\text{Pol } Q$, where Q is a subset of clausal relations, there is already a finite set $Q_0 \subseteq C\text{Inv Pol } Q$ such that

$$\text{Pol } Q = \text{Pol } Q_0.$$

Below we show that the latter statement implies $|\mathcal{CL}_D| = \aleph_0$. First, we show that if $|D| \leq \aleph_0$, then $|CR_D| \leq \aleph_0$.

Let $p, q \in \mathbb{N}_+$. Consider the set $\mathcal{R}_q^p = \{R_{\mathbf{b}}^{\mathbf{a}} \mid (\mathbf{a}, \mathbf{b}) \in D^p \times D^q\}$ of all clausal relations of arity $p + q$. The function $\varphi : D^p \times D^q \longrightarrow \mathcal{R}_q^p$ defined by

$$\varphi(\mathbf{a}, \mathbf{b}) := R_{\mathbf{b}}^{\mathbf{a}}$$

is surjective. This implies $|\mathcal{R}_q^p| \leq |D^p \times D^q| = |D^{p+q}|$. Since $|D| \leq \aleph_0$, we have $|D^{p+q}| \leq \aleph_0$ and hence, $|\mathcal{R}_q^p| \leq \aleph_0$. Thus,

$$|CR_D| = \left| \bigcup_{(p,q) \in \mathbb{N}_+^2} \mathcal{R}_q^p \right| \leq \aleph_0.$$

Second, let $\mathcal{P}_{\text{fin}}(CR_D)$ be the set of all finite subsets of CR_D . Because $|CR_D| \leq \aleph_0$, we have $|\mathcal{P}_{\text{fin}}(CR_D)| \leq \aleph_0$. Finally, consider the mapping

$$\begin{aligned} \psi : \mathcal{P}_{\text{fin}}(CR_D) &\longrightarrow \mathcal{CL}_D \\ Q_0 &\mapsto \text{Pol } Q_0. \end{aligned}$$

If we can prove that for all $\text{Pol } Q$, where Q is a subset of clausal relations, there is already a finite set $Q_0 \subseteq C\text{Inv Pol } Q$ such that

$$\text{Pol } Q = \text{Pol } Q_0,$$

we get that ψ is surjective and hence,

$$|\mathcal{CL}_D| \leq |\mathcal{P}_{\text{fin}}(CR_D)| \leq \aleph_0.$$

- P2. Is there an uncountable set of C -clones? Is there a continuum of C -clones on a set containing at least three elements?

To show that there is a continuum of C -clones one could try to show the following: Let F be a suitably chosen countable set of functions on D . Then, for all $f \in F$ one had to prove $f \notin \text{Pol } C\text{Inv}(F \setminus \{f\}) = \langle F \setminus \{f\} \rangle_C$.

In the following remark, we show that the latter statement implies $|\mathcal{CL}_D| = 2^{\aleph_0}$. In [VPT97] there is a form to prove that there is a continuum of clones on finite sets D .

7.0.5 Remark. Let F be a countable set of functions on D such that for all $f \in F$ holds $f \notin \langle F \setminus \{f\} \rangle_C$. Then for all G, H subsets of F such that $G \neq H$ we get $\langle G \rangle_C \neq \langle H \rangle_C$.

Proof. Let $F \subseteq O_D$ be countable, such that $f \notin \langle F \setminus \{f\} \rangle_C$ holds for all $f \in F$, and let $G, H \subseteq F$, where $G \neq H$. The latter implies $G \not\subseteq H$ or $H \not\subseteq G$. W.l.o.g. $G \not\subseteq H$. Hence, there is some $g \in G$ such that $g \notin H$. Since $g \notin H$, we get $H \subseteq F \setminus \{g\}$ and $\langle H \rangle_C \subseteq \langle F \setminus \{g\} \rangle_C$. Since $g \notin \langle F \setminus \{g\} \rangle_C$, $g \notin \langle H \rangle_C$ follows. Furthermore, as $g \in \langle G \rangle_C$, we get $\langle G \rangle_C \not\subseteq \langle H \rangle_C$. Hence, $\langle G \rangle_C \neq \langle H \rangle_C$. $\square$

Let F be a countable set of functions on D as above. From the previous remark, the following mapping

$$\begin{aligned} \varphi : \mathcal{P}(F) &\longrightarrow \mathcal{CL}_D \\ G &\mapsto \langle G \rangle_C, \end{aligned}$$

is injective. Hence,

$$2^{\aleph_0} = |\mathcal{P}(F)| \leq |\mathcal{CL}_D| \leq |\mathcal{L}_D| = 2^{\aleph_0}.$$

- P3. In Chapter 5 we determined all atoms in the lattice of essentially unary C -clones. A problem for future research is to determine all atoms in the lattice of C -clones.

