Equilibria in Quitting Games
and Software for the Analysis

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Introduction

In 1928 von Neumann proved, with the so called minimax theorem, the existence of an optimal strategy in two-person zero-sum games (see [40]). Later in 1944 he published together with Morgenstern the book “Theory of Games and Economic Behavior”, which can be seen as cornerstone of the analysis of games in strategic form. Some years later in 1951 Nash proved, that all finite\(^1\) non-cooperative games have an equilibrium point. We call a Nash-equilibrium a profile of strategies for each player, where one player cannot profit from changing its strategy unilaterally. Since Nash’s result is known, a lot of researchers have sought for algorithms, which find Nash-equilibria in finite games. There are various publications about the computation of them. A good overview about the existing basic algorithms and their common ground is given in [19]. For the two-player case the so-called Lemke-Howson algorithm and the van den Elzen-Talman algorithm and for the \(N\)-player case the Herings-van den Elzen algorithm and the Herings-Peeters algorithm are the most famous ones. Those algorithms are implemented, for example, in Gambit a library of game theory software (cf. [24]), which is the standard software for the analysis of finite games.

Leaving finite games – here games where the set of pure strategies for the players are finite – and turning to undiscounted stochastic games, we realize that the question whether all undiscounted stochastic games have a Nash- or \(\varepsilon\)-equilibrium for all \(\varepsilon > 0\) is still open, even if the undiscounted stochastic game is “simple”\(^2\). Such a “simple” undiscounted non-cooperative stochastic game, the so called quitting game, is the center of this thesis. The game of our interest has finitely many players, where each player has only two possible actions, namely continuing the game or quitting the game. All players make their decision simultaneously and independently of each other. If at least one player decides to quit the game, the game ends for all players and they receive a payoff, that only depends on the set of players, who did chose to quit the game. If nobody quits, the game continues to the next round. If the game never ends, the payoff to each player is zero. In 2001 the quitting game was defined in general by Solan and Vieille (see [36]).

The question whether quitting games or undiscounted stochastic games in general always have so called approximate equilibria is still open and there is no existing software for the analysis of such games, or even stochastic games with more than two players, known. This is the starting point of our work.

\(^1\)Finite means here that the set of pure strategies for the players are finite.
\(^2\)Stochastic games were considered first by Shapley in 1953 (cf. [30]). For an overview on recent results on stochastic games in general see e.g. [43].
Introduction

The thesis consists of two main parts. The first one is dedicated to questions concerning the existence of equilibria in quitting games and the second one concentrates on the computational analysis of quitting games.

We start in Chapter 1 with a review of existing results and literature, followed by the definition of quitting games in general. Beside the usual definitions we state the underlying probability space and the random variables that are used to describe the game, which are often missing in the existing literature. In Section 1.2, we consider the one-step game corresponding to the quitting game, where we focus on perfect strategies, which play a major role in the proofs of Chapter 2 and are here also used for example to prove the existence of a Nash-equilibrium in one-step games. If one repeats the one-step game finitely often we come to the finite quitting game, which is treated in Section 1.3. Vital results for the computational analysis of quitting games are treated in Section 1.4, where we work out the relations between the structure of the strategy profile of the players and the expected payoff the players achieve in the quitting game.

Chapter 2 gives an overview about results on the existence of $\varepsilon$-equilibria in quitting games, where we mainly focus on the results by Solan and Vieille (cf. [36]). Starting with symmetric quitting games and those that have dominant strategies, we consider quitting games in general. A main result of the thesis is contained in Section 2.3. It generalizes a result of Solan and Vieille. They showed that for all quitting games, where the players get a payoff equal to one if they play quit alone, and a payoff lower or equal to one if they play quit together with someone else, have an $\varepsilon$-equilibrium for all $\varepsilon > 0$. We extend this to games where it is allowed that the players receive zero if they play quit alone and a payoff lower or equal to zero if they play quit together with someone else. For the proof we need three propositions. The last one can be used to evaluate the results of an algorithm that detects cyclic $\varepsilon$-equilibria in quitting games (see Section 3.6). We give a detailed proof based on the corresponding result by Solan and Vieille, which also improves the resulting approximations on the quality of the equilibrium in that way, that it is now independent of the number of players.

Since there is no software for the analysis of quitting games, or stochastic games with more than two players, we want to make a step forward to close this gap. In Chapter 3 we provide algorithms and explain the implemented programs for symmetric quitting games, for a reduction by dominance and for the detection of a pure, instant or stationary $\varepsilon$-equilibrium. Furthermore we discuss questions that are related to an implementation of an algorithm for the detection of a cyclic $\varepsilon$-equilibrium.

While the focus in Chapter 3 is on the detection of one sample $\varepsilon$-equilibrium we consider in Chapter 4 the search for all or as many as possible equilibria. In the first part of Chapter 4, we discuss two parallelized algorithms for a search of instant and stationary $\varepsilon$-equilibria in quitting games and treat two software packages, i.e. Gambit and PHCpack, which are also helpful for the analysis of quitting games, since one can use them to analyze the corresponding one-step games or to compute stationary Nash-equilibria in quitting games.
Some general conventions

Whenever we introduce a new notation we write ‘:=’.

With $\mathbb{N} := \{1, 2, \ldots\}$ we denote the positive integers starting with 1. Furthermore $\mathbb{R}$ denotes the real numbers and $\mathbb{R}_+ := [0, \infty[$.

A vector $y \in \mathbb{R}^N$, $(N \in \mathbb{N})$, is a column vector. For the transposed vector we use $\cdot^T$.

Usually we denote by $y^n$ the $n$-th component of the vector $y$, $n = 1, \ldots, N$. Furthermore we denote with $\| \cdot \|$ the maximum norm, i.e.

$$\|y\| := \max_{i \in N} |y^i| \text{ for all } y = (y^1, \ldots, y^N)^T \in \mathbb{R}^N.$$ 

We set $\inf \emptyset := \infty$ and $\prod_{\emptyset} := 1$.

All the other notation, which are used in this thesis is listed at the end in the Index of notation.

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Chapter 1.

Quitting games

Quitting games are undiscounted sequential stochastic games with finitely many players. At any stage each player has only two possible actions, continue and quit. The game ends as soon as at least one player chooses to quit. The players then receive a payoff, which depends on the set of players that did choose to quit. If the game never ends, the payoff to each player is zero.

One field of application for quitting games could be for example in economics, where several firms, which produce the same kind of product, plan to enter a new market. They have only two possible actions. Either they choose to enter the market (represented by the action quit – quit the thoughts about doing it) or not to do it (represented by the action continue – continuing the thoughts about it). Every day each company (players) decides independently of each other to enter the market or not. If no one decides to offer their product on the new market, they have to decide the next day (or hour) again. The market entry has to be chosen carefully. If a lot of other firms enter the market, the price for the offered product is lower than the price of the product, if only a few firms enter the market. So the payoff depends on the set of players that did choose to enter.

Quitting games belong to the class of non-cooperative non-zero-sum multi-player recursive repeated games with absorbing states. A game is said to be non-cooperative (see e.g. [18]), if commitments like promises or agreements have no binding force. A state is absorbing, if the probability of leaving that state is zero for all possible pairs of actions and a game is called recursive, if all payoffs in the non-absorbing states are equal to zero (see e.g. [14]). Kohlberg 1974 proved in [20] that every two-player zero-sum absorbing game has an equilibrium payoff. Afterwards Vrieze and Thuijsman 1989 showed in [38] that every two-player non-zero-sum absorbing game has an equilibrium payoff. In 1997 Flesch, Thuijsman and Vrieze examined in [14] a three-player game with absorbing states and pointed out that the analysis of $N$-player ($N \geq 3$) games is different from any analysis used for two-player games.

Solan proved 1999 in [34] that all three-player absorbing games have approximate equilibria. In the paper Quitting Games, Solan and Vieille 2001 ([36]) defined the quitting games in generality first and proved the existence of subgame-perfect approximate equilibria under
some restrictions to the payoff function. Furthermore Solan and Vieille 2003 studied in [37] an example of a four-player quitting game, in which the simplest equilibrium strategy is periodic with period two. In The structure of non-zero-sum stochastic games 2007 ([31]), Simon showed under which properties quitting games have approximate equilibria among other things by generalization of the solution idea from Solan and Vieille. Furthermore he proved by using algebraic topology that a special class of quitting games, the so called escape games, have approximate equilibria. In 2012 Simon considered in [32] quitting games with only so called normal players using topological dynamics. Finally it is still not clear whether all quitting games have approximate equilibria or not, even for four players the question is still open.

The outline of this chapter is as follows. First we introduce the quitting game in general. We define the strategy profile for the players and construct the underlying probability space and random variables referring to a given strategy profile in order to describe mathematically, how the game is played. The expected payoff for the players plays a central role in game theory, because all players are assumed to behave rational, they try to maximize their own expected payoff. This motivates the definition of an equilibrium in the sense of Nash (see [27]), where an unilateral deviation from the strategy of one player gives her no additional profit (or not more than $\varepsilon$). We distinguish between several types of equilibria with respect to the structure of the strategy profile. In Section 1.2. the base game, the so-called one-step game, of the quitting game is considered. Here the concept of a perfect strategy profile is central. With the help of this, we show, for example, that all one-step games have at least one Nash-equilibrium. In Section 1.3., we introduce the finite quitting game. It has the same payoff-structure as the corresponding quitting game but is played only a finite number of steps and will be later used in Chapter 2 to prove that a certain kind of quitting games has an $\varepsilon$-equilibrium (see Section 2.3). Using one-step games and finite quitting games corresponding to a given quitting game, we show which relations exist between the structure of a given strategy profile and the value of the expected payoff, and how to construct new $\varepsilon$-equilibria out of given ones. This will be helpful for programming purposes (see e.g. Section 3.3).

1.1. Quitting games

As mentioned before, a quitting game is a sequential $N$-player ($N \in \mathbb{N}$) game and played as follows. In every step, each player has only two possible actions, continue and quit. The players make their decisions simultaneously, that means in particular independently from each other. The game continues as long as all players choose to play continue. If at least one player plays the action quit, the game terminates for all players. In that case, the players receive a payoff, which depends on the set of players that did choose to play the action quit. If the game never ends, the payoff to each player is zero. This leads us to the following definition.
Definition 1.1.1 ( Quitting Game). A quitting game is a tuple
\[ G = (\mathcal{N}, (r_S)_{\emptyset \subseteq S \subseteq \mathcal{N}}) \] (1.1)
where
\begin{itemize}
  \item \( \mathcal{N} = \{1, \ldots, N\} \subset \mathbb{N} \) is a finite set of players, \( N \in \mathbb{N} \),
  \item \( S \in \mathcal{P}(\mathcal{N}) \) denotes the quitting coalition and
  \item \( (r_S)_{S \in \mathcal{P}(\mathcal{N})} \) is a sequence of payoff-vectors
    \( r_S = (r^1_S, \ldots, r^N_S)^T \in \mathbb{R}^N \)
    to the players under the quitting coalition \( S \) with \( r_\emptyset := (0, \ldots, 0)^T := 0 \).
\end{itemize}

Observe that in game theory usually the number of players is written as a superscript, i.e. \( r^S_n \) denotes the payoff of player \( n \) under the quitting coalition \( S \).

Remark 1.1.2. To compare this definition of a (stochastic) game for example with [33] or [29], we have the following setting:

1. The state space is given by \( Z := \mathcal{P}(\mathcal{N}) \).
2. The action space is given by \( A := \{0, 1\}^N \), where 0 stands for \textit{continue} and 1 for \textit{quit}. That means all players have the same available actions and the actions which can be chosen are independent from the state of the game.
3. The transition law \( \tilde{t} : \mathcal{P}(\mathcal{N}) \times \mathcal{P}(\mathcal{N}) \times A \to [0, 1] \) is given by
   \[
   (z, S, a) \mapsto \tilde{t}(z|S,a) := \begin{cases} 
   1 & \text{for } S = \emptyset \text{ and } z = \{ n \in \mathcal{N} : a^n = 1 \} \\
   1 & \text{for } S \neq \emptyset \text{ and } z = S \\
   0 & \text{otherwise}
   \end{cases} 
   \]
   where \( z, S \in \mathcal{P}(\mathcal{N}) \) and \( a = (a^1, \ldots, a^N) \in A \). That means if the game is not terminated, the given state is \( S = \emptyset \). In this case the given action \( a = (a^1, \ldots, a^N) \) determines the next state, where the transition law has the value one only if \( z = \{ n \in \mathcal{N} : a^n = 1 \} \). If the game is already terminated, the current state is represented by the quitting coalition \( S \neq \emptyset \). In that case the next stage can only be \( z = S \) independent of what the players play.

Observe that the transition law is even deterministic and independent of the last state of the game and the last chosen action.
4. The payoff function is given by \( r : A \to \mathbb{R}^N \), where
   \[
   a = (a^1, \ldots, a^N)^T \mapsto r(a) := r_{\{n \in \mathcal{N} : a^n = 1\}},
   \]
   and is state-independent.
5. There is no discounting in this model.
Example 1. A typical way to describe two- or three-player quitting games is via a matrix. Consider the following two player game, where Player one is the so called row player and player two the column player.

<table>
<thead>
<tr>
<th>Player 2</th>
<th>continue</th>
<th>quit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>continue</td>
<td>(0 , -5)</td>
</tr>
<tr>
<td></td>
<td>quit</td>
<td>(-10 , -10)</td>
</tr>
</tbody>
</table>

Where $\bigcirc$ means, that the players does not receive any payoff and the game continues to the next round.

In this case the quitting game is given by

$$G = \left( \{1, 2\}, \left( r_\emptyset = 0, r_{\{1\}} = \left( \begin{array}{c} -5 \\ 0 \end{array} \right), r_{\{2\}} = \left( \begin{array}{c} 0 \\ -5 \end{array} \right), r_N = \left( \begin{array}{c} -10 \\ -10 \end{array} \right) \right) \right).$$

Now the players choose their action not with certainty, but according to a certain probability.

Definition 1.1.3 (strategy, profile). Let $G = (\mathcal{N}, (r_S)_{S \subseteq \mathcal{P}(\mathcal{N})})$ be a given quitting game. Then

- $p^n_i \in [0, 1]$ is the probability that player $n \in \mathcal{N}$ will play the action $\text{quit}$ at stage $i \in \mathbb{N},$
- the sequence $\pi^n := (p^n_i)_{i \in \mathbb{N}}$ is the strategy for player $n$ and $\Pi^n$ the set of all strategies for the player $n,$ $n \in \mathcal{N},$
- the vector $\pi := (\pi^1, \ldots, \pi^N)^T$ of strategies, respectively the sequence of vectors $\pi = (p_i)_{i \in \mathbb{N}},$ $p_i := (p^i_1, \ldots, p^i_N),$ is called (strategy) profile in the quitting game $G$ and $\Pi := \Pi^1 \times \ldots \times \Pi^N$ denotes the set of all (strategy) profiles for the given quitting game,
- $(\pi^{-n}, \tilde{\pi}^n) := (\pi^1, \ldots, \pi^{n-1}, \tilde{\pi}^n, \pi^{n+1}, \ldots, \pi^N)^T$ is the alternative strategy profile for player $n \in \mathcal{N},$ where all the players $m \in \mathcal{N} \setminus \{n\}$ play their strategy $\pi^m,$ whereas player $n$ changes her strategy and plays according to the strategy $\tilde{\pi}^n.$

Remark 1.1.4. The here defined strategy for a player is a so called “randomized Markovian strategy” (see e.g. [29]), because the decision, with which probability an action is played by the player in a state, is independent from the previous states of the game and chosen actions.

Definition 1.1.5 (subgame strategy, subgame profile). Let $G = (\mathcal{N}, (r_S)_{S \subseteq \mathcal{P}(\mathcal{N})})$ be a given quitting game and $\pi = (\pi^1, \ldots, \pi^N)^T = (p_i)_{i \in \mathbb{N}}$ a strategy profile in $G.$ For each $j \in \mathbb{N},$ $\pi^n_j := (p^n_j)_{i \geq j}$ denotes the subgame strategy for player $n \in \mathcal{N}$ induced by $\pi^n$ for the quitting game starting at time $j$ and $\pi_j := (\pi^1_j, \ldots, \pi^N_j)^T$ denotes the subgame profile induced by $\pi$ in the quitting game starting at time $j.$
1.1. Quitting games

**Definition 1.1.6** (pure, totally mixed, cyclic, stationary). Let \( \pi = (p_i)_{i \in \mathcal{N}} \) be a strategy profile in a quitting game \( G \). A strategy \( \pi^n = (p^n_i)_{i \in \mathcal{N}} \) for player \( n \in \mathcal{N} \) is

- **pure** : \( \iff \forall i \in \mathcal{N} : \ p^n_i \in \{0, 1\} \).
- **totally mixed** : \( \iff \forall i \in \mathcal{N} : \ p^n_i \in (0, 1) \).
- **cyclic** : \( \iff \exists k_0 \in \mathbb{N} : \ p^n_k = p^n_{k+k_0} \forall k \in \mathbb{N} \).
- **stationary** : \( \iff \forall i \in \mathcal{N} : \ p^n_i = p^n_1 \).

A strategy profile \( \pi \) is called pure/totally mixed/cyclic/stationary, if all strategies \( \pi^n \) are pure/totally mixed/cyclic/stationary, \( n \in \mathcal{N} \).

We end up this section by introducing the following notation:

**Notation 1.1.7.** The function \( \varrho : [0, 1]^\mathcal{N} \times \mathcal{P}(\mathcal{N}) \to [0, 1] \),

\[
(p, S) \mapsto \varrho(p, S) := \prod_{n \in S} p^n \prod_{m \in \mathcal{N}\setminus S} (1 - p^m),
\]

with \( p = (p^1, \ldots, p^N)^T \), denotes the probability that a quitting coalition \( S \) or – equivalent to that – an action \( a = (a^1, \ldots, a^N) \in A \) with \( S = \{ n \in \mathcal{N} \mid a^n = 1 \} \) is chosen under the vector \( p \).

**Proposition 1.1.8.** For all \( p = (p^1, \ldots, p^N) \in [0, 1]^\mathcal{N} \), all \( S \in \mathcal{P}(\mathcal{N}) \) and all \( i \in \mathcal{N} \)

\[
\varrho(p, S) = p^i \cdot \varrho((p^{-i}, 1), S) + (1 - p^i) \cdot \varrho((p^{-i}, 0), S)
\]

holds, where \( (p^{-i}, b) := (p^1, \ldots, p^{-i-1}, b, p^{i+1}, \ldots, p^N) \), \( b \in \{0, 1\} \).

**Remark 1.1.9** (Interpretation). Let \( p \in [0, 1]^\mathcal{N} \) be given. Consider the probability space \( (\mathcal{P}(\mathcal{N}), \mathcal{P}(\mathcal{P}(\mathcal{N})), P) \), where \( P(\{S\}) := P_p(\{S\}) := \varrho(p, S), \ S \in \mathcal{P}(\mathcal{N}) \), is the probability that the quitting coalition \( S \) is played. Fix a player \( i \in \mathcal{N} \) and define a random variable \( X : \mathcal{P}(\mathcal{N}) \to \{0, 1\} \) with

\[
X(S) := \begin{cases} 
1 & \text{for } i \in S, \ S \in \mathcal{P}(\mathcal{N}), \\
0 & \text{for } i \in \mathcal{N}\setminus S, \ S \in \mathcal{P}(\mathcal{N}).
\end{cases}
\]

Then the formula from Proposition 1.1.8 represents the law of the total probability. That means

\[
P(S) = P(S|X = 1) \cdot P(X = 1) + P(S|X = 0) \cdot P(X = 0).
\]

Furthermore we want to give an analytical proof of Proposition 1.1.8:
Proof. Case 1: Assume that player \( i \) is in the quitting coalition, i.e. \( i \in S \). Because

\[
\varrho(p, S) = \prod_{n \in S} p^n \prod_{n \in N \setminus S} (1 - p^n) = p^i \cdot \prod_{n \in S \setminus \{i\}} p^n \prod_{n \in N \setminus S} (1 - p^n),
\]

with \((p^{-i}, 1)^i = 1\) and \( p^n = (p^{-i}, 1)^n \) for all \( n \in N \setminus \{i\} \)

\[
\varrho(p, S) = p^i \cdot (p^{-i}, 1)^i \cdot \prod_{n \in S \setminus \{i\}} (p^{-i}, 1)^n \prod_{n \in N \setminus S} (1 - (p^{-i}, 1)^n)
= p^i \cdot \prod_{n \in S \setminus \{i\}} (p^{-i}, 1)^n \prod_{n \in N \setminus S} (1 - (p^{-i}, 1)^n)
= p^i \cdot \varrho((p^{-i}, 1), S)
\]

follows.

