On Ruled Surfaces in three-dimensional Minkowski Space

D I S S E R T A T I O N

Zur Erlangung des akademischen Grades

Doktor rerum naturalium
(Dr. rer. nat.)

Vorgelegt

der Fakultät Mathematik und Naturwissenschaften
der Technischen Universität Dresden

von

Emad N. Naseem Shonoda

geboren am 29. Januar 1976 in Elismailya

Gutachter: Prof. Dr. G. Weiss (TU Dresden, Institut für Geometrie)

Prof. Dr. H. Martini (TU Chemnitz, Fakultät für Mathematik)

Eingereicht am: 16.07.2010

Tag der Disputation: 13.12.2010

Acknowledgements

I would like to express deepest thanks and appreciation to my supervisor **Prof. Dr. G. Weiss** for his cooperation to suggest and propose this problem to me, which is completely new for me and for his help during the research. I benefit a lot not only from his intuition and readiness for discussing problems, but also his way of approaching problems in a structured way had a great influence on me. Of course without his care, support and invaluable guidance this thesis would not have come to light.

I wish to use this chance to congratulate all the members of the institute of Geometry, TU Dresden, who provided a familiar and friendly atmosphere.

My appreciation goes further to my family, especially, to my wife and daughters whose love and understanding activated me for better arenas of being. Special thanks to my father and mother who always accompanied me with their support and hope during my stay in Germany for increasing my activity.

Finally, I want to thank the "MINISTRY OF HIGHER EDUCATION, EGYPT" for financial support during my PhD-studies.

Abstract

In a Minkowski three dimensional space, whose metric is based on a strictly convex and centrally symmetric unit ball B, we deal with ruled surfaces Φ in the sense of E. Kruppa. This means that we have to look for Minkowski analogues of the classical differential invariants of ruled surfaces in a Euclidean space. Here, at first – after an introduction to concepts of a Minkowski space, like semi-orthogonalities and a semi-inner-product based on the so-called cosine-Minkowski function - we construct an orthogonal 3D moving frame using Birkhoff's left-orthogonality. This moving frame is canonically connected to ruled surfaces: beginning with the generator direction and the asymptotic plane of this generator g we complete this flag to a frame using the left-orthogonality defined by B; (B is described either by its supporting function or a parameter representation). The plane left-orthogonal to the asymptotic plane through generator $g(t)$ is called Minkowski central plane and touches Φ in the striction point $s(t)$ of $g(t)$. Thus the moving frame defines the Minkowski striction curve S of the considered ruled surface Φ similar to the Euclidean case. The coefficients occur5ring in the Minkowski analogues to Frenet-Serret formulae of the moving frame of Φ in a Minkowski space are called "M-curvatures" and "M-torsions". Here we essentially make use of the semi-inner product and the sine-Minkowski and cosine-Minkowski functions. Furthermore we define a *covariant differentiation* in a Minkowski 3-space using a new vector called "deformation vector" and locally measuring the deviation of the Minkowski space from a Euclidean space. With this covariant differentiation it is possible to declare an "M-geodesicc parallelity" and to show that the vector field of the generators of a skew ruled surface Φ is an M-geodesic parallel field along its Minkowski striction curve s. Finally we also define the Pirondini set of ruled surfaces to a given surface Φ. The surfaces of such a set have the M-striction curve and the strip of M-central planes in common.

Keywords: Ruled surfaces, spherical image, Kruppa's differential invariants, Kruppa-Sannia moving frame, striction curve; Minkowski space, Birkhoff orthogonality, semi-inner product, cosine- and sine-Minkowski function; M-moving frame, Frenet-Serret formulae, Minkowski curvature, Minkowski torsion, vector field, tangential vector field, directional derivative, covariant differentiation, deformation vector, second fundamental form, Gauss's equation, M-geodesic parallel field, Bonnet's theorem, Pirondini theorem.

Table of Contents

	Acknowledgements	i
	Abstract	ii
1	**Introduction**	1
1.1	Minkowski space	1
1.2	Birkhoff orthogonality	2
1.3	Inner product space	2
1.4	The aim of the dissertation	3
1.5	Organization of this dissertation	4
2	**Orthogonality in normed linear space**	6
2.1	Introduction	6
2.2	Properties of orthogonality in normed linear spaces	8
2.3	Relations between Birkhoff and Isosceles orthogonality	9
2.4	Relations between Birkhoff and 2-norm (Diminnie) orthogonality	11
2.5	Area orthogonality in normed linear space	14
2.6	Birkhoff orthogonality in Minkowski space	16
3	**Support theorem in Minkowski space**	18
3.1	Introduction	18
3.2	Dual space	18
3.3	Support function in Minkowski space	19
3.4	Volume and Mixed volume in Minkowski space	22
3.5	The isoperimetric problem in a Minkowski plane M_B^2	24
3.6	Transversality in Minkowski plane M_B^2	28
3.7	Radon plane	29
3.8	The isoperimetric problem in a higher dimensional Minkowski space M_B^n	30
4	**Trigonometry and semi-inner product in Minkowski space**	34
4.1	Introduction	34
4.2	Cosine function	34
4.3	Sine function	38
4.4	Trigonometric formulae	42
5	**Ruled surfaces in Minkowski Three-dimensional space**	48
5.1	Introduction	48
5.2	Ruled surfaces and frame construction of Minkowski 3-space	50
5.3	Striction curve in Minkowski 3-space	51
5.4	The Deformation vectors in Minkowski plane M_B^2	54
5.5	Frenet-Serret frame in Minkowski 3-space	56
6	**Geodesics in Minkowski space**	63
6.1	Introduction	63
6.2	The covariant derivative in Minkowski space	63
6.3	Parallel field in Minkowski space	66

6.4 Further theorems on ruled surfaces 68
6.4.1 Bonnet's theorem 68
6.4.2 Pirondini's theorem 69
6.4.3 Conoidal surfaces, Conoids 70
6.5 Conclusion 71

References 72
List of figures 76
Index 77
Curriculum Vitae 80
Summary 81
Erklärung 88

Chapter 1

Introduction

Minkowski geometry is the geometry of a finite dimensional (affine-) linear space possessing a norm. Usually this norm is based on a centrally symmetric convex set of this space used as unit ball (gauge ball) B. The standard book of Thompson [1] as well as the survey articles [2] and [3] cover many basic and advanced results of this geometry. The concept of orthogonality in such a Minkowski space is different from the Euclidean one and it makes sense in Minkowski spaces with a strictly convex and smooth gauge ball B. In 1934 Roberts [4] defined an orthogonality in normed spaces for the first time. After that many authors have studied other possibilities to define an orthogonality in Minkowski spaces; most of them are non-symmetric relations in contrary to Euclidean orthogonality, see e.g. Birkhoff [5], James [6-8] and Day [9]. We will focus on Birkhoff's non-symmetric "B-orthogonality concept", as its definition is a very geometric one: The supporting plane of the unit ball B at a point x, contains the lines y being "left-orthogonal" to vector x; (and then x is right-orthogonal to y). In this introductory Chapter 1 and, even more detailed, in Chapter 2 we repeat the main properties of a Minkowski space and its B-orthogonality as well as its relations to other orthogonality concepts, thereby following Thompson[1] and Alonso [10-16].

The other central topic we have to introduce here is the *differential geometry of ruled surfaces*. We will consider ruled surfaces, which are not developable. They are called "skew ruled surfaces". For such surfaces in a Euclidean 3-space Kruppa [17] and Sannia [18] have developed a differential geometric treatment by generalising the classical differential geometry of curves. This leads to a "main theorem of ruled surfaces": Given three functions (instead of two) of an arc length parameter s of a Euclidean distinguished curve c (the "striction curve"), then, disregarding positioning in space, there is exactly one surface having the given functions as Kruppa-Sannia-functions.

The aim of the dissertation is to study ruled surfaces in Minkowski spaces and coming as close as possible to an analogue of the above mentioned main theorem.

1.1 Minkowski space

In this section we will give some basic concepts related to Minkowski space which are essential for our work. Let B be a centrally symmetric, convex body in an affine three dimensional space E^3, then we can define a norm whose unit ball is B. Such a space is called Minkowski normed space. In our work we consider only Minkowski spaces with a strictly convex and smooth unit ball B. On one hand we can define the norm by using the parametric representation of the unit ball B and on the other we can define the norm by using the support function of the unit ball B at all points $x \in B$.

Furthermore, since the Euclidean 3-space is a special case of a Minkowski 3-space, we always would like to know to what extent its properties may remain valid also in a general Minkowski 3-space.

Definition 1.1: A Minkowski space M^n is a real linear space of finite dimension n and endowed with a norm $\|\cdot\|$, that is a functional such that the following properties hold for any elements x and y of the respective linear space:

- $\|x\| \geq 0 \ \forall x \in M^n$,
- $\|x\| = 0$ if and only if $x = 0$,
- $\|\lambda x\| = |\lambda| \|x\| \ \ \forall \lambda \in \mathbb{R}$,
- $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality).

1.2 Birkhoff orthogonality

In this dissertation will prefer a concept of Minkowski orthogonality due to Birkhoff [5], as it is most naturally related to the geometry of the gauge Ball B of the Minkowski space and its construction is the same as for the Euclidean case. But it is, in general, no longer a symmetric relation between linear subspaces of the Minkowski space. Explicitly this construction of "left-orthogonality" reads as follows, see Fig. 1.1: The supporting plane of the unit ball B at a point x, contains all the lines y being "left-orthogonal" to vector x; (and then x is right-orthogonal to y). We use the symbols $y \dashv x$ for y left-orthogonal x resp. $y \mathbb{Z} x$ for y right-orthogonal x.

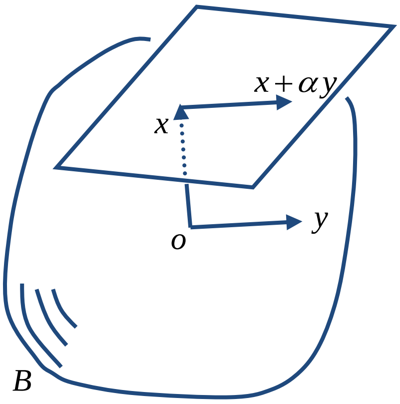

Figure 1.1

1.3 Inner product space

A (real) inner product space X is a special normed linear space with the additional structure of an inner product $\langle \cdot, \cdot \rangle$ which, for any x, y and $z \in X$, satisfies the conditions

- $\langle x,y\rangle = \langle y,x\rangle$,

- $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle \forall \alpha \in \mathbb{R}$,

- $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$,

- $\langle x,x\rangle \geq 0$, with equality if and only if $x=0$,

- $\langle x,x\rangle = \|x\|^2$.

We know that a normed linear space is not necessarily an inner product space. Therefore a real normed linear space is an inner product space if and only if each two-dimension linear subspace of it is also an inner product space. Equivalent to this we can state (see [1]) that a normed linear space is an inner product space if and only if every plane section of the unit ball B is an ellipse. For $n \geq 3$ this means that B is an ellipsoid and the Minkowski space is Euclidean.

As we cannot start with an inner product having the properties above, we need to find a so-called "semi-inner product", which is compatible with the B-orthogonality concept and the (non-Euclidean) Minkowski norm. It is known – and we will repeat this in Chapter 2 - that in each real normed linear space $(X, \|\cdot\|)$ there exists at least one semi-inner product $[\cdot,\cdot]$ which generates the norm $\|\cdot\|$, that is, $\|x\| = [x,x]^{1/2}$ for all $x \in X$, and it is unique if and only if X is smooth, see Chmielinski [19]. We shall define a semi-inner-product based on the so-called *cosine-Minkowski function* [1] and together with a sine-Minkowski function this will allow us to calculate coefficients of derivative equations of type Frenet-Serret also in a Minkowski 3-space M_B^3, in spite there is no motion group acting there.

1.4 The aim of the dissertation

Ruled surfaces can be seen as (continuous) one-parameter sets of lines in the Projective or Affine or Euclidean Line Space, (see e.g. Hlavaty [20]). But they can also be seen as two-dimensional surfaces in a Projective or Affine or Euclidean Point Space, thus having a set of straight asymptotic lines, namely their rulings, (see e.g. Kruppa [17] or Hoschek [21]). Furthermore, some characteristic properties of a ruled surface in E^3, related to the geodesic curvature and the second fundamental form of it, are given by A. Sarıoğlugil [22].

The main task of this dissertation is to consider ruled surfaces in a Minkowski three dimensional space. It turns out that it is necessary to assume that the unit ball B of this space is centrally symmetric, smooth and strictly convex. That means, the boundary ∂B contains no line segment. Analogue to the Euclidean case we will construct a B-orthonormal frame in a Minkowski three dimensional space M_B^3. This frame is based on a given oriented flag (P, g, α) of incident half-space, namely point P, half-line g and half-plane α. After some steps we get a right handed (affine) frame based on B-orthogonality.

Especially for (neither cylindrical nor conical) ruled surfaces $\Phi = \{g(t), t \in I \subset \mathbb{R}\}$ there is a canonically defined flag connected with each (oriented) generator $g(t_\circ)$. It consists of g itself, the asymptotic plane α parallel to direction vectors $g(t_\circ)$ and its derivative $\dot{g}(t_\circ)$. As the point P of the flag we use the point of contact *s* of the so-called central plane $\zeta = g \vee n_M$ with Φ. Thereby n_M denotes the line left-orthogonal to α**.** It is constructed as follows: The support plane parallel to α touches *B* in a point *Z* and unit direction vector of n_M connects the origin with this point *Z*. We just have to take care by choosing the "right" support plane of two that { $g(t_\circ)$, $\dot{g}(t_\circ)$,*OZ*}, form a right handed system.

The touching point $s(t_0)$ of $\zeta(t_0)$ with Φ is the Minkowski analogue to the Euclidean striction point of a ruling $g(t_\circ)$ and obviously it has to be called "Minkowski-striction point" of generator $g(t_\circ)$. All those points $s(t)$ form a curve S_M , the Minkowski striction curve of Φ. Along this curve we consider a moving frame consisting of the unit direction vector ***g*** of generator $g(t_\circ)$, the Minkowski central normal ***n*** (parallel *OZ*) and the Minkowski central tangent vector ***t***, which is parallel to the tangent of the M-spherical image of the ruled surface Φ at $g(t_0) \cap B$

Furthermore, using the above mentioned sine- and cosine-Minkowski functions it becomes possible to calculate the coefficients of the Frenet-Serret derivation equations of the moving frame. Thereby these coefficients, (describing a special affine transformation of the frame) can be called M-curvatures and M-torsions. It turns out that there are 4 such M-curvatures and M-torsions. Specialising *B* to an ellipsoid these M-curvatures and M-torsions tend to the usual Euclidean curvature and torsion of a ruled surface, such that the presented treatment comprises the Euclidean case, too.

Finally we adapt the classical concepts of geodesics and geodesic parallelity in Minkowski spaces, using the Gauss's equation in Minkowski space and an adaption of a covariant differentiation process, confer also [23]. For this covariant differentiation we use the (local Minkowski normal projection into the tangent planes. This M-normal projection is describes as a linear combination of the usual Euclidean normal projection and a "deviation" defined by a deviation vector measuring the deviation of the Minkowski space from a Euclidean one. We redefine the concept of a geodesically parallel field Y along a curve $c(t)$ on the surface in Minkowski 3-space. We give the fundamental condition of a curve to be geodesic using the covariant differentiation of its tangential vector fields and can prove the main result of the dissertation, that (like in the Euclidean case) the M-striction curve is distinguished among all curves of Φ by the property that the generators form a M-geodesic parallel field along this curve.

1.5 Organization of this dissertation

This dissertation in six chapters as follows. The Chapter 2 after this introductory Chapter 1 contains a survey over the orthogonality concepts of Minkowski spaces and their properties with concentration on the Birkhoff orthogonality and its relations with other orthogonalities. In Chapter 3 we collect the main ideas about the supporting theory in Minkowski spaces which will play a basic role in the following calculations. Also here we restrict ourself to Minkowski spaces with a strictly convex and smooth gauge ball *B*. In Chapter 4 we introduce the cosine and sine functions in Minkowski spaces and use them to define the unique semi-inner product in Minkowski space.

In Chapter 5 we present new results, like the Frenet-Serret derivation equations for ruled surfaces in a Minkowski space and derive generalized curvature and torsion concepts.

Finally, in Chapter 6 we focus on the concept of covariant differentiation in Minkowski spaces and the concept of geodesic parallel vector fields along a curve. Here we present also the main result about the M-striction curve formulated already at the end of 1.4. Furthermore we give a modification of the second fundamental form of a ruled surface in a Minkowski space.

There is no theorem for ruled surfaces in a (non-Euclidean) Minkowski 3-space corresponding to *Bonnet's theorem* just by simple modification of it. That means that if the curve is M-striction and M-geodesic it does not follow that it is also an isogonal trajectory of the generators. Constance of the striction angle would involve Minkowski angle measurement aside orthogonality and also for this there exist many different approaches. But for the (Euclidean) *theorem of Pirondini* considering the set of ruled surfaces with common striction strip (i.e. the striction curve plus the set of central planes) it is possible to formulate also a version in Minkowski spaces M_B^3.

Chapter 2

Orthogonality in Normed Linear Spaces

2.1. Introduction

One of the most fundamental ideas which play a basic role in Euclidean geometry is that of orthogonality. In Euclidean space a tangent to a unit sphere is perpendicular to the radius that joins the centre to the point of tangency. The concept of orthogonality in normed spaces has been studied in different ways because the unit sphere, in general, has shapes different from an ellipsoid, which defines the Euclidean space.

Orthogonality of two vectors $x \neq 0$ and $y \neq 0$ in a normed linear space $\left(X, \|\cdot\|\right)$ has been defined by various authors:

- **Roberts** [4] in (1934) has defined the orthogonality in normed space as follows:

$$x \perp_R y \Leftrightarrow \|x - \lambda y\| = \|x + \lambda y\| \ \ \forall \lambda \in \mathbb{R} \tag{2.1}$$

- **Birkhoff** [5] in (1935), **James** [6-8] and **Day** [9] from (1945-1947) from a very geometric point of view were lead to a non-symmetric orthogonality and introduced left and right orthogonality as:

$$x \dashv y \Leftrightarrow \|x\| \leq \|x + \lambda y\| \ \ \forall \lambda \in \mathbb{R} \qquad \text{(left-orthogonality)} \tag{2.2}$$

  Geometrically, this means that $x \dashv y$ if and only if the line $x + \lambda y$ supports the unit ball B at x, see Figure 2.1. The Hahn-Banach theorem then implies that $x + \lambda y$ lies in a hyperplane which supports B at x. Obviously $\dashv$ is not a symmetric relation, i.e. if $x \dashv y$, it is not means that $y \dashv x$. In fact, for dimensions three or above, the only normed spaces for which normality is symmetric are the Euclidean spaces. In dimension two, normality is symmetric for the wide class of Radon plans [24].

- **Carlsson** [25] in (1962) has studied the C-orthogonality

$$x \perp_C y \Leftrightarrow \sum_{k=1}^{m} a_k \|b_k x + c_k y\|^2 = 0, \text{ where } m \geq 2 \text{ and } a_k, b_k, c_k \in \mathbb{R} \text{ are such that}$$

$$\sum_{k=1}^{m} a_k b_k c_k \neq 0, \ \sum_{k=1}^{m} a_k b_k^2 = \sum_{k=1}^{m} a_k c_k^2 = 0 \tag{2.3}$$

Obviously C-orthogonality is not a singular concept, but a family of them. Before Carlsson's work, the following cases of such family have been considered as:

- **James** [6] in (1945): $\|x-y\|=\|x+y\| \ (x\perp_I y)$, (isosceles normal) (2.4)

- **Pythagorean** [6] in (1945): $\|x-y\|^2=\|x\|^2+\|y\|^2 \ (x\perp_P y)$ (2.5)

  Both of them are introduced by James.

- **Singer** [26] in (1957): $\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|=\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\| \ (x\perp_S y)$ (2.6)

- **Miliĉić** [27] used a functional g and introduced the following orthogonality relations:

$$\left.\begin{aligned} &x\perp_g y \Leftrightarrow g(x,y)=0,\\ &x\overset{g}{\perp} y \Leftrightarrow g(x,y)+g(y,x)=0,\\ &x\underset{g}{\perp} y \Leftrightarrow \|x\|^2 g(x,y)+\|y\|^2 g(y,x)=0 \end{aligned}\right\} \qquad (2.7)$$

where,

$$g(x,y):=\|x\|\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}, \ x,y\in X \qquad (2.8)$$

For more details on B-orthogonality, g-orthogonality and more, see J. Alonso [10-16] and L. Zheng [28].

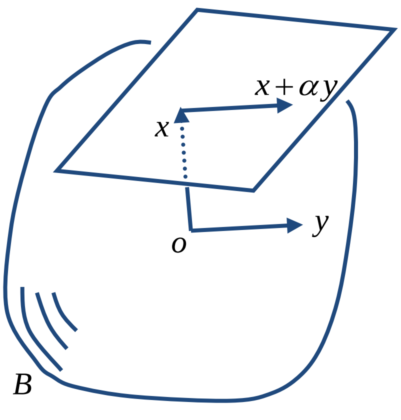

Figure 2.1

2.2. Properties of orthogonality in normed linear spaces:

In an inner product space $\left(X,\langle\cdot,\cdot\rangle\right)$, $x\perp_P$, $x\perp_I y$ and $x \dashv y$ are all equivalent to the condition $\langle x,y\rangle=0$, for which we have the usual orthogonality $x\perp y$. The inner product space is always a normed linear space with the inner product norm $\|f\|=\sqrt{\langle f,f\rangle}$. However, a normed linear space is not necessarily an inner product space, one of the pervious orthogonalities does not imply another in general.

Therefore a real normed linear space is an inner product space if and only if every two-dimension linear subspace of it is also inner product space. Therefore we can say that a normed linear space is an inner product space if and only if every plane section of the unit sphere is an ellipse.

The following collects some of the main properties of orthogonality in inner product spaces; see Alonso [15-16],

1. $\lambda x\perp \mu y$ iff either $\lambda x=0$ or $\mu y=0$ $\forall\lambda,\mu\in\mathbb{R}$ (Nondegeneracy).
2. If $x\perp y$ then $\lambda x\perp\mu y$ $\forall\lambda,\mu\in\mathbb{R}$ (Homogeneity).
3. If $x\perp y$ then $y\perp x$ (symmetry).
4. If $x\perp y$ and $x\perp z$ implies $x\perp y+z$ $\forall x,y,z\in X$ (Additivity).
5. For every $x,y\in X$, $x\neq 0$ there exist a number α such that $x\perp \alpha x+y$.

Theorem 2.1: Let X be a smooth uniformly convex normed space, and $x,y\in X-\{0\}$ be fixed linearly independent vectors. Then

1- If $z\in span\{x,y\}$ and $(y\dashv x)\wedge(z\dashv x)$, then there exists $\lambda\in\mathbb{R}$ such that $z=\lambda y$.

2- If $(z\dashv y-\alpha x)\wedge(x\dashv y-\beta x)$ then $\alpha=\beta$.

Proof: see Miliĉić [27].

Definition 2.2: Let X be a real linear space, a mapping $[\cdot,\cdot]: X\times X\to\mathbb{R}$ is called a semi-inner product on X if it satisfies the following conditions:

(i) $[x,x]\geq 0$ for all $x\in X$, and $[x,x]=0\Rightarrow x=0.$

(ii) $[z,ax+by]=a[z,x]+b[z,y]$ for all $a,b\in\mathbb{R}$ and all $x,y,z\in X$,

(iii) $[ax, y] = a[x, y]$ for all $a \in \mathbb{R}$ and $x, y \in X$,

(iv) $\left|[x, y]\right|^2 \leq [x, x][y, y]$ for all $x, y \in X$.

It is easy to see that if $[\cdot,\cdot]$ is a semi-inner product on X, then the mapping $\|\cdot\| : x \mapsto [x, x]^{1/2}$ for all $x \in X$ is a norm on X. We know that in each real normed linear space $(X, \|\cdot\|)$ there exist at least one semi-inner product $[\cdot,\cdot]$ which generate the norm $\|\cdot\|$, that is, $\|x\| = [x, x]^{1/2}$ for all $x \in X$, and it is unique if and only if X is smooth see J. Chmielinski [19].

Definition 2.3: Let $[\cdot,\cdot]$ be a semi-inner product which generates the norm of any vector $x \in X$, and let $x, y \in X$. The vector $x \in X$ is said to be orthogonal to $y \in X$ in the sense of Lumer [29] relative to the semi-inner product $[\cdot,\cdot]$ if $[x, y] = 0$. We can denote this orthogonality by $x \perp_L y$.

Proposition 2.4: Let $(X, \|\cdot\|)$ be a real normed space, and $x, y \in X$. Then $x \dashv y$ if and only if $x \perp_L y$ relative to some semi-inner product $[\cdot,\cdot]$ which generate the norm $\|\cdot\|$. The proof of this proposition can be found in S. S. Dragomir [30].

2.3 Relations between Birkhoff and Isosceles orthogonality

To understand the main difference between B-orthogonality and I-orthogonality, we introduce the constant $D(X)$ for all real normed linear space X according to J. Donghai and Wu. Senlin [31],

$$D(X) = \inf\left\{\inf_{\lambda\in\mathbb{R}} \|x + \lambda y\| : x, y \in S, x \perp_I y\right\}, \tag{2.9}$$

where S is the unit sphere of X.

Theorem 2.4: (Lower and upper bound of $D(X)$), For any real normed linear space X with dimension two or more then,

$$2\left(\sqrt{2} - 1\right) \leq D(X) \leq 1. \tag{2.10}$$

The proof is given in [31], where the case of $D(X) = 1$ is valid if and only if X is Euclidean and so B-orthogonality and I-orthogonality implies each others.

Some following examples may be useful to understand the constant $D(X)$ in Minkowski plane space.

Example 2.5: Let the Minkowski plane X be $\left(\mathbb{R}^2,\|\cdot\|_p\right)$, $1\le p<\infty$, be the symmetric $X=l_p^2$, then the constant $D(X)$ is given by the following formula,

$$D(X)=\inf\left\{\frac{\left(1+t^2\right)}{\left(1+t^q\right)^{1/q}\left(1+t^p\right)^{1/p}};t\in[0,1]\right\}. \tag{2.11}$$

Also,
$\lim_{p\to\infty} D\left(l_p^2\right)=2\left(\sqrt{2}-1\right)$ and $D\left(l_p^2\right)=D\left(l_q^2\right)$ where q is the real number such that $\frac{1}{p}+\frac{1}{q}=1$.

Proof: Let (a,b) be on the unit circle of l_p^2 then,

$$|a|^p+|b|^p=1 \tag{2.12}$$

If $x,y\in\mathbb{R}^2$ and $x=(a,b)$, $x\perp_I y$ then $y=(-b,a)$ because the space l_p^2 is symmetric plane see A. C. Thompson [1] and hence, we can assume the functions

$$f(\lambda)=\|x+\lambda y\|^p=|a-b\lambda|^p+|a+b\lambda|^p,$$

and

$$g(\lambda)=(a-b\lambda)^p+(a+b\lambda)^p.$$

Then $g(\lambda)$ intersects $f(\lambda)$ within the closed interval $\left[-\frac{b}{a},\frac{a}{b}\right]$, we can find the derivative of the function $g(\lambda)$ as $g'(\lambda)=p\left(-b(a-b\lambda)^{p-1}+a(b+a\lambda)^{p-1}\right)$.