- P4. The following open problem is related to the characterization of the Galois closed sets for $\text{Pol} - C\text{Inv}$.

From the restriction of the Galois connection $\text{Pol} - \text{Inv}$ to the Galois connection $\text{Pol} - C\text{Inv}$, we get $\text{PolInv}(F) = \langle F \rangle_{O_D} \subseteq \langle F \rangle_C$, where F is a set of operations on D . The question is to determine the new functions in

$$\langle F \rangle_C \setminus \langle F \rangle_{O_D}$$

in a constructive way that is essentially different from applying $\text{Pol } C\text{Inv}$.

- P5. Answer the open question at the end of Chapter 5, related to the characterization of the Galois closed sets for $\text{End} - C\text{Inv}$.

P6. Answer the open questions at the end of Section 6.2, related to sets of unary functions that are C -complete.

F.R. Future research:

In what follows, we mention a result on the order of certain one-generated C -clones. Although the proof of Theorem 7.0.6 below will be shown in [BV11], we consider it convenient to mention the theorem, and an example to illustrate it. First, we introduce some notation.

Notation: Let $p, q \in \mathbb{N}_+$, $\mathbf{a} = (a_1, \dots, a_p) \in D^p$ and $\mathbf{b} = (b_1, \dots, b_q) \in D^q$. For $I \subseteq \{1, \dots, p\}$ let $a_{\wedge I} := \min \{a_v \mid v \in I\}$, likewise $b_{\vee J} := \max \{b_v \mid v \in J\}$ for $J \subseteq \{1, \dots, q\}$.

Let $p, q, R_a, R_b \in \mathbb{N}_+$, $\mathbf{a} = (a_1, \dots, a_p) \in D^p$, $\mathbf{b} = (b_1, \dots, b_q) \in D^q$. Let $\{I_1^a, \dots, I_{R_a}^a\} \subseteq \mathcal{P}(\{1, \dots, p\}) \setminus \{\emptyset\}$ and $\{I_1^b, \dots, I_{R_b}^b\} \subseteq \mathcal{P}(\{1, \dots, q\}) \setminus \{\emptyset\}$ be collections of nonempty subsets of indices such that

- (1) the sets $I_1^a, \dots, I_{R_a}^a$ are pairwise different,
- (2) $\bigcup_{t=1}^{R_a} I_t^a = \{1, \dots, p\}$,
- (3) there exist $k_1^a, \dots, k_{R_a}^a \in \{1, \dots, p\}$ pairwise different such that

$$\forall t \in \{1, \dots, R_a\} : k_t^a \in I_t^a \text{ and } a_{\wedge I_t^a} = a_{k_t^a};$$

- (1') the sets $I_1^b, \dots, I_{R_b}^b$ are pairwise different,
- (2') $\bigcup_{s=1}^{R_b} I_s^b = \{1, \dots, q\}$,
- (3') there exist $k_1^b, \dots, k_{R_b}^b \in \{1, \dots, q\}$ pairwise different such that

$$\forall s \in \{1, \dots, R_b\} : k_s^b \in I_s^b \text{ and } b_{\vee I_s^b} = b_{k_s^b}.$$

Note that these conditions are trivially true if $\{I_1^a, \dots, I_{R_a}^a\}$ and $\{I_1^b, \dots, I_{R_b}^b\}$ are partitions of $\{1, \dots, p\}$ and $\{1, \dots, q\}$, respectively.

7.0.6 Theorem ([BV11]). *Let $\{I_1^a, \dots, I_{R_a}^a\}$ and $\{I_1^b, \dots, I_{R_b}^b\}$ be collections of nonempty subsets of indices satisfying the conditions mentioned above. Let also $\mathbf{c} := (a_{\wedge I_1^a}, \dots, a_{\wedge I_{R_a}^a}) \in D^{R_a}$, and $\mathbf{d} := (b_{\vee I_1^b}, \dots, b_{\vee I_{R_b}^b}) \in D^{R_b}$. Then*

$$\text{Pol } R_{\mathbf{b}}^{\mathbf{a}} \subseteq \text{Pol } R_{\mathbf{d}}^{\mathbf{c}}.$$