Case 2: Now player \( i \) is not in the quitting coalition, that means \( i \in N \setminus S \). With \((p^-i, 0)^i = 0\) and \( p^n = (p^-i, 0)^n \) for all \( n \in N \setminus \{i\} \) we obtain similarly to Case 1 that

\[
\varrho(p, S) = (1 - p^i) \cdot \prod_{n \in S \setminus \{i\}} p^n \prod_{n \in N \setminus (S \cup \{i\})} (1 - p^n)
= (1 - p^i) \cdot (1 - (p^-i, 0)^i) \cdot \prod_{n \in S} (p^-i, 0)^n \prod_{n \in N \setminus (S \cup \{i\})} (1 - (p^-i, 0)^n)
= (1 - p^i) \cdot \prod_{n \in S \setminus \{i\}} (p^-i, 0)^n \prod_{n \in N \setminus S} (1 - (p^-i, 0)^n)
= (1 - p^i) \cdot \varrho((p^-i, 0), S).
\]

Due to the fact that \( \varrho((p^{-i}, 1), S) = 0 \) for \( i \in N \setminus S \) and \( \varrho((p^{-i}, 0), S) = 0 \) for \( i \in S \), Case 1 and 2 imply the proposition. \( \square \)

**Probability space and equilibria**

Let \( G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})}) \) be the given quitting game, \( Z = \mathcal{P}(\mathcal{N}) \) the corresponding state space and \( A = \{0, 1\}^N \) the corresponding action space. Furthermore and without loss of generality let \( z = \emptyset \) be the initial state. If a strategy profile \( \pi = (p_i)_{i \in N} \in \Pi \) is given, the probability space \((\Omega, \mathcal{A}, \mathbb{P}_\pi)\) and a stochastic process \((X_k, Y_k)_{k \in \mathbb{N}}\) on \((\Omega, \mathcal{A}, \mathbb{P}_\pi)\) with values in \((Z \times A)\) are defined by\(^1\)

- \( \Omega := (Z \times A)^N \)
- \( \mathcal{A} := \mathcal{P}(Z) \otimes \mathcal{P}(A) \otimes \mathcal{P}(Z) \otimes \mathcal{P}(A) \otimes \ldots \)

\(^1\)This definition is standard, cf. [29]
1.1. Quitting games

- \( X_k(\omega) = X_k((z_1, a_1, z_2, a_2, \ldots)) := z_k \)
  (\( X_k \) denotes the random state of the system at time \( k, k \in \mathbb{N}, \omega \in \Omega \)),

- \( Y_k(\omega) = Y_k((z_1, a_1, z_2, a_2, \ldots)) := a_k \)
  (\( Y_k \) denotes the random action taken at time \( k, k \in \mathbb{N}, \omega \in \Omega \)),

- \( H_k := (X_1, Y_1, \ldots, X_k) \),
  that means \( H_k(\omega) = H_k((z_1, a_1, z_2, a_2, \ldots)) = (z_1, a_1, z_2, a_2, \ldots, z_k) \)
  (\( H_k \) describes the random history up to time \( k, k \in \mathbb{N}, \omega \in \Omega \))

- \( P(\omega) := 1 \) and
  \[ P(H_k = (z_1, a_1, z_2, a_2, \ldots, z_k)) \]
  \[ := P(X_1 = z_1) \prod_{i=1}^{k-1} l(z_{i+1}|z_i, a_i) \prod_{\{n \in \mathcal{N} \mid a^n_i = 1\}} p^n_i \prod_{\{m \in \mathcal{N} \mid a^m_i = 0\}} (1 - p^m_i) \]
  \[ = P(X_1 = z_1) \prod_{i=1}^{k-1} l(z_{i+1}|z_i, a_i) \cdot \varrho(p_i, \{ n \in \mathcal{N} \mid a^n_i = 1 \}) \]
  where \( z_i \in Z \) for all \( i = 1, \ldots, k \) and \( a_i \in A \) for all \( i = 1, \ldots, k - 1 \).

**Notation 1.1.10.** Let \( Y^n_k : \Omega \rightarrow \{0, 1\} \) denote the random action taken at time \( k \in \mathbb{N} \) of player \( n \in \mathcal{N} \). That means

\[ Y^n_k(\omega) = Y^n_k((z_1, a_1, z_2, a_2, \ldots)) := a^n_k, \]

where \( a_i = (a^1_i, \ldots, a^N_i)^T \in A \) for all \( i \in \mathbb{N} \).

With Notation 1.1.7 we have

\[ P(Y_k = a_k) = \varrho(p_k, \{ n \in \mathcal{N} \mid a^n_k = 1 \}) \]

for all \( k \in \mathbb{N} \) and \( a_k \in A \). That means \( \varrho(p_k, \cdot) \) can be interpreted as density (with respect to a counting measure) from \( Y_k \).

Let the quitting game \( G = (\mathcal{N}, (\tau_s)_{s \in \mathcal{P}(\mathcal{N})}) \) and the strategy profile \( \pi \in \Pi \) be given. Then a stopping time \( \tau : \Omega \rightarrow \mathbb{N} \cup \{+\infty\} \) concerning the filtration \( (\mathfrak{A}_k)_{k \in \mathcal{N}} \), with \( \mathfrak{A}_k := \sigma\{ Y_i : 1 \leq i \leq k \} \), is defined by

\[ \tau(\omega) := \inf \{ k \in \mathbb{N} : Y_k(\omega) \in A \setminus \{(0, \ldots, 0)^T\} \}. \quad (1.2) \]
Chapter 1. Quitting games

Definition 1.1.11 (terminating). Let $G$ be a given quitting game and $\pi = (p_i)_{i \in \mathbb{N}}$ a strategy profile in $G$. The quitting game is called terminating under $\pi$ if
\[
\prod_{i \in \mathbb{N}} \varrho(p_i, \varnothing) = 0
\]
holds.

Remark 1.1.12. Observe, that terminating does not mean that a $k \in \mathbb{N}$ has to exist such that $\varrho(p_k, \varnothing) = 0$, respectively that a player plays *quit* with certainty in one stage of the game. It only means, in other words,
\[
\sum_{i=1}^{M} \ln \varrho(p_i, \varnothing) \rightarrow -\infty \quad \text{for } M \rightarrow \infty.
\]

For example the quitting game with the strategy profile $\pi = (p_1, p_2, \ldots)$, where
\[
p_k^n = 1 - \sqrt[\frac{N}{e}] \frac{5}{k}, \quad k \in \mathbb{N}, \; \forall n \in \mathcal{N},
\]
does not terminate, although the quitting probability is strictly positive in each stage.

Definition 1.1.13 (expected payoff). Let $G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})})$ be a quitting game. Define the function $\gamma : \Pi \rightarrow \mathbb{R}^\mathcal{N}$ by
\[
\pi \mapsto \gamma(\pi) = (\gamma^1(\pi), \ldots, \gamma^N(\pi))^T := E_\pi(r(Y_\tau)|\tau < \infty),
\]
with $r$ from Remark 1.1.2, where $E_\pi$ is the expected value with respect to the probability measure $P_\pi$. Then $\gamma(\pi)$ is called the expected payoff to the players in the quitting game $G$ under the strategy profile $\pi$. $\gamma^n(\pi)$ denotes the expected payoff to player $n \in \mathcal{N}$ under the profile $\pi$.

Using the definition of $P_\pi$ and $r(\emptyset) = r_\emptyset = 0$, one obtains
\[
\gamma(\pi) = \sum_{k \in \mathbb{N}} \sum_{a_k \in A} P_\pi(\tau = k, Y_k = a_k) \cdot r(a_k)
\]
\[
= \sum_{k \in \mathbb{N}} \left( \sum_{a_k \in A} P_\pi(H_{k-1} = (\emptyset, \emptyset, \emptyset, \ldots, \emptyset), Y_k = a_k) \cdot r(a_k) \right)
\]
\[
= \sum_{k \in \mathbb{N}} \left( \prod_{i=1}^{k-1} \prod_{n \in \mathcal{N}} (1 - p_i^n) \cdot \sum_{S \in \mathcal{P}(\mathcal{N})} r_S \cdot \prod_{n \in S} p_k^n \prod_{m \in \mathcal{N}\setminus S} (1 - p_m^n) \right)
\]
\[
= \sum_{k \in \mathbb{N}} \left( \prod_{i=1}^{k-1} \varrho(p_i, \varnothing) \cdot \sum_{S \in \mathcal{P}(\mathcal{N})} r_S \cdot \varrho(p_k, S) \right).
\]
Remark 1.1.14. Another way to obtain the formula for the calculation of the expected payoff for a given profile \( \pi \) is the following:

\[
\gamma(\pi) = \sum_{k \in \mathbb{N}} P_\pi(\tau = k) \cdot E_\pi(r(Y_k) | \tau = k)
\]

\[
= \sum_{k \in \mathbb{N}} P_\pi(\tau > k - 1) \cdot E_\pi(r(Y_k))
\]

\[
= \sum_{k \in \mathbb{N}} \left( \prod_{i=1}^{k-1} g(p_i, \emptyset) \cdot \sum_{S \in \mathcal{P}(\mathcal{N})} r_S \cdot g(p_k, S) \right)
\]

Definition 1.1.15 (\( \varepsilon \)-equilibrium, Nash-equilibrium, approximate equilibria). Let \( G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})}) \) be a quitting game.

- The strategy profile \( \pi = (p_i)_{i \in \mathbb{N}} \) is an \( \varepsilon \)-equilibrium \((\varepsilon \geq 0)\) in \( G \):
  \( \iff \forall n \in \mathcal{N} \land \forall \tilde{\pi}^n \in \Pi^n : \gamma^n(\pi) \geq \gamma^n((\pi^{-n}, \tilde{\pi}^n)) - \varepsilon \).

- \( G \) has got approximate equilibria
  \( \iff \forall \varepsilon > 0 : \exists \varepsilon \)-equilibrium in \( G \).

- The strategy profile \( \pi = (p_i)_{i \in \mathbb{N}} \) is a Nash-equilibrium (or short: equilibrium) in \( G \):
  \( \iff \pi \) is an \( \varepsilon \)-equilibrium in \( G \) with \( \varepsilon = 0 \).

To be an \( \varepsilon \)-equilibrium means, that a player, who is changing her strategy alone, can only expect a profit lower or equal to \( \varepsilon \) compared to the expected payoff before the change, i.e.

\[
\gamma^n(\pi) \geq \max_{\tilde{\pi}^n \in \Pi^n} \gamma^n((\pi^{-n}, \tilde{\pi}^n)) - \varepsilon \quad \forall n \in \mathcal{N}.
\]

Furthermore, a Nash-equilibrium is an adopted maximal expected payoff, and of course having a Nash-equilibrium implies that the game has approximate equilibria.

Remark 1.1.16 (uniform \( \varepsilon \)-equilibrium). In literature one often finds another important equilibrium concept, namely the uniform \( \varepsilon \)-equilibrium, which is mentioned for completeness sake. Therefore we define for all \( i \in \mathbb{N} \) the function \( \tilde{\gamma}_i : \Pi \to \mathbb{R}^\mathcal{N} \) as follows (see [36] p. 280):

\[
\pi \mapsto \tilde{\gamma}_i(\pi) = E_\pi \left( \mathbb{1}_{\{\tau \leq i\}} r(Y_\tau) \cdot \frac{i - \tau}{i} \right).
\]

A strategy profile \( \pi \) is now called a uniform \( \varepsilon \)-equilibrium in \( G \) if:

\[
\exists i_0 \in \mathbb{N} : \forall i \geq i_0 \forall n \in \mathcal{N} \forall \tilde{\pi}^n \in \Pi^n : \quad \tilde{\gamma}_i^n(\pi) \geq \tilde{\gamma}_i^n((\pi^{-n}, \tilde{\pi}^n)) - \varepsilon
\]

Solan and Vieille proved in [36] that an \( \varepsilon \)-equilibrium \( \pi \) in a quitting game \( G \) is also a uniform \( \tilde{\varepsilon} \)-equilibrium, with \( \tilde{\varepsilon} > \varepsilon \).
Chapter 1. Quitting games

Definition 1.1.17 (cyclic / stationary / instant / subgame $\varepsilon$-equilibrium). Let $G$ be a quitting game and $\varepsilon \geq 0$. An $\varepsilon$-equilibrium $\pi = (p_i)_{i \in \mathbb{N}}$ in $G$ is called

- cyclic : $\iff \pi$ is cyclic,
- stationary : $\iff \pi$ is stationary,
- instant : $\iff \exists n \in \mathcal{N} : p^i_n = 1$ and
- subgame $\varepsilon$-equilibrium : $\iff \forall j \in \mathbb{N} : \pi_j$ is an $\varepsilon$-equilibrium in $G$.

An established procedure to analyze quitting games respectively repeated games is to dissect it into fragments. The smallest fragment is of course the one-step (quitting) game, where one considers only one stage of the game. This one-step game should be examined in the next section.

1.2. One-step games corresponding to a quitting game

In this section we introduce the base game of the quitting game. We will call it the one-step quitting game or shortly one-step game. The base game is played once by $N$ players. The players have again only the choice between two possible actions, continue and quit. They make their decisions simultaneously. If at least one player decides to play quit the players receive the same payoff like in the corresponding quitting game in dependence on the set of players that did choose to quit. If no player decides to play the action quit, the players receive a payoff $v \in \mathbb{R}^N$.

Definition 1.2.1 (One-step game). A one-step (quitting) game corresponding to a quitting game $G$ is a tuple

$$\Gamma_v := (G, v) = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})}, v),$$

where

- $G$ is the corresponding quitting game with the finite set of players $\mathcal{N} = \{1, \ldots, N\}$, the quitting coalition $S \in \mathcal{P}(\mathcal{N})$ and the family of payoff vectors $(r_S)_{S \in \mathcal{P}(\mathcal{N})}$
- $v = (v^1, \ldots, v^N) \in \mathbb{R}^N$ is the payoff to the players, if all of them choose to play the action continue otherwise they receive $r_S$, $S \in \mathcal{P}(\mathcal{N}) \setminus \{\emptyset\}$.

Remark 1.2.2. The state space, action space and the transition law from the one-step game are of course the same like in the corresponding quitting game (see therefore Remark 1.1.2). The payoff function has to be modified to $\tilde{r}_v : A \to \mathbb{R}^N$, where

$$a = (a^1, \ldots, a^N)^T \mapsto \tilde{r}_v(a) := \begin{cases} r_{\{n \in \mathcal{N} \mid a^n = 1\}} & \text{for } a \neq 0 \\ v & \text{otherwise} \end{cases}$$

As we see later the vector $v \in \mathbb{R}^N$ will mainly be used for technical reasons.

These games are also known as one-stage games ([31], p. 15) or as one-shot games ([36], p. 269).
Example 2 (Continuation of Example 1 – Prisoner’s dilemma). Back to our Example 1. The corresponding one-step quitting games are given by

<table>
<thead>
<tr>
<th></th>
<th>Player 2 (\text{continue})</th>
<th>Player 2 (\text{quit})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1 (\text{continue})</td>
<td>((v^1, v^2))</td>
<td>((-5, 0))</td>
</tr>
<tr>
<td>Player 1 (\text{quit})</td>
<td>((-5, 0))</td>
<td>((-10, -10))</td>
</tr>
</tbody>
</table>

respectively

\[ \Gamma_v = \left( \{1, 2\}, \begin{pmatrix} r_0 = 0, r_{(1)} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}, r_{(2)} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}, r_N = \begin{pmatrix} -10 \\ -10 \end{pmatrix} \end{pmatrix}, v \right). \]

Replacing \((v^1, v^2)\) by \((-1, -1)\) leads to one of the most famous one-step games the so called Prisoner’s dilemma. The payoff to the players is the time they have to stay in jail. The action \(\text{continue}\) means that the prisoner stays silent, the action \(\text{quit}\) means the prisoner confesses. The corresponding quitting game could then be interpreted in the following way: If both prisoners stay silent the examination goes on in another round. If at least one of the prisoners confesses they receive a prison sentence, depending on who of them confessed. If both of them never confess, they get no prison sentence. In praxis that means they stay in investigative custody.

Definition 1.2.3 (strategy, profile and alternative profile in the one-step game). Let \(\Gamma_v\) be a given one-step game. Then

- \(p^n \in [0, 1]\) is the strategy for player \(n, n \in \mathcal{N}\), in the one-step game \(\Gamma_v\).
- the vector \(p := (p^1, \ldots, p^N)^T \in [0, 1]^N\) is the (strategy) profile for the one-step game \(\Gamma_v\).
- \((p^{-n}, \tilde{p}^n) := (p^1, \ldots, p^{n-1}, \tilde{p}^n, p^{n+1}, \ldots, p^N)^T\) is the alternative (strategy) profile for the one-step game \(\Gamma_v\), where \(p \in [0, 1]^N\), \(\tilde{p}^n \in [0, 1]\).

Definition 1.2.4 (pure, totally mixed). Let \(\Gamma_v\) be a given one-step game and \(p \in [0, 1]^N\) a strategy profile in \(\Gamma_v\). Then \(p\) is called

- pure :\(\iff\) \(p \in \{0, 1\}^N\)
- totally mixed :\(\iff\) \(p \in (0, 1)^N\).
**Probability space and equilibria**

Let $\Gamma_v = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})}, v)$ and $p \in [0,1]^N$ be given. The corresponding probability space $(\bar{\Omega}, \bar{A}, \bar{P}_p)$ and the stochastic process $(\bar{X}_1, \bar{Y}, \bar{X}_2)$ are defined analogously to that from the quitting game, as one can see in the following table.

<table>
<thead>
<tr>
<th>One-step game</th>
<th>Quitting game</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\Omega} := Z \times A \times Z$</td>
<td>$\Omega := (Z \times A)^N$</td>
</tr>
<tr>
<td>$\bar{A} := \mathcal{P}(Z) \otimes \mathcal{P}(A) \otimes \mathcal{P}(Z)$</td>
<td>$\mathcal{A} := \bigotimes_{i \in \mathbb{N}} (\mathcal{P}(Z) \otimes \mathcal{P}(A))$</td>
</tr>
<tr>
<td>$\bar{X}_i(\omega) = \bar{X}_i((z_1, a, z_2)) := z_i$</td>
<td>$X_k(\omega) = X_k((z_1, a_1, z_2, a_2, \ldots)) := z_k$</td>
</tr>
<tr>
<td>$\bar{Y}(\omega) = (\bar{Y}^1(\omega), \ldots, \bar{Y}^N(\omega))^T = \bar{Y}((z_1, a, z_2)) := a$</td>
<td>$Y_k(\omega) = (Y_k^1(\omega), \ldots, Y_k^N(\omega))^T = Y_k((z_1, a_1, z_2, a_2, \ldots)) := a_k$</td>
</tr>
<tr>
<td>$\bar{P}_p(\bar{X}_i = \emptyset) := 1$</td>
<td>$\bar{P}_\pi(X_1 = \emptyset) := 1$</td>
</tr>
<tr>
<td>$\bar{P}_p({\omega}) = \bar{P}_p((z_1, a, z_2)) := \bar{P}_p(\bar{X}_i = z_1) \cdot \tilde{t}(z_2</td>
<td>z_1, a) \cdot \bar{g}(p, {n \in \mathcal{N} \mid a^n = 1})$,</td>
</tr>
<tr>
<td>where $i = 1, 2, z_i \in Z, \omega \in \bar{\Omega}$ and $a = (a^1, \ldots, a^N) \in A$</td>
<td>where $k \in \mathbb{N}, z_k \in Z, \omega \in \Omega$ and $a_k = (a^1_k, \ldots, a^N_k) \in A$</td>
</tr>
</tbody>
</table>

Furthermore define $\gamma_v : [0,1]^N \rightarrow \mathbb{R}^N$ by

$$p \mapsto \gamma_v(p) = (\gamma_1^v(p), \ldots, \gamma_N^v(p))^T := \mathbf{E}_p(\tilde{r}_v(\bar{Y})) = \sum_{a \in A} \bar{P}_p(\bar{Y} = a) \cdot \tilde{r}_v(a),$$

where $\mathbf{E}_p$ is the expected value with respect to the probability measure $\bar{P}_p$ and $\tilde{r}_v$ from Remark 1.1.2. Then $\gamma_v(p)$ is the expected payoff for the players in the one-step game $\Gamma_v$ under the strategy profile $p$ and $\gamma_n^v(p)$ denotes the expected payoff for player $n \in \mathcal{N}$ in the one-step game $\Gamma_v$ under the strategy profile $p$. With Notation 1.1.7 we obtain for the expected payoff $\gamma_v(p)$ the formula

$$\gamma_v(p) = \sum_{S \in \mathcal{P}(\mathcal{N})} \bar{g}(p, S) \cdot r_S + \bar{g}(p, \emptyset) \cdot v.$$
1.2. One-step games corresponding to a quitting game

**Definition 1.2.5** (ε-equilibrium, Nash-equilibrium of the one-step game). Let $\Gamma_v$ be a one-step quitting game.

- The strategy profile $p \in [0, 1]$ is an $\varepsilon$-equilibrium for $\varepsilon \geq 0$ in $\Gamma_v$:
  $$\iff \forall n \in \mathcal{N} \forall \tilde{p}^n \in [0, 1] : \gamma^n_v(p) \geq \gamma^n_v((p^{-n}, \tilde{p}^n)) - \varepsilon.$$
- The strategy profile $p$ is a Nash-equilibrium (or short: equilibrium) in $\Gamma_v$:
  $$\iff \pi \text{ is an } \varepsilon\text{-equilibrium in } \Gamma_v \text{ with } \varepsilon = 0.$$

It is already known, that all one-step games have got a Nash-equilibrium. A proof is given at the end of the next section using the so-called $\varepsilon$-perfect strategy profile.

We end up this section by proving some technical facts. First we show, how the one-step games are embedded into the quitting game referring to the calculation of the expected payoff. Recall for the expected payoff in the quitting game $G$ under a strategy profile $\pi = (p_1, p_2, \ldots)$

$$\gamma(\pi) = \sum_{k \in \mathbb{N}} \left( \prod_{i=1}^{k-1} g(p_i, \emptyset) \cdot \sum_{S \in \mathcal{P}^{(\mathcal{N})}} r_S \cdot g(p_k, S) \right)$$

$$= \sum_{S \in \mathcal{P}^{(\mathcal{N})}} r_S \cdot g(p_1, S) + g(p_1, \emptyset) \cdot \sum_{k \in \mathbb{N}} \left( \sum_{S \in \mathcal{P}^{(\mathcal{N})}} \prod_{i=1}^{k-1} g(p_{i+1}, \emptyset) \cdot r_S \cdot g(p_{k+1}, S) \right) = \gamma(\pi_2)$$

holds.

With the Definition 1.1.5 of the subgame profile,

$$\gamma(\pi) = \sum_{S \in \mathcal{P}^{(\mathcal{N})}} r_S \cdot g(p_1, S) + g(p_1, \emptyset) \cdot \gamma(\pi_2) = \gamma_{\gamma(\pi_2)}(p_1) \quad (1.5)$$

follows.

So if we have some results in one-step games, it might be possible to assign them to the corresponding quitting game.

**Remark 1.2.6.** Let $\Gamma_v$ be a given one-step game and $p \in [0, 1]^N$ a strategy profile in $\Gamma_v$. Because of

$$|\gamma^n_v(p)| \leq g(p, \emptyset) \cdot |v^n| + \sum_{S \in \mathcal{P}^{(\mathcal{N})} \setminus \{\emptyset\}} g(p, S) \cdot |r^n_S|$$

$$\leq \max \{ |v^n|, \max_{S \subseteq \mathcal{N}} |r^n_S| \}$$
Chapter 1. Quitting games

for all $n \in \mathcal{N}$, the expected payoff $\gamma_v(p)$ is bounded, i.e.

$$\gamma_v(p) \in [-\delta_v, \delta_v]^N,$$

where

$$\delta_v := \max \left\{ \max_{n \in \mathcal{N}} |v^n|, \max_{n \in \mathcal{N}, S \subseteq \mathcal{N}} |r_S^n| \right\}. \quad (1.6)$$

The following proposition shows that the expected payoff $\gamma_v(p)$ of a one-step game $\Gamma_v$ is linear in the strategy $p^n$ of a player $n$.