Without loss of generality, we can consider that $a,b\ge 0$ and $a\ge b$ and let $g'(\lambda)=0$, we have $\lambda=\frac{a-b\gamma}{a\gamma+b}$ where $\gamma=\left(\frac{a}{b}\right)^{1/(p-1)}$. Obviously $\gamma\le\frac{a}{b}$ and by the following inequality

$$\lambda-\left(-\frac{b}{a}\right)=\frac{a^2+b^2}{a(a\gamma-b)}\ge 0, \tag{2.13}$$

we have $\lambda \in \left[-\frac{b}{a}, \frac{a}{b}\right]$. Thus $g(\lambda)$ attains its minimum on the interval $\left[-\frac{b}{a}, \frac{a}{b}\right]$, and then $f(\lambda)$ attains its minimum on the same interval.

Let $t = \frac{a}{b}$ and $\lambda_o = \frac{t - t^{q-1}}{1+t^q}$, where q is the real number such that $\frac{1}{p} + \frac{1}{q} = 1$ valid. Then we have $g'(\lambda_o) = 0$ and

$$\left.\begin{aligned} g(\lambda_o) &= \left((a - b\lambda_o)^p + (b + a\lambda)^p\right) \\ &= b^p\left((t - \lambda_o)^p + (1 + t\lambda_o)^p\right) \end{aligned}\right\} \tag{2.14}$$

By using some calculations, we can easily find that

$$g(\lambda_o) = \frac{(1+t^2)^p}{(1+t^q)^{p-1}(1+t^q)}. \tag{2.15}$$

Thus

$$D(l_p^2) = \inf\left\{\frac{(1+t^2)}{(1+t^q)^{1/q}(1+t^p)^{1/p}}; t \in [0,1]\right\}. \quad \square$$

From symmetry of last equation, we have $D(l_p^2) = D(l_q^2)$ and from theorem (2.4) and equation (2.11) also, $\lim_{p\to\infty} D(l_p^2) = 2(\sqrt{2} - 1)$. For more examples see [31].

2.4 Relations between Birkhoff and 2-norm (Diminnie) orthogonality

To formulate the concept of 2-norm orthogonality and its relations, we are using a concept introduced by C. R. Diminnie [32] in 1983, this orthogonality relations of normed linear spaces are based on a concept of the area of a parallelogram, formulated by E. Silverman [33-35]. Finally Diminnie compared it with B-orthogonality.

Definition 2.5: Let F denote the set of linear functions defined on the normed linear space $(X, \|\cdot\|)$ whose norms are less than or equal to 1. Then the ```2-norm`` is defined as follows

$$\|x, y\| = \sup\left\{\begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix} : f, g \in F \text{ and } x, y \in X\right\}. \tag{2.16}$$

Where $\|x,y\|$ may be visualized as the area of the parallelogram with vertices at $0,\ x,\ y,\ x+y,$ this quantity was used by F. Sullivan [36] to obtain the convexity properties of normed linear spaces.

Lemma 2.6: The 2-norm $\|\cdot,\cdot\|$ has the following properties:

a. $\|x,y\|\geq 0$ and $\|x,y\|=0$ if and only if x and y are dependent .

b. $\|x,y\|=\|y,x\|$.

c. $\|\alpha x,y\|=|\alpha|\|x,y\|$ for all $\alpha\in\mathbb{R}$.

d. $\|x+y,z\|\leq\|x,z\|+\|y,z\|$.

e. $\|x,\alpha x+y\|=\|x,y\|$ for all $\alpha\in\mathbb{R}$.

f. If $(X,\|\cdot\|)$ is an inner-product space then, $\|x,y\|^2=\|x\|^2\cdot\|y\|^2-\langle x,y\rangle^2$.

Definition 2.7: Let $(X,\|\cdot\|)$ be a normed linear space, then we say $x\in X$ is 2-norm orthogonal to $y\in X$ with the notation $x\perp_2 y$ if and only if $\|x,y\|=\|x\|\cdot\|y\|$, see [32].

Lemma 2.8: $x\dashv y$ if and only if there is a functional $f\in F$ such that $f(x)=\|x\|$ and $f(y)=0$ i.e. $y\in f^{\perp}$, where $f^{\perp}$ is the kernel functional of f which belongs to the dual space X^*.

Lemma 2.9: If $x\dashv y$ then $\|x,y\|\geq\|x\|\cdot\|y\|$.

Proof: If $x\dashv y$, then by Lemma 2.8 and Equation (2.16), we have

$$\|x,y\|\geq\|x\|g(y)\ \text{for all}\ g\in F \tag{2.17}$$

From Hahn-Banach theorem, see A. C. Thompson [1] corollary (1.3.4 p34), $g(y)\geq\|y\|$, then the proof is completed. □

Theorem 2.10: If $x,y\in X$, $b\in\mathbb{R}$ and $x\dashv(bx+y)$, then there are real numbers a_1 and a_2 such that $a_1\leq b\leq a_2$, $x\perp_2(a_1x+y)$, and $x\perp_2(a_2x+y)$. In particular, for all $x,y\in X$, there is a real number a such that $x\perp_2(ax+y)$. Proof with details may be found in [32].

Remark: In general, the number a of theorem 2.10 need not be unique and if it is unique, $\perp_2$ is unique.

Lemma 2.11: If $\perp_2$ is additive, then it is unique.

Proof: Let $x, y \in X$ with $x \neq 0$ and suppose that $x \perp_2 (a_1 x + y)$ and $x \perp_2 (a_2 x + y)$. Since, $\perp_2$ is additive by assumption. Then $x \perp_2 (a_1 - a_2) x \Rightarrow a_1 = a_2$. □

Lemma 2.12: If $\|x, y\| \leq \|x\| \cdot \|y\|$ for all $x, y \in X$, then $\perp_2$ and B-orthogonality are equivalent.

Proof: Assume that $x, y \neq 0$ and $x \perp_2 y$ then by Definition 2.7, we have $\|x, y\| = \|x\| \cdot \|y\|$.

By using the assumption and replacing x by $x + ay$, we can find that,

$\|x + ay\| \cdot \|y\| \geq \|x + ay, y\| = \|x, y\| = \|x\| \cdot \|y\|$, which means that $\|x + ay\| \geq \|x\|$ for all $a \in \mathbb{R}$ and hence $x \dashv y$.

Conversely, if $x \dashv y$ then from Lemma 2.9, we have $\|x, y\| \geq \|x\| \cdot \|y\|$ then $\|x, y\| = \|x\| \cdot \|y\|$ $\Rightarrow x \perp_2 y$. □

Theorem 2.13: Let $(X, \|\cdot\|)$ be a normed space of dimension three or higher. Then, the following statements are equivalent.

1. $(X, \|\cdot\|)$ is an inner-product space.
2. B-orthogonality is symmetric.
3. B-orthogonality and 2-norm orthogonality are equivalent.
4. $\perp_2$ is unique.
5. $\perp_2$ is additive.

Theorem 2.14: $\perp_2$ is additive if and only if there exists no pair of vectors $x, y \in X$ such that

$$\|x, y\| = \|x - y\| \text{ and } \|x\| = \|y\| = 1. \tag{2.18}$$

Proof: see [32]

Definition 2.15: A normed space $\left(X,\|\cdot\|\right)$ is strictly convex if its convex unit sphere S has no line segments.

Lemma 2.16: If $\left(X,\|\cdot\|\right)$ is strictly convex and $\|x,y\|\le\|x\|\cdot\|y\|$ for all $x,y\in X$, then $\perp_2$ is additive.

Proof: Assume the converse that $\perp_2$ is additive, then there exist $x,y\in X$ such that (2.18) is valid. Let $z=tx+(1-t)y$, $0\le t\le 1$ then $z\in\overrightarrow{xy}$ and $\|z\|\le 1$.

Since,

$$\begin{aligned}\|z\|\cdot\|x-y\| &\ge \|z,x-y\|\\ &=\|y,x-y\|\\ &=\|x,y\|\\ &=\|x-y\|\ge\|z\|\cdot\|x-y\|\end{aligned}$$

and hence $\quad \|z\|\cdot\|x-y\|=\|x-y\|\Rightarrow\|z\|=1.$

This means that $z\in S$, which is a contradiction to the assumption that the space $\left(X,\|\cdot\|\right)$ is strictly convex, i.e. $\perp_2$ is additive. □

2.5 Area orthogonality in normed linear space

In his PhD-Thesis [10] J. Alonso in 1984 introduced the concept of area orthogonality, which satisfies all orthogonality relations in an inner product space (see [15, 16]) except the additive relation, in case the space X has dimension three or higher. The area orthogonality has been discussed in details by D. Amir [37] and B. Boussouis [38].

In this section, we will discuss some relations of area orthogonality with some of the above mentioned orthogonalities. Thereby we focus on relations with B-orthogonality, which in the following will be used to construct the geometrical frame in Minkowski three dimension spaces.

Definition 2.17: Let E be a real normed linear space, then $x\in E$ is area orthogonal to $y\in E$ with the notation $x\perp_A y$, if either $\|x\|\|y\|=0$, or x and y are linearly independent and such that the straight lines spanned by them divided the unit ball of the plane (x,y) (identified with $\mathbb{R}^2$) into four parts of equal area.

If we assume that $\tilde{S}$ be the unit sphere of the plane spanned by x and y (the intersection of oxy plane and the unit sphere S) is convex curve and it can be parameterized by the following mapping

$$\tilde{s}:\theta\in[0,2\pi]\to\tilde{s}(\theta)=(s_1(\theta),s_2(\theta))\in\tilde{S}, \tag{2.19}$$

where $\tilde{s}(\theta)$ is the point of $\tilde{S}$ and $\tilde{s}(0)\in\tilde{S}$ measured with the orientation of the plane. Thereby $\tilde{S}$ is a convex curve, then s_1 and s_2 are continuous functions of bounded variation with $0\le\theta_1<\theta_2\le 2\pi$ and $x=\|x\|\tilde{s}(\theta_1)$, $y=\|y\|\tilde{s}(\theta_2)$ are Area orthogonal if and only if

$$\int_{\theta_1}^{\theta_2}\left(s_1(\theta)ds_2(\theta)-s_2(\theta)ds_1(\theta)\right)=\frac{1}{4}\int_0^{2\pi}\left(s_1(\theta)ds_2(\theta)-s_2(\theta)ds_1(\theta)\right). \tag{2.20}$$

This relation is equivalent with the usual orthogonality when the space E is an inner product space and that it is easy to see that it is nondegenerate, continuous, homogenous and symmetric in every real normed linear space.

Proposition 2.18: For every $x,y\in E,\ x\neq 0$, there exists a unique $\alpha\in\mathbb{R}$ such that $x\perp_A(\alpha x+y)$.

Proof: Geometrically, since A-orthogonality is homogenous, then for every homogenous plane $L\subset E$ and every $x\in S\cap E$, there exists a unique $z\in S\cap E$, such that $x\perp_A z$.

Analytically, we insert the continuous and strictly monotony function $f(\lambda)$ as follows

$$f(\lambda)=\int_0^{\lambda}\left(s_1(\theta)ds_2(\theta)-s_2(\theta)ds_1(\theta)\right)-\int_{\lambda}^{\pi}\left(s_1(\theta)ds_2(\theta)-s_2(\theta)ds_1(\theta)\right), \tag{2.21}$$

for $0\le\lambda\le\pi$, where $\tilde{s}(\theta)$ is a parameterization of $S\cap E$ with $x=\tilde{s}(0)$. And hence, we can find the parameter λ_1 of the element $z\in S\cap E$, such that $f(\lambda_1)=0$, $z=\tilde{s}(\lambda_1)$ and $x\perp_A z$.

Without lose of generality, we take also any element $y\in S\cap E$, such that $z=\alpha x+y$, $\alpha\in\mathbb{R}$. □

Proposition 2.19: (Existence and uniqueness of diagonals), For every $x,y\in E\setminus\{0\}$ there exists a unique $\delta>0$ such that $x+\delta y\perp_A x-\delta y$.

Proof: see [14].

From the previous proposition, A-orthogonality is a binary relation with the property of existence of diagonals, and then the parallelogram law for pair of A-orthogonal points is a sufficient condition for the norm of the space to be induced by an inner product space. So, we have the following corollary which gives the characterization of an inner product space by the meaning of A-rectangle inequality.

Corollary 2.20: A real normed linear space E is an inner product space if and only if the following condition are satisfied,

$$x, y \in E,\ x \perp_A y \Rightarrow \|x+y\|^2 + \|x-y\|^2 \sim 2\left(\|x\|^2 + \|y\|^2\right), \tag{2.22}$$

where $\sim$ is one of the signs $\leq$ or $\geq$.

Proposition 2.21: Let E be a real normed linear space. The following properties are equivalent:

(i) E is an inner product space.

(ii) If $x, y \in E$ and $x \dashv y$ implies $x \perp_A y$.

(iii) If $x, y \in E$ and $x \perp_A y$ implies $x \dashv y$.

Proof: see [14].

2.6 Birkhoff orthogonality in Minkowski spaces

In this section, we will insert some theorems to define and discuss the existence of the Birkhoff-orthonormal basis in Minkowski spaces with respect to its unit ball B. Taylor's theorem [39] is one of the basic ideas of considering circumscribed parallelotopes to the unit ball B of the Minkowski d-dimension space X. In chapter 5, we will construct the more general orthogonal frame by using left and right orthogonality in Minkowski 3-dimensional space with respect to some special surfaces (ruled surfaces) to obtain the main idea of this dissertation.

Theorem 2.22 (*Taylor*): Let A be a closed, bounded set in $\mathbb{R}^d$ that spans $\mathbb{R}^d$. Then there exist points $x_1, x_2, ..., x_d$ in A and hyperplanes $H_1, H_2, ..., H_d$ such that

(1) For each i, $x_i \in H_i$,

(2) For each i, H_i is parallel to the span of $\{x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_d\}$,

(3) For each i, H_i supports A at x_i.

Corollary 2.23: If B is the unit ball in Minkowski space X then there exists a basis $(x_1, x_2, ..., x_d)$ such that $\|x_i\| = 1$ and $x_i \dashv x_j$ for all i and j with $i \neq j$, i.e. each pair of these basis vectors is mutually orthogonal $\left(x_i \dashv x_j \Leftrightarrow x_j \dashv x_i \forall i \neq j\right)$.

Proof: Let $(x_1, x_2, ..., x_d)$ be an ordered set given by theorem 2.22, we can take $A = B$. Since H_i supports B at x_i, we have $\|x_i\| = 1$ and hence, $x_i + x_j \in H_i$, we have $x_i \dashv x_j$ for all $i \neq j$.

Corollary 2.24: If B and B° are the unit balls in a Minkowski space X and its dual X^*, see chapter 3, then there exist bases $(x_1, x_2, ..., x_d)$ in X and $(f_1, f_2, ..., f_d)$ in X^* such that $\|x_i\| = \|f_i\| = 1$ for all i and $f_i(x_j) = \delta_{ij}$ for all i, j, where $\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j, \\ 1 \text{ if } i = j \end{cases}$ is the Dirac delta function.

Proof: Let $(x_1, x_2, ..., x_d)$ be a basis for X as in Corollary 2.18. Then define f_i on X by $f_i\left(\sum_{j=1}^{d} \alpha_j x_j\right) := \alpha_i$. Then $\|x_i\| = 1$ and $f_i(x_j) = \delta_{ij}$. Also, $\|f_i\| = 1$ follows from the definition of the hyperplane $H_i = \{x : f_i(x) = 1\}$ which support B at x_i.

Corollary 2.25: If B is the unit ball in a Minkowski space, then there is a parallelotope C with 2^d vertices at $\{\pm x_1, \pm x_2, ..., \pm x_d\}$ such that $B \subseteq C$ and each face of C supports B at the centre of the face.

To prove Theorem 2.22 and Corollary 2.25, see [1].

Chapter 3

Support theorems in Minkowski spaces

3.1 Introduction

A support theorem is a geometric version of the Hahn-Banach theorem. Each convex set K induces a real valued function called support function of K. Every support function is sublinear (convex) and conversely every sublinear function is the support function of some convex set *K*, see [1]. This function plays an importance role in the definition of surface area in Minkowski space. The polar reciprocal K° of a closed convex set K in M^n also depends on it fundamentally. We are interested also in the isoperimetric problem in Minkowski space, the solution is in general not a Minkowski sphere, but a convex surface called isoperimetrix body. In this chapter, we identify the affine point space M^n with a vector space using an arbitrary given point o as origin respective zero vector. We will give a short survey about the support function in Minkowski space and some related definitions.

In a two dimensional (normed) Minkowski space the surface area is defined by the induced norm of this space. But the norm in higher dimensional $n \geq 3$ is no longer sufficient to define the surface area. Therefore various definitions of surface area were obtained in higher dimensional Minkowski spaces see [40-44]. In this part we will focus only on the definition of surface area which is given by Busemann and Petty [43] and the Holmes-Thompson definition [1]. Of course we have more definitions given by other authors, e.g., Benson [45], the definitions of area by means of perimeter see [1] and by means of the Löwner ellipsoid. Löwner did not publish this result by himself but we can see [46] for complete results, one can find the details of these definitions in the books of Gardner [47] and Thompson [1].

3.2 Dual Space

If $\left(M^n, \|\cdot\|\right)$ is a Minkowski space, then the set of all linear functionals onto the one-dimensional normed space $\left(\mathbb{R}, |\cdot|\right)$ carries a natural norm. The resulting Minkowski space is called the dual of $\left(M^n, \|\cdot\|\right)$ and is denoted by $\left(\left(M^n\right)^*, \|\cdot\|^*\right)$. It is easy to see that the double dual (i.e., the dual of the dual) of a Minkowski space can be naturally identified with the space itself. The unit ball of $\left(\left(M^n\right)^*, \|\cdot\|^*\right)$ is said to be the polar of the unit ball of $\left(M^n, \|\cdot\|\right)$ as we see later.

Now let us denote M^n with a Minkowski metric $\left(M^n, \|\cdot\|\right) =: M_B^n$ based on a centrally symmetric unit ball B, then we can define a norm for the dual affine space $\left(M^n\right)^*$ too by

$$\|f\| := \sup\{|f(x)| : x \in B\}. \tag{3.1}$$

Definition 3.1 (Linear functional on M^n): A linear functional on affine space M^n, f, is a linear mapping from M^n to $\mathbb{R}$.

Definition 3.2: The dual space of M^n denoted by $(M^n)^*$ is the finite dimensional normed vector space of all linear functions on M^n.

Definition 3.3: If $f \in (M^n)^*$ then the subspace of M^n that is annihilated by f is denoted by $f^{\perp}$ (kernel of f). Furthermore, each translate of the kernel of a non-zero linear functional, *i.e.* $\{x \in M^n : f(x) = \alpha\}$, is called a hyperplane H in an affine space M^n.

It is easy to see that this norm $\|\cdot\|^*$ based on a unit ball $B^\circ \subset (M^n)^*$, which is related to B in the following way.

We can show that all linear maps defined on a Minkowski space are continuous. Therefore, (3.1) defines a norm on the dual space $(M^n)^*$ of all linear functionals on M^n, which coincides with the *support function* of the unit ball B and that $\{f(x) \in M^n : \|f\| \le 1\}$ coincides with the *polar reciprocal* of B, see again [1].

Then we can define the *dual unit ball* in $(M^n)^*$ by B°, i.e.

$$B^\circ := \{f \in (M^n)^* : \|f\| \le 1\}. \tag{3.2}$$

3.3 Support function in Minkowski Space

Definition 3.4: A hyperplane $H := \{x : f(x) = \alpha\}$ is called a support hyperplane of a closed convex set K if $K \cap H \ne \Phi$ and either $K \subseteq H[f,\alpha]^+ := \{x : f(x) \ge 0\}$ or $K \subseteq H[f,\alpha]^- := \{x : f(x) \le 0\}$, we can say that the linear function f supports the convex set K at each point $x \in K \cap H$.

Corollary 3.5: If M_B^n is a Minkowski space of dimension n and if $x_\circ \in \partial B$ then there exists a linear function $f_\circ$ with $\|f_\circ\| = 1$ which supports B at $x_\circ$.

Corollary 3.6: If x is a point in a Minkowski space $(M^n, \|\cdot\|)$ then

$$\|x\| := \sup\left\{\left|f(x)\right| : f \in B^{\circ}\right\}. \tag{3.3}$$

Definition 3.7: The function $h_K(f)$ defined by

$$h_K(f) := \sup\left\{f(x) : x \in K\right\}, \tag{3.4}$$

is called the support function of the convex set K. The function h_K is sublinear function on $(M^n)^*$, i.e.,

$$h_K(\lambda f) = \lambda h_K(f)\, \forall \lambda \geq 0,$$
$$h_K(f_1 + f_2) \leq h_K(f_1) + h_K(f_1),$$

what easily can be proved by applying the properties of the supremum function in (3.3).

Proposition 3.8: The support functions have some properties:

- If K is bounded then, f is bounded because it is continuous, h_K is a real valued function on $(M^n)^*$.
- If $0 \in K$ then $h_K \geq 0$ and $h_K(f) > 0\, \forall f \neq 0$ if $0 \in Int(K)$.
- If K is symmetric then h_K is an even function and $h_K(f) := \sup\left\{\left|f(x)\right| : x \in K\right\}$.
- In the special case when K is the unit ball B then h_K is the dual norm on $(M^n)^*$.
- If K_1 and K_2 have the same closure the $h_{K_1} = h_{K_2}$.
- If K, K_1 and K_2 are closed convex sets then,

    1. $h_{K_1} = h_{K_2}$ implies $K_1 = K_2$.
    2. $h_{\alpha K} = \alpha h_K$, $\alpha \geq 0$.
    3. $h_{K_1 + K_2} = h_{K_1} + h_{K_2}$.
    4. If $K_1 \subseteq K_2$ then $h_{K_1} \leq h_{K_2}$.

5. If $K' = K + \{x\}$ then $h_{K'}(f) = h_K(f) + f(x)$.

For a functional h, we can find the associated set $K_h \in (M^n)^*$ where

$$K_h := \left\{ f \in (M^n)^* : h(f) \le 1 \right\}, \tag{3.5}$$

we can easily proof that the set K_h is a non-empty closed convex set. Starting with a closed convex set $K \in M^n$ we derive the corresponding supporting function h_K and hence the corresponding set K_h as (3.5) and we obtain the so-called *polar reciprocal* of K which is denoted by K°.

Definition 3.9: If K is a closed convex set in M^n, the set K° of K is called the polar reciprocal defined by

$$K^\circ := \left\{ f \in (M^n)^* : f(x) \le 1 \forall x \in K \right\}. \tag{3.6}$$

If K is symmetric then $K^\circ := \left\{ f \in (M^n)^* : |f(x)| \le 1 \forall x \in K \right\}$ is symmetric, too.

Theorem 3.10: If K is a closed convex set in M^n with the origin $0 \in K$ then $K^{\circ\circ} = K$, where,

$$K^{\circ\circ} := \left\{ x \in M^n : f(x) \le 1 \forall f \in K^\circ \right\}, \tag{3.7}$$

For more details about the polar reciprocal see [1].

Definition 3.11: If K is a closed convex set in M_B^n with $0 \in Int(K)$ then for each element $x \in M_B^n \setminus \{0\}$ we define the *radial function* $\rho_K(x)$ of K to be the positive number such that $\rho_K(x) \bullet x \in \partial K$.

Proposition 3.12: The radial function $\rho_K(x)$, defined in definition 3.11, has the following properties;

(i) $\rho_K(\lambda x) = \lambda^{-1} \rho_K(x)$, $\lambda > 0$.

(ii) If $K_1 \subseteq K_2$ and $0 \in Int(K_i)$ then $\rho_{K_1}(x) \le \rho_{K_2}(x), i = 1, 2$.

(iii) $\rho_{\lambda K}(x) = \lambda \rho_K(x)$, $\lambda > 0$.

Theorem 3.13: If K is a closed convex set in M_B^n with $0 \in Int(K)$ then, $\rho_{K^\circ}(f) = (h_K(f))^{-1}$ and $\rho_K(x) = (h_{K^\circ}(x))^{-1}$

Proof: Since $K^\circ := \left\{ f \in (M^n)^* : h_K(f(x)) \leq 1 \forall x \in K \right\}$ we have that $f \in \partial K^\circ$ *iff* $h_K(f) = 1$, since h_K is positively homogenous function this means that $(h_K(f))^{-1} f \in \partial K^\circ$ which complete the proof of the first equation. By the same way we can complete the proof of the theorem.

3.4 Volume and Mixed volume in Minkowski Space

Definition 3.14: Let M_B^n be an n-dimensional Minkowski space with unit ball B. Then a regular Borel measure μ is called a Haar measure, see Cohn [48], on M^n if it has the following properties:

1- $\mu(K) < \infty$ for each compact subset K of M^n.

2- $\mu(P)$>0 for each open set P of M^n.

3- μ is translation invariant, i.e. $\mu(a+A) = \mu(A)$ for all $a \in M^n$, and all Borel sets $A \subseteq M^n$.

Theorem 3.15: Let M_B^n be an n-dimensional Minkowski space with unit ball B then there exists a Haar measure on M_B^n.

Theorem 3.16: Let M_B^n be an n-dimensional Minkowski space with unit ball B and if μ and ν are two Haar measures on M^n then there is a constant c such that $\mu = c\nu$.

The proof of the above two theorems can be found e.g. in [1, 48]. Haar [49] proved the existence of the left invariant measure on a locally compact group with a countable basis. This proof was extended to arbitrary locally compact groups by Kakutani [50], who also give the uniqueness property.

Proposition 3.17: If K, K_1 and K_2 are closed convex sets (Borel sets) in M_B^n then the volume functional λ have the following properties:

i- $\lambda(K) \geq 0$ with equality iff $\dim(K) \leq n-1$.

ii- If $K_1 \subseteq K_2$ then $\lambda(K_1) \leq \lambda(K_2)$.

iii- $K_2 = K_1 + \{x\}$ implies $\lambda(K_1) = \lambda(K_2)$ (λ is translation invariant).

iv- If $c \geq 0$ then $\lambda(cK) = c^n \lambda(K)$.

v- $\lambda(-K) = \lambda(K)$.

Definition 3.18: If $b_1, b_2, ..., b_n$ is a basis of the space M_B^n and if T is a non-singular linear transformation on M^n then there exists a dual transformation T^* on $(M^n)^*$ such that

$$T^* f(x) = f(Tx) \tag{3.8}$$

If T transforms a basis $b_1, b_2, ..., b_n$ in M^n to $Tb_1, Tb_2, ..., Tb_n$ then the corresponding dual bases on $(M^n)^*$ are $b_1^*, b_2^*, ..., b_n^*$ and $T^{*^{-1}} b_1^*, T^{*^{-1}} b_2^*, ..., T^{*^{-1}} b_n^*$ respectively.

If λ is a Haar measure on M^n then there is a unique dual Haar measure λ^* on $(M^n)^*$ such that

$$\lambda\left(\sum_i [0, b_i]\right) \lambda^*\left(\sum_i [0, b_i^*]\right) = 1, \tag{3.9}$$

where, $\sum_i [0, b_i]$ denote the parallelotope spanned by the basis vectors $b_1, b_2, ..., b_n$. We can see easily that the measure λ^* is independent of the choice of the basis $b_1, b_2, ..., b_n$ in M_B^n, see [1].