7.0.7 Example. Let $\mathbf{a} = (4, 2, 3, 2)$, $\mathbf{b} = (1, 2, 3, 4, 3)$. Consider, $R_a = 2$, $R_b = 3$, $\underline{I}_1^a = \{1, 2, 3\}$, $\underline{I}_2^a = \{2, 4\}$, $\underline{I}_1^b = \{1, 2, 3\}$, $\underline{I}_2^b = \{2, 3, 4\}$, $\underline{I}_3^b = \{2, 3, 5\}$, and the numbers $k_1^a = 2$, $k_2^a = 4$, $k_1^b = 3$, $k_2^b = 4$, $k_3^b = 5$ which satisfy the conditions (3) and (3') mentioned above. Then by 7.0.6, we get

$$\text{Pol } R_{\underline{1}\underline{2}\underline{3}\underline{4}\underline{3}}^{\underline{4}\underline{2}\underline{3}\underline{2}} \subseteq \text{Pol } R_{\underline{3}\underline{4}\underline{3}}^{\underline{2}\underline{2}}$$

8 Concluding remarks

In this work we studied a special set of relations, so-called clausal relations, and we described clones that are determined by sets of these relations. These clones are called C -clones.

It was shown that clausal relations meet a sufficient condition that is known to ensure polynomial time solvability of the corresponding CSP, namely that their clone of polymorphisms contains a constant operation (see Section 3.4).

Our aim was the description of C -clones. First, the description of all C -clones on the two-element set was given (see Section 3.2). However, for $|D| \geq 3$ the description of C -clones has been proven to be a lot more difficult, due to the infinitely many C -clones showed in Section 3.5, by constructing infinite descending chains of these clones. Therefore, we analysed the C -clones up to equality of their unary bijective parts, i.e. C -automorphism groups.

The main result in the description of all C -automorphism groups is that they form a Boolean lattice with $|D| - 1$ atoms (see Section 4.1). In particular, the C -automorphism groups can be characterised by permutations preserving some of their unary relations, which are intervals w.r.t. the natural order on D .

Furthermore, we studied the C -clone lattice up to equality of the corresponding endomorphism monoids, which is the lattice of C -monoids. The structure of this lattice is considerably harder to understand, since unary relations are not any more enough to determine the lattice of all relational clones given by C -monoids. However, first steps towards the description of this lattice were undertaken. First, it was shown that finite sets of clausal relations suffice to describe C -monoids (see Section 5.1). Then, the lattice of C -monoids for $|D| = 3$ was determined (see Section 5.2). After that, the co-atoms and the atoms of the C -monoid lattice were characterised (see Sections 5.3 and 5.4).

Finally, all co-atoms in the lattice of C -clones were determined. It turned out that one non-trivial clausal relation of arity two can be used to describe both a co-atom in the lattice of C -clones and a co-atom in the lattice of C -monoids (see Section 6.1).

In future investigations, we will study the structure of the C -clone lattice on a finite set D .

Notation

$f[\mathbf{r}_1, \dots, \mathbf{r}_k]$	composition of $f \in \mathcal{O}_D^{(k)}$ by tuples $\mathbf{r}_1, \dots, \mathbf{r}_k \in D^m$, $m \in \mathbb{N}_+$
$f(g_1, \dots, g_k)$	general composition of $f \in \mathcal{O}_D^{(k)}$ with $g_1, \dots, g_k \in \mathcal{O}_D^{(m)}$
$ D $	cardinality of a set D
$\text{Aut} - C\text{Inv}$	Galois connection $\text{Aut} - C\text{Inv}$
$\text{Aut} - \text{Inv}$	Galois connection $\text{Aut} - \text{Inv}$
$\text{Aut } Q$	set of weak automorphisms of a set Q of finitary relations
c_a	constant unary function with value a
c_a^k	constant k -ary function with value a
$C\text{Inv } F$	set of invariant clausal relations of a set F of finitary operations
$\mathcal{CL}_D$	lattice of all C -clones on a set D
$\mathcal{CL}_D^*$	lattice of all relational C -clones on a set D
$\langle F \rangle_C$	C -clone of operations ($\text{Pol } C\text{Inv } F$) of a set F of finitary operations
$\langle F \rangle_{CO_D^{(1)}}$	C -monoid of operations ($\text{End } C\text{Inv } F$) of a set F of unary operations
CR_D	set of all finitary clausal relations on a set D
$[Q]_C$	relational C -clone ($C\text{Inv Pol } Q$) of a set Q of finitary clausal relations
CR_D^*	set of non-trivial clausal relations on a set D
CSP	Constraint Satisfaction Problem
$[Q]_{wC}$	weak Krasner C -clone ($C\text{Inv End } Q$) of a set Q of finitary clausal relations