**Proposition 1.2.7.** Let $\Gamma_v$ be a given one-step game. Then for all $p \in [0,1]^N$ and all $n \in \mathcal{N}$

$$\gamma_v(p) = \gamma_v((p^n,0)) + p^n \cdot (\gamma_v((p^n,1)) - \gamma_v((p^n,0)))$$

holds.

**Proof.** With Proposition 1.1.8 and $\varrho((p^n,1), \emptyset) = 0$, one obtains for all $n \in \mathcal{N}$

$$\gamma_v(p) = \varrho(p, \emptyset) \cdot v + \sum_{S \in \mathcal{P}(\mathcal{N})} \varrho(p, S) \cdot r_S$$

$$= \left( p^n \cdot \varrho((p^n,1), \emptyset) + (1 - p^n) \cdot \varrho((p^n,0), \emptyset) \right) \cdot v$$

$$+ \sum_{S \in \mathcal{P}(\mathcal{N})} \left( p^n \cdot \varrho((p^n,1), S) + (1 - p^n) \cdot \varrho((p^n,0), S) \right) \cdot r_S$$

$$= (1 - p^n) \cdot \varrho((p^n,0), \emptyset) \cdot v + (1 - p^n) \sum_{S \in \mathcal{P}(\mathcal{N})} \varrho((p^n,0), S) \cdot r_S$$

$$+ p^n \sum_{S \in \mathcal{P}(\mathcal{N})} \varrho((p^n,1), S) \cdot r_S$$

$$= (1 - p^n) \left( \varrho((p^n,0), \emptyset) \cdot v + \sum_{S \in \mathcal{P}(\mathcal{N})} \varrho((p^n,0), S) \cdot r_S \right)$$

$$= \gamma_v((p^n,0)) \quad \square$$

$$+ p^n \sum_{S \in \mathcal{P}(\mathcal{N})} \varrho((p^n,1), S) \cdot r_S.$$  

$$= \gamma_v((p^n,1))$$

**Remark 1.2.8.** Because of the linearity of the expected payoff $\gamma_v(p)$ in the strategies $p^n$ ($n \in \mathcal{N}$), it is sufficient to consider the expected payoff only for pure strategies in order to determine, whether a given strategy profile in a one-step game is an equilibrium or not, since the extreme values of $\gamma_v(p)$ is for each single player attained in a boundary point.
Corollary 1.2.9. Let \( p \in [0,1]^N \) be a strategy profile in the one-step game \( \Gamma_v \). The following formulae hold:

1. \( \gamma_v((p^{-n},1)) = p^n \gamma_v\left((p^{-n},1)^{-i},1\right) + (1-p^n) \gamma_v\left((p^{-n},1)^{-i},0\right) \)
2. \( \gamma_v((p^{-n},0)) = p^n \gamma_v\left((p^{-n},0)^{-i},1\right) + (1-p^n) \gamma_v\left((p^{-n},0)^{-i},0\right) \)

for all \( i, n \in \mathcal{N}, i \neq n \).

Perfect strategy profiles

We now introduce the concept of a perfect strategy profile. It is motivated by the linearity of the expected payoff \( \gamma_v \) in the strategies of the players. The difference between the expected payoffs for a player \( n \), if he plays quit or alternatively continue with certainty, is in the interest of this concept.

Definition 1.2.10 (\( \varepsilon \)-perfect). Let \( \Gamma_v \) be a given one-step game and \( \varepsilon \geq 0 \). A strategy profile \( p \in [0,1]^N \) in \( \Gamma_v \) is called \( \varepsilon \)-perfect, if

\[
\forall n \in \mathcal{N} : \begin{cases} \gamma_v^n((p^{-n},1)) - \gamma_v^n((p^{-n},0)) \leq \varepsilon & \text{for } p^n = 0 \\ \gamma_v^n((p^{-n},1)) - \gamma_v^n((p^{-n},0)) \in [-\varepsilon, \varepsilon] & \text{for } p^n \in (0,1) \\ \gamma_v^n((p^{-n},1)) - \gamma_v^n((p^{-n},0)) \geq -\varepsilon & \text{for } p^n = 1 \end{cases}
\]

Remark 1.2.11. An alternative definition of an \( \varepsilon \)-perfect strategy profile in a one-step game can be found in [36]. The definition there uses the concept of the so-called best reply.

The question, which relations between (\( \varepsilon \))-equilibria and (\( \varepsilon \))-perfect strategy profiles exist, is answered by the following theorem.

Theorem 1.2.12. Let \( \Gamma_v \) be a given one-step game and \( \varepsilon \geq 0 \). Then the following propositions hold:

1. \( p \in [0,1]^N \) is \( \varepsilon \)-perfect for \( \Gamma_v \) \( \implies \) \( p \) is an \( \varepsilon \)-equilibrium in \( \Gamma_v \);
2. \( p \in [0,1]^N \) is an \( \varepsilon \)-equilibrium in \( \Gamma_v \) \( \implies \) \( p \) is \( \varepsilon \xi_p \)-perfect for \( \Gamma_v \),

where

\[
\xi_p := \max_{n \in \mathcal{N}} \xi^n_p \quad \text{and} \quad \xi^n_p := \begin{cases} \max\left(\frac{1}{p^n}, \frac{1}{1-p^n}\right) & \text{for } p^n \in (0,1) \\ 1 & \text{for } p^n \in \{0,1\} \end{cases}
\]
Chapter 1. Quitting games

Proof. 1.: Let $p \in [0,1]^N$ be $\varepsilon$-perfect for $\Gamma_v$. It is to show that $p$ is also an $\varepsilon$-equilibrium in $\Gamma_v$, i.e.

$$\gamma_v^n(p) \geq \max_{\tilde{p} \in [0,1]} \gamma_v^n((p^n, \tilde{p})) - \varepsilon$$

for all $n \in \mathcal{N}$.

Because of the linearity of $\gamma_v^n(p)$ with respect to $p^n$ (cf. Proposition 1.2.7), it is sufficient to show that for all $n \in \mathcal{N}$

$$\gamma_v^n(p) \geq \max_{\tilde{p} \in [0,1]} \gamma_v^n((p^n, \tilde{p})) - \varepsilon. \quad (1.7)$$

Since $p$ is $\varepsilon$-perfect in $\Gamma_v$, the Inequality (1.7) follows immediately for $p^n = 0$ and $p^n = 1$.

For $p^n \in (0,1)$ it holds either

(a) $\max_{\tilde{p} \in [0,1]} = \gamma_v^n((p^n, 1)) \geq \gamma_v^n(p) \geq \gamma_v^n((p^n, 0))$ or

(b) $\max_{\tilde{p} \in [0,1]} = \gamma_v^n((p^n, 0)) > \gamma_v^n(p) > \gamma_v^n((p^n, 1))$.

Case (a): Since $p$ is $\varepsilon$-perfect in $\Gamma_v$, we get

$$\gamma_v^n(p) \geq \gamma_v^n((p^n, 0)) \geq \gamma_v^n((p^n, 1)) - \varepsilon = \max_{\tilde{p} \in [0,1]} \gamma_v^n((p^n, \tilde{p})) - \varepsilon. \quad (1.8)$$

Case (b): Analogously to Case (a), with $p$ $\varepsilon$-perfect in $\Gamma_v$, we obtain

$$\gamma_v^n(p) > \gamma_v^n((p^n, 1)) \geq \gamma_v^n((p^n, 0)) - \varepsilon = \max_{\tilde{p} \in [0,1]} \gamma_v^n((p^n, \tilde{p})) - \varepsilon. \quad (1.9)$$

So for both cases we have (1.7).

2.: Let $p$ be an $\varepsilon$-equilibrium, that means for all $n \in \mathcal{N}$ and for all $\tilde{p} \in [0,1]$

$$\gamma_v^n(p) \geq \gamma_v^n((p^n, \tilde{p})) - \varepsilon \quad (1.10)$$

holds. If $p^n = 0$, this implies

$$\gamma_v^n(p) \geq \gamma_v^n((p^n, 1)) - \varepsilon \quad (1.11)$$

and for $p^n = 1$

$$\gamma_v^n(p) \geq \gamma_v^n((p^n, 0)) - \varepsilon. \quad (1.10)$$

Consider the case $p^n \in (0,1)$. For all $\tilde{p} \in [0,1]$, we have

$$\gamma_v^n(p) = p^n \cdot \gamma_v^n((p^n, 1)) + (1 - p^n) \cdot \gamma_v^n((p^n, 0)) \geq \gamma_v^n((p^n, \tilde{p})) - \varepsilon \quad (1.11)$$
(cf. Proposition 1.2.7). For \( \hat{p} = 1 \), (1.11) yields to
\[ (1 - p^n) \cdot \gamma_v^n((p^{-n}, 0)) - (1 - p^n) \cdot \gamma_v^n((p^{-n}, 1)) \geq -\varepsilon \]
and consequently
\[ \gamma_v^n((p^{-n}, 1)) - \gamma_v^n((p^{-n}, 0)) \leq \frac{\varepsilon}{1 - p^n}. \]

For \( \hat{p} = 0 \), we obtain with (1.11) that
\[ p^n \cdot \gamma_v^n((p^{-n}, 1)) - p^n \cdot \gamma_v^n((p^{-n}, 0)) \geq -\varepsilon \]
and therefore
\[ \gamma_v^n((p^{-n}, 1)) - \gamma_v^n((p^{-n}, 0)) \geq -\frac{\varepsilon}{p^n}. \]

Both together imply
\[ \gamma_v^n((p^{-n}, 1)) - \gamma_v^n((p^{-n}, 0)) \in \left[ -\frac{\varepsilon}{p^n}, \frac{\varepsilon}{1 - p^n} \right] \in \left[ -\varepsilon \xi_p^n, \varepsilon \xi_p^n \right], \tag{1.12} \]
where \( \xi_p^n := \max \left( \frac{1}{1 - p^n}, \frac{1}{p^n} \right) \).

Let \( M(p) := \{ n \in \mathcal{N} \mid p^n \in (0, 1) \} \) and \( \xi_p := \begin{cases} \max_{n \in M(p)} \xi_p^n & \text{if } M(p) \neq \emptyset \\ 1 & \text{otherwise} \end{cases} \).

With (1.9), (1.10) and (1.12)
\[ \forall n \in \mathcal{N}: \begin{cases} \gamma_v^n((p^{-n}, 1)) - \gamma_v^n((p^{-n}, 0)) \leq \varepsilon & \text{for } p^n = 0 \\ \gamma_v^n((p^{-n}, 1)) - \gamma_v^n((p^{-n}, 0)) \in \left[ -\varepsilon \xi_p, \varepsilon \xi_p \right] & \text{for } p^n \in (0, 1) \\ \gamma_v^n((p^{-n}, 1)) - \gamma_v^n((p^{-n}, 0)) \geq -\varepsilon & \text{for } p^n = 1 \end{cases} \]
follows. By Definition 1.2.10 and \( \xi_p \geq 1 \), \( p \) is \( \varepsilon \xi_p \)-perfect in \( \Gamma_v \). \( \square \)

**Remark 1.2.13.** With (1.12) even
\[ \forall n \in \mathcal{N}: \begin{cases} \gamma_v^n((p^{-n}, 1)) - \gamma_v^n((p^{-n}, 0)) \leq \varepsilon & \text{for } p^n = 0 \\ \gamma_v^n((p^{-n}, 1)) - \gamma_v^n((p^{-n}, 0)) \in \left[ -\frac{\varepsilon}{p^n}, \frac{\varepsilon}{1 - p^n} \right] & \text{for } p^n \in (0, 1) \\ \gamma_v^n((p^{-n}, 1)) - \gamma_v^n((p^{-n}, 0)) \geq -\varepsilon & \text{for } p^n = 1 \end{cases} \]
holds.

**Corollary 1.2.14.** Let \( \Gamma_v \) be a given one-step game. Then
1. \( p \in [0, 1]^n \) is \( (0-) \)-perfect for \( \Gamma_v \) \iff \( p \in [0, 1]^n \) is a Nash-equilibrium in \( \Gamma_v \)
2. \( p \in \{0, 1\}^n \) is \( \varepsilon \)-perfect for \( \Gamma_v \) \iff \( p \in \{0, 1\}^n \) is an \( \varepsilon \)-equilibrium in \( \Gamma_v \)
Existence of a Nash-equilibrium in a one-step game

We now prove, that every one-step game has a Nash-equilibrium. Therefore we use Brouwer’s Fixed-Point Theorem.

**Theorem 1.2.15** (Brouwer’s Fixed-Point Theorem). Suppose that $M$ is a nonempty, convex, compact subset of $\mathbb{R}^N$, where $N \geq 1$, and that $f : M \to M$ is a continuous mapping. Then $f$ has a fixed point.

With this we prove the following theorem:

**Theorem 1.2.16.** Let $G$ be a given quitting game. Every one-step game $\Gamma_v = (G, v)$, $v \in \mathbb{R}^N$, corresponding to the quitting game $G$ has at least one Nash-equilibrium.

**Proof.** The proof is similar to that from Nash in [27], but given here for the special case of a one-step game. The set $[0, 1]^N$ of all profiles of a one-step game is nonempty, convex and compact. Define the map $\Phi : [0, 1]^N \to \mathbb{R}^{N \times 2}$ with $p \mapsto \Phi(p) = (\Phi^1(p), \ldots, \Phi^N(p))$ and $\Phi^n(p) = (\Phi^n_0(p), \Phi^n_1(p))$ where

$$\Phi^n_j(p) := \max \left\{ 0, \gamma^n_v((p^{-n}, j)) - \gamma^n_v(p) \right\}, \quad j \in \{0, 1\}, n \in \mathcal{N}.$$

The function $\Phi$ is non-negative and continuous. $\Phi(p)$ is a matrix, which has two entries for each player. The first one $\Phi^n_0(p)$ is either the non-negative difference between the expected payoff to the player $n$ if she plays continue with certainty – the other players play according to $p$ – and the expected payoff to the player under the profile $p$, or zero. The second one $\Phi^n_1(p)$ is analogously either the non-negative difference between the expected payoff to the player if she plays quit with certainty and the expected payoff to the player under the profile $p$, or zero. For the case that for example $\Phi^n_0(p)$ is positive, the player $n$ would get a higher expected payoff, if she plays continue instead of $p^n$. So one next step is to decrease the quitting probability from player $n$.

Therefore we define another function $h : \mathbb{R}^2_+ \to [0, 1]$

$$x = (x_1, x_2) \mapsto h(x) = \frac{x_2}{x_1 + x_2},$$

which is a normalization function and the continuous function $f : [0, 1]^N \to [0, 1]^N$, $p \mapsto f(p) = (f^1(p), \ldots, f^N(p))$, where

$$f^n(p) := h((1 - p^n) + \Phi^n_0(p), p^n + \Phi^n_1(p)).$$

---

3See [47].

4John Nash showed in [27] that every finite game has an equilibrium. This theorem is an application of his theorem and will be proven in the same way.
According to Brouwer’s Fixed-point Theorem, \( f \) has got at least one fixed-point \( \hat{p} \). It remains to show that \( \hat{p} \) is a Nash-equilibrium in \( \Gamma_v \).

\[
f(\hat{p}) = \hat{p} \iff \Phi^n_0(\hat{p}) = \Phi^n_0(\hat{p}) = 0 \quad \forall n \in \mathcal{N}
\]

\[
\iff \gamma^n_v((\hat{p}^{-n},0)) \leq \gamma^n_v(\hat{p}) \land \gamma^n_v((\hat{p}^{-n},1)) \leq \gamma^n_v(\hat{p}) \quad \forall n \in \mathcal{N}.
\]

And with the linearity of the expected payoff \( \gamma^n_v(\hat{p}) \) in \( \hat{p}^n \)

\[
\forall n \in \mathcal{N} : \begin{cases} 
\gamma^n_v((\hat{p}^{-n},1)) - \gamma^n_v((\hat{p}^{-n},0)) \leq 0 & \text{for } \hat{p}^n = 0 \\
\gamma^n_v((\hat{p}^{-n},1)) - \gamma^n_v((\hat{p}^{-n},0)) = 0 & \text{for } \hat{p}^n \in (0,1) \\
\gamma^n_v((\hat{p}^{-n},1)) - \gamma^n_v((\hat{p}^{-n},0)) \geq 0 & \text{for } \hat{p}^n = 1
\end{cases}
\]

follows, which implies that \( \hat{p} \) is 0-perfect in \( \Gamma_v \). With Corollary 1.2.14, \( \hat{p} \) is a Nash-equilibrium in \( \Gamma_v \). \( \Box \)

### Influence of variation in the strategy profile

At this point we want to study the influence of a variation in one component of the strategy profile \( p \in [0,1]^N \) for a given quitting game \( \Gamma_v \). Therefore define \( \hat{p} \in [0,1]^N \) as follows

\[
\hat{p} := \hat{p}_{m,\lambda}(p) := (p^{-m}, (1 - \lambda)p^m + \lambda), \tag{1.13}
\]

where \( p \in [0,1]^N, \lambda \in [0,1] \) and \( m \in \mathcal{N} \).

That means, \( \hat{p}^m \) is a convex combination of \( p^m \) and the pure strategy 1, which corresponds to the action \textit{quit}. It is obvious, that for \( \lambda = 0 \) one obtains \( \hat{p} = p \) and for \( \lambda = 1 \) \( \hat{p} = (p^{-m}, 1) \).

**Theorem 1.2.17.** Let \( \Gamma_v \) be a given one-step game, \( \lambda \in [0,1], p \in [0,1]^N \) and \( m \in \mathcal{N} \) an arbitrary but fixed chosen player. Then the following hold:

1. \( g(\hat{p}, \emptyset) = (1 - \lambda)g(p, \emptyset) \)

That means, the probability that all players play continue under \( \hat{p} \) is for the \( \lambda \)-fold smaller of the continue-probability under \( p \).

2. \( \gamma_v(\hat{p}) = (1 - \lambda) \cdot \gamma_v(p) + \lambda \cdot \gamma_v((p^{-m}, 1)) \)

3. \( \| \gamma_v(\hat{p}) - \gamma_v(p) \| \leq \lambda \cdot (1 - p^m) \cdot (r_{\text{max}} + \delta_v), \)

where \( r_{\text{max}} := \max\{|r^n_S| \mid n \in \mathcal{N}, S \in \mathcal{P}(\mathcal{N}) \} \) and \( \delta_v = \max\{ \max_{n \in \mathcal{N}} |v^n|, r_{\text{max}} \} \)

4. If \( p \in [0,1]^N \) is \( \eta \)-perfect in \( \Gamma_v \) (\( \eta \geq 0 \)) and if \( p^m \in (0,1] \) for the given player \( m \in \mathcal{N} \), then \( \hat{p} = \hat{p}_{m,\lambda} \) is \( \tilde{\eta} \)-perfect in \( \Gamma_v \), with \( \tilde{\eta} := \max(2\lambda r_{\text{max}} + (1 - \lambda)\eta, \eta) \).
Chapter 1. Quitting games

Proof. 1.: The definition of $\hat{p}$ (cf. (1.13)) implies

$$g(\hat{p}, \emptyset) = \prod_{n \in A}(1 - \hat{p}^n) = (1 - (1 - \lambda)p^m - \lambda) \cdot \prod_{n \in A \setminus \{m\}} (1 - p^n)$$

$$= (1 - \lambda) \cdot \prod_{n \in A} (1 - p^n)$$

$$= (1 - \lambda) \cdot g(p, \emptyset).$$

2.: With Proposition 1.2.7 and the definition of $\hat{p}$ we have

$$\gamma_v(\hat{p}) = \hat{p}^m \cdot \gamma_v((\hat{p}^{-m}, 1)) + (1 - \hat{p}^m) \cdot \gamma_v((\hat{p}^{-m}, 0))$$

$$= ((1 - \lambda)p^m + \lambda) \cdot \gamma_v((\hat{p}^{-m}, 1)) + (1 - \lambda)(1 - p^m) \cdot \gamma_v((p^{-m}, 0))$$

$$= (1 - \lambda) \cdot \left(p^m \cdot \gamma_v((\hat{p}^{-m}, 1)) + (1 - p^m) \cdot \gamma_v((p^{-m}, 0))\right) + \lambda \cdot \gamma_v((p^{-m}, 1))$$

$$= (1 - \lambda) \cdot \gamma_v(p) + \lambda \cdot \gamma_v((p^{-m}, 1)). \quad (1.14)$$

3.: Using (1.14), one obtains

$$\|\gamma_v(\hat{p}) - \gamma_v(p)\| = \|\lambda \cdot \gamma_v(p) + \lambda \cdot \gamma_v((p^{-m}, 1)) - \gamma_v(p)\|$$

$$= \lambda \cdot \|\gamma_v((p^{-m}, 1)) - \gamma_v(p)\|$$

$$= \lambda \cdot \|\gamma_v((p^{-m}, 1)) - (p^m \gamma_v((p^{-m}, 1)) - (1 - p^m) \gamma_v((p^{-m}, 0)))\|$$

$$= \lambda \cdot \|(1 - p^m)(\gamma_v((p^{-m}, 1)) - \gamma_v((p^{-m}, 0)))\|$$

$$= \lambda \cdot (1 - p^m) \cdot \|\gamma_v((p^{-m}, 1)) - \gamma_v((p^{-m}, 0))\|.$$

Because player $m$ plays quit with certainty in the alternative strategy profile $(p^{-m}, 1)$,

$$\gamma_v^n((p^{-m}, 1)) = \sum_{S \in \mathcal{P}(\mathcal{A})} g((p^{-m}, 1), S) \cdot r^n_S \in [-r_{max}, r_{max}]$$

for all $n \in \mathcal{N}$ holds with $r_{max} = \max_{n \in \mathcal{N}, S \in \mathcal{P}(\mathcal{A})} |r^n_S|$, which implies

$$\|\gamma_v(\hat{p}) - \gamma_v(p)\| \leq \lambda \cdot (1 - p^m) \cdot (r_{max} + \delta_v),$$

where $\delta_v = \max\{\max_{n \in \mathcal{N}} |v^n|, r_{max}\}$.