Theorem 3.19: If $K = c_1 K_1 + c_2 K_2 + ... + c_r K_r$ is a closed convex set in M_B^n, with $c_r \geq 0$ then

$$\begin{aligned} \lambda(K) &= \lambda(c_1 K_1 + c_2 K_2 + ... + c_r K_r) \\ &= \sum V\left(K_{i_1}, K_{i_2}, ..., K_{i_n}\right) \cdot c_{i_1} c_{i_2} ... c_{i_n} \end{aligned} \tag{3.10}$$

i.e. $\lambda(K)$ is a homogenous polynomial of degree d in the c's.

The coefficients $V\left(K_{i_1}, K_{i_2}, ..., K_{i_n}\right)$ are known as mixed volumes. The following proposition represents some importance properties of the mixed volumes.

Proposition 3.20: If $K_1 = K_2 = ... = K_n = K$ then $\lambda(K) = V(K_1, K_2, ..., K_n)$ satisfy the following

1- $V(K_1, K_2, \ldots, K_n) \geq 0$ with strict inequality iff for each $i = 1, 2, \ldots, n$ we can find line segments $[x_i, x_i']$ in K_i such that the vectors $y_i = x_i' - x_i$ span M_B^n.

2- If $K_i \subseteq K_i'$ then $V(K_1, K_2, \ldots, K_i, \ldots, K_n) \leq V(K_1, K_2, \ldots, K_i', \ldots, K_n)$.

3- If $K_i' = K_i + \{x\}$ then $V(K_1, K_2, \ldots, K_i, \ldots, K_n) = V(K_1, K_2, \ldots, K_i', \ldots, K_n)$.

4- If $\alpha \geq 0$ then $V(K_1, K_2, \ldots, \alpha K_i, \ldots, K_n) = \alpha V(K_1, K_2, \ldots, K_i, \ldots, K_n)$.

5- If $K_i = K_i' + K_i''$ then

$$V(K_1, K_2, \ldots, K_i, \ldots, K_n) = V(K_1, K_2, \ldots, K_i', \ldots, K_n) + V(K_1, K_2, \ldots, K_i'', \ldots, K_n).$$

Definition 3.21: We abbreviate $V(\underbrace{K_1, K_1, \ldots, K_1}_{\text{n-i.times}}, \underbrace{K_2, K_2, \ldots, K_2}_{\text{i.times}})$ by $V\left(K_1^{(n-i)}, K_2^{(i)}\right)$. For $K_2 =: [u]$ the line segment $[u] := \{\lambda u \mid \lambda \in [0,1]\}$ and a convex set $K \subset M_B^n$, let us consider the functional $\sigma_{K,u} := n \cdot V\left(K^{(n-1)}, [u]\right)$. The functional $\sigma_{K,u}$ is an even function because $[u] = [-u]$, positively homogeneous and subadditive and hence is the support function of a convex set (Projection body) of K in $(M^n)^*$ which is denoted by $\Pi(K) \subseteq (M^n)^*$.

Theorem 3.22: If K_1 and K_2 are two convex bodies in M_B^n then

$$V^n\left(K_1^{(n-1)}, K_2\right) \geq \lambda^{n-1}(K_1)\lambda(K_2). \tag{3.11}$$

Proof: see [1]

3.5 The isoperimetric problem in a Minkowski plane M_B^2

The isoperimetric problem in Minkowski plane stated as: what is the simply closed curve that contains a given area such that its length is a minimum? Of course, the solution in E^2 is the circle and in fact this result will generalize to the n-sphere in E^n. We have that for any simple closed curve $C \subset E^2$ with length ℓ and enclosed area A, ℓ is a minimum when C is a circle. Thus, we have the isoperimetric inequality $\ell \geq \sqrt{4\pi A}$. In E^3 that inequality states that if S is the surface area of a compact convex body of volume V, then $S^3 \geq 36\pi V^2$ with equality if and only if the body is a ball.

For Minkowski planes the isoperimetric problem may be stated as: among all simple closed curves of given Minkowski length, we try to find the enclosing largest area. Here the Minkowski length of a closed curve can also be interpreted as the mixed area of it and the polar reciprocal of the Minkowskian unit circle with respect to the Euclidean unit circle rotated through 90° see Martini and Mustafaev [51]. Busemann [40,41] between 1947 and 1949 has presented the solution in the case of Minkowski plane and Benson [45] gives also another calculation of it. Busemann showed that the circles in an antinorm (norm dual in a certain case to the norm of an arbitrary Minkowski plane) are the solutions to the isoperimetric problem in a Minkowski plane, also the anticircles are circles only when the circles are Radon curves [24] see also [52].

We consider the Minkowski plane M_B^2 with unit ball B, assume the convex body K whose Minkowski length of its boundary ∂K is $\mu(\partial K)$. Suppose that the smooth convex body K, then the outward unit Euclidean normal to K at x is that unique linear function f_x such that $f_x(y_x)=0, |f_x|=1$ and $f_x(z)\le f_x(x)\forall z\in K$, where y_x is the unit tangent vector of K at x. This means that function attains its maximum at x such that $f_x(x)=\|x\|\|f_x\|$.

If ∂K is parameterized by

$$\partial K=\{x=x(t):t\in[\alpha,\beta]\}, \tag{3.12}$$

then we can write the Euclidean and Minkowski elements of arc length at x respectively as follows

$$d\lambda(x):=|x'(t)|dt, \tag{3.13}$$

$$d\mu_B(x):=\|x'(t)\|dt, \tag{3.14}$$

and we can get the Minkowski length of ∂K as

$$\begin{aligned}\mu_B(\partial K)&=\int_{\partial K}d\mu_B(x)=\int_\alpha^\beta\|x'(t)\|dt\\&=\int_\alpha^\beta\frac{|x'(t)|}{\rho_B(y_x)}dt=\int_{\partial K}\sigma(f_x)d\lambda(x)\end{aligned} \tag{3.15}$$

where $\sigma(f_x)=\dfrac{1}{\rho_B(y_x)}$ is the reciprocal of the radial function $\rho_B(y_x)$ of the unit ball, which is the ratio of the Minkowski and Euclidean length in the direction of the Kernel $f_x^\perp$ of f_x.

By using chapter 2 in [1] theorem 2.3.13 we can easily get that if the function $\sigma(f_x)$ is extended to all of the dual plane $(M_B^2)^*$ by positive homogeneity and if it is convex function then the integral $\frac{1}{2}\int_{\partial K}\sigma\left(f_x\right)d\lambda\left(x\right)$ obtains the two dimensional mixed volume $V\left(K,I_B\right)$, where I_B is that convex set of which the extended function $\sigma\left(f_x\right)$ is the support function.

If M_B^2 and $\left(M^2\right)^*$ are known then From theorem 3.13 the reciprocal of the radial function of B is the support function of B°, i.e. $\sigma\left(f_x\right)=\rho_B\left(y_x\right)^{-1}=h_{B^\circ}\left(y_x\right)$ see Figure 3.1, which is convex and is the support function of B° rotated through a quarter turn because of the change from f_x to y_x in the argument. However $I_B\subseteq M_B^2$ where $B^\circ\subseteq\left(M_B^2\right)^*$.

We can rewrite (3.15) as

$$\mu_B\left(\partial K\right)=\int_{\partial K}\sigma\left(f_x\right)d\lambda\left(x\right)=2V\left(K,I_B\right). \tag{3.16}$$

We have calculate the last integration over the boundary of the smooth convex body K, without loss of generality the last formula is valid for all convex body because the number of the singular points on ∂K is countable which does not affect of the value of the integration.

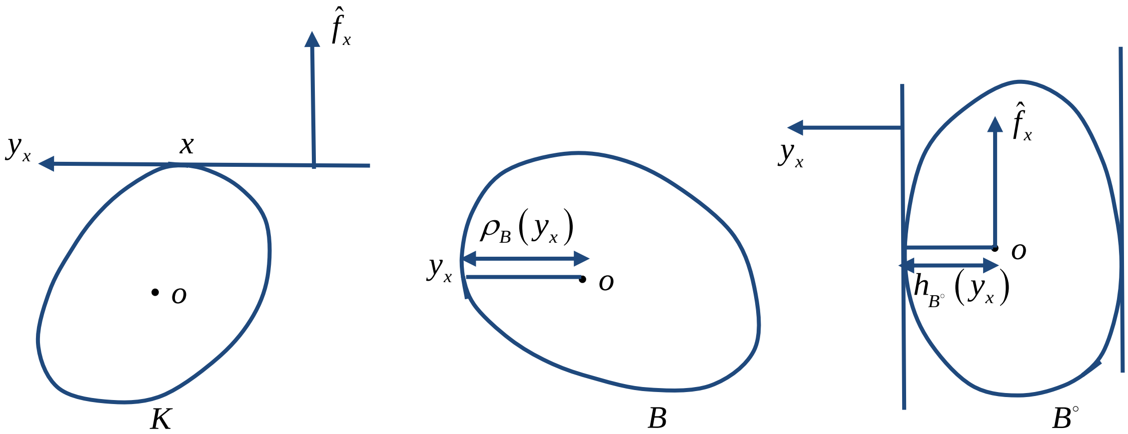

Figure 3.1

Theorem 3.23: If γ_1 and γ_2 are two convex curves from a to b with γ_1 lies inside γ_2 then

$$\mu_B\left(\gamma_1\right)\le\mu_B\left(\gamma_2\right). \tag{3.17}$$

Proof: Since γ_1 and γ_2 are two convex curves and γ_1 lies inside γ_2 then $K_1\subseteq K_2$ where $K_i=conv\left(\gamma_i\right)$ for $i=1,2$. By using Proposition 3.20 (3), $V\left(K_1,I_B\right)\le V\left(K_2,I_B\right)$ which complete the proof with (3.16).

Theorem 3.24: If K is a convex body in the Minkowski space M_B^2 with area equal to that I_B then $\mu_B(\partial I_B) \le \mu_B(\partial K)$ with equality iff K is a translate of I_B.

Proof: Using (3.16) we get $\mu_B(\partial K) = 2V(K, I_B)$, putting $K = I_B$ then $\mu_B(\partial I_B) = 2\lambda(I_B)$, from the assumption $\lambda(K) = \lambda(I_B)$ and (3.11) we have $V^2(K, I_B) \ge \lambda(K)\lambda(I_B) = (\lambda(I_B))^2$. Therefore, $\mu_B(\partial K) = 2V(K, I_B) \ge 2\lambda(I_B) = \mu_B(\partial I_B)$. □

From the above theorem, once we have the minimal perimeter I_B, we can get the minimal area of any convex body K as a translate of fixed multiple of I_B. Among the homothetic image of I_B, we need to specify a unique one which is called isoperimetric $\tilde{I}_B$. Since I_B is centrally symmetric, then $\tilde{I}_B$ can be taken to be centred at o. Now we denote the Minkowski area by $\bar{\mu}_B$ to be distinguishing with the Minkowski length μ_B, we also normalized the isoperimetric $\tilde{I}_B$ as

$$\mu_B(\partial \tilde{I}_B) = 2\bar{\mu}_B(\tilde{I}_B). \tag{3.18}$$

By the same way as in the ratio of the Minkowski and Euclidean length, we define the area ratio $\bar{\sigma}_B$ as

$$\bar{\sigma}_B := \frac{\text{Minkowski area in } M_B^2}{\text{Euclidean area in } M_B^2} = \frac{\bar{\mu}}{\bar{\lambda}}. \tag{3.19}$$

According to consider the isoperimetric $\tilde{I}_B$ as a multiple of the perimeter I_B we have $\tilde{I}_B = \alpha I_B$, using relation (3.16) and (3.18) to find the value of the factor α, then we get

$$\bar{\sigma}_B = \frac{\bar{\mu}_B(\tilde{I}_B)}{\bar{\lambda}(\tilde{I}_B)} = \frac{\mu_B(\partial \tilde{I}_B)}{2\alpha^2 \bar{\lambda}(I_B)} = \frac{\mu_B(\partial \tilde{I}_B)}{\alpha \mu_B(\partial \tilde{I}_B)} = \alpha^{-1}. \tag{3.20}$$

This deduce that $\tilde{I}_B = \bar{\sigma}_B^{-1} I_B$.

We may normalize the unit ball area as Busemann [42] to be $\mu(B) := \pi$ and easily verify that

$$\bar{\mu}_B(\tilde{I}_B) = \frac{\lambda(B)\lambda^*(B^\circ)}{\pi}, \tag{3.21}$$

where $\lambda(K)\lambda^*(K^\circ)$ is called the volume product of any closed convex body K. i.e. the Minkowski isoperimetric area is the normalized volume product of the unit ball B.

We can also do the converse [53] by normalizing the Minkowski isoperimetric area by

$$\bar{\mu}_B(B) := \frac{\lambda(B)\lambda^*(B^\circ)}{\pi}.$$

In this case $\tilde{I}_B = \pi\left(\lambda^*(B^\circ)\right)^{-1} I_B$ and we have $\mu_B\left(\partial\tilde{I}_B\right) = 2\pi$, i.e. using (3.18) we may measure an angle either by twice the area of the sector of $\tilde{I}_B$ or by the length of the perimeter of $\tilde{I}_B$ which it cuts off.

3.6 Transversality in Minkowski plane $\mathbf{M_B^2}$

Proposition 3.25: If M_B^2 is a Minkowski plane and if P is a parallelogram with one vertex at the origin o and spanned by two vectors x and y then

$$\mu_B(P) = \|x\|\,\tilde{f}(y), \tag{3.22}$$

where $\tilde{f}(y) = \dfrac{\sigma_B \hat{f}}{\sigma(\hat{f})}$ and $\hat{f}$ is the Euclidean unit linear functional such that $\hat{f}(x) = 0$ and $\hat{f}(y) > 0$.

Proof: From the properties of the functional $\hat{f}$, the perpendicular Euclidean height of P is given by $\hat{f}(y)$ then the Euclidean area can be given as

$$\lambda(P) = |x|\,\hat{f}(y), \tag{3.23}$$

as we mentioned in the last section, we get

$$\lambda(P) = \frac{\|x\|\,\hat{f}(y)}{\sigma(\hat{f})} = \|x\|\,\bar{f}(y), \tag{3.24}$$

where $\bar{f}(y) := \dfrac{\hat{f}}{\sigma(\hat{f})}$ is a unit vector relative to I_B°, and hence

$$\mu_B(P) = \sigma_B\lambda(P) = \|x\|\sigma_B\frac{\hat{f}}{\sigma(\hat{f})} = \|x\|\,\tilde{f}(y), \tag{3.25}$$

where $\tilde{f}(y) := \sigma_B \dfrac{\hat{f}}{\sigma(\hat{f})}$ is a unit vector relative to $\tilde{I}_B^{\circ}$, therefore $\tilde{f}$, like Euclidean one, is the linear functional such that $\tilde{f}(x) = 0$, $\tilde{f}(y) > 0$ and $\|\tilde{f}\|_{\tilde{I}} = h_{\tilde{I}}(\tilde{f}) = 1$.

Definition 3.26: Transversal: As the above definition of the function $\tilde{f}$ we say that y is *transversal* to x with the notation $y \nless x$ if $\tilde{f}(y) = \|y\|_{\tilde{I}}$, i.e. if $\tilde{f}$ supports $\tilde{I}$ at $\dfrac{y}{\|y\|_{\tilde{I}}}$. Then from (3.25) we get

$$\mu_B(P) = \|x\|_B \|y\|_{\tilde{I}}, \tag{3.26}$$

3.7 Radon plane

In this section we shall describe an importance curve that is Radon curve or Radon norm, introduced by Radon in 1916 [24]. It is one of centrally symmetric closed convex curve in the plane. Using this curve we can define the unit ball of the Radon plane. Almost properties and results in Euclidean plane are valid for Radon planes too, e.g., the triangle and parallelogram area formulas, the area formula of a polygon circumscribed about a circle and certain isoperimetric inequalities [52].

The definition of the left-orthogonality (2.2) is not in general a symmetric relation. In the case of Radon plane the relation $\dashv$ is symmetric. The construction of Radon curve is presented by Radon [24]; Birkhoff [5] and Day [9] also have given constructions of it in terms of polarity and a quarter rotations with respect to some Euclidean structure. Martini and Swanepoel [52] presented it using only an “affine” bilinear form.

It is clear from the definition of the isoperimetrix I_B in section 3.5 that for every ball B we get $I_B^2 = B$. As we see later, in the case of Radon plane we can consider that $I_B = \alpha B$ for some multiple α depending on the Euclidean structure.

Now we will give some selected facts and theorems which distinguishing Radon plane. We refer to some references without all proofs.

Theorem 3.27: (Radon [24]): A unit circle in a Minkowski plane is a Radon curve if and only if B-orthogonality is symmetric.

Corollary 3.28 (Busemann [40]): A norm is Radon if and only if it equals a multiple of its antinorm $I_B = \alpha B$.

Theorem 3.29: ([7]): If for any $x, y \neq 0$ yields $x \dashv \lambda x + y$ and $\mu y + x \dashv y$, then $0 \leq \lambda\mu \leq 2$.

Proof: Without loss of generality $\lambda\mu \neq 1$, from the definition of B-orthogonality we have $\left\| x + \frac{\mu}{1-\lambda\mu}(\lambda x + y) \right\| \geq \|x\|$ and $\|\mu y + x - \mu y\| \geq \|\mu y + x\|$. Then we get,

$\|x + \mu y\| \geq |1 - \lambda\mu| \|x\|$ and $\|x\| \geq \|\mu y + x\| \Rightarrow |1 - \lambda\mu| \leq 1 \Rightarrow 0 \leq \lambda\mu \leq 2$. □

Lemma 3.30: ([7]) For any linearly independent x, y there exist $\alpha \in \mathbb{R}$ such that $x \dashv \alpha x + y$. Furthermore, $|\alpha| \leq \|y\| / \|x\|$.

Theorem 3.31: ([2], [7]): A norm is Radon if and only if the following condition holds: For any $x, y \neq 0$, $x \dashv \lambda x + y$ and $y \dashv \mu y + x$, then $\lambda\mu \geq 0$.

Lemma 3.32: ([7]): If $x, y \neq 0$ are two elements in a Minkowski plane M_B^2 fulfilling $x \dashv \lambda x + y$ and $x \dashv \alpha x + y$, $\alpha < \lambda$, then $x \dashv \mu x + y$ if $\alpha \leq \mu \leq \lambda$.

Theorem 3.33: ([40], [7]): A plane is of Radon type if and only if the isoperimetrix I_B is a circle.

3.8 The isoperimetric problem in a higher dimensional Minkowski space $\mathbf{M_B^n}$

In this section we discuss the concept of surface area in different methods when $n \geq 3$ for a given n-dimensional Minkowski space. We have two well-known definitions of the surface area. The first one is obtained by Busemann [41] and the second one is called the Holmes-Thompson definition [1]. As in the two dimensional space the question is, what is the extreme value of the surface area of the unit ball in a n-dimensional Minkowski space, when $n \geq 3$?.

One of the most distinguishing differences between $\tilde{I}_B$ in M_B^2 and M_B^n is that in the higher dimensional space $n \geq 3$ the shape of the isoperimetrix depends on the definition of the area which is not unique.

We assume M_B^n be an n-dimensional Minkowski space with unit ball B which is centrally symmetric convex body. The unit sphere denoted by ∂B is the boundary of the unit ball.

For each d-dimensional subspace M^d of M_B^n we may insert the notation $\mu_B^d\left(B \cap M^d\right)$ which is the d-dimensional measure of the relative unit ball in M^d. We have the centrally symmetric convex body generated by the intersection of the unit ball B and the subspace M^d. For each dimension d, we define a function γ on all centrally symmetric convex bodies as

$$\mu_B^d\left(B \cap M^d\right) = \gamma\left(B \cap M^d\right). \tag{3.22}$$

By another way, a Minkowski space M_B^n possesses a Haar measure μ_B, as we mentioned in the two dimensional space, and this measure is unique up to multiplication of the Lebesgue measure by a constant. Now we insert the following ratio for all subspace M^d,

$$\sigma_B^d\left(M^d\right) := \frac{\mu_B^d(U)}{\lambda^d(U)}, \tag{3.23}$$

where U is a measurable subset of M^d. Substitute (3.22) into (3.23) and take $U = B \cap M^d$, we get

$$\gamma\left(B \cap M^d\right) = \sigma_B^d\left(M^d\right)\lambda^d\left(B \cap M^d\right). \tag{3.24}$$

We can replace, in the following definitions, for simplicity the measure λ instead of λ^d, the unit ball B_d instead of $B \cap M^d$, σ_B instead of σ_B^d and μ_B instead of μ_B^d.

Definition 3.34: If K is a convex body in a d-dimensional subspace M^d of M_B^n, then the d-dimensional Busemann volume of K is defined by

$$\mu_B^{Bus}(K) = \frac{\varepsilon_d}{\lambda(B_d)}\lambda(K), \text{ i.e., } \sigma_B\left(M^d\right) = \frac{\varepsilon_d}{\lambda(B_d)} \tag{3.25}$$

Definition 3.35: If K is a convex body in a d-dimensional subspace M^d of M_B^n, then the d-dimensional Holmes-Thompson volume of K is defined by

$$\mu_B^{HT}(K) = \frac{\lambda(K)\lambda^*\left(B_d^\circ\right)}{\varepsilon_d}, \text{ i.e., } \sigma_B\left(M^d\right) = \frac{\lambda^*\left(B_d^\circ\right)}{\varepsilon_d} \tag{3.26}$$

Note that these definitions coincide with the standard notion of volume if the space is Euclidean, and if E_d is a d-dimensional ellipsoid (M^d is Euclidean space) then

$$\gamma\left(E_d\right) := \varepsilon_d. \tag{3.27}$$

From the basic calculation of the ellipsoid volume in d-dimensional Euclidean space, we get

$$\varepsilon_d := \frac{\pi^{d/2}}{\Gamma(1 + d/2)}, \tag{3.28}$$

where $\Gamma(x)$ is the usual special Gamma function.

Let ϕ be a surface in $\mathbb{R}^d$ with the property that at each point $x \in \phi$ there exist a unique tangent hyperplane, and f_x is the Euclidean unit tangent vector to the surface at x. Then the Minkowski surface area of ϕ is defined by

$$\mu_B(\phi) := \int_\phi \sigma_B(f_x)\, dS(x). \tag{3.29}$$

For the Busemann surface area, $\sigma_B(f_x)$ is given by

$$\sigma_B(f_x) = \frac{\varepsilon_{d-1}}{\lambda\left(B \cap f_x^\perp\right)}. \tag{3.30}$$

For the Holmes-Thompson area, $\sigma_B(f_x)$ is given by

$$\sigma_B(f_x) = \frac{\lambda^*\left(\left(B \cap f^\perp\right)^\circ\right)}{\varepsilon_{d-1}}. \tag{3.31}$$

If K is a convex body in a d-dimensional subspace M^d with unit ball B, then the Minkowski surface area, as (3.16), of K can also be defined by

$$\mu_B(\partial K) = dV\left(K^{(d-1)}, I_B\right). \tag{3.32}$$

Since σ_B is the support function of I_B, ρ is the radial function of I_B°, then we can define the so-called intersection body of the convex body K as follows.

Definition 3.36: (Lutwak [54]) If K is a convex body in a d-dimensional subspace M^d with unit ball B, then the body whose radial function in a given direction is equal the area of the cross-section of K e-perpendicular to that direction is called the intersection body of K, denoted by IK.

Therefore, for the Busemann measure we get

$$I_B^{Bus} = \varepsilon_{d-1}(IB)^\circ. \tag{3.33}$$

Among the homothetic images of I_B we can specify a unique one, called the isoperimetrix $\tilde{I}_B$ in d-dimensional see again [1], by the same way as in two dimensional spaces, (3.18) can be generalized as follows

$$\mu_B\left(\partial \tilde{I}_B\right) = d\,\overline{\mu}_B\left(\tilde{I}_B\right). \tag{3.34}$$

For Holmes-Thompson measure, I_B is defined by

$$I_B^{HT} = \frac{\Pi\left(B^\circ\right)}{\varepsilon_{d-1}}, \tag{3.35}$$

where $\Pi\left(B^\circ\right)$ is the projection body of the dual unit ball B° see definition 3.21.

We have also that

$$\tilde{I}_B = \sigma_B^{-1} I_B. \tag{3.36}$$

Therefore, the isoperimetrix for the Busemann measure is defined by

$$\tilde{I}_B^{Bus} = \frac{\lambda(B)}{\varepsilon_d} I_B^{Bus}, \tag{3.37}$$

and the isoperimetrix for the Holmes-Thompson measure is defined by

$$\tilde{I}_B^{HT} = \frac{\varepsilon_d}{\lambda^*\left(B^\circ\right)} I_B^{HT}. \tag{3.38}$$

Geometric and isoperimetric inequalities for the Busemann and Holmes-Thompson definitions of volume and surface area in Minkowski spaces can be found in [55,56]. The article [51], too, collects basic inequalities related to isoperimetric problems in Minkowski spaces.

Chapter 4

Trigonometry and semi-inner product in Minkowski space

4.1 Introduction

In this chapter we will discuss trigonometric functions in Minkowski space. Such functions were defined for first time by Busemann [44] and updated by Thompson [1]. These functions are connected to the concept of B-orthogonality and transversality, see chapter two and three. The concept of the angle in Minkowski space is dependent on the position of the angle and not only on the size of it. Therefore, we define those functions using the most convenient unique linear function which attains its norm at exactly one member of the space. The cosine and sine Minkowski functions are of two variables dependent on the sort of those variables, cosine function is defined sufficiently when the first variable is a point in the Minkowski space, but the sine function is more general an odd function because it may be defined using a hyperplane as the first argument. We define the Minkowski semi-inner product of two vectors, see definition 2.2, using the cosine Minkowski function which may be useful in the core ideas in the later chapters of that dissertation. Finally, we need to discus some important relations between those functions and prove some trigonometric formulae which are in somehow looks like the Euclidean one. Consequently, we found that the sine function in two dimensional Minkowski space is defined in terms of the cosine function between a vector and the normal to the hyperplane.

4.2 Cosine function

In this section, we construct the more suitable definition of the cosine Minkowski function in M_B^n between two vectors $x, y \in M_{B_2}^2 \subseteq M_B^n$, where $B_2 = B \cap M^2$ is the unit ball of the subspace M^2 spanned by the two vectors x and y. Mathematically, as we see later, this function dependent on the unique linear functional if the unit ball is smooth at $x/\|x\|$ and hence this function is a non symmetric function.

Continuously, we will give some interesting properties of this function connected to the concept of Birkhoff orthogonality which is mentioned in chapter two.