$\Delta_D^{(2)}$	binary identity relation on D
$\text{diag}(D)$	set of all diagonal relations together with the empty relation
D	finite set $\{0, 1, \dots, n-1\}$
$R_{\mathbf{a}^\alpha}^{\mathbf{b}^\alpha}$	dual clausal relation of arity $p+q$ on a set D
$\emptyset$	emptyset
$\text{End} - C\text{Inv}$	Galois connection $\text{End} - C\text{Inv}$
$\text{End} - \text{Inv}$	Galois connection $\text{End} - \text{Inv}$
$\text{End } Q$	set of endomorphisms of a set Q of finitary relations
$e_i^{(k)}$	k -ary projection on the i -th coordinate ($i \in \{1, \dots, k\}$)
$\langle F \rangle_{O_D}$	clone generated by $F \subseteq O_D$
$F \leq O_D$	$F \subseteq O_D$ is a clone on D
$f \triangleright \varrho$	finitary operation f preserves a finitary relation ϱ
id_D	identity mapping on a set D
$\text{Im}(f)$	image $f[A]$ of a function $f : A \longrightarrow B$
$\text{Int } R_{\mathbf{b}}^{\mathbf{a}}$	intervals belonging to $R_{\mathbf{b}}^{\mathbf{a}}$
$\text{Inv } F$	set of invariant relations of a set F of finitary operations
$\text{Inv}^{(m)} F$	set of m -ary relations in $\text{Inv } F$
J_D	set of all projections on a set D
$\mathcal{L}_D$	lattice of all clones on a set D
$\mathcal{L}_D^*$	lattice of all relational clones on a set D
$\nabla_D^{(2)}$	full binary relation on $D, D \times D$
$\mathbb{N}_+$	the set of positive natural numbers
$\mathbb{N}_{\geq 2}$	the set of natural numbers greater or equal to 2
O_D	set of all finitary operations on a set D
$O_D^{(k)}$	set of all k -ary operations on a set D

$\text{Pol} - C\text{Inv}$	Galois connection $\text{Pol} - C\text{Inv}$
$\text{Pol} - \text{Inv}$	Galois connection $\text{Pol} - \text{Inv}$
$\text{Pol}^{(k)} Q$	set of k -ary operations in $\text{Pol } Q$
$\text{Pol } Q$	set of polymorphisms of a set Q of finitary relations
$\mathcal{P}(D)$	powerset of D
$\text{pr}_i(\mathbf{R}_{\mathbf{b}}^{\mathbf{a}})^C$	projection on the i -th coordinate ($i \in \{1, \dots, p\}$) of $(\mathbf{R}_{\mathbf{b}}^{\mathbf{a}})^C$
$[Q]_{\mathbf{R}_D}$	relational clone generated by $Q \subseteq \mathbf{R}_D$
$Q \leq \mathbf{R}_D$	$Q \subseteq \mathbf{R}_D$ is a relational clone on D
$\mathbf{R}_D$	set of all finitary relations on a set D
$\mathbf{R}_b^a$	binary clausal relation on D
$\mathbf{R}_{\mathbf{b}}^{\mathbf{a}}$	clausal relation of arity $p + q$ on a set D
$(\mathbf{R}_{\mathbf{b}}^{\mathbf{a}})^C$	complement of a clausal relation of arity $p + q$ on a set D
$\mathbf{R}_{0,q}^{1,p}$	clausal relation of arity $p + q$ on the two-element set
$\mathbf{R}_D^{(m)}$	set of all m -ary relations on a set D
$\mathbf{S}_D$	set of all permutations on a set D
V_{α}	contravariant substitution functor applied to a mapping $\alpha : m \longrightarrow n, m, n \in \mathbb{N}_+$
W_{α}	covariant substitution functor applied to a mapping $\alpha : m \longrightarrow n, m, n \in \mathbb{N}_+$

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Affirmation

- (a) Hereby I declare that I wrote the present thesis on my own, I indicated citations. I did not use any other means than indicated. This thesis has not been presented to any other examination board in this or a similar form, neither in Germany nor in any other country.
- (b) The present thesis has been produced since April 2007 at the Institut für Algebra, Department of Mathematics, Faculty of Science, TU Dresden under the supervision of Prof. Dr. rer. nat. habil. R. Pöschel.
- (c) There have been no prior attempts to obtain a PhD at any university.
- (d) I accept the requirements for obtaining a PhD (Promotionsordnung) of the Faculty of Science of the TU Dresden, issued March 20, 2000 with the changes in effect since April 16, 2003 and July 17, 2008.

Erklärung

- (a) Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig erarbeitet, verfaßt und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.
- (b) Die vorliegende Dissertation wurde im April 2007 am Institut für Algebra, Fachrichtung Mathematik, Fakultät Mathematik und Naturwissenschaften, Technische Universität Dresden unter der Betreuung von Prof. Dr. rer. nat. habil. R. Pöschel. angefertigt.
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Dresden, den 25. Februar 2011