4.: For $\lambda = 0$ or $p^m = 1$, $\hat{p} = p$ follows and therefore $\hat{p}$ is $\eta$-perfect in $\Gamma_v$ in that cases. For $\lambda \in (0, 1]$ and $p^m \in (0, 1)$, we have to show that

$$\forall n \in \mathcal{N} : \begin{cases} \gamma^n_v((\hat{p}^{-n}, 1)) - \gamma^n_v((\hat{p}^{-n}, 0)) \leq \tilde{\eta} & \text{for } \hat{p}^{-n} = 0 \\ \gamma^n_v((\hat{p}^{-n}, 1)) - \gamma^n_v((\hat{p}^{-n}, 0)) \in [-\eta, \eta] & \text{for } \hat{p}^{-n} \in (0, 1) \\ \gamma^n_v((\hat{p}^{-n}, 1)) - \gamma^n_v((\hat{p}^{-n}, 0)) \geq -\tilde{\eta} & \text{for } \hat{p}^{-n} = 1 \end{cases} \quad (1.15)$$
with $\tilde{\eta} = \max(2\lambda r_{\text{max}} + (1 - \lambda)\eta, \eta)$.

Case 1: Consider player $m$. Since $p$ is $\eta$-perfect and $p^m \in (0, 1)$ we have
\[
\gamma_v((\hat{p}^m, 1)) - \gamma_v((\hat{p}^m, 0)) = \gamma_v((p^m, 1)) - \gamma_v((p^m, 0)) \in [-\eta, \eta]
\]
and therefore the second inequality from (1.15) for $\hat{p}^m \in (0, 1)$ respectively the last inequality of (1.15) for $\hat{p} = 1$.

Case 2: Consider player $n \in \mathcal{N} \setminus \{m\}$. Since $\hat{p}$ refers to player $m$, $(\hat{p}^n, b) = (\hat{p}^m, b)$ holds for all $b \in [0, 1]$. Using equation (1.14), one obtains
\[
\gamma_v^n((\hat{p}^n, b)) = (1 - \lambda) \cdot \gamma_v^n((p^n, b)) + \lambda \cdot \gamma_v^n((p^n, b)^{-m}, 1)).
\]
This implies
\[
\begin{align*}
\gamma_v^n((\hat{p}^n, 1)) - \gamma_v^n((\hat{p}^n, 0)) &= (1 - \lambda) \cdot \gamma_v^n((p^n, 1)) + \lambda \cdot \gamma_v^n((p^n, 1)^{-m}, 1)) - (1 - \lambda) \cdot \gamma_v^n((p^n, 0)) \\
&= (1 - \lambda) \cdot (\gamma_v^n((p^n, 1)) - \gamma_v^n((p^n, 0))) \\
&\quad + \lambda \left(\gamma_v^n((p^n, 1)^{-m}, 1)) - \gamma_v^n((p^n, 0)^{-m}, 1))\right). \\
\end{align*}
\]
We now observe three different cases:
(a) $p^n = 0$:
\[
p^n = 0 \land p \text{ $\eta$-perfect} \implies \gamma_v^n((p^n, 1)) - \gamma_v^n((p^n, 0)) \leq \eta
\]
With the use of (1.16) and (1.17) we get
\[
\begin{align*}
\gamma_v^n((\hat{p}^n, 1)) - \gamma_v^n((\hat{p}^n, 0)) &= (1 - \lambda) \cdot \eta + \lambda \cdot \left(\gamma_v^n((p^n, 1)^{-m}, 1)) - \gamma_v^n((p^n, 0)^{-m}, 1))\right) \\
&\leq (1 - \lambda) \cdot \eta + \lambda \cdot \left(\left|\gamma_v^n((p^n, 1)^{-m}, 1))\right| + \left|\gamma_v^n((p^n, 0)^{-m}, 1))\right|\right) \\
&\leq (1 - \lambda) \cdot \eta + \lambda \cdot (r_{\text{max}} + r_{\text{max}}) \\
&= 2\lambda r_{\text{max}} + (1 - \lambda) \cdot \eta.
\end{align*}
\]
(b) $p^n \in (0, 1)$:
\[
p^n \in (0, 1) \land p \text{ $\eta$-perfect} \implies \gamma_v^n((p^n, 1)) - \gamma_v^n((p^n, 0)) \in [-\eta, \eta]
\]
Similar to the previous case, we get
\[
\gamma_v^n((\hat{p}^n, 1)) - \gamma_v^n((\hat{p}^n, 0)) \leq 2\lambda r_{\text{max}} + (1 - \lambda) \cdot \eta.
\]
Under usage of (1.18) and formula (1.16), one obtains
\[
\gamma_v^n((\hat{p}^{-n}, 1)) - \gamma_v^n((\hat{p}^{-n}, 0)) \\
\geq -(1 - \lambda) \cdot \eta - \lambda \cdot \left( |\gamma_v^n((p^{-n}, 1)^{-m}, 1))| + |\gamma_v^n((p^{-n}, 0)^{-m}, 1))| \right) \\
\geq -2\lambda r_{\text{max}} - (1 - \lambda) \cdot \eta
\] (1.19)
and therefore
\[
\gamma_v^n((\hat{p}^{-n}, 1)) - \gamma_v^n((\hat{p}^{-n}, 0)) \in [-2\lambda r_{\text{max}} + (1 - \lambda) \cdot \eta, 2\lambda r_{\text{max}} + (1 - \lambda) \cdot \eta].
\]

(c) \( p^n = 1: \)
\[
p^n = 1 \land p \eta\text{-perfect} \implies \gamma_v^n((p^{-n}, 1)) - \gamma_v^n((p^{-n}, 0)) \geq -\eta
\] (1.20)

Consequently formulae (1.16), (1.19) and (1.20) imply
\[
\gamma_v^n((\hat{p}^{-n}, 1)) - \gamma_v^n((\hat{p}^{-n}, 0)) \geq -(2\lambda r_{\text{max}} + (1 - \lambda) \cdot \eta).
\]

\[\text{Remark 1.2.18. (To Theorem 1.2.17 3.)}\]

1. Let \( \Gamma_v \) be a given one-step game with \( v \in [-2r_{\text{max}}, 2r_{\text{max}}], p \in [0, 1]^N, \) where \( p^m \in (0, 1) \) for at least one player \( m \in \mathcal{N}, \) a strategy profile in \( \Gamma_v \) and \( \lambda \in (0, 1). \)

Solan and Vieille state in [36] the following estimate:
\[
\|\gamma_v(\hat{p}) - \gamma_v(p)\| \leq 2\lambda r_{\text{max}},
\] (1.21)
with \( \hat{p} \) and \( r_{\text{max}} \) like before.

**Counter-example:** Consider the following one-step game \( \Gamma_v \) with \( v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \)

<table>
<thead>
<tr>
<th>Player 1 continue</th>
<th>Player 2 continue</th>
<th>Player 2 quit</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1 , 2)</td>
<td>(0 , 1)</td>
<td>(-1 , -0.5)</td>
</tr>
</tbody>
</table>

\( \implies r_{\text{max}} = \max\{|r^n_S| \mid n \in \mathcal{N}, S \in \mathcal{P}(\mathcal{N})\} = 1 \) and \( \max_{n \in \mathcal{N}} |v^n| = 2 \)

\( \implies \delta_v = 2 \)

Obviously \( p = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is one (and the only) equilibrium in \( \Gamma_v \) with the expected payoff \( \gamma_v(p) = v. \)

Let \( \lambda = 0.1 \) be given, then \( g(p, \emptyset) = 1 > 1 - \lambda = 0.9. \) Furthermore let \( \hat{p}_{\lambda,1} \) be defined like before, that means
\[
\hat{p}_{\lambda,1} = \hat{p} = \begin{pmatrix} p^1 + \lambda(1 - p^1) \\ p^2 \end{pmatrix} = \begin{pmatrix} 0 + 0.1 \times 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}.
\]
From this
\[ \| \gamma_v(\hat{p}) - \gamma_v(p) \| = \left\| \begin{pmatrix} 1 \\ 1.7 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\| = 0.3 = \lambda(r_{\text{max}} + \delta_v) \]
\[ > 2\lambda r_{\text{max}} = 2 \times 0.1 \times 1 = 0.2 \]
follows. So estimate (1.21) does not hold.

The counter-example shows that the estimate in Theorem 1.2.17 3. is sharp.

2. Interpretation: Let \( \Gamma_v \) be a given one-step game, \( \lambda \in [0, 1] \) and \( p \in [0, 1]^N \) with \( p^m \in (0, 1) \) for at least one player \( m \in \mathcal{N} \). If the continue probability of one player \( m \) (\( m \in \mathcal{N} \)) is decreased by the \( \lambda \)-fold, then the expected payoff of the players changes maximal at \( \lambda(r_{\text{max}} + \delta_v) \) for a component.

**Remark 1.2.19** (To Theorem 1.2.17 4.). We want to answer the question, why \( p^m = 0 \) is not allowed in Theorem 1.2.17 4. Assume that \( p^m = 0 \), then \( \hat{p}^m = \lambda \in (0, 1) \). In order to prove that \( \hat{p} \) is \( \tilde{\eta} \)-perfect, we have to show \( \gamma_v^m((\hat{p}^m - n, 1)) - \gamma_v^m((\hat{p}^m - n, 0)) \in [-\tilde{\eta}, \tilde{\eta}] \).

On the one hand, \( p \eta \)-perfect implies
\[ \gamma_v^m((\hat{p}^m - n, 1)) - \gamma_v^m((\hat{p}^m - n, 0)) = \gamma_v^m((p - m, 1)) - \gamma_v^m((p - m, 0)) \leq \eta \leq \tilde{\eta}, \]
but on the other hand, we only have
\[ \gamma_v^m((\hat{p}^m - n, 1)) - \gamma_v^m((\hat{p}^m - n, 0)) \geq -| \gamma_v^m((\hat{p}^m - n, 1)) | - | \gamma_v^m((\hat{p}^m - n, 0)) | \]
\[ \geq -r_{\text{max}} - \delta_v. \]

However this is true for all \( p \in [0, 1]^N \) in \( \Gamma_v \), where \( p^m = 0 \). A better a priori estimate is not possible.

Observe that the proof that
\[ \forall n \in \mathcal{N} \setminus \{ m \} : \begin{cases} \gamma_v^m((\hat{p}^m - n, 1)) - \gamma_v^m((\hat{p}^m - n, 0)) \leq \tilde{\eta} & \text{for } \hat{p}^n = 0 \\ \gamma_v^m((\hat{p}^m - n, 1)) - \gamma_v^m((\hat{p}^m - n, 0)) \in [-\tilde{\eta}, \tilde{\eta}] & \text{for } \hat{p}^n \in (0, 1) \\ \gamma_v^m((\hat{p}^m - n, 1)) - \gamma_v^m((\hat{p}^m - n, 0)) \geq -\tilde{\eta} & \text{for } \hat{p}^n = 1 \end{cases} \]
will remain unaffected from this case.

**Corollary 1.2.20.** With Theorem 1.2.17 4., it follows immediately that, if \( p \in [0, 1]^N \) with \( p^m \in (0, 1) \), for at least one \( m \in \mathcal{N} \), is \((0-)\)perfect in \( \Gamma_v \) (and therefore an equilibrium in \( \Gamma_v \)), \( \hat{p} \) is \( 2\lambda r_{\text{max}} \)-perfect in \( \Gamma_v \).
Chapter 1. Quitting games

Up to now we only analyzed the influence of the shift from the quitting probability of a player \( m \in \mathcal{N} \) into direction quit with certainty. For completeness we consider also the other case, where the quitting probability of player \( m \) is shifted into direction continue.

**Lemma 1.2.21.** Let \( \Gamma_v \) be a given one-step game, \( \mu \in [0,1] \), \( p \in [0,1]^N \) and \( m \in \mathcal{N} \) an arbitrary but fixed chosen player. Define

\[
\tilde{p} := \tilde{p}_{m,\mu}(p) := (p^{-m}, (1-\mu)p^m).
\]

Then the following holds:

1. \( \varrho(\tilde{p}, \emptyset) = \varrho(p, \emptyset) + \mu p^m \varrho((p^{-m}, 0)) \);
2. \( \gamma_v(\tilde{p}) = \gamma_v(p) + \mu p^m (\gamma_v((p^{-m}, 0)) - \gamma_v((p^{-m}, 1))) \);
3. \( \|\gamma_v(\tilde{p}) - \gamma_v(p)\| \leq \mu p^m (r_{\text{max}} + \delta_v) \);
4. If \( p \in [0,1]^N \) is \( \eta \)-perfect in \( \Gamma_v \) (\( \eta \geq 0 \)) and \( p^m \in [0,1) \) for the given player \( m \in \mathcal{N} \), then \( \tilde{p} = \tilde{p}_{m,\mu} \) is \( \tilde{\eta} \)-perfect in \( \Gamma_v \) with \( \tilde{\eta} = \eta + \mu p^m (2\delta_v + 2r_{\text{max}}) \).

**Proof.** The proof of this properties is similar to the proof of Theorem 1.2.17.

One application of Theorem 1.2.17 and Lemma 1.2.21 is given in Section 3.4.2, where we will use them in order to determine an optimal step size for an algorithm, which detects \( \varepsilon \)-equilibria in one-step games.

**Payoff vector \( v \)**

Let a quitting game \( G \) be given. We already mentioned that every one-step game \( \Gamma_v \), with \( v \in \mathbb{R}^N \) arbitrary, has at least one \( \varepsilon \)-equilibrium. An interesting question is now, whether there exists a payoff vector \( v \in \mathbb{R} \) for every probability vector \( p \in [0,1]^N \), such that \( p \) is an \( \varepsilon \)-equilibrium in \( \Gamma_v \).

**Theorem 1.2.22.** Let \( G \) be a given quitting game, \( \varepsilon \geq 0 \). For every vector \( p \in [0,1)^N \), a vector \( v(p, \varepsilon) \in \mathbb{R}^N \) exists, such that \( p \) is an \( \varepsilon \)-equilibrium in \( \Gamma_{v(p, \varepsilon)} = (G, v(p, \varepsilon)) \).

**Proof.** For \( p \) being an \( \varepsilon \)-equilibrium in a one-step game \( \Gamma_v \), with \( v \in \mathbb{R}^N \), \( p \) has to satisfy the equilibrium condition

\[
\forall n \in \mathcal{N}: \gamma^n_v(p) \geq \max \left\{ \gamma^n_v((p^{-n}, 0)), \gamma^n_v((p^{-n}, 1)) \right\} - \varepsilon.
\]

(1.22)

Given a player \( n \), we distinguish the following four cases:

1. All players play continue for sure, i.e. \( p = 0 \).
2. Only player \( n \) plays quit with a positive probability, i.e \( p^n > 0 \) and \( (p^{-n}, 0) = 0 \).
3. Player $n$ plays \textit{continue} with certainty and at least one other player plays \textit{quit} with a positive probability, i.e. $p^n = 0$ and $p \neq 0$.

4. Player $n$ and at least one other player play \textit{quit} with a positive probability, i.e. $p^n \in (0,1)$ and $(p^{-n}, 0) \neq \emptyset$.

1. For $p = \emptyset$, the Inequality 1.22 leads immediately to the condition $v^n(p, \varepsilon) \geq r^n_{\{n\}} - \varepsilon$.

2. For $p^n > 0$ and $(p^{-n}, 0) = \emptyset$,

$$
\gamma^n_{v(p,\varepsilon)}(p) = r^n_{\{n\}} \geq v^n(p, \varepsilon) - \varepsilon \quad \Rightarrow \quad v^n(p, \varepsilon) \geq r^n_{\{n\}} - \varepsilon.
$$

3. For $p^n = 0$ and $p \neq \emptyset$,

$$
\gamma^n_{v(p,\varepsilon)}(p) \geq \gamma^n_{v(p,\varepsilon)}((p^{-n}, 1)) - \varepsilon = \gamma^n_0((p^{-n}, 1)) - \varepsilon
$$

has to hold. This is equivalent to\textsuperscript{5}

$$
v^n(p, \varepsilon) \cdot g(p, \varnothing) + \sum_{S \in \mathcal{P}(\mathcal{V})} g(p, \varnothing) \cdot r^n_S \geq \gamma^n_0((p^{-n}, 1)) - \varepsilon
$$

$$
\iff \quad v^n(p, \varepsilon) \cdot g(p, \varnothing) + \gamma^n_0(p) \geq \gamma^n_0((p^{-n}, 1)) - \varepsilon
$$

$$
\iff \quad v^n(p, \varepsilon) \geq \frac{\gamma^n_0((p^{-n}, 1)) - \gamma^n_0(p)}{g(p, \varnothing)} - \frac{\varepsilon}{g(p, \varnothing)}.
$$

4. For $p^n \in (0,1)$ and $(p^{-n}, 0) \neq \emptyset$, on the one hand $\gamma^n_{v(p,\varepsilon)}(p) \geq \gamma^n_{v(p,\varepsilon)}((p^{-n}, 1)) - \varepsilon$ should be satisfied. This is the case, if and only if

$$
p^n \gamma^n_{v(p,\varepsilon)}((p^{-n}, 1)) + (1 - p^n) \gamma^n_{v(p,\varepsilon)}((p^{-n}, 0)) \geq \gamma^n_{v(p,\varepsilon)}((p^{-n}, 1)) - \varepsilon
$$

$$
\iff \quad (1 - p^n) \gamma^n_{v(p,\varepsilon)}((p^{-n}, 0)) \geq (1 - p^n) \gamma^n_{v(p,\varepsilon)}((p^{-n}, 1)) - \varepsilon
$$

$$
\iff \quad \gamma^n_{v(p,\varepsilon)}((p^{-n}, 0)) \geq \gamma^n_{v(p,\varepsilon)}((p^{-n}, 1)) - \frac{\varepsilon}{(1 - p^n)}.
$$

With

$$
\gamma^n_{v(p,\varepsilon)}((p^{-n}, 0)) = v^n(p, \varepsilon) \cdot g((p^{-n}, 0), \varnothing) + \sum_{\emptyset \neq S \in \mathcal{P}(\mathcal{V}\setminus\{n\})} g((p^{-n}, 0), \varnothing) \cdot r^n_S
$$

and $\gamma^n_{v(p,\varepsilon)}((p^{-n}, 1)) = \gamma^n_{v(p,\varepsilon)}((p^{-n}, 1))$, one obtains

$$
v^n(p, \varepsilon) \geq \frac{\gamma^n_0((p^{-n}, 1)) - \gamma^n_0((p^{-n}, 0))}{g((p^{-n}, 0), \varnothing)} - \frac{\varepsilon}{(1 - p^n) g((p^{-n}, 0), \varnothing)}.
$$

\textsuperscript{5}Observe that $r_\varnothing = 0$. 

1.2. \textbf{One-step games corresponding to a quitting game}
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On the other hand, the following condition has to hold:

\[ \gamma_v^n(p) \geq \gamma_v^n((p^n, 0)) - \varepsilon \]
\[ \iff p^n \gamma_v^n((p^n, 1)) + (1 - p^n) \gamma_v^n((p^n, 0)) \geq \gamma_v^n((p^n, 0)) - \varepsilon \]
\[ \iff -p^n \gamma_v^n((p^n, 0)) \geq -p^n \gamma_v^n((p^n, 1)) - \varepsilon \]
\[ \iff \gamma_v^n((p^n, 0)) \leq \gamma_v^n((p^n, 1)) + \frac{\varepsilon}{p^n} \]

and finally with the same arguments as before,

\[ v^n(p, \varepsilon) \leq \frac{\gamma_0^n((p^n, 1)) - \gamma_0^n((p^n, 0))}{\varrho((p^n, 0), \emptyset)} + \frac{\varepsilon}{p^n \varrho((p^n, 0), \emptyset)} = v^n_p + \frac{\varepsilon}{\varrho(p, \{n\})} \]

In summary: \( v^n(p, \varepsilon) \in \left[ v^n_p - \frac{\varepsilon}{\varrho(p, \emptyset)}; v^n_p + \frac{\varepsilon}{\varrho(p, \{n\})} \right] \) has to hold.

Remark 1.2.23.

1. Let \( p \in (0, 1)^N \) be given. For \( \varepsilon = 0 \) – that means in the case of Nash-equilibria – the continue payoff vector \( v(p, 0) = (v^1(p, 0), \ldots, v^N(p, 0)) \) is uniquely determined by

\[ v^n(p, 0) = v^n_p = \frac{\gamma_0^n((p^n, 1)) - \gamma_0^n((p^n, 0))}{\varrho((p^n, 0), \emptyset)} \]
\[ = r^n_{\{n\}} + \sum_{\emptyset \neq S \subseteq \mathcal{P}(\mathcal{N} \setminus \{n\})} \varrho((p^n, 0), \emptyset) \cdot (r_S^n - r_S^n_{S \cup \{n\}}) \].

2. What happens if we expand the interval \([0, 1)^N\) to \([0, 1]^N\) and allow the players to play quit with certainty?

Case 1: Suppose there are at least two players, who play \textit{quit} with certainty. In that case the probability that all players play the action \textit{continue} is zero, even if we exclude one of the players by consideration of the alternative profile of that player. This implies \( \gamma_v(p) = \gamma_0(p) \) or \( \gamma_v^n((p^n, 0)) = \gamma_0^n((p^n, 0)) \), for all \( v \in \mathbb{R}^N \), that means the expected payoff in this one-step game under \( p \) or, respectively, under \( (p^n, 0) \) is independent from the continue payoff \( v_p \) for all players \( n \). Therefore a given profile \( p \in [0, 1]^N \), where at least two players play \textit{quit} with certainty, is an \( \varepsilon \)-equilibrium, if the payoff structure of the corresponding quitting game is, accordingly, irrespective of \( v \).
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Case 2: Suppose there is only one player \( n \), who plays *quit* with certainty. For all players \( m \in \mathcal{N}, m \neq n \), one has the same argument as in the case before: Their expected payoffs are independent from the continue payoff \( v(p, \varepsilon) \). But for player \( n \),

\[
\gamma^n_v(p) = \gamma^0_v(p) = \gamma^0_w((p^{-n}, 1)) \geq \gamma^n_v((p^{-n}, 0)) - \varepsilon
\]

has to hold, which is equivalent to

\[
v^n(p, \varepsilon) \leq v^n_p + \frac{\varepsilon}{\rho((p^{-n}, 0), \emptyset)}
\]

with \( v^n_p \) from (1.23) for \((p^{-n}, 0) \neq 0 \) and \( v^n(p, \varepsilon) \leq r^n_{\{n\}} + \varepsilon \) otherwise.