From section 3.5, for all Minkowski space M_B^n with strictly convex smooth unit ball B, we have for all $x \in M_B^n$, $x \neq 0$, up to a positive scalar factor, an unique linear functional f_x attains its maximum at x. i.e,

$$f_x(x) = \|x\| \|f_x\|, \tag{4.1}$$

Definition 4.1 (Minkowski cosine function): For all two vectors $x, y \in M_B^n$, $y \neq 0$, the Minkowski cosine function from x to y, denoted by $\mathrm{cm}(x, y)$, is defined by

$$\mathrm{cm}(x,y) := \frac{f_x(y)}{\|y\| \|f_x\|}. \tag{4.2}$$

Substituting (4.1) into (4.2) we get

$$\mathrm{cm}(x,y) := \frac{\|x\| f_x(y)}{\|y\| f_x(x)}. \tag{4.3}$$

More general, if M^d is a subspace of M_B^n then we can define the cosine Minkowski functional

$$\mathrm{cm}(x, M^d) := \max\left\{\mathrm{cm}(x,y) \middle| y \in M^d - \{0\}\right\}. \tag{4.4}$$

The smoothness of the unit ball B is a sufficient condition of $\mathrm{cm}(x,y) \forall y \in M_B^n$ to be exist. Of course we need only the subspace M^2 spanned by the two vectors x and y with the intersected unit ball $B_{M^2} = B \cap M^2$.

In the following proposition we will give some interesting properties of the cosine function connecting with the B-orthogonality, we use it later to define the well defined semi-inner product in Minkowski space.

Proposition 4.2: For all $x_1, x_2 \in M_B^n$, $x_1, x_2 \neq 0$ we have

(i) $\mathrm{cm}(\alpha x_1, \beta x_2) = \mathrm{cm}(x_1, x_2) \forall \alpha, \beta > 0$.

(ii) $\mathrm{cm}(x_1, -x_2) = -\mathrm{cm}(x_1, x_2)$.

(iii) If f_{x_1} supports B at x_1 then $\mathrm{cm}(-x_1, x_2) = -\mathrm{cm}(x_1, x_2)$.

(iv) $\mathrm{cm}(x_1, x_1) = 1$.

(v) For all $x_1 \neq x_2$, $|\mathrm{cm}(x_1, x_2)| \leq 1$ with equality *iff* the line segment $\left[x_1/\|x_1\|, x_2/\|x_2\|\right] \subset B$.

(vi) $\mathrm{cm}(x_1, x_2) = 0$ *iff* $x_1 \dashv x_2$ and so $x_2 \in f_{x_1}^{\perp}$, where $f_{x_1}^{\perp}$ is the Kernel of the supporting function f_{x_1} which supports B at x_1.

The proof of this proposition can become directly from the fact that the function f_x in (4.2) is linear.

Definition 4.3: In Minkowski space M_B^n, we define the Minkowski semi-inner product of two vectors $x_1, x_2 \in M_B^n$ as follows:

$$\langle x_1, x_2 \rangle_M = \frac{f_{x_1}(x_2)}{f_{x_1}(x_1)} \|x_1\|^2. \tag{4.5}$$

By substitute (4.3) into (4.5), we have

$$\langle x_1, x_2 \rangle_M = \|x_1\| \|x_2\| cm(x_1, x_2), \tag{4.6}$$

This definition satisfied the conditions in definition 2.2 which are proved in details in the following proposition.

Proposition 4.4: The Minkowski semi-inner product $\langle \cdot, \cdot \rangle_M : M_B^n \times M_B^n \to \mathbb{R}$ has the following properties for all $x, x_1, x_2, x_3 \in M_B^n$:

(i) $\langle x_1, x_2 \rangle_M \neq \langle x_2, x_1 \rangle_M \ \forall x_1, x_2 \in M_B^n$.

(ii) $\langle x_1, ax_2 + bx_3 \rangle_M = a \langle x_1, x_2 \rangle_M + b \langle x_1, x_3 \rangle_M$, for all $a, b \in \mathbb{R}$ and all $x_1, x_2, x_3 \in M_B^n$.

(iii) $\langle cx_1, x_2 \rangle_M = c \langle x_1, x_2 \rangle_M$ and $\langle x_1, dx_2 \rangle_M = d \langle x_1, x_2 \rangle_M \ \forall x_1, x_2 \in M_B^n$ and $c, d \in \mathbb{R}$.

(iv) $\langle x, x \rangle_M \geq 0$ and $\langle x, x \rangle_M = 0$ *iff* $x = 0$.

(v) $\left| \langle x_1, x_2 \rangle_M \right|^2 \leq \|x_1\|^2 \|x_2\|^2 \ \forall x_1, x_2 \in M_B^n$.

Proof: For simplicity, we replace the function f_{x_1} by f_1, from the linearity of this function we get $f_{x_1} = f_{ax_1} \forall x_1 \in M_B^n, a \in \mathbb{R}$. It is clear from the definition (4.5), (4.6) and the proposition 4.2 that the statement (i) is valid.

(ii) $\text{L.H.S} = \langle x_1, ax_2 + bx_3 \rangle_M = \dfrac{f_1(ax_2 + bx_3)}{f_1(x_1)} \|x_1\|^2$

$$= \frac{af_1(x_2) + bf_1(x_3)}{f_1(x_1)} \|x_1\|^2$$
$$= a\langle x_1, x_2 \rangle_M + b\langle x_1, x_3 \rangle_M . \quad \square$$

(iii) The first part

$$\text{L.H.S} = \langle cx_1, x_2 \rangle_M = \frac{f_1(x_2)}{f_1(cx_1)} \|cx_1\|^2$$

$$= \frac{|c|^2 f_1(x_2)}{cf_1(x_1)} \|x_1\|^2$$
$$= c\langle x_1, x_2 \rangle_M . \quad \square$$

The second part

$$\text{L.H.S} = \langle x_1, dx_2 \rangle_M = \frac{f_1(dx_2)}{f_1(x_1)} \|x_1\|^2$$

$$= \frac{df_1(x_2)}{f_1(x_1)} \|x_1\|^2$$
$$= d\langle x_1, x_2 \rangle_M . \quad \square$$

(iv) It is easy to show that $\langle x, x \rangle_M = \|x\|^2 \geq 0$ and it is equal zero *iff* $x = 0$.

(v) Let $x_1, x_2 \in M_B^n$, $x_1 \neq x_2$ and using proposition 4.2 (v) then

$$\left|\langle x_1, x_2 \rangle_M\right|^2 = \|x_1\|^2 \|x_2\|^2 \left|\text{cm}(x_1, x_2)\right|^2$$
$$\leq \|x_1\|^2 \|x_2\|^2 . \square$$

Definition 4.3 is a suitable definition of the Minkowski semi-inner product of two vectors $x_1, x_2 \in M_B^n$. From the proposition 4.4, the mapping $\langle \cdot, \cdot \rangle_M : M_B^n \times M_B^n \to \mathbb{R}$ satisfies all conditions of definition 2.2, then the mapping $\|\cdot\| : x \mapsto \sqrt{\langle x, x \rangle_M}$ for all $x \in M_B^n$ is a norm on M_B^n. We know that in each Minkowski space M_B^n there exist at least one semi-inner product $[\cdot, \cdot]$ which generate the norm $\|\cdot\|$, that is, $\|x\| = [x, x]^{1/2}$ for all $x \in M_B^n$, and it is unique if and only if the space is smooth see J. Chmielinski [19]. Therefore, if the unit ball B of the space M_B^n is smooth then the semi-inner product (4.5) is the unique one.

As we see in the proposition 4.2 (vi) the semi-inner product in each Minkowski space M_B^n is compatible with the Birkhoff orthogonality (2.2), i.e. $\langle x_1, x_2 \rangle_M = 0 \ iff \ x_1 \dashv x_2$ and so $x_2 \in f_{x_1}^{\perp}$, where $f_{x_1}^{\perp}$ is the Kernel of the supporting function f_{x_1} which supports B at x_1.

4.3 Sine function

The sine function, like the Euclidean one, is odd function and it is related to the area and volume measures. The main idea is trying to define the absolute value of it and find the sign later. Using the Euclidean volume of the parallelotope we insert the Euclidean sine function and using the ratio of the Euclidean and Minkowski volumes we can find the insert the absolute value of the Minkowski sine function.

Let M_B^n be an n-dimensional Minkowski space based on the centrally symmetric convex unit ball B, H be a hyperplane in M_B^n with $0 \in H$ and suppose f be a non zero linear function such that $f(H)=0$, then the Euclidean volume of the parallelotope spanned by the parallelotope P in H and the element $x \in M_B^n$ with $f(x) \neq 0$, which is denoted by $[x]+P$ is given by

$$\lambda^n([x]+P) = \lambda^{n-1}(P)|x|\sin(H,x)$$

$$= \lambda^{n-1}(P)\frac{|f(x)|}{|f|}. \tag{4.7}$$

Assume that the ratio of the Minkowski and Euclidean volume in M_B^n is σ_B^n and the ratio of the Minkowski and Euclidean area in H is $\dfrac{\sigma(f)}{|f|}$, then we get

$$\mu^n([x]+P) = \sigma_B^n \lambda^{n-1}(P)\frac{|f(x)|}{|f|}$$

$$= \sigma_B^n \mu^{n-1}(P)\frac{|f|}{\sigma(f)}\frac{|f(x)|}{|f|}$$

$$\mu^n([x]+P) = \|x\| \mu^{n-1}(P)\left\{\frac{\sigma_B^n |f(x)|}{\|x\|\sigma(f)}\right\}. \tag{4.8}$$

Comparing (4.8) with the Euclidean volume of $[x]+P$ we get

$$\mu^n([x]+P) = \|x\| \mu^{n-1}(P)\operatorname{sm}(H,x). \tag{4.9}$$

Therefore,

$$\left|\mathrm{sm}(H,x)\right| = \frac{\left|f(x)\right|}{\|x\|\tilde{\sigma}(f)}, \tag{4.10}$$

where $\tilde{\sigma}$ is the support function of $\tilde{I}(B) = \left(\sigma_B^n\right)^{-1} I_B$, therefore $\tilde{\sigma}$ is the norm in $\left(M_B^n\right)^*$ induced by the isoperimetrix $\tilde{I}_B$ in M_B^n, see chapter 3.

Now we need to choice the sign of the function $\mathrm{sm}(H,x)$ which is equivalent to choice the direction of the normal to H, sign of $f(x)$. We need an orientation of M_B^n and a basis of H. Therefore, we assume the basis vectors $\{x_1, x_2, ..., x_n\}$ then it has a sign as + or – depending on whether the orientation of the basis agrees with that of M_B^n or not, i.e, the orientation of the parallelotope spanned by the basis vectors in the same order or not. Consider $\{h_1, h_2, ..., h_{n-1}\}$ be a basis of H and if $x \in M_B^n$ with $f(x) \neq 0$, then the ordered set $\{h_1, h_2, ..., h_{n-1}, x\}$ is a basis for M_B^n. We can choose the sign of the $f(x)$ (the sign of $\mathrm{sm}(H,x)$) to be same as the sign of the basis $\{h_1, h_2, ..., h_{n-1}, x\}$. We can say that the sign of the function $\mathrm{sm}(H,x)$ is positive if the orientation of the basis f $\{h_1, h_2, ..., h_{n-1}, x\}$ agrees with the orientation M_B^n and negative otherwise.

Using the pervious ideas we can define the sine Minkowski function between any hyperplane through the origin H and any other non-zero vector $x \in M_B^n$ as follows.

Definition 4.5: Minkowski sine function: Let H be a hyperplane through the origin and x be a non-zero vector in an oriented Minkowski space M_B^n with centrally symmetric unit ball B. Consider also a basis $\{h_1, h_2, ..., h_{n-1}\}$ of the hyperplane H. Then the Minkowski sine function $\mathrm{sm}(H,x)$ is

$$\mathrm{sm}(H,x) := \frac{f(x)}{\|x\|\tilde{\sigma}(f)}, \tag{4.11}$$

where $f \in \left(M_B^n\right)^*$ is a linear function such that $f^{\perp} = H$ whose sign(sign$\left(f(x)\right)$ as the basis $\{h_1, h_2, ..., h_{n-1}, x\}$ for M_B^n and $\tilde{\sigma}$ is the support function of $\tilde{I}(B) = \left(\sigma_B^n\right)^{-1} I_B$, therefore $\tilde{\sigma}$ is the norm of f in $\left(M_B^n\right)^*$ induced by the isoperimetrix $\tilde{I}_B$ in M_B^n.

<u>General and special cases</u>:

- In the case of $\dim M_B^n = 2$ (Minkowski plane) then sine Minkowski function of ordered pairs of vectors is

$$\operatorname{sm}(x,y) = -\operatorname{sm}(y,x), \tag{4.12}$$

where the minus sign coming from the change of the orientation.

- If H is a hyperplane and L is a subspace of M_B^n the we can define $\operatorname{sm}(H,L)$ as

$$\operatorname{sm}(H,L) := \max\left\{\operatorname{sm}(H,x) \middle| x \in L \setminus \{0\}\right\}. \tag{4.13}$$

- If L and N are subspaces of M_B^n with $\dim(L+N) = \max\{\dim L, \dim N\} + 1$, for simplicity we assume that $\max\{\dim L, \dim N\} = \dim L$ then

$$\operatorname{sm}(L,N) := \max\left\{\frac{f(x)}{\tilde{\sigma}_{L+N}(f)} \middle| x \in N \cap B\right\}, \tag{4.14}$$

where $f \in (L+N)^*$, $f^{\perp} = L$ and $\tilde{\sigma}_{L+N}(f)$ is the norm of f in $(L+N)^*$ induced by the isoperimetrix of the space $L+N$ with unit ball $B \cap L+N$.

Let now M^2 be a subspace spanned by the non-zero vectors x_1 and x_2 of the space M_B^n with unit ball generated by intersecting the convex centrally symmetric unit ball B with the space M^2 ($B_2 = B \cap M^2$). Consider the parallelogram P spanned by x_1 and x_2 then

$$\lambda(P) = |x_1||x_2|\sin(x_1,x_2) = |x_1|\frac{f_1(x_2)}{|f_1|}, \tag{4.15}$$

where $f_1 \in (M_B^n)^*$, $f_1(x_1) = 0$ and the sign is comes from the orientation of the subspace, therefore

$$\mu(P) = \sigma_{B_2}\lambda(P) = \sigma_{B_2}|x_1|\frac{f_1(x_2)}{|f_1|}. \tag{4.16}$$

After some calculations we can find the more suitable formula for the Minkowski area of the parallelogram P as follows

$$\mu(P) = \|x_1\|\|x_2\|\operatorname{sm}(x_1,x_2), \tag{4.17}$$

and

$$\operatorname{sm}(x_1,x_2) = \frac{f_1(x_2)}{\|x_2\|\tilde{\sigma}(f_1)}, \tag{4.18}$$

where $\tilde{\sigma}(f_1)=\dfrac{\sigma(f_1)}{\sigma_{B_2}}$ and $\sigma(f_1)=\dfrac{\|x_1\|\,|f_1|}{|x_1|}$ for which $f_1(x_1)=0$. Simply, if we normalize both x_1 and f_1 to be Euclidean unit vectors then the support function of the isoperimetrix I is the same as the support function of the dual unit ball B° rotated through a "quarter turn".

Definition 4.6: For each hyperplane H in M_B^n we have

$$\alpha(H) := \sup\left\{\mathrm{sm}(H,x)\,\middle|\, x\in M_B^n\right\}, \tag{4.19}$$

and for each vector $x\in M_B^n$ let

$$\alpha(x) := \sup\left\{\mathrm{sm}(H,x)\,\middle|\, H \text{ is a hyperplane in } M_B^n\right\}. \tag{4.20}$$

Therefore using the definitions 4.5 and 4.6 we get the supremum of the function $\dfrac{f(x)}{\|x\|}$ is $\|f\|$ i.e. f attains its norm at x which means f supports B as required then

$$\alpha(H)=\frac{\|f\|}{\tilde{\sigma}(f)}=\frac{\|f\|}{\|f\|_{\tilde{I}^{\circ}}}, \tag{4.21}$$

where $f^{\perp}=H$ and supremum $\dfrac{f(x)}{\tilde{\sigma}(f)}$ is $\|x\|_{\tilde{I}}$ since $\tilde{\sigma}$ is the norm dual to $\|x\|_{\tilde{I}}$ i.e. f support $\tilde{I}$ at $\dfrac{x}{\|x\|_{\tilde{I}}}$ then

$$\alpha(x)=\frac{\|x\|_{\tilde{I}}}{\|x\|}. \tag{4.22}$$

We can say that the vector $x\in M_B^n$ is normal to the hyperplane H *iff* $|\mathrm{sm}(H,x)|=\alpha(H)$ and it is transversal to the hyperplane H *iff* $|\mathrm{sm}(H,x)|=\alpha(x)$.

Proposition 4.7: The radial function $\rho_B(x)$ of $\partial\tilde{I}$ is $\alpha(x)^{-1}$ where $\|x\|=1$ because $x\in\partial B$.

Proof: Since $\rho_B(x)x\in\partial\tilde{I} \Leftrightarrow \|\rho_B(x)x\|_{\tilde{I}}=1 \Leftrightarrow \rho_B(x)=\dfrac{1}{\|x\|_{\tilde{I}}}=\alpha(x)^{-1}$. □

From (4.22) we have the fact that the isoperimetrix $\tilde{I}$ coincides with the unit ball B *iff* $\alpha=1$.

4.4 Trigonometric formulae

In this part we will give some familiar formulae of the sine and cosine Minkowski functions in the two dimensional space and we assume also that the unit ball B is centrally symmetric smooth and strictly convex. First we give the law of sine, later we will define the so-called normal basis on the Minkowski space M_B^2. We will receive that, as in ordinary trigonometry, cosine Minkowski is closely related to the sine Minkowski function; in fact the relation is quite similar.

Theorem 4.8: Let $x, y, z \in M_B^2$ are three vectors such that $x + y + z = 0$, then the sine's law is valid,

$$\frac{|\mathrm{sm}(x, y)|}{\|z\|} = \frac{|\mathrm{sm}(y, z)|}{\|x\|} = \frac{|\mathrm{sm}(z, x)|}{\|y\|}. \tag{4.23}$$

Proof: We consider the triangle whose sides are x, y and $-z$. The Minkowski area of this triangle can be given using (4.17) by three ways to get

$$\frac{1}{2}\|x\|\|y\||\mathrm{sm}(x, y)| = \frac{1}{2}\|y\|\|z\||\mathrm{sm}(y, z)| = \frac{1}{2}\|z\|\|x\||\mathrm{sm}(z, x)|, \ \square \tag{4.24}$$

what completes the proof.

If we assume that the two dimensional space M_B^2 is oriented and each pairs (x, y), (y, z) and (z, x) is positively oriented then the sign in (4.24) may be neglected.

Definition 4.9 (The M-normal basis): If M_B^2 is a Minkowski plane with strictly convex centrally symmetric unit ball B then for each $x \in \partial B$ with $\|x\| = 1$, we define the left normal vector to x with the notation $x^{\dashv}$ which is unique vector if the following conditions are valid

(i) $\tilde{\sigma}(x^{\dashv}) = 1$, i.e. $x^{\dashv} \in \partial\tilde{I}(B)$,

(ii) The pair $(x, x^{\dashv})$ is positively oriented,

(iii) $x \dashv x^{\dashv}$, i.e. $\forall f \in (M_B^2)^*$ such that $f(x) = \|x\|\|f\|$ we have $f(x^{\dashv}) = 0$.

The pair $(x, x^{\dashv})$ is called a normal basis for M_B^2 see Figure 4.1.

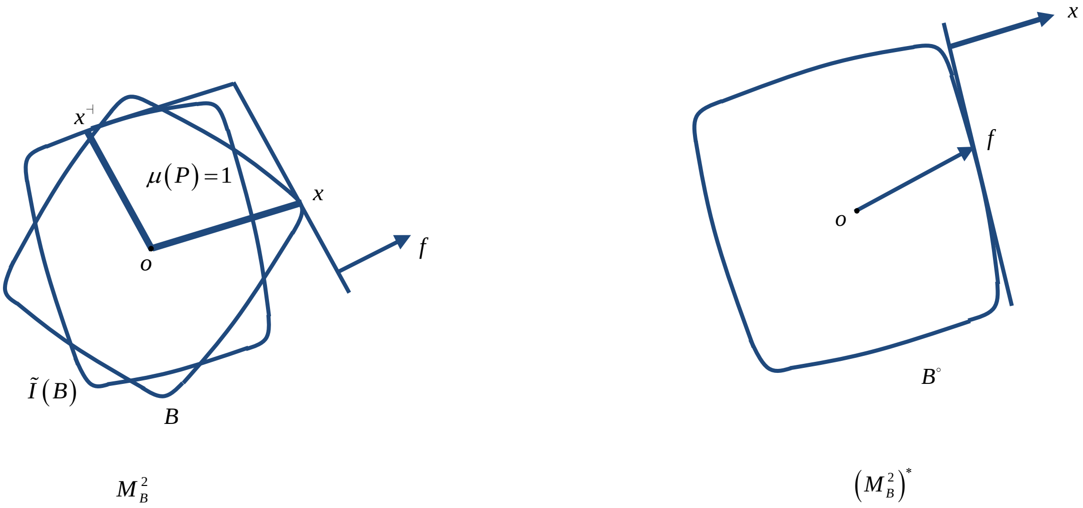

Figure 4.1

Proposition 4.10: If $\left(x_1, x_1^{\dashv}\right)$ is a normal basis then there exists a unique linear function g such that $g\left(x_1\right)=0$ and $g\left(x_1^{\dashv}\right)=\tilde{\sigma}\left(g\right)=\|g\|_{\tilde{I}^{\circ}}=1$, $\operatorname{sm}\left(x_1, x_2\right)=\pm g\left(x_2\right)$ where $x_2 \in \partial B$, the sign comes from the orientation of the pair $\left(x_1, x_2\right)$.

Proof: see [1 chapter 8]

Proposition 4.11: If $\left(x, x^{\dashv}\right)$ is a normal basis then the Minkowski area of the parallelogram spanned by x and $x^{\dashv}$ is equal to 1.

Proof: If $Q(x)$ denotes the required parallelogram which is spanned by x and $x^{\dashv}$ then from (4.17) we get

$$\mu\left(Q(x)\right)=\|x\|\left\|x^{\dashv}\right\|\operatorname{sm}\left(x, x^{\dashv}\right)=\left\|x^{\dashv}\right\|\operatorname{sm}\left(x, x^{\dashv}\right). \tag{4.25}$$

Using the function g as in the proposition 4.10, we have

$$\begin{aligned}\mu\left(Q(x)\right)&=\left\|x^{\dashv}\right\|\operatorname{sm}\left(x, x^{\dashv}/\left\|x^{\dashv}\right\|\right)\\&=\left\|x^{\dashv}\right\|g\left(x^{\dashv}/\left\|x^{\dashv}\right\|\right)\\&=g\left(x^{\dashv}\right)=1. \square\end{aligned}$$

Theorem 4.12: For all non-zero elements $x, y \in M_B^2$ we have

$$\operatorname{cm}(x, y)\operatorname{cm}(y, x)+\alpha\left(x^{\dashv}\right)^{-1}\alpha\left(y^{\dashv}\right)^{-1}\operatorname{sm}(x, y)\operatorname{sm}\left(x^{\dashv}, y^{\dashv}\right)=1. \tag{4.26}$$

Proof: Without loss of generality, we can assume that $\|x\|=\|y\|=1$ because the cosine and sine functions are independent of scalar multiples arguments. We assume two normal bases $(x, x^{\dashv})$ and $(y, y^{\dashv})$ for M_B^2 and try to calculate the 2×2 matrix $T := (t_{ij})$ such that $\begin{pmatrix} y \\ y^{\dashv} \end{pmatrix} = T\begin{pmatrix} x \\ x^{\dashv} \end{pmatrix}$; it has to preserve the Minkowski area because of proposition 4.11, therefore $\det(T)=1$. We define that matrix by

$$y = t_{11}x + t_{12}x^{\dashv}, \tag{4.27}$$

$$y^{\dashv} = t_{21}x + t_{22}x^{\dashv}. \tag{4.28}$$

Since $\det(T)=1$ then

$$T^{-1} = \begin{pmatrix} t_{22} & -t_{12} \\ -t_{21} & t_{11} \end{pmatrix}. \tag{4.29}$$

If f_x is a linear function, with $\|f_x\|=1$, supports the unit ball B at x then applying f_x to the both sides of (8.27) an using definition 4.9, we get

$$\begin{aligned} f_x(y) &= t_{11}f_x(x) + t_{12}f_x(x^{\dashv}) \\ &= t_{11}, \end{aligned}$$

therefore,

$$t_{11} = \mathrm{cm}(x, y). \tag{4.30}$$

Similarly, we take the function g as in proposition 4.10 and apply it to the both sides of (8.6) to get

$$g(y) = t_{11}g(x) + t_{12}g(x^{\dashv}) = t_{12},$$

then,

$$t_{12} = \mathrm{sm}(x, y). \tag{4.31}$$

Form (4.29) we have

$$x = t_{22}y - t_{12}y^{\dashv}. \tag{4.32}$$

If f_y is a linear functional likewise f_x with norm 1 and supports the unit ball B at y, we can apply this function to (4.32) then we get similarly

$$t_{22} = \mathrm{cm}(y,x). \tag{4.33}$$

Now we apply the function f_x to (4.28) to get

$$t_{21} = f_x\left(y^{\dashv}\right). \tag{4.34}$$

Since $f_x\left(x^{\dashv}\right)=0$ then from (4.11) we have

$$\mathrm{sm}\left(x^{\dashv}, y^{\dashv}\right) = \frac{-f_x\left(y^{\dashv}\right)}{\left\|y^{\dashv}\right\|\tilde{\sigma}(f_x)}, \tag{4.35}$$

the minus sign comes from the orientation of $x^{\dashv}$ and $y^{\dashv}$, hence

$$t_{21} = -\left\|y^{\dashv}\right\|\tilde{\sigma}(f_x)\,\mathrm{sm}\left(x^{\dashv}, y^{\dashv}\right). \tag{4.36}$$

From the definition 4.6 and the fact that $\left\|y^{\dashv}\right\|_{\tilde{I}} = 1$ then

$$t_{21} = -\alpha\left(y^{\dashv}\right)^{-1}\alpha\left(x^{\dashv}\right)^{-1}\mathrm{sm}\left(x^{\dashv}, y^{\dashv}\right). \tag{4.36}$$

And hence, (4.30), (4.31), (4.33) and (36) give the matrix T as follows

$$T = \begin{pmatrix} \mathrm{cm}(x,y) & \mathrm{sm}(x,y) \\ -\bar{\alpha}_{x,y}\,\mathrm{sm}\left(x^{\dashv}, y^{\dashv}\right) & \mathrm{cm}(y,x) \end{pmatrix}, \tag{4.37}$$

where,

$$\bar{\alpha}_{x,y} = \alpha\left(y^{\dashv}\right)^{-1}\alpha\left(x^{\dashv}\right)^{-1}. \tag{4.38}$$

Therefore the formula (4.26) is valid from the fact that $\det(T)=1$.

Corollary 4.13: We can replace in (4.26) the functions $\alpha\left(x^{\dashv}\right)^{-1}$ and $\alpha\left(y^{\dashv}\right)^{-1}$ by $\left\|x^{\dashv}\right\|$ and $\left\|y^{\dashv}\right\|$ respectively.

Proof: Since all functions in (4.26) are homogenous functions of degree zero then we can suppose that $x, y \in \partial \tilde{I}$ then the proof is clear from (4.22).