A similar theorem can be formulated for \( \varepsilon \)-perfect profiles \( p \in [0, 1]^N \).

**Theorem 1.2.24.** Let \( G \) be a given quitting game, \( \varepsilon \geq 0 \). For every vector \( p \in [0, 1]^N \), a vector \( \hat{v}(p, \varepsilon) \in \mathbb{R}^N \) exists, such that \( p \) is an \( \varepsilon \)-perfect profile in \( \Gamma_{\hat{v}(p, \varepsilon)} = (G, \hat{v}(p, \varepsilon)) \).

**Proof.** If \( p \) should be \( \varepsilon \)-perfect in a one-step game \( \Gamma_v \), with \( v \in \mathbb{R}^N \), \( p \) has to satisfy the condition

\[
\forall n \in \mathcal{N} : \begin{cases}
\gamma^n_v((p^{-n}, 1)) - \gamma^n_v((p^{-n}, 0)) \leq \varepsilon & \text{for } p^n = 0 \\
\gamma^n_v((p^{-n}, 1)) - \gamma^n_v((p^{-n}, 0)) \in [-\varepsilon, \varepsilon] & \text{for } p^n \in (0, 1).
\end{cases}
\]

We distinguish between the following four cases:

1. \( p^n = 0 \) and a) \( p = 0 \) or b) \( p \neq 0 \);
2. \( p^n \in (0, 1) \) and a) \( (p^{-n}, 0) = 0 \) or b) \( (p^{-n}, 0) \neq 0 \).

1.a): It immediately follows that \( \hat{v}^n(p, \varepsilon) \) has to be chosen such that

\[
\hat{v}^n(p, \varepsilon) \geq r^n_{\{n\}} - \varepsilon.
\]

1.b): This case is equivalent to Case 3. from the proof of Theorem 1.2.22, i.e.

\[
\hat{v}^n(p, \varepsilon) \geq v^n_p - \frac{\varepsilon}{\rho(p, \emptyset)}, \text{ with } v^n_p = \frac{\gamma^0_v((p^{-n}, 1)) - \gamma^0_v((p^{-n}, 0))}{\rho((p^{-n}, 0), \emptyset)}.
\]

2.a): Because \( \gamma^n_v((p^{-n}, 1)) = r^n_{\{n\}} \) and \( \gamma^n_v((p^{-n}, 0)) = \hat{v}(p, \varepsilon) \), we get

\[
r^n_{\{n\}} - \hat{v}^n(p, \varepsilon) \in [-\varepsilon, \varepsilon] \iff \hat{v}^n(p, \varepsilon) \in [r^n_{\{n\}} - \varepsilon, r^n_{\{n\}} + \varepsilon].
\]

2.b): For \( p^n \in (0, 1) \) additional to

\[
\hat{v}(p, \varepsilon) \geq v^n_p - \frac{\varepsilon}{\rho(p, \emptyset)}
\]
from Case 1.b), \( \tilde{v}(p, \varepsilon) \) has to be such that 
\[
\gamma_n(\tilde{v}(p, \varepsilon))((p^{-n}, 1)) - \gamma_n(\tilde{v}(p, \varepsilon))((p^{-n}, 0)) \geq -\varepsilon.
\]
With 
\[
\gamma_n(\tilde{v}(p, \varepsilon))((p^{-n}, 1)) = \gamma_0((p^{-n}, 1)) \quad \text{and} \quad \gamma_n(\tilde{v}(p, \varepsilon))((p^{-n}, 0)) = \tilde{v}(p, \varepsilon) \cdot \varrho((p^{-n}, 0), \emptyset) + \gamma_0((p^{-n}, 0)),
\]
we obtain 
\[
\tilde{v}^n(p, \varepsilon) \geq v_p^n + \frac{\varepsilon}{\varrho((p^{-n}, 0), \emptyset)}.
\]
In summary:
\[
\tilde{v}^n(p, \varepsilon) \in \left[v_p^n - \frac{\varepsilon}{\varrho((p^{-n}, 0), \emptyset)}; v_p^n + \frac{\varepsilon}{\varrho((p^{-n}, 0), \emptyset)}\right].
\]

1.3. Finite quitting games

In this section we consider a finite part of a quitting game or, to say it in other words, we look at a finitely many often, say \( t \) times, repeated one-step quitting game, where the players receive no payoff, if all of them decide to play \textit{continue} during the \( t \) stages, but a payoff \( v \in \mathbb{R}^N \) at the end of the \( t \) stages. Benoit and Krishna motivated the study of finitely repeated games in the following way (see [2]): “The possibility that noncooperative equilibria of a repeated game may involve choices that do not form equilibria of the underlying one-shot game has been recognized for a long time. In a repeated setting, players can condition their behavior at any stage of the game on the observed past behavior of other players. As a result, a player may behave in a way that is not in his or her short run interests because any attempt to realize short run gains may lead to future losses if other players retaliate. Some of these equilibria may be more lucrative for all players than any equilibria of the one-shot game.”

\textbf{Definition 1.3.1} (Finite quitting game). A finite quitting game corresponding to a given quitting game \( G \) is a tuple
\[
G_{I,v} := (G, I, v) = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})}, I, v)
\]
where
- \( G \) is the corresponding quitting game with the finite set of players \( \mathcal{N} = \{1, \ldots, N\} \), the quitting coalition \( S \in \mathcal{P}(\mathcal{N}) \) and the family of payoff vectors \( (r_S)_{S \in \mathcal{P}(\mathcal{N})} \),
- \( I = \{t_1, \ldots, t_2\} \subset \mathbb{N} \) is the set of stages, \( t_1, t_2 \in \mathbb{N}, t_1 < t_2 \),
- \( v = (v^1, \ldots, v^N) \in \mathbb{R}^N \) is the payoff to the players, if all of them choose to play the action \textit{continue} during all the \( t \) stages, otherwise they receive \( r_S \).
Definition 1.3.2 (strategy, profile in the finite quitting game). Let $G_{I,v} := (G, I, v)$ be a given finite quitting game. Then define

- $p^n_i \in [0,1]$ as probability that player $n \in \mathcal{N}$ will play the action $\text{quit}$ at stage $i \in I$,
- the sequence $\varphi^n_i := \varphi^n = (p^n_i)_{i \in I}$ as strategy for the player $n \in \mathcal{N}$,
- the vector $\varphi_I := \varphi = (\varphi^1, \ldots, \varphi^N)^T$ as (strategy) profile for the finite quitting game and
- $(\varphi^{-n}, \tilde{\varphi}^n) := (\varphi^1, \ldots, \varphi^{n-1}, \tilde{\varphi}^n, \varphi^{n+1}, \ldots, \varphi^N)$ as alternative strategy profile for player $n$, where all the other players $m \in \mathcal{N} \setminus \{n\}$ play according to their strategy $\varphi^m$, whereas player $n$ changes her strategy and plays according to the strategy $\tilde{\varphi}^n \in [0,1]^{N \times |I|}$.

Remark 1.3.3. The state space, action space and the transition law from the finite quitting game are of course the same as in the corresponding quitting game (see Remark 1.1.2).

### Probability space and equilibria

Now let a finite quitting game $G_{I,v} = (G, I, v)$ and a strategy profile $\varphi$ in $G_{I,v}$ be given. Without loss of generality, we assume that $I := \{1, \ldots, t\}, \ t \in \mathbb{N}$, furthermore let $z = \emptyset$ be the initial state. A probability space $(\Omega_I, \mathcal{A}_I, P_\varphi := P_{I,v})$ and a stochastic process $(\tilde{X}_k, \tilde{Y}_k)_{k \in I}$ on $(\Omega_I, \mathcal{A}_I, P_\varphi)$ with values in $(Z \times A)$ are defined by

- $\Omega_I := (Z \times A)^{|I|}$
- $\mathcal{A}_I := \bigotimes_{k \in I} (\mathcal{P}(Z) \otimes \mathcal{P}(A))$
- $\tilde{X}_k(\omega) = \tilde{X}_k((z_1, a_1, \ldots, z_l, a_l)) := z_k, \ k \in I, \ \omega \in \Omega_I$,
- $\tilde{Y}_k(\omega) = \tilde{Y}_k((z_1, a_1, \ldots, z_l, a_l)) := a_k, \ k \in I, \ \omega \in \Omega_I$,
- $\tilde{Y}_k = (\tilde{Y}_k^1, \ldots, \tilde{Y}_k^N)^T$
- $\tilde{H}_k := (\tilde{X}_1, \tilde{Y}_1, \ldots, \tilde{X}_k)$,
  that means $\tilde{H}_k(\omega) = \tilde{H}_k((z_1, a_1, \ldots, z_l, a_l)) = (z_1, a_1, z_2, a_2, \ldots, z_k)$
  $k \in I, \ \omega \in \Omega_I$
- $P_\varphi(\tilde{X}_1 = \emptyset) := 1$ and
- $P_\varphi(\tilde{H}_k = (z_1, a_1, z_2, a_2, \ldots, z_k))$
  $:= P_\varphi(\tilde{X}_1 = z_1) \prod_{l=1}^{k-1} \tilde{t}(z_{l+1} | z_l, a_l) \cdot \varphi(p_l, \{n \in \mathcal{N} \mid a^n_l = 1\})$,

where $z_l \in Z$ for all $l \in I$ and $a_l = (a^1_l, \ldots, a^N_l) \in A$ for all $l = 1, \ldots, t - 1$ with the transition law $\tilde{t}$ from Remark 1.1.2.
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Let the finite quitting game \( G_{I,v} = (G, \pi, I = \{1, \ldots, t\}, v) \) and the strategy profile \( \varphi \) in \( G_{I,v} \) be given. Then a stopping time \( \tau_I : \Omega_I \to I \cup \{+\infty\} \) concerning the filtration \( (\mathcal{F}_k)_{k \in I} \) with \( \mathcal{F}_k := \sigma\{\tilde{Y}_n : 1 \leq k \leq n\}, k \in I \) is defined by

\[
\tau_I(\omega) := \inf \{ k \in I : \tilde{Y}_k(\omega) \in \{0,1\}^N \setminus \{0\}\}. \tag{1.25}
\]

The expected payoff for the players in a finite quitting game \( G_{I,v} = (G, I = \{1, \ldots, t\}, v) \) under the strategy profile \( \varphi = (p_1, \ldots, p_t) \) is given by

\[
g(\varphi) = (g(\varphi)_1, \ldots, g(\varphi)_N)^T := g_{I,v}(\varphi) = \mathbb{E}_\varphi(\varphi(\tilde{Y}_{\tau_I})^I_{\{\tau_I=I\}} + v_{\{\tau_I=+\infty\}})
= \sum_{k=1}^{t-1} \prod_{l=1}^{k-1} \mathbb{P}_\varphi(\tilde{Y}_l = \emptyset) \sum_{a_k} \mathbb{P}_\pi(\tilde{Y}_k = a_k) \cdot r_{\{n \in N : a_n = 1\}} + \prod_{l=1}^{t} \mathbb{P}_\varphi(\tilde{Y}_l = \emptyset) \cdot v
= \sum_{k=1}^{t-1} \prod_{l=1}^{k-1} g(p_l, \emptyset) \sum_{S \in \mathcal{P}(N) \setminus \{\emptyset\}} g(p_k, S) \cdot r_S + \prod_{l=1}^{t} g(p_l, \emptyset) \cdot v,
\]

where \( \mathbb{E}_\varphi \) is the expected value with respect to the probability measure \( \mathbb{P}_\varphi \).

Similar to the one-step game respectively the quitting game, we define the \( \varepsilon \)-equilibrium for the finite quitting game.

**Definition 1.3.4** (\( \varepsilon \)-equilibrium in the finite quitting game). Let \( G_{I,v} = (G, I, v) \) be a given finite quitting game. A strategy profile \( \varphi \) is called \( \varepsilon \)-equilibrium in \( G_{I,v} \), \( \varepsilon \geq 0 \), if and only if

\[
\forall n \in N, \hat{\varphi}^n \in [0,1]^{|I|} : \ g^n(\varphi) \geq g^n((\varphi^{-n}, \hat{\varphi}^n)) - \varepsilon. \tag{1.26}
\]

**Lemma 1.3.5.** In a finite quitting game, the requirement (1.26) can be replaced by

\[
\forall n \in N, \hat{\varphi}^n \in \{0,1\}^{\lfloor t/2 \rfloor} : \ g^n(\varphi) \geq g^n((\varphi^{-n}, \hat{\varphi}^n)) - \varepsilon. \tag{1.27}
\]

**Proof.** The proof uses mainly the finiteness of the game and the linearity property of the one-step quitting game.

Let the finite quitting game \( G_{I,v} = (G, I = \{1, \ldots, t\}, v) \) be given. In order to show that \( \varphi = (p_1, \ldots, p_t) \) is an \( \varepsilon \)-equilibrium in the finite quitting game for a given player \( n \in N \), one has to verify that

\[
g^n(\varphi) \geq \max_{\hat{\varphi}^n \in [0,1]^{\lfloor t/2 \rfloor}} g^n((\varphi^{-n}, \hat{\varphi}^n)) - \varepsilon = \max_{(\hat{p}_1^n, \ldots, \hat{p}_t^n) \in \{0,1\}^{\lfloor t/2 \rfloor}} g^n((\varphi^{-n}, (\hat{p}_1^n, \ldots, \hat{p}_t^n))) - \varepsilon. \tag{1.28}
\]
Because of
\[
g^n(\varphi, \tilde{\varphi}) = \sum_{S \in \mathcal{P}(N) \setminus \{\varnothing\}} \varphi(p_1^n, \tilde{p}_1^n, S) \cdot r^n_S + \varphi(p_1^n, \tilde{p}_1^n, \varnothing) \left( \sum_{S \in \mathcal{P}(N) \setminus \{\varnothing\}} \varphi(p_2^n, \tilde{p}_2^n, S) \cdot r^n_S \right) + \varphi(p_2^n, \tilde{p}_2^n, \varnothing) \left( \sum_{S \in \mathcal{P}(N) \setminus \{\varnothing\}} \varphi(p_{t-1}^n, \tilde{p}_{t-1}^n, S) \cdot r^n_S \right) + \varphi(p_{t-1}^n, \tilde{p}_{t-1}^n, \varnothing) \left( \sum_{S \in \mathcal{P}(N) \setminus \{\varnothing\}} \varphi(p_t^n, \tilde{p}_t^n, S) \right) r^n_S + \varphi(p_t^n, \tilde{p}_t^n, \varnothing) v^n(S) \right) \right),
\]
the Inequality (1.28) is equivalent to
\[
g^n(\varphi) \geq \max_{\tilde{p}_1^n \in [0,1]} \left( \sum_{S \in \mathcal{P}(N) \setminus \{\varnothing\}} \varphi(p_1^n, \tilde{p}_1^n, S) \cdot r^n_S + \varphi(p_2^n, \tilde{p}_2^n, \varnothing) \left( \sum_{S \in \mathcal{P}(N) \setminus \{\varnothing\}} \varphi(p_2^n, \tilde{p}_2^n, S) \cdot r^n_S \right) + \varphi(p_{t-1}^n, \tilde{p}_{t-1}^n, \varnothing) \left( \sum_{S \in \mathcal{P}(N) \setminus \{\varnothing\}} \varphi(p_t^n, \tilde{p}_t^n, S) \right) r^n_S + \varphi(p_t^n, \tilde{p}_t^n, \varnothing) v^n(S) \right) - \varepsilon.
\]
Since the expected payoff \( \gamma^n_v((p_t^n, \tilde{p}_t^n)) \) for the one-step game \( \Gamma_v \) under the strategy profile \((p_t^n, \tilde{p}_t^n)\) is linear in \( \tilde{p}_t^n \), the maximum is attained either in \( \tilde{p}_t^n = 0 \) or in \( \tilde{p}_t^n = 1 \) (or in both for the case that \( \gamma^n_v((p_t^n, \tilde{p}_t^n)) \) is constant in \( \tilde{p}_t^n \)). This leads us to the following two cases:

Case 1: The maximum of \( \gamma^n_v((p_t^n, \tilde{p}_t^n)) \) is attained in \( \tilde{p}_t^n = 0 \). Then
\[
\max_{\tilde{p}_{t-1}^n \in [0,1]} \left( \sum_{S \in \mathcal{P}(N) \setminus \{\varnothing\}} \varphi((p_{t-1}^n, \tilde{p}_{t-1}^n), S) r^n_S + \varphi((p_{t-1}^n, \tilde{p}_{t-1}^n), \varnothing) \cdot \max_{\tilde{p}_t^n \in [0,1]} \gamma^n_v((p_t^n, \tilde{p}_t^n)) \right)
= \max_{\tilde{p}_{t-1}^n \in [0,1]} \left( \sum_{S \in \mathcal{P}(N) \setminus \{\varnothing\}} \varphi((p_{t-1}^n, \tilde{p}_{t-1}^n), S) r^n_S + \varphi((p_{t-1}^n, \tilde{p}_{t-1}^n), \varnothing) \cdot \gamma^n_v((p_t^n, 0)) \right)
= \max_{\tilde{p}_{t-1}^n \in [0,1]} \gamma^n_v((p_{t-1}^n, \tilde{p}_{t-1}^n))
= \max_{\tilde{p}_{t-1}^n \in [0,1]} \gamma^n_v((p_{t-1}^n, \tilde{p}_{t-1}^n)).
\]
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Case 2: The maximum of $\gamma^n(i((p_{t-1}^{-n}, \tilde{p}_t^n)))$ lies in $\hat{p}_t^n = 1$. Then
\[
\max_{\hat{p}_{t-1}^n \in [0,1]} \left( \sum_{S \in \mathcal{P}(N) \backslash \{\emptyset\}} \varrho((p_{t-1}^{-n}, \hat{p}_{t-1}^n), S) r_S^n + \varrho((p_{t-1}^{-n}, \hat{p}_{t-1}^n), \emptyset) \cdot \max_{\hat{p}_t^n \in [0,1]} \gamma^n(i((p_{t-1}^{-n}, \hat{p}_t^n))) \right)
= \max_{\hat{p}_{t-1}^n \in [0,1]} \left( \sum_{S \in \mathcal{P}(N)} \varrho((p_{t-1}^{-n}, \hat{p}_{t-1}^n), S) r_S^n + \varrho((p_{t-1}^{-n}, \hat{p}_{t-1}^n), \emptyset) \sum_{S \in \mathcal{P}(N)} \varrho((p_{t-1}^{-n}, \hat{p}_t^n), S) r_S^n \right)
= \max_{\hat{p}_{t-1}^n \in [0,1]} \gamma^n(i((p_{t-1}^{-n}, 1))) ^\gamma_{\varrho}(i(p_{t-1}^{-n}, 1))
= \max_{\hat{p}_{t-1}^n \in [0,1]} \gamma^n(i((p_{t-1}^{-n}, \hat{p}_{t-1}^n)))
\]
for every $\hat{v} \in \mathbb{R}^n$.

The same type of arguments is used for the other $\hat{p}_k$, $k = 1, \ldots, t - 1$. So the maximizing strategies for player $n$ can be found recursively, and there will be always a maximizing strategy for player $n$, which is a pure strategy, that means, which is an element of $\{0, 1\}^t$.

Remark 1.3.6. 1. Although there will be always a pure (expected) payoff-maximizing strategy for a player – given the strategies for the other players – in general, an equilibrium with only pure strategies need not exist.

2. Due to the structure of quitting games respectively finite quitting games, it is clear, that if $\hat{\varphi}^n = (\hat{p}_1^n, \ldots, \hat{p}_t^n) \neq (0, \ldots, 0)$ is one of the pure maximizing strategies for player $n$ in the given finite quitting game $G_{I, v}$, then all strategies $\hat{\varphi}^n := (\hat{p}_1^n, \ldots, \hat{p}_j^n, \hat{p}_{j+1}^n, \ldots, \hat{p}_t^n)$ with $\hat{p}_1 = \ldots = \hat{p}_{j-1} = 0$, $\hat{p}_j = 1$ and $\hat{p}_k^n \in [0,1]$ for all $k \in j + 1, \ldots, t$ are maximizing strategies for player $n$, as well.

At the end of this section, we show how finite quitting games can be embedded into a given quitting game with respect to the calculation of the expected payoff. Therefore consider the finite quitting game $G_{I, v} = (G, I, v)$ and a strategy profile $\varphi = \varphi_I$ in $G_{I, v}$. The expected payoff to the players can be calculated as
\[
 g_{I, v}(\varphi) = E_{\varphi}(r(\tilde{Y}_T) 1_{\{\tau_I \in I\}} + v 1_{\{\tau_I = +\infty\}})
= E_{\varphi}(r(\tilde{Y}_T) 1_{\{\tau_I \in I\}}) + E_{\varphi}(v 1_{\{\tau_I = +\infty\}})
= E_{\varphi}(r(\tilde{Y}_T) 1_{\{\tau_I \in I\}}) + P_{\varphi}(\tau_I = +\infty) \cdot v
= E_{\varphi}(r(\tilde{Y}_T) 1_{\{\tau_I \in I\}}) + (1 - P_{\varphi}(\tau_I \in I)) \cdot v. \tag{1.29}
\]
Now let a strategy profile \( \pi = (p_1, p_2, \ldots) \) for \( G \) be given. We shift the interval \( I \) and consider the finite quitting game
\[
G_{I,v} = (G, \{i, \ldots, j\}, v),
\]
which starts at the time \( i \) and ends at \( j \), \( i \leq j \). If the continue payoff \( v \) is equal to the expected payoff from the quitting game starting at time \( j + 1 \) under the subgame profile \( \pi_{j+1} \), that means \( v = \gamma(\pi_{j+1}) \), then for the expected payoff to the players under a strategy profile \( \pi \) in the finite game \( G_{I,v} \),
\[
g_{I,v}(\varphi) = \mathbb{E}_\varphi(r(\hat{\tau}_I) \mathbb{1}_{\{\tau_I \leq k\}}) + (1 - \mathbb{P}_\varphi(\tau_I \in I)) \gamma(\pi_{j+1})
\]
holds.