Theorem 4.14: If $x, y, z \in M_B^2$ are non-zero vectors we have

(i) $$\operatorname{cm}(x,z) = \operatorname{cm}(y,z)\operatorname{cm}(x,y) - \alpha\left(x^{\dashv}\right)^{-1}\alpha\left(y^{\dashv}\right)^{-1}\operatorname{sm}(y,z)\operatorname{sm}\left(x^{\dashv}, y^{\dashv}\right),$$

(ii) $$\operatorname{sm}(x,z) = \operatorname{cm}(y,z)\operatorname{sm}(x,y) + \operatorname{sm}(y,z)\operatorname{cm}(y,x),$$

(iii) $$\alpha\left(y^{\dashv}\right)\operatorname{sm}\left(x^{\dashv}, z^{\dashv}\right) = \alpha\left(x^{\dashv}\right)\operatorname{sm}\left(y^{\dashv}, z^{\dashv}\right)\operatorname{cm}(x,y) + \alpha\left(z^{\dashv}\right)\operatorname{sm}\left(x^{\dashv}, y^{\dashv}\right)\operatorname{cm}(z,y).$$

Proof: If $T:\begin{pmatrix} x \\ x^{\dashv} \end{pmatrix} \to \begin{pmatrix} y \\ y^{\dashv} \end{pmatrix}$ and $S:\begin{pmatrix} y \\ y^{\dashv} \end{pmatrix} \to \begin{pmatrix} z \\ z^{\dashv} \end{pmatrix}$ then $ST:\begin{pmatrix} x \\ x^{\dashv} \end{pmatrix} \to \begin{pmatrix} z \\ z^{\dashv} \end{pmatrix}$ where,

$$T = \begin{pmatrix} \operatorname{cm}(x,y) & \operatorname{sm}(x,y) \\ -\bar{\alpha}_{x,y}\operatorname{sm}\left(x^{\dashv}, y^{\dashv}\right) & \operatorname{cm}(y,x) \end{pmatrix},$$

$$S = \begin{pmatrix} \operatorname{cm}(y,z) & \operatorname{sm}(y,z) \\ -\bar{\alpha}_{y,z}\operatorname{sm}\left(y^{\dashv}, z^{\dashv}\right) & \operatorname{cm}(z,y) \end{pmatrix},$$

$$\begin{aligned} ST &= \begin{pmatrix} \operatorname{cm}(x,z) & \operatorname{sm}(x,z) \\ -\bar{\alpha}_{x,z}\operatorname{sm}\left(x^{\dashv}, z^{\dashv}\right) & \operatorname{cm}(z,x) \end{pmatrix} \\ &= \begin{pmatrix} \operatorname{cm}(x,y) & \operatorname{sm}(x,y) \\ -\bar{\alpha}_{x,y}\operatorname{sm}\left(x^{\dashv}, y^{\dashv}\right) & \operatorname{cm}(y,x) \end{pmatrix}\begin{pmatrix} \operatorname{cm}(y,z) & \operatorname{sm}(y,z) \\ -\bar{\alpha}_{y,z}\operatorname{sm}\left(y^{\dashv}, z^{\dashv}\right) & \operatorname{cm}(z,y) \end{pmatrix}, \end{aligned}$$

Therefore the statements (i), (ii) and (iii), can be found directly by comparing the component of the matrix ST.

The following proposition gives the direct relation between the cosine and sine Minkowski functionals which looks like the usual complement law in Euclidean Space.

Proposition 4.15: If $x, y \in M_B^2$ are non-zero vectors then

$$\operatorname{sm}\left(x^{\dashv}, y\right) = \alpha\left(x^{\dashv}\right)\operatorname{cm}(x,y). \tag{4.39}$$

Proof: It is clear from the definition of the cosine and sine Minkowski functional that if f_x supports B at x, then

$$\operatorname{cm}(x, y) = \frac{f_x(y)}{\|y\| \|f_x\|},$$

$$\operatorname{sm}(x^{\dashv}, y) = \frac{f_x(y)}{\|y\| \tilde{\sigma}(f_x)}.$$

Using (4.21) we get

$$\alpha(x^{\dashv}) = \frac{\|f_x\|}{\tilde{\sigma}(f_x)},$$

By substituting the value of $\tilde{\sigma}(f_x)$ into the previous formulae we can complete the proof.

Chapter 5

Ruled Surfaces in Minkowski Three-dimensional space

5.1 Introduction

The study of ruled surface in Euclidean space $\mathbb{R}^3$ is classical subject in differential geometry. Recently such surfaces occur also in some modern areas in mathematics (e.g. Projective Geometry, Computer Aided Geometric Design, see e.g. [57] and [58]. Line Geometry, especially the theory of ruled surfaces has been applied to Kinematics and Robotics and to spatial mechanisms in $\mathbb{R}^3$ [59]. Tracing back to 19th century works of Steiner [60] the study of closed one-parameter sets of spatial motions became an interesting topic in Kinematics and Robotics, see e.g. E.A. Weiß [61], H.R. Müller [62]. Global differential geometric concepts for ruled surfaces stem from A. Holditch [63] and J. Hoschek [21]; Pottmann [57] and [58] are more recent publications to this theme.

A ruled surface can be generated by sweeping a line through space. Developable surfaces are special cases of ruled surfaces [58], they allow a (locally) isometric mapping to a part of the Euclidean plane. Cylinders, cones and tangent surfaces of curves are (the only) examples of developable surfaces. They consist of *torsal generators* only, i. e. generators, where the tangent plane in each regular point is the same plane. Each generator has at most one singular point, which is called *cusp*, the regular points are *parabolic* points. Consequently the Gaussian curvature of these developable ruled surfaces is zero in every regular point. Non developable ruled surfaces are called *skew* surfaces. They may contain isolated torsal generators, too. Non torsal generators are called skew and they consist of regular points only; all the points of a skew generator are hyperbolic points, i.e. there the Gaussian curvature of the skew surface is negative. The concepts “regular”, “skew”, “torsal”, ”hyperbolic” and “parabolic” belong to Projektive Geometry. Thus they are the same if we consider a ruled surface in Euclidean space or in Minkowski spaces.

Ruled surfaces were investigated by G. Monge, who established the partial differential equation satisfied by all ruled surfaces, see also A. Sarıoğlugil [22]. Ruled surfaces considered as one-parameter set of lines (whereby “lines” are the basic elements of the so-called line space) have been investigated by V. Hlavaty [20], E.A.Weiß [61] and J. Hoschek [21]. We will follow a treatment of ruled surfaces due to E. Kruppa [17], who considered them a two-dimensional surface in the Euclidean space and found, that they, in some sence, generalise the theory of space curves.

We aim at investigating ruled surfaces Φ in an Minkowski 3-space M_B^3 in the sense of Kruppa. For this we need to define an orthonormal frame in an Minkowski 3-space M_B^3; (for this we use the Birkhoff orthogonality concept). There is special curve on the ruled surface uniquely defined by the property that its points are the touching points of one plane of the frame, the *central plane*. This distinguished curve is named *central curve* or *striction curve*. The normed direction vectors of the generators of Φ describe the *spherical image* of Φ, which is, in general, a curve on the Euclidean or the Minkowski unit sphere *B*. In the following we exclude *cylindrical surfaces* from our considerations, as their spherical image is just a point and not a regular curve.

Finely we will define a so-called *deformation vector* which describes the deviation of the Minkowski space from a Euclidean space. Then we use it to define the Kruppa-Sannia frame in M_B^3 connected to the striction curve $c(t)$ of Φ to formulate the Frenet-Serret derivation equations.

Like in the Euclidean case an orthogonal frame in an Minkowski 3-space with Birkhoff orthogonality, see chapter 2, is based on a given (oriented) flag (P, g, α) of incident half-spaces, namely point P, half-line g and half-plane α, (see Figure 5.1). We translate this flag such that P becomes the centre of the unit ball B and we intersect g and α with B, receiving intersection point G and arc *a*. The half-tangent *t* of arc *a* at *G* translated through *P* represents the 2nd leg of the frame, it intersects B in T and $\overrightarrow{PG} =: e_1$ and $\overrightarrow{PT} =: e_2$ are left-orthogonal unit vectors. Translating α such that it touches B gives two possibilities for a point of contact N. We choose this one, such that $\{e_1, e_2, \overrightarrow{PN} =: e_3\}$ forms a right handed (affine) frame. For this frame we therefore have $e_1 \dashv e_2, e_3 \dashv e_1$ and $e_3 \dashv e_2$.

Especially for (not cylindrical) ruled surfaces $\Phi = \{g(t), t \in I \subset \mathbb{R}\}$ there is a canonically defined flag connected with each (oriented) generator $g(t_\circ)$. It consists of g itself, the asymptotic plane α parallel to direction vectors $g(t_\circ)$ and $\dot{g}(t_\circ)$. As the point P of the flag we use the point of contact of the so-called central plane $\zeta = g \vee n_M$, whereby n_M denotes the Minkowski-normal vector to α. This point s obviously has to be called "Minkowski-striction point" of generator $g(t_\circ)$. All those points $P(t)$ form a curve S, the Minkowski striction curve. In the following we want to show that the characteristic property of an Euclidean striction line namely that the generators form a geodesic parallel field along S, also yields in the Minkowski case.

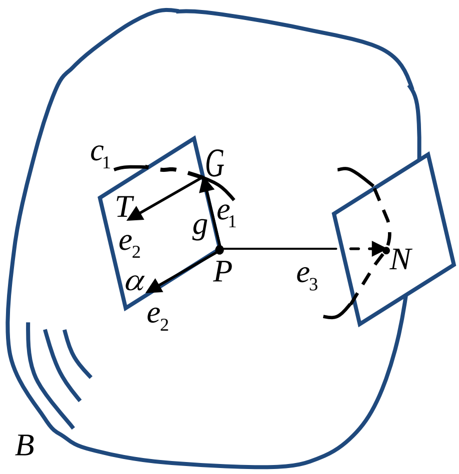

Figure 5.1

5.2 Ruled surfaces and frame construction of Minkowski 3-space

Definition 5.1: A *ruled surface* $\phi(u,v)$ is a surface that can be swept by moving a line in the space which can be parameterized in the following way:

$$\phi(u,v)=\left\{x(u,v)\middle|x=P(u)+v\,e(u)\right\} \tag{5.1}$$

In (5.1) the function $P(u)$ describes a so-called *director curve*, and $e(u)\neq 0$ is the direction vector of the (oriented) generator or ruling $R(u)\subset\Phi$ and we norm $e(u)$ in the sense of the Minkowski norm with respect to the given (smooth and strictly convex) unit ball B. A ruling is called *torsal* iff at all its regular points we have the same unit normal vector (same tangent plane). The analytic condition for a torsal ruling is

$$\det\left(\dot{P}(u),e(u),\dot{e}(u)\right)=0 \tag{5.2}$$

A ruled surface with only torsal rulings is a developable surface. This is a surface which can be mapped isometrically into the Euclidean plane. The surfaces which may have a finite number of torsal rulings, but all other generators are non-torsal, are called skew surfaces. It is well known that all points of a non-torsal ruling R are regular and two different points have also different tangent planes, (see Figure 5.2). Any plane through R is a tangent plane at some point (which might be at infinity in case that the tangent plane is the so-called asymptotic plane) see Pottmann and Wallner [58]. A torsal generator $R(u)\subset\Phi$ is called *cylindrical,* if $\{e(u),\dot{e}(u)\}$ are linear dependent. In the following we will consider only those non-empty and continuous parts of a ruled surface, which are free of cylindrical generators.

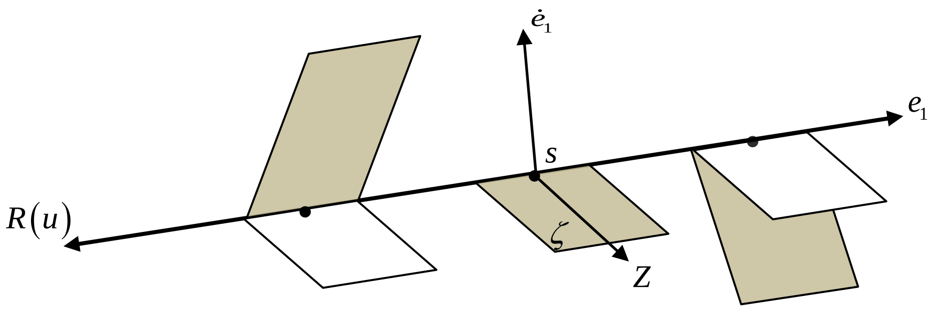

Figure 5.2

The purpose of the following algorithm is to find or to define local invariants of ruled surfaces on the basis of a non-symmetric (left) orthogonality concept, in analogy to the classical Kruppa-invariants [17] of ruled surfaces in the Euclidean 3-space. This construction is of course depending on the shape of the unit ball of the Minkowski space which can be defined by using either the support function, see chapter 3, or the vector representation. We will present a method for constructing the Minkowski frame on the unit ball and show, how this can be applied to design the ruled surface.

To construct that frame, we begin with the unit vector $e(u)$ parallel to the (oriented) ruling $R(u)$ and fix it at the centre of the unit ball B, and hence we find the vector $\dot{e}(u)$ which is tangent to the spherical image of Φ and therefore contained in the supporting plane (tangent plane) of B at the end point p of $e(u)$, see Figure 5.3. a, b.

Now, we have two vectors $e(u)$ and $\dot{e}(u)$ spanning a plane parallel to the asymptotic plane $\alpha(u)$ of $R(u)$. Thereby $e(u) \dashv \dot{e}(u)$ and $\alpha(u)$ touches Φ at the point of infinity of $R(u)$. The supporting plane $\bar{\alpha}$ of B parallel to α touches B in a point Z. Therewith we have the third vector Z, the *central tangent vector*, which is left-orthogonal to $e(u)$ and $\dot{e}(u)$. Now we have a B-orthomormal frame ($o; e\dot{e}Z$) with $e(u) \dashv \dot{e}(u), Z \dashv e(u)$ and $Z \dashv \dot{e}(u)$. The plane $\zeta(u) \supset R(u)$ parallel to the plane spanned by $\{e(u), Z(u)\}$ is called the M-*central plane* of $R(u)$.

5.3 Striction curve in Minkowski 3-space

Definition 5.2: The M-*striction point* $s(u)$ of a ruling $R(u)$ of a skew ruled surface Φ in M_B^3 is the touching point of the M-central plane $\zeta(u)$ of $R(u)$ with Φ. Thereby $\zeta(u)$ is the plane through $R(u)$ left-orthogonal to the asymptotic plane $\alpha(u)$ of $R(u)$. The set of all M-striction points is called the ´´M-striction curve´´ of Φ.

One can parameterize the asymptotic plane as follows:

$$\alpha(u,v) = P(u) + \lambda e(u) + \mu \dot{e}(u) \quad . \tag{5.3}$$

It is the tangent plane at the ideal point of $R(u)$ (if we think of the projective extension of the (affine) Minkowski space. The general tangent plane τ of the ruled surface $\phi(u,v)$ is spanned by the partial derivative vectors x_u and x_v as

$$\tau(u,v) = P(u) + \lambda\left(\dot{P}(u) + v\dot{e}(u)\right) + \mu e(u) \tag{5.4}$$

Moreover, we have the unit ball B which is described either by supporting function or vector representation e.g. $b(\phi,\psi)$ (the unit vector representation) which is central symmetric ($b(-\phi,-\psi) = -b(\phi,\psi)$).

Let ε be the tangent plane of the unit ball B which is described as

$$\varepsilon(\phi,\psi) = b(\phi,\psi) + \lambda b_\phi + \mu b_\psi \tag{5.5}$$

The asymptotic plane α parallel to the tangent plane ε of the unit ball B then,

$$\left.\begin{aligned}\det\left(b_{\phi},e(u),\dot{e}(u)\right)=0\\ \det\left(b_{\psi},e(u),\dot{e}(u)\right)=0\end{aligned}\right\} \tag{5.6}$$

By solving these equations, we can find the corresponding parameters $\phi_{\circ}(u)$ and $\psi_{\circ}(u)$ which give the vector $b(\phi_{\circ},\psi_{\circ})$ to be parallel to the central tangent vector and hence we can span the central plane ζ as

$$\zeta(u)=P(u)+\beta e(u)+\gamma b\left(\phi_{\circ}(u),\psi_{\circ}(u)\right) \tag{5.7}$$

In terms of the ruled surface, the central tangent vector Z is tangent to the surface at the central point or striction point. So, the main question to express a suitable formula of the striction curve of the ruled surface in Minkowski space is how we can specialise u and v in tangent plane τ such that $(u_{\circ},v_{\circ})$ is point of contact of ζ .

To answer this pervious question, we assume the central plane ζ with fixed ruling at parameter $u_{\circ}$

$$\zeta(u_{\circ})=P(u_{\circ})+\beta e(u_{\circ})+\gamma b\left(\phi_{\circ}(u_{\circ}),\psi_{\circ}(u_{\circ})\right) \tag{5.8}$$

The striction point $S_M(u_{\circ},v_{\circ})=P(u_{\circ})+v_{\circ}e(u_{\circ})$ with parameter $v_{\circ}$ calculated from the fact that the central tangent vector $b(u_{\circ})$ lies in the tangent plane. Then,

$$\det\left(e,\dot{P}+v_{\circ}\dot{e},b\right)=0, \tag{5.9}$$

where $b(u_{\circ})$ is dependent only on the parameter $u_{\circ}$. Therefore, we obtain that,

$$v_{\circ}=-\frac{\det\left(e,\dot{P},b\right)}{\det\left(e,\dot{e},b\right)}. \tag{5.10}$$

Inserting (5.10) into the equation of striction curve to obtain

$$S_M(u_{\circ})=P(u_{\circ})-\frac{\det\left(e,\dot{P},b\right)}{\det\left(e,\dot{e},b\right)}e(u_{\circ}). \tag{5.11}$$

In the special case of Euclidean space the central tangent vector $b(u_{\circ})$ is $e(u_{\circ})\times\dot{e}(u_{\circ})$ and hence,

$$v_{\circ}=-\frac{\dot{P}\cdot\dot{e}}{\dot{e}^2}, \tag{5.12}$$

and the striction curve in the Euclidean space is

$$S_E(u_\circ) = P(u_\circ) - \frac{\dot{P}\cdot\dot{e}}{\dot{e}^2}e(u_\circ). \tag{5.13}$$

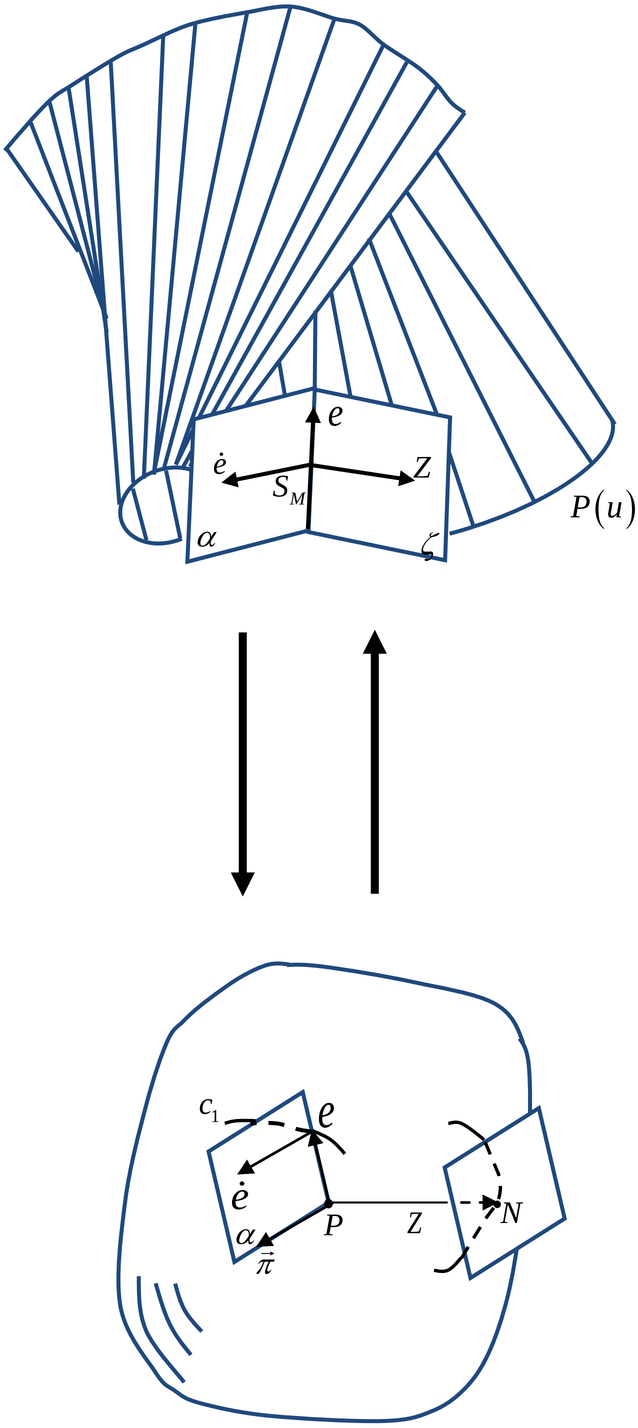

Figure 5.3 a, b

5.4 Deformation vectors in a Minkowski plane $\mathbf{M}_{\mathbf{B}}^{\mathbf{2}}$

In this section we define a new vector called *Deformation vector* which will help us to find the Minkowski version of the Frenet-Serret formulae for the moving frame of a ruled surface in a Minkowski space.

Definition 5.3 (Deformation vector): Let $M_{B_2}^2$ be a Minkowski plane with smooth and strictly convex centrally symmetric unit ball B_2. Then the deformation vector $\tilde{x}$ of any normed vector $x \in M_{B_2}^2$ is defined as

$$\tilde{x} = \frac{\left(x^{\dashv}\right)^{\dashv}}{\left\|\left(x^{\dashv}\right)^{\dashv}\right\|} . \tag{5.14}$$

Note that the pair $\left(x, x^{\dashv}\right)$ is a B-orthonormal basis for $M_{B_2}^2$, see Figure 5.4.

Theorem 5.4: Let $M_{B_2}^2$ be a Minkowski plane with smooth, strictly convex and centrally symmetric unit ball B_2. Let $y := x^{\dashv}\left\|x^{\dashv}\right\|^{-1}$ be any normed vector of $M_{B_2}^2$, then its left-orthogonal vector $y^{\dashv}$ is described by the formula

$$y^{\dashv} = \left\|x^{\dashv}\right\|\left\{-x + y\, cm\left(x^{\dashv}, x\right)\right\}.$$

Proof: Let the vector $x \in M_{B_2}^2$, and the normal basis $\left(x, x^{\dashv}\right)$. If we assume $y \in M_{B_2}^2$, $y = \dfrac{x^{\dashv}}{\left\|x^{\dashv}\right\|}$, then, we have the deformation vector which can be considered as a linear combination of the normal basis $\left(x, x^{\dashv}\right)$.

$$y^{\dashv} = A_1 x + A_2 x^{\dashv}. \tag{5.15}$$

Multiply both sides of (5.15) by $x^{\dashv}$ from left as Minkowski semi-inner product (4.6), we have

$$\left\langle x^{\dashv}, y^{\dashv}\right\rangle_M = \left\langle x^{\dashv}, A_1 x + A_2 x^{\dashv}\right\rangle_M , \tag{5.16}$$

by using the definition and additive property of the Minkowski semi-inner product and $\left\langle x^{\dashv}, y^{\dashv}\right\rangle_M = 0$. Then

$$0 = A_1 \left\langle x^{\dashv}, x\right\rangle_M + A_2 \left\|x^{\dashv}\right\|^2 , \tag{5.17}$$

and hence,

$$A_2 = -\frac{A_1 cm\left(x^{\dashv}, x\right)}{\left\|x^{\dashv}\right\|}. \tag{5.18}$$

We know that Minkowski plane $M_{B_2}^2$ is spanned by the pairs $\left(x, x^{\dashv}\right)$ or $\left(y, y^{\dashv}\right)$, therefore the Minkowski area $\mu\left(P\right)$ of the parallelogram P spanned by y and $y^{\dashv}$ is

$$\mu\left(P\right) = \|y\|\left\|y^{\dashv}\right\| sm\left(y, y^{\dashv}\right) = \left\|y^{\dashv}\right\| sm\left(y, y^{\dashv}\right). \tag{5.19}$$

Since, $sm\left(y, y^{\dashv}\right) = \frac{1}{\left\|y^{\dashv}\right\|}$, then by substituting $y^{\dashv}$ from equation (5.15) into (5.18), we have

$$\begin{aligned} \mu\left(P\right) &= \frac{1}{\left\|y^{\dashv}\right\|} = sm\left(y, y^{\dashv}\right) \\ &= sm\left(y, A_1 x + A_2 x^{\dashv}\right) \end{aligned} \tag{5.20}$$

Using theorem 4.14 (ii), we have the fact that, for all $x_1, x_2, x_3 \in M_{B_2}^2 - \{0\}$ we have

$$sm\left(x_1, x_3\right) = cm\left(x_2, x_3\right) sm\left(x_1, x_2\right) + cm\left(x_2, x_1\right) sm\left(x_2, x_3\right). \tag{5.21}$$

If we take $x_1 = y$, $x_2 = A_1 x$ and $x_3 = y^{\dashv}$, then (5.20) becomes

$$\begin{aligned} \frac{1}{\left\|y^{\dashv}\right\|} = sm\left(y, y^{\dashv}\right) &= cm\left(A_1 x, y^{\dashv}\right) sm\left(y, A_1 x\right) + cm\left(A_1 x, y\right) sm\left(A_1 x, y^{\dashv}\right) \\ &= -cm\left(x, y^{\dashv}\right) \frac{1}{\left\|x^{\dashv}\right\|}. \end{aligned} \tag{5.22}$$

Now it is easy to calculate the vector $y^{\dashv}$ by multiplying again both sides of (5.15) by x from left as Minkowski semi-inner product, we have

$$A_1 = \left\|y^{\dashv}\right\| cm\left(x, y^{\dashv}\right) = -\left\|x^{\dashv}\right\|. \tag{5.23}$$

Therefore, substituting (5.22) into (5.23) and into (5.18), we find that

$$y^{\dashv} = \left\|x^{\dashv}\right\|\left\{-x + y\, cm\left(x^{\dashv}, x\right)\right\}. \; \square \tag{5.24}$$

Without loss of generality, we can use the notation $\tilde{x}_y$ instead of the notation $\dfrac{y^{\dashv}}{\left\| x^{\dashv} \right\|}$

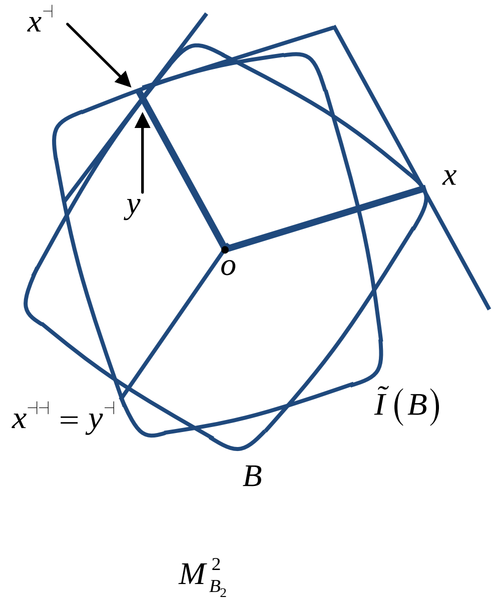

Figure 5.4

5.5 Frenet-Serret frame in Minkowski 3-space:

Let $e(s_1)$ be a unit vector which depends only on one parameter s_1. By attaching this vector at the origin O of a fixed (affine) frame of a Minkowski space M_B^3 we receive the spherical image c_1 of the ruled surface in consideration Φ at the unit sphere B and we call the cone $O \vee c_1$ the "direction cone" of Φ. Without loss of generality we can assume that the parameter s_1 is a "Minkowski arc length parameter" of the curve c_1, , i.e. the derivation vector $e'(s_1)$ is normed all over de definition interval of s_1. The direction cone of Φ takes the form

$$\chi = v\, e(s_1) \tag{5.25}$$

Based on the right handed B-orthonormal (affine) frame $\{e, \pi, z\}$ we have $e \dashv e'$, $z \dashv e$ and $z \dashv e'$, and therefore $\langle e, e' \rangle_M = 0, \langle z, e \rangle_M = 0$ and $\langle z, e' \rangle_M = 0$.

The derivatives of the vectors $\{e, e', z\}$ must be linear combinations of these vectors. The formulae for these expressions are usually called the Frenet-Serret derivation equations of a moving frame.

We can assume that the vector $e'(s_1) = \pi$ is defined by the first equation of the three Frenet-Serret equations. We state that the third equation must be the same for Minkowski cases as well as for the Euclidean case, because the unit vector $e(s_1)$ is left-orthogonal to the derivative vector $e'(s_1)$ as in the Euclidean case.

The derivative of the unit vector z can be obtained as a linear combination of the three B-orthonormal vectors $\{e,\pi,z\}$ as follows,

$$z' = B_1 e + B_2 \pi + B_3 z . \tag{5.26}$$

Multiplying both sides of (5.26) by z from left as Minkowski semi-inner product and using the properties of it we get $B_3 = 0$, where $z \dashv z'$ (z' lies in the supporting plane of the unit ball B at z). Therefore

$$z' = B_1 e + B_2 \pi . \tag{5.27}$$

By the same manner, we multiply both sides of (5.27) by e, we can find $B_1 = \langle e, z' \rangle_M$. Then

$$z' = \langle e, z' \rangle_M \, e + B_2 \pi . \tag{5.28}$$

By using the deformation vector $\tilde{e}_\pi$ of the vector $\vec{e}$ in the plane $(o\vec{e}\vec{\pi})$, the vector $\vec{\pi}$ becomes

$$\pi = \frac{e + \tilde{e}_\pi}{cm(\pi, e)} . \tag{5.29}$$

Then, we can rewrite (5.28) as follows

$$z' = \left[\langle e, z' \rangle_M + \frac{B_2}{cm(\pi, e)} \right] e + B_2 \frac{\tilde{e}_\pi}{cm(\pi, e)} . \tag{5.30}$$

Multiply again both sides of (5.30) by π from left as Minkowski semi-inner product, we can easily compute the constant $B_2 = \|z'\| \{ cm(\pi, z') - cm(e, z') cm(\pi, e) \}$, we have from (5.28)

$$z' = \|z'\| \{ cm(e, z') e + [cm(\pi, z') - cm(e, z') cm(\pi, e)] \pi \} . \tag{5.31}$$

By the same method, we can assume that the vector π' lies in the plane which contains the vectors $\tilde{e}_\pi$ and $\tilde{z}_\pi$, hence we can describe it as linear combination of that vectors,

$$\pi' = d_1 \tilde{e}_\pi + d_2 \tilde{z}_\pi , \tag{5.32}$$

where

$$\left.\begin{aligned}\tilde{e}_\pi &= -e + \pi\, cm(\pi, e),\\ \tilde{z}_\pi &= -z + \pi\, cm(\pi, z)\end{aligned}\right\}. \tag{5.33}$$

It is clear to calculate the constants d_1 and d_2 as follows

$$\left.\begin{aligned}d_1 &= \frac{\|\pi'\|}{\|\tilde{e}_\pi\|}\frac{cm(\tilde{e}_\pi, \pi') - cm(\tilde{z}_\pi, \pi')cm(\tilde{e}_\pi, \tilde{z}_\pi)}{1 - cm(\tilde{z}_\pi, \tilde{e}_\pi)cm(\tilde{e}_\pi, \tilde{z}_\pi)},\\ d_2 &= \frac{\|\pi'\|}{\|\tilde{z}_\pi\|}\frac{cm(\tilde{z}_\pi, \pi') - cm(\tilde{e}_\pi, \pi')cm(\tilde{z}_\pi, \tilde{e}_\pi)}{1 - cm(\tilde{z}_\pi, \tilde{e}_\pi)cm(\tilde{e}_\pi, \tilde{z}_\pi)}\end{aligned}\right\}. \tag{5.34}$$

Then (5.32) becomes

$$\begin{aligned}\pi' = \|\pi'\| \Bigg\{ &\frac{cm(\tilde{e}_\pi, \pi') - cm(\tilde{z}_\pi, \pi')cm(\tilde{e}_\pi, \tilde{z}_\pi)}{-H(\tilde{z}_\pi, \tilde{e}_\pi)\|\tilde{e}_\pi\|}\vec{e} + \\ &\Bigg\{\frac{\left(cm(\tilde{e}_\pi, \pi') - cm(\tilde{z}_\pi, \pi')cm(\tilde{e}_\pi, \tilde{z}_\pi)\right)cm(\pi, e)}{H(\tilde{z}_\pi, \tilde{e}_\pi)\|\tilde{e}_\pi\|} + \\ &\frac{\left(cm(\tilde{z}_\pi, \pi') - cm(\tilde{e}_\pi, \pi')cm(\tilde{z}_\pi, \tilde{e}_\pi)\right)cm(\pi, z)}{H(\tilde{z}_\pi, \tilde{e}_\pi)\|\tilde{z}_\pi\|}\Bigg\}\vec{\pi} + \\ &\frac{cm(\tilde{z}_\pi, \pi') - cm(\tilde{e}_\pi, \pi')cm(\tilde{z}_\pi, \tilde{e}_\pi)}{-H(\tilde{z}_\pi, \tilde{e}_\pi)\|\tilde{z}_\pi\|}\, z\Bigg\}\end{aligned} \tag{5.35}$$

where, $H(\tilde{z}_\pi, \tilde{e}_\pi) = 1 - cm(\tilde{z}_\pi, \tilde{e}_\pi)cm(\tilde{e}_\pi, \tilde{z}_\pi)$.

Now, we consider $x(s)$ describing the space curve c with arc length s. The tangent vector $t = x'$ can be moved into the unit sphere S to obtain the spherical image c_1, the cone (oc_1) has generators which are parallel to the tangent of the curve c. As for the previous construction, we have the right handed orthonormal (affine) frame $\{t, h, b\}$, h is called the principal normal vector and b is the binormal, without loss of generality, we can consider that the derivatives of those vectors are unit vectors and the plane $(t\,h)$ is the osculating plane, the plane $(h b)$ is the normal plane. By using the derivative of these vectors as before, we have

$$\frac{dt}{ds_1} = h, \tag{5.36}$$

$$\frac{dh}{ds_1} = \frac{\mathrm{cm}\left(\tilde{b}_h, h'\right)\mathrm{cm}\left(\tilde{t}_h, \tilde{b}_h\right) - \mathrm{cm}\left(\tilde{t}_h, h'\right)}{H\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{t}_h\right\|} t +$$
$$\left\{ \frac{\left(\mathrm{cm}\left(\tilde{t}_h, h'\right) - \mathrm{cm}\left(\tilde{b}_h, h'\right) cm\left(\tilde{t}_h, \tilde{b}_h\right)\right)\mathrm{cm}\left(h, t\right)}{H\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{t}_h\right\|} + \right.$$
$$\left. \frac{\left(\mathrm{cm}\left(\tilde{b}_h, h'\right) - \mathrm{cm}\left(\tilde{t}_h, h'\right)\mathrm{cm}\left(\tilde{b}_h, \tilde{t}_h\right)\right)\mathrm{cm}\left(h, b\right)}{H\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{b}_h\right\|} \right\} h + \tag{5.37}$$
$$\frac{\mathrm{cm}\left(\tilde{t}_h, h'\right)\mathrm{cm}\left(\tilde{b}_h, \tilde{t}_h\right) - \mathrm{cm}\left(\tilde{b}_h, h'\right)}{H\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{b}_h\right\|} b$$

$$\frac{db}{ds_1} = \mathrm{cm}\left(t, b'\right) t + \left[\mathrm{cm}\left(h, b'\right) - \mathrm{cm}\left(t, b'\right)\mathrm{cm}\left(h, t\right)\right] h\,. \tag{5.38}$$

We insert the following abbreviations and notations:

$\dfrac{ds_1}{ds} =: \chi \rightarrow$ M-curvature,

$-\mathrm{cm}\left(h, b'\right) =: \chi_2 \rightarrow$ conical curvature,

$\mathrm{cm}\left(t, b'\right) =: \chi_3 \rightarrow$ second conical curvature,

$\mathrm{cm}\left(\tilde{b}_h, h'\right) =: \tilde{\chi}_4 \rightarrow$ third conical curvature,

$\mathrm{cm}\left(\tilde{t}_h, h'\right) =: \tilde{\chi}_5 \rightarrow$ fourth conical curvature .

Multiplying both sides of the equations (5.36), (5.37) and (5.38) with the pervious functions, we similarly get Minkowski analogues to the classical torsion functions as follows:

$\chi\chi_2 =: \tau_1 \rightarrow$ M-torsion,

$\chi\chi_3 =: \tau_2 \rightarrow$ second torsion,

$\chi\tilde{\chi}_4 =: \tilde{\tau}_3 \rightarrow$ third torsion,

$\chi\tilde{\chi}_5 =: \tilde{\tau}_4 \rightarrow$ fourth torsion.

Therewith we can rewrite the Minkowski Frenet-Serret Formulae of any curve $x(s)$ in the final form as

$$t' = \chi h, \tag{5.39}$$

$$h' = \frac{\tilde{\tau}_3 \operatorname{cm}\left(\tilde{t}_h, \tilde{b}_h\right) - \tilde{\tau}_4}{H\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{t}_h\right\|} t + \left\{ \frac{\left(\tilde{\tau}_4 - \tilde{\tau}_3 \operatorname{cm}\left(\tilde{t}_h, \tilde{b}_h\right)\right)\operatorname{cm}(h,t)}{H\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{t}_h\right\|} + \frac{\left(\tilde{\tau}_3 - \tilde{\tau}_4 \operatorname{cm}\left(\tilde{b}_h, \tilde{t}_h\right)\right)\operatorname{cm}(h,b)}{H\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{b}_h\right\|} \right\} h + \frac{\tilde{\tau}_4 \operatorname{cm}\left(\tilde{b}_h, \tilde{t}_h\right) - \tilde{\tau}_3}{H\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{b}_h\right\|} b \tag{5.40}$$

$$b' = \tau_2 t - \left[\tau_1 + \tau_2 \operatorname{cm}(h,t)\right] h. \tag{5.41}$$

The coefficient functions of the M-Frenet-Serret formulae are the 2nd, 3rd and 4th curvatures and torsions. They have no geometric meaning in general but we can find such a meaning for some special unit balls *B*.

<u>The Euclidean case:</u>

In the case of an ellipsoid as unit ball, which refers to the Euclidean space, we can easily go to the usual Frenet-Serret formulae, whereby the M-curvatures and M-torsions occur as follows:

M-curvature = E-curvature $\left(\chi = \chi^e\right)$,

M-conical curvature = E-conical curvature $\left(\chi_2 = \chi_2^e\right)$,

2nd conical curvature $\chi_3 = 0$,

3rd conical curvature $\tilde{\chi}_4 = -\chi_2^e$,

4th conical curvature $\tilde{\chi}_5 = 1$,

M-torsion = E-torsion $\left(\tau_1 = \tau^e\right)$,

2nd torsion $\tau_2 = 0$,

3rd torsion $\tilde{\tau}_3 = -\tau^e$,

4th torsion $\tilde{\tau}_4 = \chi^e$ = E-curvature.

Therefore we get

$$\begin{pmatrix} t' \\ h' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \chi^e & 0 \\ -\chi^e & 0 & \tau^e \\ 0 & -\tau^e & 0 \end{pmatrix} \begin{pmatrix} t \\ h \\ b \end{pmatrix}. \tag{5.42}$$

In addition to this direct limitation we could consider cases, where some of the Frenet-Serret frame vectors are mutually B-orthonormal. Also here we can expect simplified Frenet-Serret-equations. By using other M-invariants, e.g. linear combinations of the above mentioned, we can also find some simplifications, as we state in the following

Corollary 5.5: The Frenet-Serret formulae (5.39-41) can be written as

$$\begin{pmatrix} t' \\ h' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \chi & 0 \\ -\bar{\tau} & \bar{\tau}\,\mathrm{cm}(h,t) + \bar{\tau}_1\,\mathrm{cm}(h,b) & \bar{\tau}_1 \\ \tau_2 & -\left(\tau_1 + \tau_2\,\mathrm{cm}(h,t)\right) & 0 \end{pmatrix} \begin{pmatrix} t \\ h \\ b \end{pmatrix}, \tag{5.43}$$

whereby $\bar{\tau}$ and $\bar{\tau}_1$ are the following functions of the 3rd and 4th Minkowski torsions:

$$\bar{\tau} = \chi \frac{\mathrm{sm}\left(\tilde{b}_h, \vec{h}'\right)}{\mathrm{sm}\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{t}_h\right\|}\ ,\ \bar{\tau}_1 = \chi \frac{\mathrm{sm}\left(\tilde{t}_h, h'\right)}{\mathrm{sm}\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{b}_h\right\|} \tag{5.44}$$

Proof: Using (4.12), theorem 4.14 (i) and the fact that the three vectors h', $\tilde{t}_h$ and $\tilde{b}_h$ lie in the same plane then we have

$$\mathrm{cm}\left(\tilde{t}_h, h'\right) - \mathrm{cm}\left(\tilde{b}_h, h'\right)\mathrm{cm}\left(\tilde{t}_h, \tilde{b}_h\right) = \alpha(\tilde{b}_h^{\dashv})^{-1}\alpha(\tilde{t}_h^{\dashv})^{-1}\,\mathrm{sm}\left(\tilde{b}_h, h'\right)\mathrm{sm}\left(\tilde{b}_h^{\dashv}, \tilde{t}_h^{\dashv}\right)$$

$$= \frac{1}{\chi}\left(\tilde{\tau}_4 - \tilde{\tau}_3\,\mathrm{cm}\left(\tilde{t}_h, \tilde{b}_h\right)\right) \tag{5.45}$$

and

$$\mathrm{cm}\left(\tilde{t}_h, h'\right)\mathrm{cm}\left(\tilde{b}_h, \tilde{t}_h\right) - \mathrm{cm}\left(\tilde{b}_h, h'\right) = \alpha(\tilde{b}_h^{\dashv})^{-1}\alpha(\tilde{t}_h^{\dashv})^{-1}\,\mathrm{sm}\left(\tilde{t}_h, h'\right)\mathrm{sm}\left(\tilde{b}_h^{\dashv}, \tilde{t}_h^{\dashv}\right)$$

$$= \frac{1}{\chi}\left(\tilde{\tau}_3 - \tilde{\tau}_4\,\mathrm{cm}\left(\tilde{b}_h, \tilde{t}_h\right)\right) \tag{5.46}$$

From (4.26) we get

$$H\left(\tilde{b}_h, \tilde{t}_h\right) = \alpha(\tilde{b}_h^{\dashv})^{-1}\alpha(\tilde{t}_h^{\dashv})^{-1}\,\mathrm{sm}\left(\tilde{b}_h, \tilde{t}_h\right)\mathrm{sm}\left(\tilde{b}_h^{\dashv}, \tilde{t}_h^{\dashv}\right) \tag{5.47}$$

Therefore, we can simplify (5.40) as follows

$$h' = -\chi \frac{\mathrm{sm}\left(\tilde{b}_h, \vec{h}'\right)}{\mathrm{sm}\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{t}_h\right\|} t + \left\{ \chi \frac{\mathrm{sm}\left(\tilde{b}_h, \vec{h}'\right)}{\mathrm{sm}\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{t}_h\right\|}\mathrm{cm}(h,t) + \right.$$

$$\left. \chi \frac{\mathrm{sm}\left(\tilde{t}_h, h'\right)}{\mathrm{sm}\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{b}_h\right\|}\mathrm{cm}(h,b) \right\} h + \chi \frac{\mathrm{sm}\left(\tilde{t}_h, h'\right)}{\mathrm{sm}\left(\tilde{b}_h, \tilde{t}_h\right)\left\|\tilde{b}_h\right\|} b. \tag{5.48}$$

From the initial assumptions we have

$$h' = -\bar{\tau} t + \left\{ \bar{\tau} \operatorname{cm}(h,t) + \bar{\tau}_1 \operatorname{cm}(h,b) \right\} h + \bar{\tau}_1 b. \tag{5.49}$$

Hence, the Frenet-Serret formulae take the more simple form as stated in Cor. 5.5.

Theorem 5.6: Given a Minkowski space M_B^3 and a B-orthonormal Frenet-Serret frame of a curve or ruled surface. If the derivative of the principal normal vector h' lies in the rectifying plane $(t\,b)$ and $t \perp b$, then $h \perp t$ implies $h \perp b$ which means that M_B^3 is Euclidean and $h' \parallel t$.

Proof: Since h' lies in the rectifying plane then using (5.40) we have

$$\|\tilde{b}_h\|\left(\tilde{\tau}_4 - \tilde{\tau}_3 \operatorname{cm}\left(\tilde{t}_h, \tilde{b}_h\right)\right)\operatorname{cm}(h,t) + \|\tilde{t}_h\|\left(\tilde{\tau}_3 - \tilde{\tau}_4 \operatorname{cm}\left(\tilde{b}_h, \tilde{t}_h\right)\right)\operatorname{cm}(h,b) = 0. \tag{5.50}$$

Assume that $h \perp t$, i.e. $\operatorname{cm}(h,t) = 0$ then from (5.52) we get $\left(\tilde{\tau}_3 - \tilde{\tau}_4 \operatorname{cm}\left(\tilde{b}_h, \tilde{t}_h\right)\right)\operatorname{cm}(h,b) = 0$

which implies two possibilities, first of all that $\operatorname{cm}(h,b) = 0 \Rightarrow h \dashv b$ which means $h \perp b$ which is the Euclidean case. Finally, we have $\tilde{\tau}_3 - \tilde{\tau}_4 \operatorname{cm}\left(\tilde{b}_h, \tilde{t}_h\right) = 0$ and by using the definitions of the Minkowski torsions and curvatures we have

$$\operatorname{cm}\left(\tilde{b}_h, h'\right) = \operatorname{cm}\left(\tilde{t}_h, h'\right)\operatorname{cm}\left(\tilde{b}_h, \tilde{t}_h\right). \tag{5.51}$$

By using theorem 4.14 (i) and (5.53) we get

$$\operatorname{sm}\left(\tilde{t}_h, h'\right)\operatorname{sm}\left(\tilde{b}_h^{\dashv}, \tilde{t}_h^{\dashv}\right) = 0. \tag{5.52}$$

If $\operatorname{sm}\left(\tilde{b}_h^{\dashv}, \tilde{t}_h^{\dashv}\right) = 0$, using (4.39) we get $\operatorname{cm}\left(\tilde{b}_h, \tilde{t}_h^{\dashv}\right) = 0 \Rightarrow \tilde{b}_h \dashv \tilde{t}_h^{\dashv} \Rightarrow \tilde{b}_h \parallel \tilde{t}_h$, which contradicts to the assumption. Therefore we have only $\operatorname{sm}\left(\tilde{t}_h, h'\right) = 0$ and since $\tilde{t}_h \parallel t$ then $h' \parallel t$ □.

Chapter 6

Geodesics in Minkowski space

6.1 Introduction

For a (linear) Euclidean space geodesics are just straight lines. On a curved (and sufficiently smooth) surface a geodesic is a curve that represents the extremely value of a distance function. This concept holds therefore in all normed spaces, especially in those under consideration. In the Euclidean space, extremal means ‘‘minimal’’, thus geodesics are paths of minimal arc length or, in other words, they are the locally shortest paths between two distinct surface points. Besides this inner geometric definition there is an outer geometric definition, too: The osculating planes of a geodesic on a surface contain the surface normals and thus geodesics are the curves along which the geodesic curvature vanishes.

In this part, using an equivalent to Gauss’s equation in Minkowski spaces, we define an M-geodesic parallel field Y [23] along a curve $c(t)$ on the surface in Minkowski 3-space. We give the fundamental condition of a curve to be M-geodesic using the covariant differentiation of its tangential vector fields.

From e.g. [58] we know the fact that the generators of the skew ruled surface along the striction curve in Euclidean space are geodesically parallel and that also the converse is true. In a Minkowski space we find a similar result, but we must redefine the parallel field along the curve with respect to the B-orthogonality and demand additional conditions.

We find that there is no equivalent to Bonnet’s theorem for ruled surfaces in Minkowski spaces based on the definition of M-striction curve (5.11) as this would involve an angle measure.

Finally we formulate an analogue to the theorem of Pirondini see Müller-Krames [64] considering the set of ruled surfaces with common striction strip (i.e. the striction curve plus the set of central planes) it is possible to formulate a version in Minkowski spaces M_B^3.

6.2 The covariant derivative in Minkowski space

In Euclidean space we are lead to the concepts of the directional derivative of functions and vector fields. When calculating the standard derivative of a tangential vector field of a surface we receive a vector field, which, in general, is no longer tangential to the given surface. By splitting the derivative vector into a tangential and a normal component one can define the so-called covariant derivative as the tangential component alone. This covariant derivative obtained in this manner has some important properties, e.g. it is an inner-geometric concept of the surface and therefore belongs to the intrinsic geometry of it, see [23].

By a parallel vector field we mean a vector field with the property that the vectors at different points are geodesic parallel in the sense of Levi-Civita. In Euclidean space, there is an affinity from one tangential bundle to the neighbouring one canonically defined by the Levi-Civita connection between these bundles. A given vector of the first bundle and its corresponding one of the second bundle are called geodesic parallel.

Figure 6.1

Lemma 6.1: Projection onto the tangent space: Let us consider a surface ϕ in Euclidean space E_n and let V be a vector in E_n. The projection πV of V onto the tangent space T has coordinates given by

$$(\pi V)_i = g^{ik}\left(V.\frac{\partial}{\partial x^k}\right), \tag{6.1}$$

where $\left(g^{ik}\right)$ is the matrix inverse of $\left(g_{ik}\right)$ and $g_{ik} = \left(\partial / \partial x^i\right)\cdot\left(\partial / \partial x^k\right)$ are the coefficients of first fundamental form in Euclidean space.

Theorem 6.2: Let $\psi(u,v)$ be a surface in 3-dim space and V be a vector field on it, then the Minkowski projection $\pi' V$ of V onto the Tangent plane $\tau(u,v)$ are given by

$$(\pi' V)_i = g^{ik}\left\{\left(V.\frac{\partial}{\partial x^k}\right) - \left(V^{\dashv}.\frac{\partial}{\partial x^k}\right)\right\}, \tag{6.2}$$

Proof: We can represent V by two ways as

$$V = \pi V + V^{\perp} \tag{6.3}$$

and

$$V = \pi' V + V^{\dashv} \tag{6.4}$$

where $V^{\perp}$ and $V^{\dashv}$ are the component of V which are normal to the tangent plane in the Euclidean and Minkowski sense respectively. We write $\frac{\partial}{\partial x^k}$ as e_k and write

$$\left.\begin{aligned} \pi V &= a_1 e_1 + a_2 e_2, \\ \pi' V &= a_1' e_1 + a_2' e_2 \end{aligned}\right\} \tag{6.5}$$

where a_i and a_i' are the desired local coordinates of πV and $\pi' V$ respectively. Let $V^{\dashv} = b_1 e_1 + b_2 e_2 + b_3 n_e$, here n_e denotes the normalized surface vector and hence,

$$V = (a_1' + b_1) e_1 + (a_2' + b_2) e_2 + b_3 n_e, \tag{6.6}$$

and so,

$$V \cdot e_1 = (a_1' + b_1) e_1 \cdot e_1 + (a_2' + b_2) e_2 \cdot e_1 + 0,$$

$$V \cdot e_2 = (a_1' + b_1) e_1 \cdot e_2 + (a_2' + b_2) e_2 \cdot e_2 + 0.$$

We can rewrite the last equations as

$$\left.\begin{aligned} V \cdot e_1 &= (a_1' + b_1) g_{11} + (a_2' + b_2) g_{21}, \\ V \cdot e_2 &= (a_1' + b_1) g_{12} + (a_2' + b_2) g_{22} \end{aligned}\right\}, \tag{6.7}$$

which can be written in the matrix form as

$$(V \cdot e_i) = (a_i' + b_i)(g_{ki}) \Rightarrow (\pi' V)_i = g^{ik} \left\{ \left(V . \frac{\partial}{\partial x^k} \right) - \left(V^{\dashv} . \frac{\partial}{\partial x^k} \right) \right\}.$$

Therewith we have shown that the Minkowski projection of any vector field onto the tangent plane is described by the difference between the Euclidean normal projection and the Minkowski normal projection of it.

Definition 6.3: Covariant Differentiation: If U and V are two tangential vector fields defined on the surface ϕ then

$$\nabla_U V = (D_U V)^{Tang.} = D_U V - \langle D_U V, n_e \rangle n_e \tag{6.8}$$

called covariant differentiation of V in the direction of U where n_e is the Euclidean normal of the surface and $D_U V$ is the directional derivative of V in the direction of U, therefore the normal part is described by the second fundamental form of ϕ, i.e., $\langle D_U V, n_e \rangle = II(U,V)$.

And hence, we can rewrite (6.8) as,

$$D_U V = \nabla_U V + II(U,V) n_e, \tag{6.9}$$

and we note that equation (6.8) may be called *Gauss equation* as it induces the covariant differentiation.

Theorem 6.4: The Minkowski second fundamental form of two tangential vector fields is the Euclidean second fundamental form multiplied by the component of Euclidean normal vector in the direction of the Minkowski normal, i.e

$$II'(U,V) = II(U,V) \langle n_M, n_e \rangle_M, \tag{6.10}$$

where $II'(U,V)=\langle n_M, D_U V\rangle_M$ is the Minkowski second fundamental form and n_M is the Minkowski unit normal vector.

Proof: The Minkowski covariant differentiation of V in the direction of U can be written as the following form $(\nabla_U V)_M = D_U V - II'(U,V)n_M$ then,

$$D_U V = (\nabla_U V)_M + II'(U,V)n_M \tag{6.11}$$

Now, if we take the Minkowski semi-inner product of both sides of (6.11) with n_M from left, we get, $\langle n_M, D_U V\rangle_M = 0 + II'(U,V)$. And hence, we can define the Minkowski second fundamental form as $II'(U,V)=\langle n_M, D_U V\rangle_M$.