Observe that in general \( \varphi_{\{i, \ldots, j\}} \) need not be equal to \( (p_i, \ldots, p_j) \), but if this is the case, i.e. \( \varphi = \varphi_{\{i, \ldots, j\}} = (p_i, \ldots, p_j) \), that means \( \varphi \) is a part of \( \pi \), then
\[
g_{I,v}(\varphi) = \mathbb{E}_\varphi(r(\hat{\tau}_I) \mathbb{1}_{\{\tau_I \leq k\}} + \gamma(\pi_{k+1}) \mathbb{1}_{\{\tau_I > k\}})
\]
for all \( k \in I \) and furthermore
\[
g_{I,v}(\varphi) = \mathbb{E}_\pi(r(\tau) \mathbb{1}_{\{\tau \leq j\}} + \gamma(\pi_{j+1}) \mathbb{1}_{\{\tau > j\}} | \tau \geq i) = \gamma(\pi_i),
\]
where \( \tau_I \) is the stopping time corresponding to the finite game and \( \tau \) is the stopping time of the quitting game.

These technical facts play an important role in the proof of Proposition 2.3.8.

### 1.4. Relations between the structure of profile \( \pi \) and the expected payoff in quitting games

In this section we consider a quitting game \( G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})}) \) and a strategy profile \( \pi = (p_1, p_2, \ldots) \) in this game. We study the relations between the structure of \( \pi \) and the expected payoff to the players under the strategy profile \( \pi \). The following questions will be treated:
Chapter 1. Quitting games

1. Assume that at least one stage $i \in \mathbb{N}$ exists, such that all players play $continue$ with certainty, i.e. $p_i = 0$.
   a) How does the value of the expected payoff change, if we remove or insert a finite number of zero-vectors to the given one?
   b) If $\pi$ is an $\epsilon$-equilibrium ($\epsilon \geq 0$), is it possible to construct new $\epsilon$-equilibria out of $\pi$ by erasing or adding a finite number of zero vectors to the given one?

2. Assume that one stage $i \in \mathbb{N}$ exists, such that one player plays quit with certainty.
   a) How far does this simplify the calculation of the expected payoff to the players under $\pi$?
   b) If $\pi$ is an $\epsilon$-equilibrium ($\epsilon \geq 0$), is it possible to construct new $\epsilon$-equilibria out of $\pi$?

3. Let $\pi$ be a stationary strategy profile. How does this affects the calculation of the expected payoff and the proof that a given profile is an $\epsilon$-equilibrium?

Before concerning the questions in detail, we introduce the following notation:

**Notation 1.4.1.** Denote

- $c := (0, 0, \ldots)$ the strategy in the quitting game $G$, where a player chooses to play the action $continue$ with certainty all the time,
- $q := (1, 1, \ldots)$ the strategy in the quitting game $G$, where a player chooses to play the action $quit$ with certainty all the time,
- $q_i := (0, \ldots, 0, 1, 0, \ldots)$ the strategy in the quitting game $G$, where a player plays the action $continue$ all the time except of stage $i \in \mathbb{N}$, in which the action $quit$ is played with certainty,
- $c := 0$,
- $q := 1 = (1, \ldots, 1)^T \in [0, 1]^N$.

**1. Removing and inserting zero-vectors**

Firstly we show, that the value of the expected payoff does not change, if one removes or inserts finitely many zero-vectors.

**Lemma 1.4.2.** Let $G$ be a quitting game and $\pi$ a strategy profile in $G$. If a set of consecutive stages $I = \{i, \ldots, j\}$ exists, such that $p_k = c$ for all $k \in I$, then $\gamma(\pi) = \gamma(\hat{\pi})$ for all

$$\hat{\pi} := \left(p_1, \ldots, p_{i-1}, c, \ldots, c, p_{j+1}, p_{j+2}, \ldots\right),$$

$l \in \mathbb{N}_0$. 

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1.4. Relations between the structure of profile $\pi$ and the expected payoff in quitting games

**Proof.** As shown in Table 1.1, we divide the profile $\pi$ into three parts. The first one is $\varphi_L := (p_1, \ldots, p_{i-1})$ with $L := \{1, \ldots, i-1\}$, the second one is $\varphi_I = (p_i, \ldots, p_j) = (c, \ldots, c)$ and the third $\pi_{j+1}$. Furthermore we set $\varphi := \emptyset$, and let $l \in \mathbb{N}_0$ be arbitrary but fixed.

$$
\gamma(\pi) = \mathbb{E}_\pi \left( r(Y_\tau)I_{\{\tau \in L\}} + r(Y_\tau)I_{\{\tau \in I\}} + r(\tilde{Y}_\tau)I_{\{\tau < \tau \leq \infty\}} \right)
= \mathbb{E}_\pi \left( r(Y_\tau)I_{\{\tau \in L\}} \right) + \mathbb{E}_\pi \left( r(Y_\tau)I_{\{\tau \in I\}} + r(\tilde{Y}_\tau)I_{\{\tau = +\infty\}} \right)
= \mathbb{E}_{\varphi_L} \left( r(\tilde{Y}_\tau)I_{\{\tau \in I\}} \right) + \mathbb{E}_{\varphi_I} \left( r(\tilde{Y}_\tau)I_{\{\tau \in I\}} + r(\pi_{j+1})I_{\{\tau = +\infty\}} \right)
= g_{\varphi_L}(\varphi_L) + \prod_{l=1}^{i-1} \rho(p_k, \emptyset) g_{l, \gamma(\pi_{j+1})}(\varphi_I)
$$

This, (1.29) and $P_{\varphi_I}(\tau_I \in I) = 0$ yield

$$
\gamma(\pi) = g_{\varphi_L}(\varphi_L) + \prod_{l=1}^{i-1} \rho(p_k, \emptyset) \gamma(\pi_{j+1})
= g_{\varphi_L}(\varphi_L) + \prod_{l=1}^{i-1} \rho(p_k, \emptyset) g_{(i, \ldots, i+l-1, \gamma(\pi_{j+1}))}((c, \ldots, c))
= \gamma(\hat{\pi}).
$$

Now suppose, that $\pi$ is an $\varepsilon$-equilibrium and that $\pi$ contains at least one block of zero vectors. The question how to construct other $\varepsilon$-equilibria for the given quitting game out of the given profile is answered through the next lemma.

**Lemma 1.4.3.** Let $G$ be a given quitting game, $I = \{i, \ldots, j\}$, $i \leq j$, a set of consecutive stages and $\pi = (p_1, p_2, \ldots)$ an $\varepsilon$-equilibrium in $G$, where $p_k = c$ for all $k \in I$, $\varepsilon \geq 0$. Then every profile $\hat{\pi}$, where

$$
\hat{\pi} := (p_1, \ldots, p_{i-1}, c, \ldots, c, p_{j+1}, p_{j+2}, \ldots),
$$

(1.31)

$l \in \mathbb{N}_0$, is an $\varepsilon$-equilibrium in $G$.

**Proof.** We show that the equilibrium condition

$$
\gamma^n(\hat{\pi}) \geq \max_{\pi^n \in [0,1]^n} \gamma^n((\hat{\pi}^{-n}, \bar{\pi}^n)) - \varepsilon
$$

(1.32)

holds for all players $n \in \mathcal{N}$ and all $l \in \mathbb{N}_0$, with $\hat{\pi}$ from (1.31).

The previous Lemma 1.4.2 gives us that $\gamma(\pi) = \gamma(\hat{\pi})$ for all $\hat{\pi}$ like in (1.31). Define $L := \{1, \ldots, i-1\}$ and $\varphi_L := (p_1, \ldots, p_{i-1})$, $\varphi_I := (p_i, \ldots, p_j) = (c, \ldots, c)$ as decomposition
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of $\pi$ together with $\hat{\rho}_L^n := (\hat{\rho}_1^n, \ldots, \hat{\rho}_{i-1}^n) \in [0,1]^{|I|-1}$ and $\hat{\rho}_L^n := (\hat{\rho}_1^n, \ldots, \hat{\rho}_j^n) \in [0,1]^{|I|}$ as decomposition of $\hat{\pi}^n$. $\pi$ being an $\varepsilon$-equilibrium implies

$$\gamma^n(\pi) \geq \max_{\tilde{\pi}^n \in [0,1]^{|\pi|}} \gamma^n(\tilde{\pi}^n) - \varepsilon$$

$$= \max_{\tilde{\pi}^n \in [0,1]^{|\pi|}} \left( g_{L,L}^n(\tilde{\pi}, \hat{\rho}) + \prod_{k=1}^{i-1} g((\hat{\rho}_k^n, \hat{\rho}_k^n), \emptyset) \cdot \left( g_{L,L}^n(\tilde{\pi}, \hat{\rho}) + \prod_{k=1}^{j}(1 - \hat{\rho}_k) \gamma^n((\hat{\pi}_{j+1}^n, \hat{\rho}_{j+1}^n)) \right) \right) - \varepsilon$$

(1.33)

for all $n \in \mathcal{N}$. We distinguish two cases. In the first case, we show (1.32) for $l \in \mathbb{N}$ and in the second for $l = 0$.

Case 1: Let $l \in \mathbb{N}$ be arbitrary but fixed. Denote $J := \{i, i + 1, \ldots, i + l - 1\}$, $\hat{\rho}_J := (\hat{\rho}_i, \ldots, \hat{\rho}_{i+l-1}) = (e, \ldots, e)$ and $\hat{\rho}_J^n := (\hat{\rho}_1^n, \ldots, \hat{\rho}_{i+l-1}^n) \in [0,1]^{|J|}$. Observe that $\hat{\pi}$ differs from $\pi$ only in the interval $J$, which is longer or shorter than the counterpart $I$, thus it is sufficient to focus on that part referring to the maximization of the expected payoff. For $b \in \mathbb{R}^N$ arbitrary but fixed we have on the one hand

$$\max_{\hat{\rho}_J \in [0,1]^{|I|}} g_{L,L}^n((\tilde{\pi}, \hat{\rho}_J^n)) + \prod_{k=1}^{j}(1 - \hat{\rho}_k) \cdot b$$

$$= \max_{\hat{\rho}_J \in [0,1]^{|J|}} r_{I,n}^n \left( \sum_{k=1}^{j} (1 - \hat{\rho}_k) \hat{\rho}_k^n + \prod_{k=1}^{j}(1 - \hat{\rho}_k) \cdot b \right)$$

(1.34)

for the strategy $\pi$ and on the other hand

$$\max_{\hat{\rho}_J \in [0,1]^{|J|}} g_{J,J}^n((\tilde{\pi}, \hat{\rho}_J^n)) + \prod_{k=1}^{i+l-1}(1 - \hat{\rho}_k) \cdot b$$

$$= \max_{\hat{\rho}_J \in [0,1]^{|J|}} r_{J,n}^n \left( \sum_{k=1}^{i+l-1} (1 - \hat{\rho}_k) \hat{\rho}_k^n + \prod_{k=1}^{i+l-1}(1 - \hat{\rho}_k) \cdot b \right)$$

(1.35)

for the strategy $\hat{\pi}$, where we keep in mind, that only player $n$ may quit with a positive probability, while the other players play continue with certainty.

Because of the structure of (1.34) and (1.35), a strategy $\hat{\rho}_J^n$ from player $n$ is a maximizing argument of (1.35) if, and only if $\hat{\rho}_J^n = (\hat{\rho}_1^n, \ldots, \hat{\rho}_j^n)$ with $\hat{\rho}_1^n := 1 - \prod_{k=1}^{i+l-1}(1 - \hat{\rho}_k^n)$ and $\hat{\rho}_{i+l} = \ldots = \hat{\rho}_j^n = 0$ is a maximizing argument for (1.34).
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This implies

$$\gamma^n(\hat{\pi}) = \gamma^n(\pi) \geq \max_{\tilde{\pi} \in [0,1]^n} \gamma^n((\pi^-\tilde{n}, \tilde{\pi}^n)) - \varepsilon = \max_{\tilde{\pi}^n \in [0,1]^n} \gamma^n((\tilde{\pi}^-\tilde{n}, \tilde{\pi}^n)) - \varepsilon.$$ 

Case 2: Let $l = 0$. Then $\hat{\pi} = (p_1, \ldots, p_{i-1}, p_{j+1}, \ldots)$. With (1.33),

$$\gamma^n(\hat{\pi}) = \gamma^n(\pi) \geq \max_{\tilde{\pi} \in [0,1]^n} \left( g_{I,0}^n((\varphi_{I}^n, \varphi_{I}^n)) + \prod_{k=1}^{i-1} \varrho((p_k^{-n}, \tilde{p}_k^n), \emptyset) \cdot \left( g_{I,0}^n((\varphi_{I}^n, \varphi_{I}^n)) + \prod_{k=i}^{j} (1 - \tilde{p}_k) \gamma^n((\tilde{\pi}_{j+1}^{-n}, \tilde{\pi}_{j+1}^n)) \right) - \varepsilon \right)$$

holds. Because $(\varphi_{I}^n, (0, \ldots, 0))$ contains only zeros, $g_{I,0}^n((\varphi_{I}^n, (0, \ldots, 0))) = 0$ and $\prod_{k=i}^{j} (1 - \tilde{p}_k) = 1$, therefore

$$\gamma^n(\hat{\pi}) \geq \max_{(\tilde{p}_1, \ldots, \tilde{p}_{i-1}, 0, \ldots, 0, \tilde{p}_{j+1}, \ldots) \in [0,1]^n} \left( g_{I,0}^n((\varphi_{I}^n, \varphi_{I}^n)) + \prod_{k=1}^{i-1} \varrho((p_k^{-n}, \tilde{p}_k^n), \emptyset) \cdot \gamma^n((\tilde{\pi}_{j+1}^{-n}, \tilde{\pi}_{j+1}^n)) \right) - \varepsilon$$

follows. This finishes the proof.

Lemma 1.4.3 pointed out that one may extend, reduce or even eliminate a given block of zero-vectors without changing the value of the expected payoff. Furthermore if $\pi$ was an $\varepsilon$-equilibrium, the resulting strategy profiles are $\varepsilon$-equilibria as well. But the converse implication is not true, in general. This means that it is not possible to insert a finite block of zero vectors at an arbitrary position in a given $\varepsilon$-equilibrium $\pi$ and get an $\varepsilon$-equilibrium profile again. We illustrate this with a counter example:
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Example 3. Consider the two-player quitting game $G$ given by:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>continue</th>
<th>quit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 2</td>
<td>$\varnothing$</td>
<td>$(1, -1)$</td>
</tr>
<tr>
<td></td>
<td>$(10, 1)$</td>
<td>$(5, 5)$</td>
</tr>
</tbody>
</table>

The stationary strategy profile $\pi = (p, p, \ldots)$ with $p = (1, 1)^T$ is an equilibrium in $G$ because

$$
\gamma^n(\pi) = 5 \geq \max_{\tilde{\pi}^n \in \Pi^n} \gamma^n((\pi^{-n}, \tilde{\pi}^n)) \\
= \max_{\tilde{p}_i^n \in [0,1]} \tilde{p}_i^n \cdot 5 + (1 - \tilde{p}_i^n) \cdot 1 = 5 \quad n = 1, 2.
$$

Now we insert a zero vector at the first position of $\pi$, such that $\hat{\pi} := (\hat{p}_1, \hat{p}_2, \ldots) = (c, p, p, \ldots)$. On the one hand, $\gamma(\pi) = \gamma(\hat{\pi})$ (cf. Lemma 1.4.2), but on the other hand,

$$
\gamma^1(\pi) = 5 \leq \max_{\hat{\pi}^1 \in \Pi^1} \gamma^1((\hat{\pi}^{-1}, \hat{\pi}^1)) \\
= \max_{(\hat{p}_1^1, \hat{p}_2^1) \in [0,1]^2} \hat{p}_1^1 \cdot 10 + (1 - \hat{p}_1^1) \cdot (\hat{p}_2^1 \cdot 5 + (1 - \hat{p}_2^1) \cdot 1) = 10.
$$

2. A player plays quit with certainty

The next lemma refers to the second question (see 2.a) stated at the beginning of this section.

Lemma 1.4.4. Let $G$ be a given quitting game and $\pi$ a strategy profile in $G$. If there exist a player $n \in \mathcal{N}$ and a stage $i \in \mathbb{N}$, such that $p_i^n = 1$, then $\gamma(\pi) = \gamma(\hat{\pi})$ for all

$$
\hat{\pi} := (p_1, \ldots, p_i, \hat{p}_{i+1}, \hat{p}_{i+1}, \ldots),
$$

with $\hat{p}_{i+k} \in [0, 1]^N$ arbitrary for all $k \in \mathbb{N}$.

Proof. $p_i^n = 1$ implies $\{i < \tau\} = \varnothing$ (cf. 1.2) and thus

$$
\gamma(\pi) = \mathbb{E}_n(r(Y_\tau)\mathbb{I}_{\{\tau \leq i\}} + r(Y_\tau)\mathbb{I}_{\{i < \tau < \infty\}}) \\
= \mathbb{E}_n(r(Y_\tau)\mathbb{I}_{\{\tau \leq i\}}) \\
= \mathbb{E}_n(r(Y_\tau)\mathbb{I}_{\{\tau \leq i\}}) \\
= \mathbb{E}_n(r(Y_\tau)\mathbb{I}_{\{\tau \leq i\}} + r(Y_\tau)\mathbb{I}_{\{i < \tau < \infty\}}) \\
= \gamma(\hat{\pi}).
$$

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**Remark 1.4.5.** Lemma 1.4.2 and Lemma 1.4.4 can be formulated and proven analogously for finite quitting games – observe the change of the length of the interval $I$ and therewith the change of the length of the finite game in the first case.

Consider now part b) of the second question from the beginning.

**Lemma 1.4.6.** Let $G$ be a given quitting game and $\pi = (p_1, p_2, \ldots)$ an $\epsilon$-equilibrium in $G$, where two players $n, u \in \mathcal{N}$ and two stages $i, j \in \mathbb{N}$, $i \leq j$, exist, such that $p^n_i = p^u_j = 1$.

Then every profile $\tilde{\pi}$, where

$$\tilde{\pi} := (p_1, \ldots, \hat{p}_i, \hat{p}_{j+1}, \hat{p}_{j+2}, \ldots), \quad \hat{p}_{j+k} \in [0, 1]^N, \quad k \in \mathbb{N},$$

is an $\epsilon$-equilibrium in $G$.

**Proof.** Again we have to show, that for all player $m \in \mathcal{N}$

$$\gamma^m(\tilde{\pi}) \geq \max_{\hat{\pi}^n \in [0, 1]^N} \gamma^m((\tilde{\pi}^{-n}, \hat{\pi}^n)) - \epsilon$$

holds.

First consider all players $m \neq n$. Denote $\ell := \{1, \ldots, i\}$, $\varphi_{L} := (p_1, \ldots, p_i)$ and let $v \in \mathbb{R}^N$ be arbitrary. Using the structure of $\tilde{\pi}$, especially that $\hat{p}^n_i = p^n_i = 1$, and that $\pi$ is an $\epsilon$-equilibrium in $G$, one gets

$$\gamma^m(\tilde{\pi}) = \gamma^m(\pi) \geq \max_{\hat{\pi}^n \in [0, 1]^N} \gamma^m((\pi^{-m}, \hat{\pi}^m)) - \epsilon$$

$$= \max_{(\hat{p}^m_i, \ldots, \hat{p}^m_i) \in [0, 1]^i} g^m_{L,v}((\varphi_L^{-m}, (\hat{p}^m_i, \ldots, \hat{p}^m_i))) - \epsilon$$

$$= \max_{\hat{\pi}^n \in [0, 1]^N} \gamma^m((\pi^{-m}, \hat{\pi}^m)) - \epsilon$$

The proof for player $n$ is similar to that for the other players, taking into account that player $u$ plays *quit* with certainty at stage $j$, i.e. $\hat{p}^u_j = p^u_j = 1$, such that one has to choose $\ell := \{1, \ldots, j\}$.

**Corollary 1.4.7.** Let $G$ be a given quitting game and $\pi = (p_1, p_2, \ldots)$ a pure $\epsilon$-equilibrium in $G$, $\epsilon \geq 0$, where at least two players play quit with certainty. Denote by $i$ the first stage, where the first player $n \in \mathcal{N}$ plays quit, and by $j$ the first stage, where the second player $m \in \mathcal{N} \setminus \{n\}$ plays quit. Then the profile $\tilde{\pi} = (p_1, p_j, \hat{p}_3, p_4, \ldots)$ with $\hat{p}_k \in \{0, 1\}^N$, $2 < k \in \mathbb{N}$, is a pure $\epsilon$-equilibrium in $G$. 

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3. Stationary strategy profiles

Let $\pi = (p, p, \ldots)$ with $p \in [0,1]^N$ be a stationary strategy profile in $G$. In general the expected payoff for the players under a given profile was calculated by

$$\gamma(\pi) = \sum_{S \in \mathcal{P}(N)} r_S \cdot \varrho(p_1, S) + \varrho(p_1, \emptyset) \cdot \gamma(\pi_2).$$

For a stationary profile $\gamma(\pi) = \gamma(\pi_k)$, for all $k \in \mathbb{N}$,

$$\gamma(\pi) = \sum_{S \in \mathcal{P}(N)} r_S \cdot \varrho(p, S) + \varrho(p, \emptyset) \cdot \gamma(\pi) = \gamma_{\gamma(n)}(p)$$

respectively

$$\gamma(\pi) = \begin{cases} 0 & \text{if } p = c \\ \frac{1}{1-\varrho(p, \emptyset)} \cdot \sum_{S \in \mathcal{P}(N)} r_S \cdot \varrho(p, S) & \text{otherwise} \end{cases}$$

follows.