The Euclidean unit normal vector can be written as

$$n_e = \pi' n_e + (n_e)^{\dashv} n_M \tag{6.12}$$

where, $\|\pi' n_e\|$ and $(n_e)^{\dashv}$ are the tangential and normal Minkowski components respectively. Thus, from (6.12) follows $(n_e)^{\dashv} = \langle n_M, n_e\rangle_M$. Inserting (6.12) into (6.9) and comparing the result with (6.11) one receives

$$\left.\begin{aligned} (\nabla_U V)_M &= \nabla_U V + II(U,V)\cdot \pi' n_e, \\ II'(U,V) &= II(U,V)\langle n_M, n_e\rangle_M . \end{aligned}\right\}, \tag{6.13}$$

$$II'(U,V) = II(U,V)\|n_e\| cm(n_M, n_e) \tag{6.14}$$

and thereof follows finally

$$\langle n_M, D_U V\rangle_M = \langle D_U V, n_e\rangle \|n_e\| cm(n_M, n_e). \tag{6.15}$$

6.3 Parallel fields in Minkowski space

At first we recall the Euclidean case:

Definition 6.5: let $c(t)$ be a curve on a surface ϕ in an Euclidean space with tangent $\dot{c}(t)$ and let Y be a tangential vector field along ϕ that is C^∞ on $c(t)$. The field Y is geodesic parallel along the curve $c(t)$, if (and only if) $\nabla_{\dot{c}} Y = 0$ along $c(t)$. The regular curve $c(t)$ is a geodesic on ϕ, if (and only if) $\nabla_{\dot{c}} \dot{c} = 0$, i.e. if its tangents $\dot{c}(t)$ are geodesic parallel along $c(t)$ or its geodesic curvature vanishes.

Other (equivalent) characterizations of geodesics in a Euclidean space are that their osculating planes contain the surface normals or that they are locally the shortest paths between two distinct surface points.

In the following we modify this definition such that it makes sense in a Minkowski space. One possibility would be to define the Minkowski auto-parallel curves, i. e. the M-geodesics on a surface Φ, by the property that their osculating planes contain the M-normals of Φ respectively. For the moment we still can decide to choose the lines left-orthogonal or the one right-orthogonal to Φ as M-normals or even a (fixed) linear combination of both normals. A similar approach can be used for defining M-geodesic parallelity.

Let us first consider Definition 6.6A and its consequences:

Definition 6.6A: Assuming that $V(t)$ is a tangent vector field attached to the curve $c(t)=x(u(t))$ of a surface Φ in a Minkowski space, then it is an "M$_L$-geodesic parallel field" along $c(t)$ if $\dot{V}(t)$ is *left-orthogona*l to the surface at the point $x(t)$.

In other words, let $c(t)$ be a curve on a surface ϕ in Minkowski space with tangent $\dot{c}(t)$ and let Y be a tangential vector field along ϕ that is C^{∞} on $c(t)$, then the field Y is M$_L$-geodesic parallel along the curve $c(t)$, if $\left(\nabla_{\dot{c}}Y\right)_M=0$ along $c(t)$. The curve $c(t)$ is an M$_L$-geodesic, if $\left(\nabla_{\dot{c}}\dot{c}\right)_M=0$, i.e. if its tangents $\dot{c}(t)$ are M$_L$-geodesic parallel along $c(t)$.

Lemma 6.7A: A curve $c(t)$ on a surface ϕ in Minkowski space is M$_L$-geodesic iff $D_{\dot{c}}\dot{c}$ is left-orthogonal to the surface ϕ i.e. $\nabla_{\dot{c}}\dot{c}=-\left\langle D_{\dot{c}}\dot{c},n_e\right\rangle\cdot\pi' n_e$.

Proof: Let $c(t)$ be a curve on a surface ϕ in an Minkowski space, the Gauss equation in Minkowski sense implies $\left(\nabla_{\dot{c}}\dot{c}\right)_M=D_{\dot{c}}\dot{c}-\left\langle n_M,D_{\dot{c}}\dot{c}\right\rangle_M n_M$. Thus $\left(\nabla_{\dot{c}}\dot{c}\right)_M=0$ iff $D_{\dot{c}}\dot{c}$ is in the direction of the Minkowski normal vector n_M ($D_{\dot{c}}\dot{c}\,||\,n_M$). By using equation (6.13), we receive $\nabla_{\dot{c}}\dot{c}=-\left\langle D_{\dot{c}}\dot{c},n_e\right\rangle\cdot\pi' n_e$, which completes the proof.

Theorem 6.8A: In a Minkowski 3-space M_B^3 the generators of the skew ruled surface along the M-striction curve S_M are M$_L$-geodesic parallel.

Proof: From the definition of the striction points $s(t_0)$ as the touching point of the central plane $\zeta(t_0)$ with Φ follows by Definition 6.6A that $\dot{e}(u)\dashv\zeta(u)$. Thus $\dot{e}(u)\dashv e(u)$ and because of $e(u)\dashv\dot{e}(u)$ (see Figure 5.3 a, b) $e(t),\dot{e}(t)$ must be mutually left-orthogonal. Therewith B has the property that along the spherical image of Φ B-orthogonality is a symmetric relation. As we consider arbitrarily chosen ruled surfaces B-orthogonality has to be globally symmetric and B must be an ellipsoid. Based on Definition 6.6A we restrict ourselves to the Euclidean case only.

Right-orthogonality is not declared yet and it seems impossible to find a definition of a right-orthogonal line of a surface in any point P of it. For ruled surfaces we can find a naturally defined right normal at least in the striction points as follows:

Definition 6.9: The asymptotic plane α and the central plane ζ are right-orthogonal planes per definition, as $z\dashv e\wedge z\dashv\dot{e}$ and $e\dashv\dot{e}$. Thus we can put $\dot{e}\vdash\zeta$ and call it the right-orthogonal vector of the central plane.

When we construct the left-orthogonal vector y to ζ, we find that in general $y\neq\dot{e}$ (we orient y such that $\{e,y,z\}$ forms a right handed frame).

Then, within the pencil of planes around a generator $g(t)$ we can define a deviation vector $d(t)$ by

Definition 6.10: Let $y(t)$ be the touching point of the unit ball B with a plane $\bar{\zeta} \parallel \zeta$, $\zeta \parallel$ central plane of a generator $g(t)$ of a ruled surface such that its vector $y(t)$ together with the direction vector $e(t)$ of $g(t)$ and the central vector $z(t)$ forms a right handed affine frame $\{o; e(t), y(t), z(t)\}$. Then $y(t) - \dot{e}(t) =: d(t)$ is the "*deviation vector*" between the left- and right- B-orthonormals of the central plane.

We now start with right-orthogonality and modify Def. 6.6A to

Definition 6.6B: Let $V(t)$ be a tangent vector field attached to the curve $c(t) = x(u(t))$ of a surface Φ in a Minkowski space, then it is an "M_R-geodesic parallel field" along $c(t)$ if $\dot{V}(t)$ is *right-orthogonal* to the surface at the point $x(t)$.

Now we must modify Lemma 6.7A, too:

Lemma 6.7B: A curve $c(t)$ on a ruled surface ϕ in Minkowski space is M_R-geodesic iff $D_{\dot{c}}\dot{c}$ is right-orthogonal to the surface ϕ

Proof: Now we reformulate the *Gauss equation* (6.11) for ruled surface to be

$$D_U V = (\nabla_U V)_{M_R} + II''(U,V) n_{M_R} \tag{6.11}$$

where, n_{M_R} is the right-normal unit vector to the ruled surface ϕ at the striction point and $II''(U,V)$ is the corresponding second fundamental form. Then a curve $c(t)$ on the surface ϕ in Minkowski space is M_R-geodesic iff $(\nabla_U V)_{M_R}$ vanishes, i.e. $D_{\dot{c}}\dot{c}$ is right-orthogonal to the surface ϕ.

With Def. 6.6B and Lemma 6.7B the formulation of Theorem 6.8A need not be changed:

Theorem 6.8B = Theorem 6.8A: In a Minkowski 3-space M_B^3 the generators of the skew ruled surface along the M-striction curve S_M are M_R-geodesic parallel.

Proof: This time $\dot{e}(u)$ and $\zeta(u)$ must be right-orthogonal with the consequence that $e(u) \dashv \dot{e}(u) \wedge Z(u) \dashv \dot{e}(u)$. This is valid because of $Z(u) \dashv e(u), \dot{e}(u)$ per definition of the M-central tangent. Based on Definition 6.6B we indeed have an extension of a classical Euclidean result to any Minkowski space with a smooth and strictly convex (centrally symmetric) gauge ball *B*.

6.4 Further theorems on ruled surfaces

6.4.1 Bonnet's theorem

In the introductory Chapter 1 we mentioned that there is no theorem corresponding to *Bonnet's theorem* in the Minkowski 3-space. The Euclidean Bonnet's theorem [17,64] reads as follows:

If c is a curve on a ruled skew surface and has two of the three following properties,

- c is an isogonal trajectory of the generators, i.e. $\measuredangle \dot{c} e_1 = const$, where e_1 is the Euclidean unit vector in the direction of the generator,
- c is a geodesic curve,
- c is striction line,

then c has also the third property.

A direct translation to a theorem in Minkowski spaces would demand a concept of isogonality there. This would involve an angle measure in the Minkowski space. Like for the concept of orthogonality there are of course many different ways to define an angle measure in Minkowski spaces and it would be worthwhile to investigate each of the usual definitions, whether it allows an analogue statement to the Euclidean theorem of Bonnet. As this lengthy discussion would exceed this dissertation we save this topic to another occasion just mentioning that "isogonality" could also mean "constant cosine-Minkowski" $cm(e(t)\dot{c}(t))$ of the pair of lines (generator, tangent of striction curve) .

6.4.2 Pirondini's theorem

The Euclidean Theorem of Pirondini reads as follows.

Theorem 6.11: Let a (skew) ruled surface Φ be given in Euclidean 3-space. It defines the "striction stripe" consisting of the striction curve $c = \{s(u)\}$ and the developable surface Δ enveloped by all central planes $\zeta(u)$ of Φ. If we rotate each generator $e(u)$ around its striction point $s(u)$ in its central plane $\zeta(u)$ by a fixed angle measure $\alpha \in \mathbb{R}$, then we receive a new surface $\Psi(\alpha) := \{ f(u,\alpha) \mid f(u,\alpha) := e(u)^{\alpha} \}$ having again (c,Δ) as its striction stripe.

Again an angle measure is essential. The proof of the Euclidean Pirondini theorem makes use of the property that the generators are geodesic parallel along c and adding a constant angle to the striction angle $\sigma(u) = \angle e(u)\dot{s}(u)$ does not affect this property. Amore or less elementary proof can be formulated as follows:

Develop the striction stripe (c,Δ) into a plane so that the curve c becomes a plane curve $\bar{C}$ say. Then the generators $\{e(u)\}$ will occur as a pencil of parallel lines $\bar{e}(u)$, because they are geodesic parallel and development does not affect this intrinsic geometric property. Rotating each line $\bar{e}(u)$ by a fixed angle α delivers again a pencil of parallel lines $\bar{f}(u)$, which, by the inverse of the development operation, stem from a new ruled surface with the same striction stripe (c,Δ).

To formulate a Minkowski geometric version of Theorem 6.11 we at first redefine what we mean with "fixed Minkowski rotation angle":

As the direction vectors of the new generators $f(u)$ are in any case linear combinations of the (unit) vectors $e(u), \dot{s}(u)$, i.e. $f(u) = \lambda(u).e(u) + \mu(u).\dot{s}(u)$, we might formulate

Definition 6.12: By "fixed Minkowski rotation angle" of generators $e(u)$ around their M-striction point $s(u)$ in their M-central plane $\zeta(u)$ we mean that the new unit vector $f(u)$ resulting by this M-rotation is described by $f(u) = \lambda.e(u) + \mu.\dot{s}(u)$ with constant coefficient functions λ, μ.

With this Def. 6.12 we formulate

Theorem 6.13: Let a (skew) ruled surface Φ be given in a Minkowsi space with smooth, strictly convex gauge ball B. It defines the "striction stripe" consisting of the striction curve $c = \{s(u)\}$ and the developable surface Δ enveloped by all central planes $\zeta(u)$ of Φ. If we rotate each generator $e(u)$ around its striction point $s(u)$ in its central plane $\zeta(u)$ by a fixed Minkowski rotation angle measure $(\lambda, \mu) \in \mathbb{R}^2$, then we receive a new surface $\Psi(\lambda, \mu) := \{ f(u; \lambda, \mu) \mid f(u; \lambda, \mu) := \lambda e(u) + \mu \dot{s}(u)\}$ having again (c, Δ) as its M-striction stripe.

Proof: Consider a ruled surface ϕ in a Minkowski 3-space M_B^3 with M-striction line S_M (5.11), rotate all generators g about its central point (the point of contact) in the central plane ζ through an angle (in general not constant) $\theta(t)$. We get the corresponding generator $\bar{g}$ where it lies in the central plane ζ then we have,

$$\bar{g}(t) = \lambda(t)e + \mu(t)b, \tag{6.16}$$

where the vector $\bar{g}(t)$ is unit vector in the direction of the rotated generators, from (5.11) we choose the M-striction line S_M as the director curve then we have,

$$\det\left(e, \dot{S}_M, b\right) = 0. \tag{6.17}$$

With respect to the new surface Ψ, we consider again S_M as the director curve and use (5.11) we get

$$\bar{S}_M(u_\circ) = S_M(u_\circ) - \frac{\det\left(\bar{g}, \dot{S}_M, \bar{b}\right)}{\det\left(\bar{g}, \dot{\bar{g}}, \bar{b}\right)} \bar{g}(u_\circ), \tag{6.18}$$

since,

$$\det\left(\bar{g}, \dot{S}_M, \bar{b}\right) = \lambda(t)\det\left(\vec{e}, \dot{S}_M, \bar{b}\right) + \mu(t)\det\left(b, \dot{S}_M, \bar{b}\right) = 0. \tag{6.18}$$

This means from (5.18) that $\bar{S}_M(u_\circ) = S_M(u_\circ)$ which complete the proof.

Of course the functions $\lambda(t)$ and $\mu(t)$ in Euclidean version must be constant because the rotation angle is constant. We note also that the angle $\theta(t)$ in the previous proof is measured in Euclidean space.

6.4.3 Conoidal surfaces, Conoids

Conoidal surfaces have generators, which are parallel to a fixed plane ρ, what means that their generators meet a directrix line at infinity. Conoids are conoidal surfaces possessing a proper directrix line in addition. This means that conoids belong to a (special) hyperbolic line congruence, where one of the focal lines is a line at infinity. Both concepts, conoidal surfaces and conoids, are of affine geometric nature and one can consider them in a Minkowski space, too.

Because of the existence of a directrix plane ρ the spherical image of such a ruled surface Φ must be an arc belonging to the planar intersection of the gauge ball *B* with a diameter plane parallel to ρ. Therewith follows that al M-central tangents of Φ have the same unit direction vector *Z* and the M-striction curve is the contour line (shadow contour) of Φ with respect to a parallel projection parallel to *Z*.
Analogue to the Euclidean case one can define an "M-straight conoid" Φ by the property that the proper directrix line of Φ is parallel to *Z*.

6.5 Conclusion

We have shown that examples of Euclidean properties of skew ruled surfaces Φ (at least in most cases) can be translated into properties of Φ in a Minkowski space. Of course there remain many open questions. In this dissertation we e.g. omitted those problems involving angle measures. By presenting just one topic of these problems, namely the theorem of Pirondini, we want to point out the arbitrariness of finding and using a suitable angle concept.

Furthermore we omitted to discuss Minkowski analogues of the important Euclidean first order invariant called *pitch*. And also the discussion of special ruled surfaces remains to another occasion. For example, it would be interesting to know, how surfaces with constant M-curvatures and M-torsions look like.
In this sense this dissertation is a first step in the research of ruled surfaces in Minkowski spaces.

References

[1] *A. C. Thompson*, **Minkowski Geometry**, New York Encyclopedia of Math. and Its Applications 63, Cambridge Univ. Press, Cambridge, (1996).

[2] *H. Martini, K. J. Swanepoel, G. Weiss*, **The geometry of Minkowski spaces – a survey. Part I**, Expo. Math. 19, 97-142, (2001).

[3] *H. Martini, K. J. Swanepoel*, **The geometry of Minkowski spaces – a survey. Part II**, Expo. Math. 22, 93-144, (2004).

[4] *B. D. Roberts*, **On the geometry of the abstract vector spaces**, Tohoku Math. J. 39, 42-59, (1934).

[5] *G. Birkhoff*, **Orthogonality in linear metric spaces**, Duke Math. J. 1, 169–172, (1935).

[6] *R. C. James*, **Orthogonality in normed linear spaces**, Duke Math. J. 12, 291-302, (1945).

[7] *R. C. James*, **Orthogonality and linear functionals in normed linear spaces**, Trans. Amer. Math. Soc. 61, 265–292, (1947).

[8] *R. C. James*, **Inner products in normed linear spaces**, Bull. Amer. Math. Soc. 53, 559-566, (1947).

[9] *M. M. Day*, **Some characterizations of inner product spaces**, Trans. Amer. Math. Soc. 62, 320-337, (1947).

[10] *J. Alonso*, **Ortogonalidad en espacios normados**. **(Spanish) [Orthogonality in normed spaces]**, Dissertation, Univ. de Extremadura, Cáceres, (1984).

[11] *J. Alonso*, **Some results on Singer orthogonality and characterizations of inner product spaces**, Arch. Math. (Basel) 61, no. 2, 177-182, (1993).

[12] *J. Alonso*, **Uniqueness properties of isosceles orthogonality in normed linear spaces**, Ann. Sci. Math. Québec 18, no. 1, 25-38, (1994).

[13] *J. Alonso*, **Some properties of Birkhoff and isosceles orthogonality in normed linear spaces. Inner product spaces and applications**, 1-11, Pitman Res. Notes Math. Ser., 376, Longman, Harlow, (1997).

[14] *J. Alonso*, C. Benítez, **Area orthogonality in normed linear spaces,** Arch. Math. (Basel) 68, no. 1, 70-76, (1997).

[15] *J. Alonso*, C. Benítez, **Orthogonality in normed linear spaces: a survey. I. Main properties,** Extracta Math. 3, no. 1, 1-15, (1988).

[16] *J. Alonso, C. Benítez*, **Orthogonality in normed linear spaces: a survey. II. Relations between main orthogonalities**, Extracta Math. 4, no. 3, 121-131, (1989).

[17] *E. Kruppa*, **Analytische und konstruktive Differentialgeometrie**, Springer-Verlag, Wien, (1957).

[18] *G. Sannia*, **Una rappresentatione intrinseca delle rigate**, Giorn. d. Mat. 31-47, (1957).

[19] *J. Chmielinski*, **On an ε-Birkhoff orthogonality**, J. Inequal. Pure and Appl. Math. 6 (3), Art. 79, (2005).

[20] *V. Hlavaty*, **Differentielle Liniengeometrie**, Uitg P. Noorfhoff, Groningen, (1945).

[21] *J. Hoschek*, **Integralinvarianten von Regelflächen**, Arch. Math. XXIV, 218-224, (1973).

[22] *A. Sarıoğlugil, A. Tutar*, **On ruled surfaces in Euclidean space E^3**, Int. J. of Contemp. Math. Sci. Vol. 2, no. 1-4, 1-11, (2007).

[23] *W. Kühnel*, **Differentialgeometrie. Kurven-Flächen-Mannigfaltigkeiten. 2.**, überarbeitete Auflage. vieweg: Braunschweig/Wiesbaden, (2003).

[24] *J. Radon*, **Über eine besondere Art ebener Kurven**, Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. 68, 23–28, (1916).

[25] *S. O. Carlsson*, **Orthogonality in normed linear spaces**, Ark. Mat. 4, 297-318, (1962).

[26] *I. Singer*, **Unghiuri abstracte si functii trigonometrice in spatii Banach**, Acad. Republ. Popul. Romine, Bul. Sti., Sect. Sti. Mat. Fiz. 9, 29 – 42, (1957).

[27] *P. M. Miliĉić*, **Sur le g-angle dans un espace norme**, Mat. Vesnik, 45, 43-48, (1993).

[28] *L. Zheng, Z. Yadong*, **Singer orthogonality and characterizations of inner product spaces**, Arch. Math. 55, 588-594, (1990).

[29] *G. Lumer*, **Semi–inner product spaces**, Trans. Amer. Math. Soc. 100, 29–43, (1961).

[30] *S. S. Dragomir, J. J. Koliha*, **Two mappings related to semi-inner products and their applications in geometry**, Appl. Math. 45, no. 5, 337-335, (2000).

[31] *J. Donghai, Wu. Senlin*, **Quantitative characterization of the difference between Birkhoff orthogonality and isosceles orthogonality**, J. Math. Anal. Appl. 323, 1-7, (2006).

[32] *C. R. Diminnie*, **A new orthogonality relation for normed linear spaces**, Math. Nachr. 114, 197-203, (1983).

[33] *E. Silverman*, **Definitions of area for surfaces in metric spaces**, Rivista Mat. Univ. Parma 2, 47-76, (1951).

[34] *E. Silverman*, **An intrinsic property of Lebesque area**, Rivista Mat. Univ. Parma 2, 195-201, (1951).

[35] *E. Silverman*, **Set functions associated with Lebesque area**, Pacific J. Math. 243-260, (1962).

[36] *F. Sullivan*, **A generalization of uniformly rotund Banach spaces**, Can. J. Math. 81, 628-636, (1979).

[37] *D. Amir*, **Characterizations of inner product spaces**, Basel, (1986).

[38] *B. Boussouis*, **Orthogonalité et caractérisation des espaces préhilbertiens**, Ph. D. Thesis, Université Sidi Mohamed Ben Abdellah, Fès, Morocco, (1995).

[39] *A. E. Taylor*, **A geometric theorem and its application to biorthogonal systems**, Bull. Amer. Math. Soc. 53, 614-616, (1947).

[40] *H. Busemann*, **The isoperimetric problem in the Minkowski plane**, Amer. J. Math. 69, 863-871, (1947).

[41] *H. Busemann*, **The isoperimetric problem for Minkowski area**, Amer. J. Math. 71, 743-762, (1949).

[42] *H. Busemann*, **The foundations of Minkowskian geometry**, Comment. Math. Helv. 24, 156-187, (1950).

[43] *H. Busemann, C.M. Petty*, **Problems on convex bodies**, Math. Scand. 4, 88–94, (1956).

[44] *A. C. Thompson*, **Applications of various inequalities to Minkowski geometry**, Geom. Dedicata, 46, 215–231, (1993).

[45] *R. V. Benson*, **Euclidean Geometry and Convexity**, McGraw-Hill, New York, (1966).

[46] *L. Danzer, D. Laugwitz, H. Lenz*, **Über das Löwnersche Ellipsoid und sein Analogon unter den einem Eikörper einbeschriebenen Ellipsoiden**, Arch. Math. 8, 214-219, (1957).

[47] *R. J. Gardner*, **Geometric Tomography**, Encyclopedia of Math. and Its Applications 54, Cambridge Univ. Press, New York, (1995).

[48] *D. L. Cohn*, **Measure Theory**, Birkhäuser, Basel, (1980).

[49] *A. Haar*, **Der Massbegriff in der Theorie der kontinuierlichen Gruppen**, Ann. Of Math. (2) 34, 147-169, (1933).

[50] *S. Kakutani*, **On the uniqueness of Haar's measure**, Proc. Imp. Acad. Tokyo, 14, 27-31, (1938).

[51] *H. Martini, Z. Mustafaev*, **On Isoperimetric inequalities in Minkowski spaces**, J. Inequal. and Appl. Article ID 697954, 18 page, (2010).

[52] *H. Martini, K. J. Swanepoel*, **Antinorms and Radon curves**, Aeq. Math. 72, no. 1-2, 110-138, (2006).

[53] *R. D. Holmes, A. C. Thompson*, **N-dimensional area and content in Minkowski spaces**, Pacific J. Math. 85, 77-110, (1979).

[54] *E. Lutwak*, **Intersection bodies and dual mixed volumes**, Adv. Math. 71, 232-261, (1988).

[55] *Z. Mustafaev*, **On Some isoperimetric inequalities for the Holmes-Thompson definitions of volume and surface area in Minkowski spaces**, J. Inequal. Pure and Appl. Math. 5 (1), Art. 17, (2004).

[56] *Z. Mustafaev*, **On Busemann surface area of the unit ball in Minkowski spaces**, J. Inequal. Pure and Appl. Math. 7 (1), Art. 19, (2006).

[57] *H. Pottmann*, **Holditch .Sicheln**, Arch, Math. 44, 373-378, (1985).

[58] *H. Pottmann, J. Wallner*, **Computational line geometry**, Springer-Verlag, Berlin Heidelberg, (2001).

[59] *K. Sprott, B. Ravani*, **Kinematic generation of ruled surfaces**, Adv. Comp. Math. 17, 115-133, (2002).

[60] J. Steiner, Ges. werke, Berlin, (1881.1882).

[61] *E. A. Weiß*, **Einführung in die Liniengeometrie und Kinematik**, Teubner, Leipzig, (1937).

[62] *H. R. Müller*, **Kinematik**. W. de Gruyter (1963).

[63] *A. Holditch*, **Geometrical Theorem**, Q. J. Pure Appl. Math. 2, (1858).

[64] *E. Müller*, *J. L. Krames,* **Vorlesungen über Darstellende Geometrie. Band III: Konstruktive Behandlung der Regelflächen**, bearbeitet von *Dr. Joseph Leopold Krames.* Leipzig and Wien, *Franz Deuticke*, (1931).