In the next lemma we want to show, that it is comparatively easy to verify, whether a given stationary strategy profile is an $\varepsilon$-equilibrium or not. We use the fact, that given that all players except player $n$ play a stationary strategy, there always exists a stationary strategy for player $n$, which maximizes her expected payoff. This is because the maximizing problem for player $n$ can be seen as a Markov decision problem and it is well known that there always exists a stationary strategy which solves it (see e.g. [3] or [18] p.87ff.).

**Lemma 1.4.8.** Let $G$ be a given quitting game and $\pi = (p, p, \ldots)$ with $p \in [0,1]^N$ a stationary strategy profile in $G$. Then $\pi$ is an $\varepsilon$-equilibrium ($\varepsilon \geq 0$) in $G$

$$\iff \forall n \in \mathcal{N} : \gamma^n(\pi) \geq \gamma^n((\pi^{-n}, c)) - \varepsilon \land \gamma^n(\pi) \geq \gamma^n((\pi^{-n}, q)) - \varepsilon.$$

**Proof.** Consider a player $n \in \mathcal{N}$. Let $p \neq (c^{-n}, b)$ with $b \in [0,1]$, otherwise the proof is trivial. The expected payoff for this player is given by

$$\gamma^n(\pi) = \frac{1}{1 - \varrho(p, \emptyset)} \sum_{S \in \mathcal{P}(N)} \varrho(p, S)r^n_S$$

$$= \frac{1}{1 - (1 - p^n)\varrho((p^{-n}, 0), \emptyset)} \cdot \left( p^n \cdot \sum_{S \in \mathcal{P}(N \setminus \{n\})} \varrho((p^{-n}, 1), S \cup \{n\})r^n_{S \cup \{n\}} \right)$$

$$+ (1 - p^n) \cdot \sum_{S \in \mathcal{P}(N \setminus \{n\})} \varrho((p^{-n}, 0), S)r^n_S.$$
Observe that \( p \neq (c^{-n}, b) \) with \( b \in [0, 1] \) implies \( \varrho(p, \emptyset) < 1 \).

Differentiation leads to

\[
\frac{\partial \gamma^n(\pi)}{\partial p^n} = \frac{1}{(1 - (1 - p^n)\varrho((p^{-n}, 0), \emptyset))^2} \cdot \left( (\mu_n - \kappa_n) \cdot \left( 1 - (1 - p^n)\varrho((p^{-n}, 0), \emptyset) \right) \right. \\
\left. - \left( p^n \cdot \mu_n + (1 - p^n) \cdot \kappa_n \right) \cdot \varrho((p^{-n}, 0), \emptyset) \right) \\
= \frac{1}{(1 - (1 - p^n)\varrho((p^{-n}, 0), \emptyset))^2} \cdot \left( \mu_n - \kappa_n - (1 - p^n)\varrho((p^{-n}, 0), \emptyset)\mu_n \\
+ (1 - p^n)\varrho((p^{-n}, 0), \emptyset)\kappa_n - p^n\varrho((p^{-n}, 0), \emptyset) \cdot \mu_n \\
- (1 - p^n)\varrho((p^{-n}, 0), \emptyset)\kappa_n \right) \\
= \frac{1}{(1 - (1 - p^n)\varrho((p^{-n}, 0), \emptyset))^2} \cdot \left( \mu_n - \kappa_n - \varrho((p^{-n}, 0), \emptyset)\mu_n \right).
\]

The term

\[
\mu_n - \kappa_n - \varrho((p^{-n}, 0), \emptyset)\mu_n
\]

is independent from \( p \) and therefore constant. Furthermore for \( p^n \in [0, 1] \) and \( \varrho((p^{-n}, 0), \emptyset) < 1 \),

\[
(1 - (1 - p^n)\varrho((p^{-n}, 0), \emptyset))^2 > 0
\]

follows. This implies that the slope of \( \gamma^n(\pi) \) is either positive or negative in \( p^n \). Therefore the maximum of the expected payoff for a player \( n \in \mathcal{N} \) is always reached by a pure stationary strategy.

**Remark 1.4.9.** Lemma 1.4.8 implies that in order to verify, whether a stationary strategy profile \( \pi \) is an \( \varepsilon \)-equilibrium in \( G \) or not, it is sufficient to consider only pure strategies as alternatives for a player.

**Remark 1.4.10.** For two-player recursive repeated games with absorbing states, Flesch, Thuijsman and Vrieze showed in [15] that a stationary \( \varepsilon \)-equilibrium always exists. That means all two-player quitting games have a stationary \( \varepsilon \)-equilibrium, \( \varepsilon > 0 \). On the other hand, they found an example of a three-player quitting game, which has no stationary \( \varepsilon \)-equilibrium (see [14]). The example is the following:

<table>
<thead>
<tr>
<th>Player 3</th>
<th>c</th>
<th>Player 2</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>c</td>
<td>c</td>
<td>(0, 1, 3)</td>
</tr>
<tr>
<td></td>
<td>q</td>
<td>(1, 3, 0)</td>
<td>(1, 0, 1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Player 3</th>
<th>q</th>
<th>Player 2</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>c</td>
<td>(3, 0, 1)</td>
<td>(1, 1, 0)</td>
</tr>
<tr>
<td></td>
<td>q</td>
<td>(0, 1, 1)</td>
<td>(0, 0, 0)</td>
</tr>
</tbody>
</table>
Chapter 1. Quitting games

After answering all three questions stated at the beginning of this section, let us look at consequences for the search of equilibria: If one searches for \( \varepsilon \)-equilibria it is sufficient to consider profiles where either all players play \textit{continue} all the time, or where at least one player plays \textit{quit} with a positive probability at the beginning – blocks with zero-vectors, i.e. stages where all players play \textit{continue} with certainty, can be ignored. Concerning stationary strategy profiles: In order to find a stationary \( \varepsilon \)-equilibrium, it is sufficient to search for vectors \( p \in [0,1]^N \), such that

\[
\forall n \in \mathcal{N} : \gamma^n(\pi) \geq \gamma^n((\pi^{-n}, c)) - \varepsilon \quad \land \quad \gamma^n(\pi) \geq \gamma^n((\pi^{-n}, q)) - \varepsilon
\]

holds.

The next chapter is dedicated to existence theorems for equilibria in quitting games.
Chapter 2.

Equilibria in Quitting games

In this chapter, we present important results referring to the existence of \((\varepsilon-\text{)}\text{-equilibria}\) in quitting games. The first one was proved by Solan and Vieille in [36]. They showed that every symmetric quitting games possesses a Nash-equilibrium (see Section 2.1). In Section 2.2 we make a short excursion to dominant respectively dominated strategies. After defining, what this means in the context of quitting games, we show that if at least one player exists, who has quit as dominant strategy, then the quitting game has a Nash-equilibrium. Section 2.3 plays a major role in this chapter. We present again a generalized version of a theorem by Solan and Vieille published in [36]. They showed that for all quitting games with \(r_{\{n\}}^n = 1\) for all \(n \in \mathcal{N}\) and \(r_S^n \leq 1\) for all \(n \in \mathcal{N}\) and every \(S\) such that \(n \in S\), a cyclic subgame \(\varepsilon\)-equilibrium exists. We extend this to \(r_{\{n\}}^n \geq -\varepsilon\) where \(\varepsilon\) is specified in the theorem. The proof of this is divided into three parts. We mention the first two parts and give a short (sketch of the) proof for it, in order to figure out, where the requirements to the game structure are needed. For the last part we use a version of the proof by Solan and Vieille, with even better estimates for the resulting equilibrium.

We end this chapter with Section 2.4., where we state some equivalent formulations for the existence of (cyclic) approximate equilibria by Robert Simon published in [31].

2.1. Symmetric quitting games

Symmetric quitting games are defined as follows:

**Definition 2.1.1 (Symmetric quitting game).** Let \(G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})})\) be a given quitting game. Then \(G\) is called symmetric, if and only if two families of numbers \((\alpha_k)_{k \in \mathcal{N}} \subset \mathbb{R}\) and \((\beta_k)_{k \in \mathcal{N}} \subset \mathbb{R}\) exist, such that

\[
r_S^n = \begin{cases} 
\alpha_{|S|} & \text{for } n \in S \\
\beta_{|S|} & \text{for } n \notin S
\end{cases}
\]

where \(\beta_N := 0\).
Chapter 2. Equilibria in Quitting games

That means in a symmetric quitting game, the payoff to a player depends only on the size of \( S \) and whether a player belongs to \( S \) or not.

Solan and Vieille stated in [36]:

**Proposition 2.1.2.** Every symmetric quitting game has a pure, stationary (0-)equilibrium.

The proof is also stated in [36]. We only mention how the equilibria are constructed with respect to the payoff families \((\alpha_k)_{k \in \mathbb{N}}\) and \((\beta_k)_{k \in \mathbb{N}}\) from a given symmetric quitting game \( G \). One has to distinguish three cases:

1. \( \alpha_1 \leq 0 \):
   - The strategy profile \( \pi = (\pi^1, \ldots, \pi^N)^T \), where \( \pi^n = c \) for all \( n \in \mathcal{N} \) is an equilibrium in \( G \).

2. \( \exists k \in \{1, \ldots, N - 1\} : \alpha_k+1 \leq \beta_k \land \alpha_k \geq \beta_k-1, \) where \( \beta_0 := 0 \):
   - The strategy profile \( \pi = (\pi^1, \ldots, \pi^N)^T \), where \( \pi^1 = \ldots = \pi^k = q \) and \( \pi^{k+1} = \ldots = \pi^N = c \) is a stationary equilibrium in \( G \).

3. Neither 1. nor 2. hold:
   - The strategy profile \( \pi = (\pi^1, \ldots, \pi^N)^T \), where \( \pi^1 = \ldots = \pi^N = q \) is a stationary equilibrium in \( G \).

As a consequence of the proof, one can construct an easy algorithm which determines an equilibrium (or all equilibria) for a given symmetric quitting game. Such an algorithm will be stated later in Chapter 4. For completeness, an algorithm, which checks, whether a given quitting game is symmetric or not, is also mentioned.

### 2.2. Quitting games with dominant strategies

In this section, we consider so-called dominant respectively dominated strategies and show, how they are related to \((\varepsilon-)\)equilibria in quitting games. But first the definition:

**Definition 2.2.1** (weakly) dominant/-dominated strategy). Let \( G \) be a given quitting game and \( \bar{\pi}^n \in \Pi^n \) a strategy for player \( n \in \mathcal{N} \). Then \( \bar{\pi}^n \) is called

- (weakly) dominant \( \iff \forall \pi \in \Pi : \gamma^n((\pi^{-n}, \bar{\pi}^n)) \geq \gamma^n(\pi) \)
- (weakly) dominated \( \iff \forall \pi \in \Pi : \gamma^n((\pi^{-n}, \bar{\pi}^n)) \leq \gamma^n(\pi) \)

Consider the quitting game \( G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})}) \). Assume that all players \( n \in \mathcal{N} \) own a (weakly) dominant strategy \( \bar{\pi}^n \), then the strategy profile \( \bar{\pi} := (\bar{\pi}^1, \ldots, \bar{\pi}^N)^T \) is obviously a Nash-equilibrium in \( G \).
2.2. Quitting games with dominant strategies

Proposition 2.2.2. Let $G$ be a given quitting game and $\pi^m$ a dominant strategy for player $m$, $m \in \mathcal{N}$. Then either $\pi^m$ is pure and stationary, or there exists at least another dominant strategy $\hat{\pi}^m$ for player $m$, which is pure and stationary.

Proof. Because $\pi^m$ is dominant for all $\pi \in \Pi$, 

$$\gamma^n((\pi^{-n}, \pi^n)) \geq \gamma^n(\pi) \quad (2.1)$$

holds.

Take $\pi = (c, c, \ldots)$. Formula (2.1) implies 

$$\gamma^n((\pi^{-n}, \pi^n)) = \sum_{k \in \mathbb{N}} \bar{p}_k^m \cdot r_{\{m\}}^m \cdot \prod_{i=1}^{k-1} (1 - \bar{p}_i^m) \geq \max\{r_{\{m\}}^m, 0\}. \quad (2.2)$$

Now choose $\pi = (p_1, p_2, \ldots)$ such that $p_1$ is pure and that additionally at least one player $n \neq m$ plays quit with certainty, i.e. $p_1 = \{0, 1\}^N$ and $p_1^m = 1$. Because $n$ plays quit with certainty we could observe the one-step game $\Gamma_2$ instead of the quitting game and have for player $m$

$$\gamma^n((\pi^{-m}, \pi^m)) = \gamma^n_2((p_1^{-m}, \bar{p}_1^m)) = \bar{p}_1^m \cdot r_{S \cup \{m\}}^m + (1 - \bar{p}_1^m) \cdot r_{S}^m,$$

where $S = \{k \in \mathcal{N} \setminus \{m\} \mid p_k = 1\}$, observe that $\pi$ is pure. This and inequality (2.1) lead to 

$$\bar{p}_1^m \cdot r_{S \cup \{m\}}^m + (1 - \bar{p}_1^m) \cdot r_{S}^m \geq r_{S \cup \{m\}}^m \quad (2.3)$$

for $p^m = 1$ or 

$$\bar{p}_1^m \cdot r_{S \cup \{m\}}^m + (1 - \bar{p}_1^m) \cdot r_{S}^m \geq r_{S}^m \quad (2.4)$$

for $p^m = 0$.

Finally we obtain with (2.2), (2.3) and (2.4) three cases depending on the value of $\bar{p}_1^m$:

1. $\bar{p}_1^m \in (0, 1)$ \(\implies\) $r_{S \cup \{m\}}^m = r_{S}^m$ for all $S \in \mathcal{P}(\mathcal{N} \setminus \{m\})$ with $|S| = 1$ and $r_{\{m\}}^m \geq 0$

2. $\bar{p}_1^m = 0$ \(\implies\) $r_{S \cup \{m\}}^m \leq r_{S}^m$ for all $S \in \mathcal{P}(\mathcal{N} \setminus \{m\})$

3. $\bar{p}_1^m = 1$ \(\implies\) $r_{S \cup \{m\}}^m \geq r_{S}^m$ for all $S \in \mathcal{P}(\mathcal{N} \setminus \{m\})$

So the value of $\pi^m$ gives us statements about the payoff structure from $m$. In the first case the payoffs to player $m$ are independent from her behavior, as long as some other player plays quit with certainty, otherwise she would prefer to quit alone, especially if $r_{\{m\}}^m > 0$. This implies that playing quit with certainty all the time is weakly dominant as well. In the second case, the structure of the payoff family is that way, that player $m$ always gets a higher (or equal) payoff, if she plays continue with certainty, which makes playing always continue for sure dominant for $m$. The third case is similar to the second one, with quit instead of continue.

Observe, that the implications of the three cases are time-independent. \(\square\)
For completeness, we define strongly dominant respectively dominated strategies as well, where we slightly modify this definition by concerning the corresponding one-step game.

**Definition 2.2.3** (strongly dominant/dominated strategy). Let $G$ be a given quitting game and $\tilde{\pi}^n = (p^n, p^n, \ldots) \in \Pi^n$ a strategy for player $n \in N$. Then $\tilde{\pi}^n$ is called

- strongly dominant : $\iff \forall p \in [0, 1]^N, p^n \neq \bar{p}^n : \gamma^n_{\tilde{G}}((p^{-n}, \bar{p}^n)) > \gamma^n_{\tilde{G}}(p)$
- strongly dominated : $\iff \forall p \in [0, 1]^N, p^n \neq \bar{p}^n : \gamma^n_{\tilde{G}}((p^{-n}, \bar{p}^n)) < \gamma^n_{\tilde{G}}(p)$

**Remark 2.2.4.**
1. The modification is motivated by the fact that, in the case of an equivalent definition to weakly dominant resp. dominated strategies, only continuing all the time would be a possible strongly dominant strategy (cf. (2.2)).
2. Let $G$ be a given quitting game and $\tilde{\pi}^m$ a dominant strategy for player $m, m \in N$. Then $\tilde{\pi}^m$ is pure (and stationary by definition). The proof is similar to the proof of Proposition 2.2.2, where we have “>” resp. “<” instead of “≥” resp. “≤”. Furthermore this implies that, if playing continue all the time is a strongly dominant strategy for a player, then playing quit all the time is a strongly dominated strategy for her and vice versa.
3. Assume that $\pi$ is a Nash-equilibrium in $G$ and $\tilde{\pi}^m$ is a strongly dominant strategy for player $m$, then $\pi^m = \tilde{\pi}^m$. This holds for all Nash-equilibria in $G$.

**Proposition 2.2.5.** Let $G$ be a given quitting game and $\tilde{\pi}^m = (\bar{p}_1^m, \bar{p}_2^m, \ldots)$ a (weakly) dominant strategy for player $m$, where $\bar{p}_1^m = 1$. Then $G$ has at least one $(0\text{-})$-equilibrium.

*Proof.* W.l.o.g. we assume that $\tilde{\pi}^N$ with $\bar{p}_1^N = 1$ is a dominant strategy for player $N$. For player $N$, nothing is to prove as long as she is playing according to $\tilde{\pi}^m$ in the equilibrium profile. Consider the players $n \neq N$. $N$ plays quit with certainty in the first stage implies

$$\gamma^n(\pi) = \gamma^n_{\tilde{G}}(p_1)$$

or

$$\gamma^n((\pi^{-n}, \tilde{\pi}^n)) = \gamma^n_{\tilde{G}}((p_1^{-n}, \bar{p}_1^n)), \quad \tilde{\pi}^n = (\bar{p}_1^n, \bar{p}_2^n, \ldots) \in \Pi^n,$$

respectively, for all $\pi = (p_1, p_2, \ldots)$, where $p_i^N = \bar{p}_i^N, i \in \mathbb{N}$. This means, that the players $n$ are situated in a one-step game given by $\Gamma_v = (\hat{N} := \{1, \ldots, N - 1\}, (\hat{\pi}_{\hat{s}})_{\hat{s} \in p(\hat{s})}, v)$, where $\hat{\pi}_{\hat{s}} := r_{\hat{s} \cup \{N\}}^n$ and $v_n := r_n^n$ for $n \in \hat{N}$. As mentioned in Theorem 1.2.15, every one-step game has got an $(0\text{-})$-equilibrium. Let $\hat{p}$ be such an equilibrium in the one-step game $\Gamma_v$, then all $\pi = (p_1, p_2, \ldots)$ with $p_1^{-N} = \hat{p}, p_1^m = 1$ and $p_i \in [0, 1]^N$ for all $i = 2, 3, \ldots$ are $(0\text{-})$-equilibria in $G$. \qed

**Remark 2.2.6.** The resulting equilibrium from Proposition 2.2.5 is an instant equilibrium.

We come back to strictly dominant strategies later in Chapter 3, where we use them – if at least one player has such a strategy – for a game reduction.


2.3. Quitting games in general

This section deals with an important theorem by Solan and Vieille stated in [36]. Simon for example used that theorem to prove existence predications for quitting games in general (see [31] page 18, Theorem 3).

We quote the theorem from [36]:

**Theorem 2.3.1.** Let be \( \varepsilon > 0 \). Every quitting game \( G \) that satisfies the following has a cyclic subgame \( \varepsilon \)-equilibrium:

1. \( r_{\{n\}}^n = 1 \) for every \( n \in \mathcal{N} \);
2. \( r_S^n \leq 1 \) for every \( n \in \mathcal{N} \) and every \( S \) such that \( n \in S \).

The first condition in this theorem is achievable by scaling the family of payoff vectors, supposed that \( r_{\{n\}}^n \) is positive for all players \( n \in \mathcal{N} \). The second condition is the more restrictive one and implies that the players would prefer to \textit{quit} alone instead of quitting together with anyone else. Observe that the payoff to the players, if they are not in the quitting coalition, is not restricted.

**Remark 2.3.2.** Two-player quitting games, with the structure from Theorem 2.3.3, i.e. \( r_{\{n\}}^n = 1 \) and \( r_{\{1\}}^1, r_{\{2\}}^2, r_{\{1,2\}}^n \in (-\infty, 1] \) for \( n = 1, 2 \), always have an instant Nash-equilibrium. The argumentation is the following: Consider the two-player quitting game

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>continue</td>
<td>( (b, 1) )</td>
</tr>
<tr>
<td>quit</td>
<td>( (d_1, d_2) )</td>
</tr>
</tbody>
</table>

where \( a, b \in \mathbb{R} \) and \( d_1, d_2 \in (-\infty, 1] \). We observe three cases:

1. \( b \leq d_1 \land a \leq d_2 \): Then playing \textit{quit} all the time is a dominant strategy for player one and Proposition 2.2.5 is applicable.
2. \( b > d_1 \): Consider the profile \( \pi = (p, p, \ldots) \) with \( p = (c, q)^T \). For player one,
   \[
   \gamma^1((\pi^{-1}, \tilde{\pi}^1)) = \tilde{p}_1^1 \cdot d_1 + (1 - \tilde{p}_1^1) \cdot b \leq b = \gamma^1(p),
   \]
   holds for all \( \tilde{\pi}^1 = (\tilde{p}_1^1, \tilde{p}_2^1, \ldots) \in [0, 1]^\mathcal{N} \). \( \pi \) is a stationary profile and for player two,
   \[
   \gamma^2((\pi^{-2}, c)) = 0 \quad \text{and} \quad \gamma^2((\pi^{-2}, q)) = 1 = \gamma^2(\pi)
   \]
   hold, which imply with Lemma 1.4.8 that \( \pi \) is an instant Nash-equilibrium for \( G \).
3. \( a > d_2 \) is analogous to 2.
Chapter 2. Equilibria in Quitting games

Here we prove the following theorem, which allows that the payoff $r^n_{(n)}$ may also be zero respectively negative.

**Theorem 2.3.3.** Let $G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})})$ be a given quitting game and $\varepsilon > 0$. If

1. $r^n_{(n)} \geq -\varepsilon$ for all players $n \in \mathcal{N}$ and
2. $r^n_{(n)} \geq r^n_S$ for all $S \in \mathcal{P}(\mathcal{N})$ with $n \in S$

hold, then $G$ has a (cyclic subgame) $\tilde{\varepsilon}$-equilibrium, where

$$\tilde{\varepsilon} := \max \{ \varepsilon + 2\varepsilon r_{max}(\varepsilon + \varepsilon^{b-a} + \varepsilon^{a} + 4 \cdot \varepsilon^{b}), 2\varepsilon^{1-b-d} + 4\varepsilon^{1-b} + 15\varepsilon_{r_{max}} \cdot \varepsilon^{a} + \varepsilon \}$$

with $a, b, d \in (0, 1)$ such that $b > a$, $d > a$ and $(1 - \varepsilon^{a})^{1/d} \leq \varepsilon$.