List of figuers

1.1 B-orthogonality of two lines: y is left-orthogonal to x ($x \dashv y$) 2

2.1 B-orthogonality, $x \dashv y$ where the line $x + \lambda y$ supports the unit ball B at x 7

3.1 Smooth convex body K and the radial function $\rho_B\left(y_x\right)$ of the unit ball B which is the support function of the dual unit ball B° 26

4.1 The pair $\left(x, x^\dashv\right)$ is a normal basis for M_B^2 43

5.1 Birkhoff orthogonality on a Minkowski three dimensional unit ball B 49

5.2 All points of a non-torsal ruling R possess a different tangent plane 50

5.3 a,b Orthonormal frame construction, the unit vector $e(u)$ in the direction of the ruling $R\left(u\right)$, the vector $\dot{e}(u)$ which is contained in the supporting plane (tangent plane) at the point of contact P and the central tangent vector Z 53

5.4 Deformation vector $x^{\dashv\dashv}$ of the vector x in the Minkowski two-dimensional unit ball B_2 56

6.1 Parallel fields on the Torus 64

Index

2-norm, *see* Diminnie orthogonality

A

affine
– dual space, *18*, *19*
– frame, *3*, *49*, *56*, *58*
– hyperplane, *19*
– point space, *3*, *18*
– space, *18*
antinorm, *25*
area orthogonality, *14*
asymptotic plane, *4*, *49*, *50*, *51*, *67*

B

binormal vector, *58*
Birkhoff orthogonality, *2*, *16*, *38*
Bonnet’s theorem, *63*, *68*
Busemann
– area, *27*, *32*
– measure, *31*
– volume, *32*

C

Carlsson orthogonality, *6*
central
– plane, *48*
– tangent vector, *51*
centrally symmetric
– convex curve, *26*
– unit ball, *1*, *18*
Conoidal surfaces, *70*
covariant differentiation, *63*, *65*, *66*

D

deformation vector, *54*
Diminnie orthogonality, *11*
dirac delta function, *17*
direction vector, *4*, *48*
directional derivative, *63*
dual
– Haar measure, *23*
– plane, *25*
– space, *12*, *18*
– unit ball, *19*

E

Euclidean striction curve, *48*, *53*

F

Frenet-Serret formulae, *54*
Frenet-Serret frame, *56*, *62*

G

Gauss’s equation, *65*, *68*
generator, *3*, *50*, *58*
geodesic
– curvature, *63*, *66*
– parallel fields, *49*, *64*, *66*, *67*, *68*
g-orthogonality, *7*

H

Haar measure, *22*
Hahn-Banach theorem, *6*, *12*, *17*

Holmes-Thompson
– area, *32*
– measure, *32, 33*
– volume, *31*
hyperplane, *19*

I

inner-product
– norm, *8*
– space, *2*, *8*
isogonal trajectory, *69*
isoperimetric
– area, *27*
– inequality, *24*
– problem, *24*, *30*
isoperimetrix body, *18, 28*
isosceles orthogonality, *7, 9*

K

kernel function, *12, 36*
Kruppa-invariant, *50*

L

Lebesgue measure, *31*
Löwner ellipsoid, *18*
Lumer orthogonality, *9*

M

Minkowski
– 2nd curvature, *59*
– 2nd torsion, *59*
– 3rd curvature, *59*
– 3rd torsion, *59*
– 4th curvature, *59*
– 4th torsion, *59*
– Area, *27*, *40*, *42*, *55*
– cosine function, *3, 34, 35*
– curvature, *59*
– geodesic curve, *67*
– length, *24, 25*
– normal vector, *49*
– projection, *64*
– second fundamental form, *65*
– semi-inner product, *8, 36*
– sine function, *38*
– striction curve, *see* striction line
– surface area, *see* surface area
– torsion, *59*
mixed
– area, *24*
– volume, *23*, *25*
mutually orthogonal, *16, 60, 67*

N

normal
– basis, *42, 54*
– plane, *58*

O

orthonormal frame, *3, 48, 53*
osculating plane, *58*

P

parallel vector field, *64*, *66*, *67*
perimeter, *see* isoperimetric
Pirondini's theorem, *69*
polar reciprocal, *19*, *21*, *25*

principal normal vector, *58*
projection body, *24, 33*
Pythagorean orthogonality, *7*

R

radial function, *21, 25, 26, 32, 41*
Radon plane, *29*
Roberts orthogonality, *6*
Ruled surface, *48, 49*

S

Sannia frame, *49*
semi-inner product, *8*
Singer orthogonality, *7*
smooth convex body, *25*
spherical image, *4, 48, 58, 67, 71*
striction
– angle, *69*
– line, *48, 49, 52, 63, 67, 68, 69*
– strip, *69*
strictly convex unit ball, *1, 3, 14*
support
– plane, *1, 2, 6, 7, 51, 57*
– theorem, *18*
– function, *18, 19, 20, 32, 39, 41, 50*
surface area, *18, 24, 30, 32*

T

tangent plane, *4, 48, 50, 51, 64*
tangential vector field, *see* vector field
torsal, *48, 50*
transversal, *28*

V

vector field, *63, 65, 66, 67*
volume product, *27*

Lebenslauf

Family name	Shonoda
First name	Emad
Date of Birth	29-01-1967- Elismailya, Egypt
Nationality	Egyptian
Marital status	Married
Work address	TU Dresden, Fachrichtung Mathematik-Institut für Geometrie. Willersbau, Zi.: C8, Zellescher Weg 12-14, 01069 Dresden, Germany
Tel	+49-351-46334484 & +49-176-53434213
Fax	+49-351-46336027
E-mail	En_shonoda@yahoo.de
September 1982 – May 1993	Primary and Secondary School in Elismailya
May 1993	General Secondary Education Certificate from “Elismailya” Experimental School in Elismailya
September 1993 – June 1997	B.Sc. studies in the Faculty of Science, Suez canal University
August 1997	B.Sc. Degree in Science from the Mathematics Dept., Faculty of Science, Suez canal University
January 1999 – July 2005	Assistant in the Mathematics Dept., Faculty of Education, Suez canal University, Port-Said
October 1999	The final examination in the qualifying subjects for the M.Sc. studies, Zagazig University
January 2000 – May 2005	Research work for the M.Sc. with topic of research: “Quantum Statistical properties of Radiation“
25 June 2005	M.Sc. Degree in Mathematics from the Mathematics Dept., Faculty of Science, Zagazig University
August 2005	Assistant Lecturer in the Mathematics Dept., Faculty of Education, Suez canal University, Port-Said
December 2006	Ph.D. student at the Institute for Geometry – TU Dresden.

Dresden in ....   Emad Shonoda

Emad Shonoda

Summary of the Dissertation

"Ruled Surfaces in three-dimensional Minkowski Space"

For the topic treated in the dissertation the place of action is an affine 3-space E^3 endowed with a metric which is ruled by a centrally symmetric, convex body B as unit ball. Such a space is called *Minkowski normed space.* We will consider only Minkowski spaces with a strictly convex and smooth unit ball B. This means that the boundary ∂B contains no line segment and each point of B has a unique supporting plane and is therefore regular in the sense of differential geometry.

We aim at studying *ruled surfaces* in such a Minkowski space M_B following thereby E. Kruppa's treatment of ruled surfaces in a Euclidean 3-space. According to Kruppa a ruled surface can be uniquely described by three so-called *Kruppa-invariants* (curvature, torsion and striction as functions of the arc length parameter of the striction curve of the considered surface). Thus the program is fixed, namely to find substitutes of the Kruppa invariants and the striction curve for ruled surfaces in a Minkowski space M_B and geometric properties of those substitutes. It will turn out that we find several types of curvatures and torsions and that we have to modify the usual covariant differentiation in a way that connects the topic also to the so-called "relative differential geometry".

The first thing to do is to find a useful substitute for the Euclidean (Cartesian) moving frame, i.e. to replace the Euclidean concepts of orthogonality by one defined by B. For a Euclidean space orthogonality stems from the polarity of an oval quadric, the unit sphere B, and is symmetric and motion- (and similarity-) invariant. In Minkowski geometry one has to look for suitable substitutes for this concept and references provide a large number of more or less reasonable Minkowski orthogonalities. The most frequently used definitions are due to Birkhoff [5] and to James [6-8]. It turns out that these orthogonalities are not symmetric. We will use the Birkhoff orthogonality as the one closest to geometry. We have to distinguish a line a left-orthogonal to another b from b being right- (but in general not left-) orthogonal to a, (in signs: $a \dashv b$, $b \dashv a$), and the same symbols are used for the orthogonality relations of lines and planes and of planes and planes. If M_B is a normed linear space with unit ball B and if $x, y \in M_B$ then, we say that x is left orthogonal (Birkhoff orthogonal) to y ($x \dashv y$), if $\|x+\alpha y\| \geq \|x\|$ for all α in $\mathbb{R}$. If x is left-normal to y, it does not follow that y is left-normal to x. In fact, for dimensions three or above the only normed spaces for which normality is symmetric are the Euclidean spaces. In dimension two, normality is symmetric for the wide class of Radon planes [24].

We collect the main results of this dissertation briefly:

- Definition of the Minkowski striction curve (M-striction curve) and the left-orthogonal moving frame of a ruled surface. The striction point S of a generator g is declared as the touching point of the central plane of g, the latter being left-orthogonal to the asymptotic plane of g.

- Definition of a semi-inner product in M_B (following Thompson [1]) and with this the functions Cosine-Minkowski and Sine-Minkowski.
- Definition and calculation of Minkowski substitutes for the classical Euclidean Kruppa-curvature and –torsion as coefficients of the system of Frenet-Serret equations.
- Modification of the covariant differentiation in M_B and definition of a "deformation vector".
- Characterisation of the M-striction curve as the uniquely defined curve, along which the generators of the ruled surface form a Minkowski-geodetic parallel field.
- Definition of a Pirondini set of ruled surfaces having the M-striction curve and the strip of central planes in common.

The dissertation is structured as follows:

Chapter 1 is an introductory chapter with an explanation of the problem and giving basic definitions and presenting the aim of the dissertation.

Chapter 2 presents some of the different orthogonality concepts in normed linear spaces. For example the orthogonalities which are given by Roberts [4] in (1934), Birkhoff [5] between 1945-1947, Carlsson [25] in (1962) and Miliĉić [27] and we show relations between these orthogonalities. Based on the Birkhoff orthogonality concept we define a semi-inner product in a real linear space which later will be fundamental (in chapter 5).

Chapter 3 contains a summary about the support theorem and some related definitions in Minkowski space which play an importance role for the definition of the Minkowski surface area. The polar reciprocal K° of a closed convex set K in M_B^n also depends on it fundamentally. As a topic aside we collect results concerning the *isoperimetric problem* in Minkowski space. Its solution $\tilde{I}_B$ is, in general, not a Minkowski sphere, but a convex surface called *isoperimetrix*. We discuss the concept of surface area following different approaches in for a given *n*-dimensional Minkowski space ($n \geq 3$). There are mainly two definitions of the surface area, one by Busemann [41] and the other by Holmes-Thompson [1]. For M_B^n, $n \geq 3$, the shape of the isoperimetrix depends on the definition of the area, which is not unique.

Chapter 4 presents the trigonometric functions in Minkowski space which are defined for first time by Busemann [42] and updated by Thompson [1]. The concept of the angle in Minkowski space is dependent on the position of the angle and not only on the size of it. Therefore, we define those functions using the most convenient unique linear function which attains its norm at exactly one member of the space. The cosine and sine Minkowski functions are of two variables dependent on the sort of those variables. These functions are connected to the concept of B-orthogonality and transversality. We define the Minkowski semi-inner product of two vectors, using the cosine Minkowski function which may be useful in the core ideas in the later chapters of that dissertation. Also we insert some importance trigonometric formulae in the two dimensional space which can be used later.

Chapter 5 contains the construction of an Minkowski orthonormal frame using Birkhoff orthogonality in M_B^3. Like in Euclidean case it is based on a given (oriented) flag (P,g,α) of incident half-spaces, namely point P, half-line g and half-plane α. We translate this flag such that P becomes the centre of the unit ball B and we intersect g and α and B, receiving intersection point G and an arc a. The half-tangent t of arc a at G translated through P represents the 2nd leg of the frame, it intersects B in T and $\overrightarrow{PG} =: e_1$ and $\overrightarrow{PT} =: e_2$ are left-orthogonal unit vectors. Translating α such that it touches B gives two possibilities for a point of contact N. We choose the one, such that $\{ e_1, e_2, \overrightarrow{PN} =: e_3 \}$ forms a right handed (affine) frame (see Figure 5.1 in the dissertation). For this frame we therefore have

$$e_1 \dashv e_2 \ , \ e_3 \dashv e_1 \ , \ e_3 \dashv e_2 \ .$$

Especially for (non-cylindrical) ruled surfaces $\Phi = \{ g(t), t \in I \subset \mathbb{R} \}$ there is a canonically defined flag connected with each (oriented) generator $g(t_0)$. It consists of g itself, the asymptotic plane α parallel to direction vectors $g(t_\circ)$ and $\dot{g}(t_\circ)$. As the point P of the flag we use the point of contact of the so-called central plane $\zeta = g \vee n_M$, whereby n_M denotes the Minkowski-normal vector to α. This point P:=s obviously has to be called "Minkowski-striction point" of generator $g(t_\circ)$. All those points $P(t)$ form a curve S, the Minkowski striction curve (see Figure 5.3 a,b of dissertation).

Furthermore we define the Minkowski covariant differentiation on the ruled surface. It is the product of the usual differentiation and the (local) Minkowski-normal projection. This Minkowski-normal projection deviates from the Euclidean normal projection by a "deformation vector" and and with this vector it is possible to find Minkowski-Frenet-Serret formulae for ruled surfaces. In these formulae the occurring coefficients mean M-curvatures and M-Torsions. If B especially is an ellipsoid, the Frenet-equations and coefficients become those for the Euclidean case.

Chapter 6: Using the Minkowski covariant differentiation and the Minkowski version of Gauss's and Weingarten's derivative equations we can define the geodesic parallel field Y along a curve $c(t)$ on the surface. and we can formulate the fundamental condition for a curve to be Minkowski geodesic.

In Euclidean space the striction curve S is characterised by the fact that it is the unique curve such that the generators form a geodesic parallel field along S. In a Minkowski space this is also true, if we redefine the concept "parallel field" along the curve with respect to the B-orthogonality. Here additional conditions have to be used, too.

There is no theorem corresponding to *Bonnet's theorem* in the Minkowski 3-space, that means that if the curve is M-striction and M-geodesic it does not follow that it is also an isogonal trajectory of the generators. Constance of the striction angle would involve Minkowski angle measurement aside orthogonality and also for this there exist many different approaches. But for the (Euclidean) *theorem of Pirondini* considering the set of ruled surfaces with common striction strip (i.e. the striction curve plus the set of central planes) it is possible to formulate a version in Minkowski spaces M_B^3.

References and Figures mentioned in this text please see the dissertation itself.

Emad Shonoda
Kurzfassung der Dissertation

"Ruled Surfaces in three-dimensional Minkowski Space"

(Regelflächen im 3-dimensionalen Minkowski Raum)

Schauplatz des Gegenstandes, den diese Dissertation behandelt, ist ein affiner 3-Raum mit einer Metrik, di durch einen zentralsymmetrischen, convexen Eichball B definiert wird. Ein solcher Raum heißt *Minkowski-Raum*. Wir werden nur solche Minkowski-Räume benützen, bei denen B noch zusätzlich glatt und streng convex ist. Jede Stützebene von B berührt B also in genau einem Punkt und der Rand ∂B von B ist eine im Sinne der Differentialgeometrie reguläre (geschlossene) Fläche.

Wir studieren Regelflächen in einem solchen Minkowski-Raum M_B^3 und folgen dabei der auf E. Kruppa zurückgehenden Behandlungsweise von Regelflächen eines euklidischen Raumes. Gemäß Kruppa ist eine Regelfläche durch drei Invariantenfunktionen, (die vom Bogenlängen-Poarameter dereiner ausgezeichneten Flächenkurve, der Striktionskurve, abhängenden Kruppa'schen Funktionen *Krümmung*, *Torsion* und *Striktion*) bis auf Bewegungen im euklidischen 3-Raum eindeutig bestimmt. Damitz ist das Arbeitsprogramm der Dissertation fixiert: Man suche nach Analoga der Kruppa-Invarianten für Regelflächen eines Minkowski-Raumes und studiere ihre geometrischen Eigenschaften. Es zeigt sich, dass wir auf mehrere Krümmungs- und Torsionsfunktionen stoßen und dass wir die kovariante Differentiantion von Vektorfeldenr im Sinne einer „relativen Differentialgeometrie" zu modifizieren haben.

Als erstes muss ein im Sinne der Metrik des M_B^3 orthonormiertes Begleit-Dreibein für Regelflächen erklärt werden. Im euklidischen Fall stammt die Orthogonalität (und die Normiertheit) vom Polarsystem einer ovalen Quadrik, der euklidischen Einheitssphäre. In M_B^3 ist eine große Zahl von unterschiedlichen Orthogonalitätsbegriffen definiert worden, worunter diejenige von Birkhoff [5], die auch wir zur Grundlage nehmen, die gängigste ist. Die Birkhoff-Orthogonalität ist in nicht-euklidischen Minkowski-Räumen keine symmetrische Relation! Wir haben die Begriffe „a ist links-orthogonal zu b" ($a \dashv b$) und „a ist rechts-orthogonal zu b" (was gleichbedeutend mit $b \dashv a$ ist) zu unterscheiden. Wir verwenden die nämlichen Symbole auch für die Orthogonalitätsrelationen zwischen Geraden und Ebenen bzw. Ebenen und Ebenen. Sei M_B^3 ein Mikowski-Raum mit dem Eich-Ball B und $x, y \in M_B^3$ dann heißt x links-orthogonal (Birkhoff orthogonal, B-orthogonal) zu y ($x \dashv y$), wenn gilt $\|x+\alpha y\| \geq \|x\|$ for all $\alpha \in \mathbb{R}$. Aus $x \dashv y$ folgt i. a. nicht $y \dashv x$. In der Tat, für Dimension 3 und höher sind die einzigen Minkowski-Räume mit symmetrischer Orthgogonalität die euklischen Räume. Hingegen ist für Minkowski-Ebenen die Orthogonalität symmetrisch (genau) für die große Klasse der sogenannten Radon-Ebenen [24].

Wir fassen die Hauptresultate dieser Dissertation wie folgt zusammen:

- Definition der Minkowski-Strictionskurve (M-striction curve) and eines B-orthgogonalen Begleitbeins einer Regelfläche Der Striktionspunkt S einer Erzeugenden g der Regelfläche ist dabei erklärt als Berührpunkt der sogenannten Zentralebene von g, die ihrerseits links-orthogonal zur asymptotischen Ebene von g ist.

- Definition eines *semi-inneren Produkts* in M_B^3 (- dabei folgen wir Thompson [1]) und damit Erklärung einer „*Cosine-Minkowski*-Funktion“ und einer „*Sine-Minkowski*-Funktion“.
- Definition and Berechnung der Minkowski-geometrischen Analoga der klassischen euklidischen Invarianten (Krümmung, Torsion) von Kruppa als Koefficienten des Systems von Ableitungsgleichungen von Frenet-Serret.
- Modifikation der covarianten Differentiation in M_B^3 und Definition eines “Deformations-Vektors”.
- Characterisierung der M-Strictionskurve als jene eindeutig bestimmte Flächenkurve, uniquely defined curve, längs der die Erzeugenden der Regelfläche geodätische parallel sind.
- Definition der Pirondini-Schar von Regelflächen zu einer gegebenen (nicht-zylindrischen) Regelfläche. Es sind dies alle jene Flächen, die mit der Ausgangsfläche den Striktionsstreifen (bestehend aus der Striktionskurve und der von den Zentralebenen eingehüllten Torse) gemeinsam haben.

Gliederung der Dissertation:

Chapter 1 ist ein einführendes Kapitel, in welchem das zu behandelnde Problem erklärt wird und grundlegende Definitionen dargelegt werden.

Chapter 2 gibt eine Zusammenstellung der Orthogonalitätsbegriffe in Minkowski-Räumen Beispiele solcher Orthogonalitätsbegriffe sind etwa die von Roberts [4] (1934), Birkhoff [5] (1945-1947), Carlsson [25] (1962) und Miliĉić [27] stammenden. Sämtliche dieser Begriffe müssen sich auf den Eichball *B* stützen, weshalb sich auch Beziehungen zwischen diesen Begriffen aufzeigen lassen. Auf der Basis der Birkhoff Orthogonaliät definieren wir ein semi-inneres Produkt, das später (in Chapter 5) grundlegend wird.

Chapter 3 enthält eine Zusammenstellung der durch die Stützfunktion des Eichballs *B* erklärten Begriffsbildungen, wie sie zur Definition eines Flächeninhaltbegriffes in Minkowski erforderlich sind. Der polar-reziproke Körper K° einer konvexen Menge K in M_B^n hängt ebenfalls davon ab. Als ein Gegenstand am Rande wird das Isoperimetrie-Problem erwähnt. Das Lösungsgebilde $\tilde{I}_B$ ist im allgemeinen keine Minkowski-Sphäre, sondern eine Konvexe Fläche, die sogenannte I*soperimetrix*. Wir führen verschiedene Konzepte der Inhaltsdefinition in einem *n*-dimensionalen Minkowski-Raum an ($n \geq 3$), wobei hauptsächlich auf die Definitionen des Flächeninhaltes durch Busemann [41] and durch Holmes-Thompson [1] eingegangen wird. For M_B^n, $n \geq 3$, hängt die Gestalt der Isoperimetrix von der Definition der Flächenmaßes ab und ist demnach nicht eindeutig.

Chapter 4 stellt trigonometrische Funktionen in Minkowski-Räumen vor; solche sind erstmalig von Busemann [42] eingeführt und später von Thompson [1] modifiziert worden. Der Winkelmaß-Begriff in Minkowski-Räumene hängt nicht allein von der Gestalt des Winkels ab, sondern auch von seinen Position. Wir definieren deshalb soche Winkelfunktionen durch eine eindeutig erklärte lineare Funktion, die ihre Norm für ein bestimmtes Exemplar eines Winkels annimmt. Die cosine- und sine- Minkowski-Funktionen beschreiben Größe und Lage eines solchen Winkels. Diese trigonometrischen Funktionen sind natürlich von *B* abhängig. Damit erklären wir das *Minkowski-semi-innere Produkt* zweier

Vektoren, welches sich für diese Dissertation als fundamental für die Berechnung von Vektor-Abhängigkeiten erweist.

Chapter 5 ist das erste der Liniengeometrie in M_B^3 gewidmete Kapitel. Es stellt die Definition und Handhabung des B-orthonormierten Begleit-Dreibeins einer Regelfläche und deren M-Striktionskurve vor. Wie im euklidischen Fall wird dieses Begleit-Dreibein von der Flagge (P,g,α) bestehend aus der Erzeugenden g, ihrem Fernpunkt P und dessen Tangentialebene α , der sogenannten asymptotischen Ebene, abgeleitet. Wir verschieben diese Flagge parallel, sodas P zum Mittelpunkt O von B wird. Der normierte Richtungsvektor e_1 der Erzeugenden g ist das 1.Bein des Dreibeins; durchläuft g die Regelfläche Φ, so entsteht so das "sphärische Bild" von Φ auf B. Dessen normierter Tangentenvektor e_2 (er ist der Ableitungsvektor des Erzeugendenvektors) bestimmt dabei das 2.Bein und spannt mit $\vec{e}_1$ eine zur asymptotischen Ebene α parallele Ebene auf. Es gilt dabei $e_1 \dashv e_2$. Verschiebt man α so, dass B berührt wird, ergeben sich zunächst zwei Möglichkeiten. Wir wählen diejenige aus, sodass der Ortsvektor zum Berührbunkt N die beiden bisherigen Beinvektoren zu einem Rechtsdreibein $\{e_1, e_2, \overline{PN} =: e_3\}$ (Fig. 5.1 der Dissertation). Wir haben also insgesamt

$$e_1 \dashv e_2 \ , \ e_3 \dashv e_1 \ , \ e_3 \dashv e_2 \ .$$

Speziell für (nicht-zylindrische) Regelflächen Φ = $\{ g(t), t \in I \subset \mathbb{R} \}$ haben wir mit obiger Konstruktion ein B-orthonormiertes Begleitbein konstruiert. Dessen 1. und 2. Beinvektor sind demnach Linearkombinationen des (i.a. nicht normierten) Erzeugendenrichtungsvektors $g(t_\circ)$ und dessen Ableitungsvektors $\dot{g}(t_\circ)$. Der dritte Beinvektor spannt mit g die sogenannte M-Zentralebene $\zeta = g \vee n_M$ auf, wobei $n_M = \lambda e_3$ den Minkowski-Normalvektor zu α bezeichnet. Der Berührpunkt $P(t)=s$ von ζ mit Φ muss dann offensichtlich "Minkowski-Striktionspunkt" der Erzeugenden $g(t_\circ)$ genannt werden. Alle diese Punkte $P(t)$ bilden die Kurve S, die M-Striktionskurve von Φ, (Fig.5.3 a,b).

Des Weiteren erklären wir eine covariante Differentiation auf Flächen in Minkowski Räumen. Wie im Euklidischen ist sie das Produkt aus der gewöhnlichen Differentiation und der (lokalen, links-orthogonalen) Minkowski-Normalprojektion. Letztere lässt sich in zwei Komponenten zerlegen, wovon eine die euklidische Normalprojektion ist und die zweite eine tangentiale Scherung bestimmt durch einen „Deviation Vector", der eben die Abweichung des Minkowski-Raumes von einem euklidishcen Raum beschreibt. Damit ist auch die Bestimmung der Koeffizientenfunktionen der Frenet-Serret-Ableitungsgleichungen für Regelflächen möglich. Die in diesen Gleichungen auftretenden Koeffizienten werden sinngemäß als M-Krümmungen und M-Torsionen bezeichnet. Speziell im Fall, dass B ein Ellipsoid ist, gehen die allgemeinen Frenet-Serret-Formeln in die bekannten euklidischen über.

Chapter 6: Unter Verwendung der oben genannten covarianten Differentiation und der Minkowski-Versionen der Formeln von Gauß und Weingarten läßt sich eine geodätische Parallelverschiebung auf einer Fläche im Mikowski-Raum erklären und insbesondere ein M-geodätisches Parallelvektorfeld Y längs einer Flächenkurve $c(t)$ definieren. Ebenso kann man die fundamentale Bedingung dafür herleiten, dass eine bestimmte Kurve eine M-Geodätische der Fläche ist.

Im Euklidischen ist die Striktionskurve S einer (windschiefen) Regelfläche dadurch charakterisiert, dass die Regelflächenerzeugenden längs ihr ein geodätisches Parallelfeld bilden. In Minkowski-Räumen ist das ebenso richtig, wenn man die Begriffe „Parallelfeld" und „covariante Differentiation" etwas weiter modifiziert. (Bei Links-Normalprojektion ist die M-geodaetische Parallelität der Erzeugenden längs der M-Striktionskurve nur dann gegeben, wenn die Orthogonalität symmetrisch ist, es sich also um einen euklidischen Raum handelt. Für Regelflächen lässt sich auch eine Rechts-Normalprojektionlängs der M-Striktionslinie erklären und damit eine neue covariante Differentiation begründen.)

Die Diskussion weitere liniengeometrische Sätze, die für Regelflächen im euklidischen Raum gelten, auf ihre Gültigkeit in Minkowski-Räumen zu untersuchen, ist naheliegend. Zum Beispiel gibt es in Minkowski-Räumen zunächst kein dem Theorem von Bonnet entsprechdes Theorem. (Es würde besagen, dass wenn eine Kurve einer Regelfläche zwei von den drei Eigenschaften (sie ist Striktionslinie, Geodätische, Isogonaltrajektorie der Erzeugenden) hat, so hat sie auch die dritte. Zuerst müsste ein mit der B-Orthogonalität kompatibler Winkelbegriff in M_B^3 eingeführt und der Begriff „Isogonalität" erklärt werden. Da es auch für die Erklärung des Winkelmaßes mehrere Zugänge gibt, verdient der Satz non Bonnet eine gesonderte Behandlung, die nicht im Rahmen dieser Dissertation erfolg. Hingegen lässt sich für den (euklidischen) *Satz von Pirondini,* der die Menge aller Regelflächen betrachtet, die den Striktionsstreifen (bestehend aus der Striktionslinie und der von den Zentralebenen eingehüllten Torse) in Minkowski-Räume übertragen M_B^3. Auch hier enthält die ursprüngliche Formulierung des Satzes zwar einen Winkelmaßbegriff, der aber umgangen werden kann.

Ein kurzer Ausblick beschließt die Dissertation.

Die hier zitieren **Literaturangaben** und **Figuren** beziehen sich auf die Dissertation und sind hier nicht gesondert aufgelistet.

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe. Die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Erklärung

Die Dissertation mit dem Thema „**On ruled surfaces in Minkowski three dimensional space**“ wurde an der Technischen Universität Dresden an der Fakultät für Mathematik und Naturwissenschaften im Institut für Geometrie unter der Betreuung von **Prof. Dr. G. Weiß** angefertigt.

Ich erkenne die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der TU Dresden vom 20.03.2000, in der geänderten Fassung mit Gültigkeit vom 16.04.2003 und vom 01.10.2008 an.

Dresden, den Unterschrift