**Remark 2.3.4.** Let $l \in \mathbb{N}$ be the length of the cyclic subgame profile mentioned in the previous theorem, then we have:

$$\tilde{\varepsilon} := \begin{cases} \eta_{l} := \eta_{l}(a, b) := \varepsilon + 2\varepsilon r_{max}(\varepsilon + \varepsilon^{b-a} + \varepsilon^{a} + 4 \cdot \varepsilon^{b}) & \text{for } l = 1 \\ \eta_{l} := \eta_{l}(a, b, d) := 2\varepsilon^{1-b-d} + 4\varepsilon^{1-b} + 15\varepsilon_{r_{max}} \cdot \varepsilon^{a} + \varepsilon & \text{for } l > 1 \end{cases}$$

The main structure of the proof is similar to the proof of Theorem 2.3.1 by Solan and Vieille and divided into three parts, represented by the Propositions 2.3.5, 2.3.7 and 2.3.8. But before quoting the mentioned propositions, another notation is needed.

Let $V$ be a subset of $\mathbb{R}^N$ and $\varepsilon \in [0, 1)$ be given. $\psi_\varepsilon$ denotes a correspondence\footnote{Let $K$ and $L$ be sets. A correspondence $J : K \rightharpoonup L$ is a subset $J$ of $K \times L$ and one defines for all $k \in K$: $J(k) := \{ l | (k, l) \in J \}$. It is not assumed a priori that $J(k) \neq \emptyset$ for all or any particular $k \in K$.} from $V$ into $V$, where

$$\psi_\varepsilon(v) := \psi_{\varepsilon, V}(v) \overset{(2.5)}{=} \{ \gamma_v(p) | \gamma_v(p) \in V, p \in [0, 1]^N, p \text{ is } 2\varepsilon r_{max}-\text{perfect}, \emptyset(p, \emptyset) \leq 1 - \varepsilon \}.$$ 

So $\psi_\varepsilon$ maps a vector $v \in V$ to a set of expected payoffs (in $V$) from a one-step game $\Gamma_v$, where the payoffs are generated by profiles which are $2\varepsilon r_{max}$-perfect in $\Gamma_v$ and where the game quits with a probability of at least $\varepsilon$.

**Proposition 2.3.5.** Let $\varepsilon \in [0, 1)$ be given. Define

$$V := \{ v \in [-2\varepsilon r_{max}, 2\varepsilon r_{max}]^N | \exists n \in \mathcal{N} : v^n \leq r^n_{(n)} \}.$$ 

Assume that for every $v \in V$, an equilibrium $p$ in $\Gamma_v$ exists, such that either

a) $p = c$ (that means all players choose continue) or
b) $p \neq c$ and $\gamma^n_v(p) \leq r^n_{(n)}$ for some $n \in \mathcal{N}$ with $p^n > 0$.

Then $\psi_\varepsilon(v) \neq \emptyset$ for all $v \in V$. 

\footnote{Let $K$ and $L$ be sets. A correspondence $J : K \rightharpoonup L$ is a subset $J$ of $K \times L$ and one defines for all $k \in K$: $J(k) := \{ l | (k, l) \in J \}$. It is not assumed a priori that $J(k) \neq \emptyset$ for all or any particular $k \in K$.}
Remark 2.3.6. 1. In comparison with Proposition 2.2 from [36], the set \( V \) is modified. The requirement \( \exists n \in \mathcal{N} : v^n \leq 1 \) from [36] is replaced by \( \exists n \in \mathcal{N} : v^n \leq r^n_{\{n\}} \) and additionally to that, instead of \( \gamma^n_v(p) \leq 1 \) now \( \gamma^n_v(p) \leq r^n_{\{n\}} \) has to be assumed.

2. Let \( G \) be a quitting game, which satisfies the second requirement of Theorem 2.3.3, and \( \Gamma_v \) a corresponding one-step game to \( G \) with \( v \in V \). Then a strategy profile \( p \in [0,1]^N \) exists, which satisfies the assumptions of Proposition 2.3.5.

Proof. That every one-step game has got an equilibrium is a well-known fact. Let \( p \) be a Nash-equilibrium in \( \Gamma_v \), then either \( p = c \) or \( p \neq c \) holds. For the second case, it remains to show, that a player \( n \) with \( p^n > 0 \) and \( \gamma^n_v(p) \leq r^n_{\{n\}} \) exists. Since \( p \neq (0,\ldots,0)^T \), at least one player \( n \in \mathcal{N} \) exists with \( p^n > 0 \).

Assume that \( p^n = 1 \). With assumption 2 of Theorem 2.3.3, one obtains

\[
\gamma^n_v(p) = \sum_{S \in \mathcal{P}(\mathcal{N})} g(p,S) r^n_S \\
= \sum_{S \subseteq \mathcal{N} \setminus \{n\}} g(p,S \cup \{n\}) r^n_{S \cup \{n\}} \\
\leq \sum_{S \subseteq \mathcal{N} \setminus \{n\}} g(p,S \cup \{n\}) r^n_{\{n\}} \\
\leq r^n_{\{n\}}.
\]

Consider the case \( p^n \in (0,1) \). Because \( p \) is an equilibrium in \( \Gamma_v \), \( p \) is also (0-)perfect in \( \Gamma_v \). This implies \( \gamma^n_v(p^{-n},1) - \gamma^n_v(p^{-n},0) = 0 \), i.e. \( \gamma^n((p^{-n}, b)) \) is constant in \( b \in [0,1] \), and equally to the case above,

\[
\gamma^n_v(p) = \gamma^n_v(p^{-n},1) \leq r^n_{\{n\}}
\]

follows.

3. The assumptions of Proposition 2.3.5 are chosen in such a way that the estimate in Remark 1.2.19 holds for \( p = c \) or, in the case \( p \neq c \), to guarantee that \( p^n \in (0,1] \).

Before giving details on the proof of the last one, we quote the other two propositions which are used to proof the theorem.

Proposition 2.3.7. \(^2\) Let \( \varepsilon \in (0,1) \) be given. If a compact set \( V \) exists such that \( \psi_v(v) \neq \emptyset \) for all \( v \in V \), then a cyclic profile \( \pi = (p_i)_{i \in \mathbb{N}} \) in \( G \) exists, such that for every \( i \in \mathbb{N} \):

1. \( G \) is terminating under each subgame profile \( \pi_i = (p_j)_{i \leq j \in \mathbb{N}} \) induced by \( \pi \); and
2. \( p_i \) is \( (2r_{\max} + 2)\varepsilon \)-perfect in \( \Gamma_{\gamma(\pi_{i+1})} \).

\(^2\)See [36] Proposition 2.3. p. 270.
Chapter 2. Equilibria in Quitting games

Proposition 2.3.8. Let $G = (\mathcal{N}, (r_S)_{S \in P(N)})$ be a given quitting game, $\varepsilon \in (0,1)$ and $r^*_n \geq -\eta_\varepsilon$ for all $n \in \mathbb{N}$ and $S \in \mathcal{N}$ with $n \in S$. Furthermore let $\pi = (p_i)_{i \in \mathbb{N}}$ be a strategy profile in $G$ and assume that the following properties hold for every $i$:

1. $G$ is terminating under every subgame strategy profile $\pi_i$.
2. $p_i$ is $\varepsilon$-perfect in $\Gamma_{\gamma(\pi_{i+1})}$.

Then either $\pi$ is a subgame $\tilde{\eta}_\varepsilon$-equilibrium, or there exists a stationary $\eta_\varepsilon$-equilibrium in $G$, where

$$\tilde{\eta}_\varepsilon := \tilde{\eta}_\varepsilon(a, b, d) := 2\varepsilon^{1-b-d} + 4\varepsilon^{1-b} + 15r_{\max} \cdot \varepsilon^a + \varepsilon \quad \text{and}$$

$$\eta_\varepsilon := \eta_\varepsilon(a, b) := \varepsilon + 2r_{\max} \cdot (\varepsilon + \varepsilon^{b-a} + \varepsilon^a + 4 \cdot \varepsilon^b),$$

with $a, b, d \in (0,1)$, $b > a$, $d > a$ and $(1 - \varepsilon^a)^{1/e^d} \leq \varepsilon$.

The next three sections are dedicated to the three propositions, which are used to prove Theorem 2.3.3. First we prove Proposition 2.3.5 at length by using the known results about one-step games and their strategy profiles.

Proof of Proposition 2.3.5

Proof. Let $v \in V = \{ v \in [-2r_{\max}, 2r_{\max}]^N \mid \exists n \in \mathcal{N} : v_n \leq r^*_n \}$ and $\varepsilon \in [0,1)$ be arbitrary but fixed. The aim is to construct a $\hat{p} \in [0,1]^N$ with $\gamma_v(\hat{p}) \in \psi_\varepsilon(v)$ (see (2.5)).

It holds $\psi_\varepsilon(v) \neq \emptyset$, if a strategy profile $p \in [0,1]^N$ in $\Gamma_v$ exists, such that

(i) $\gamma_v(p) \in V$,
(ii) $p$ is $2\varepsilon r_{\max}$-perfect in $\Gamma_v$ and
(iii) $\varrho(p, \emptyset) \leq 1 - \varepsilon$.

Now let $p$ be an (0-)equilibrium in $\Gamma_v$ that satisfies the assumptions of the proposition. If $p = c$, then $\gamma_v(p) = v \in V$, i.e. (i). For $p \neq c$, (i) holds by assumption b) of the proposition. Furthermore Corollary 1.2.14 (see p. 17) implies that $p$ is even 0-perfect in $\Gamma_v$, thus (ii) is fulfilled. For $p = c$, we have $\varrho(p, \emptyset) = 1 \not\geq 1 - \varepsilon$ and for $p \neq c$, $\varrho(p, \emptyset) < 1$ but not necessarily $\varrho(p, \emptyset) \leq 1 - \varepsilon$, if $\varepsilon > 0$. Otherwise with $\varepsilon = 0$, (iii) follows.

So assume that $\varepsilon \in (0,1)$. Based on the given strategy profile $p$, a new profile $\hat{p} \in [0,1]^N$ like in Theorem 1.2.17 (see p. 19) for the one-step game $\Gamma_v$ will be constructed, such that $\varrho(\hat{p}, \emptyset) \leq 1 - \varepsilon$ holds. Afterwards we show that this profile $\hat{p}$ satisfies (i) and (ii).
2.3. Quitting games in general

Construction of \( \hat{p} \): Fix a player \( m \) with \( v^m = r^m_{\{m\}} \), if \( p = (0, \ldots, 0)^T \), or with \( p^m > 0 \) and \( \gamma^m_v(p) \leq r^m_{\{m\}} \) otherwise\(^3\) and set \( \hat{p} \) like in (1.13) (see p. 19), i.e.

\[
\hat{p}^n = \begin{cases} 
(1 - \varepsilon) \cdot p^n + \varepsilon & \text{for } n = m \\
 p^n & \text{for } n \neq m
\end{cases}.
\]

Theorem 1.2.17 1. implies

\[ g(\hat{p}, \emptyset) = (1 - \varepsilon) \cdot g(p, \emptyset) \leq 1 - \varepsilon. \]

Now we prove that (i) and (ii) from the beginning of this proof hold for \( \hat{p} \):

(i) We have to show that \( \gamma_v(\hat{p}) \in V = \{ v \in [-2r_{\text{max}}, 2r_{\text{max}}]^N \mid \exists n \in \mathcal{N} : v^n \leq r^n_{\{m\}} \} \).

Remark 1.2.6 (see p. 13) and \( \delta_s \leq 2r_{\text{max}} \) imply \( \gamma_v(\hat{p}) \in [-2r_{\text{max}}, 2r_{\text{max}}]^N \). Consider the chosen player \( m \). Because \( p \) is an equilibrium in \( \Gamma_v \),

\[
\gamma^m_v(p) \geq \gamma^m_v((p^{-m}, \hat{p}^m)) = \gamma^m_v(\hat{p}),
\]

and with the requirements to the choice of player \( m \),

\[
r^m_{\{m\}} \geq \gamma^m_v(p) \geq \gamma^m_v(\hat{p}) \tag{2.6}
\]

follows, thus \( \gamma_v(\hat{p}) \in V \).

(ii) It is to prove that \( \hat{p} \) is \( 2\varepsilon r_{\text{max}} \)-perfect in \( \Gamma_v \).

Case 1: \( p^m \in (0, 1] \). With Theorem 1.2.17 4. and \( p \) (0--)perfect in \( \Gamma_v \), it follows immediately, that \( \hat{p} \) is \( 2\varepsilon r_{\text{max}} \)-perfect in \( \Gamma_v \).

Case 2: \( p^m = 0 \) (i.e. \( p = c \)).

(a) Consider player \( m \). With the choice of \( m \),

\[
\gamma^m_v((\hat{p}^{-m}, 1)) - \gamma^m_v((\hat{p}^{-m}, 0)) = \gamma^m_v((p^{-m}, 1)) - \gamma^m_v((p^{-m}, 0)) = r^m_{\{m\}} - v^m = 0
\]

follows.

(b) Consider player \( n \neq m \). With Theorem 1.2.17 4. and Remark 1.2.19, we have

\[
\forall n \in \mathcal{N} \setminus \{m\} : \begin{cases} 
\gamma^v_n((\hat{p}^{-n}, 1)) - \gamma^v_n((\hat{p}^{-n}, 0)) \leq 2\varepsilon r_{\text{max}} & \text{for } \hat{p}^n = 0 \\
\gamma^v_n((\hat{p}^{-n}, 1)) - \gamma^v_n((\hat{p}^{-n}, 0)) \in [-2\varepsilon r_{\text{max}}, 2\varepsilon r_{\text{max}}] & \text{for } \hat{p}^n \in (0, 1) \\
\gamma^v_n((\hat{p}^{-n}, 1)) - \gamma^v_n((\hat{p}^{-n}, 0)) \geq -2\varepsilon r_{\text{max}} & \text{for } \hat{p}^n = 1
\end{cases}
\]

(a) and (b) imply that \( \hat{p} \) is \( 2\varepsilon r_{\text{max}} \)-perfect in the one-step game \( \Gamma_v \).

Finally \( \hat{p} \in \psi_e(v) \) and \( \psi_e(v) \neq \emptyset \) for all \( v \in V \). \( \square \)

Remark 2.3.9. The second requirement from Theorem 2.3.3 was not needed for this proof.

\(^3\)Let \( p = (0, \ldots, 0)^T \) be the given equilibrium in \( \Gamma_v \). Since \( v \in V \), a player \( m \in \mathcal{N} \) with \( v^m \leq r^m_{\{m\}} \) exists. Because \( p \) is an equilibrium in \( \Gamma_v \), \( v^m = r^m_{\{m\}} \) follows.
Proof of Proposition 2.3.7

We want to give a short sketch of the proof for Proposition 2.3.7, which is stated in [36] more detailed. It is mainly based on a partitioning of the compact set $V$.

In the first step, the set $V$ is partitioned into a family $(V_j)_{j \in K}$, $K \subset \mathbb{N}$, of disjoint sets with a diameter smaller than $\varepsilon^2$. Secondly one determines the image-set $\hat{V}_j := \bigcup_{v \in V_j} \psi_\varepsilon(v)$ of each $V_j$ and chooses one representing element $\hat{v}_j$ for each set $\hat{V}_j$, $j \in K$. Because $|K| < \infty$, a cycle of representing elements $(\bar{v}_1,\ldots,\bar{v}_{k+1})$, $k \in \mathbb{N}$, exists, such that

(i) $\bar{v}_1 = \bar{v}_{k+1}$,

(ii) $\bar{v}_j \in \{ \hat{v}_j \mid j \in K \}$ and

(iii) $\bar{v}_{i+1} \in \hat{V}_m = \bigcup_{v \in V_m} \psi_\varepsilon(v)$, where $\bar{v}_i$ is the representing element of the set $V_m$, i.e $\bar{v}_i \in V_m$.

Using (iii), one determines a sequence $(\bar{p}_1,\ldots,\bar{p}_k)$, such that $\bar{v}_{i+1} = \gamma_v(\bar{p}_i)$, where $v \in V_m$ with $\bar{v}_i \in V_m$ and $\bar{p}_i$ $2\varepsilon r_{\max}$-perfect, $\varrho(\bar{p}_i,\emptyset) \leq 1 - \varepsilon$. With the help of this sequence $(\bar{p}_1,\ldots,\bar{p}_k)$, a profile $\pi$ is constructed, which fulfills the postulated properties. The proof of the last fact is similar to the proof of Lemma 3.6.1, which is helpful in the context of detecting cyclic $\varepsilon$-equilibria in quitting games.

Proof of Proposition 2.3.8

Introductory remark

The Proposition 2.3.8 is a generalized version of Proposition 2.4 by Solan and Vieille stated in [36] on page 270, where this proposition reads:

Let $\pi = (p_i)_{i \in \mathbb{N}}$ be a profile in $G$. Assume that the following properties hold for every $i$:

1. $G$ is terminating under each subgame profile $\pi_i$ induced by $\pi$; and
2. $p_i$ is $\varepsilon$-perfect in $\Gamma_{\gamma(\pi_{i+1})}$.

Then either $\pi$ is a subgame $\varepsilon^{1/6}$-equilibrium, or there is a stationary $\varepsilon^{1/6}$-equilibrium.

The proof in the original paper of Solan and Vieille provides two more exact estimates for the $\varepsilon^{1/6}$-equilibria. Namely either $\pi$ is a subgame $\zeta_\varepsilon$-equilibrium, or a stationary $\zeta_\varepsilon$-equilibrium in $G$, where

\begin{align}
\zeta_\varepsilon := \zeta_\varepsilon(a, b, d) &:= 4\varepsilon^{1-b-d} + 2r_{\max} \cdot \varepsilon^a (7N + 1) + \varepsilon \quad \text{and} \\
\zeta_\varepsilon &:= \zeta_\varepsilon(a, b) := 2r_{\max} \cdot (\varepsilon + \varepsilon^{b-a} + \varepsilon^a + N \varepsilon^b) + \varepsilon,
\end{align}
with $a, b, d \in (0, 1)$, $b > a$, $d > a$, $(1 - \varepsilon^a)^{1/\varepsilon} \leq \varepsilon$.

The estimates (2.7) and (2.8) could be improved for a number of players greater or equal four. Furthermore it figures out, that the proof of the proposition by Solan and Vieille needs some more requirements, which are not mentioned in their proposition. One of the most restrictive one is $r^n_{\{n\}} = 1$ for all $n \in \mathcal{N}$.

Here the proof will be shown for the less restrictive assumption $r^n_{\{n\}} \geq -\eta_\varepsilon$ for all $n \in \mathcal{N}$ and $S \in \mathcal{N}$ with $n \in S$ together with the improved estimates, which are even independent of the number of players.

**Proof of Proposition 2.3.8**

The structure of the proof is similar to the proof by Solan and Vieille in [36], but differs in some details, which are commented by remarks or footnotes. Proof structure:

1. The stage set $\mathbb{N}$ is divided into disjoint consecutive blocks of stages.
2. The blocks are classified into two types.
3. Depending on the distribution of the blocks, a stationary $\eta_\varepsilon$-equilibrium will be constructed, or
4. it will be shown, that the given strategy profile is an $\eta_\varepsilon$-equilibrium in $G$ by the consideration of finite quitting games. Here it will be proved:
   a) that deviation from the given strategy to a strategy, where the player plays *quit* with certainty in at least one stage in the finite quitting game, respectively
   b) that deviation from the given strategy to a strategy, where the player plays *continue* all the time in the finite quitting game is not profitable.
   c) Finally, because of the given block structure, one can extend the estimate for the finite quitting games to a global estimate for the whole quitting game.

1. **Partitioning of the stage set $\mathbb{N}$**

First the stages of the game are divided into disjoint consecutive blocks of stages.

Let $\pi = (p_i)_{i \in \mathbb{N}}$ be a given strategy profile and $\tau$ the stopping time of the game with respect to $\pi$ (cf. (1.2)). Fix $a, b, d \in (0, 1)$ and $\varepsilon > 0$ sufficiently small, such that the following properties hold\(^4\)

1. $b > a$, $d > a$ and
2. $(1 - \varepsilon^a)^{1/\varepsilon} \leq \varepsilon$.

\(^4\)For every $\varepsilon \in (0, 1)$ a combination of $a$, $b$ and $d$ exists, for which the stated properties hold. Solan and Vieille stated in [36] a rule for choosing $a$, $b$ and $d$, which not necessarily leads to a successful choice. It is better to choose first $d \in (0, 1)$, then calculate $a$ as $a = \ln(1 - \varepsilon^d) / \ln \varepsilon$ and finally choose $b \in (0, 1)$, with $b > a$. 

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Define \((B_k)_{k \in \mathbb{N}}\) as partition of \(\mathbb{N}\), where \(B_k := \{i_k, \ldots, i_{k+1} - 1\}\) with \(i_1 := 1\) and

\[ i_{k+1} := \inf \{ i \in \mathbb{N} \mid i > i_k \land P_\pi(\tau < i | \tau \geq i_k) \geq \varepsilon^a \} \]

for all \(k \in \mathbb{N}\).

\[
\begin{align*}
\mathbb{N} &= \{1, 2, \ldots, i_2 - 1, i_2, i_2 + 1, \ldots, i_3 - 1, \ldots, i_k, i_k + 1, \ldots, i_{k+1} - 1, \ldots\} \\
=:_B_ & 1 \\
\vdots & =:_B_ k \\
\end{align*}
\]

The sets \(B_k, k \in \mathbb{N}\), are defined such that the probability that at least one player plays the action \(\text{quit}\) during the stages \(i_k, \ldots, i_{k+1} - 1\) is greater than or equal to \(\varepsilon^a\), whereas the probability that at least one player plays the action \(\text{quit}\) during the stages \(i_k, \ldots, i_{k+1} - 2\) is strictly lower than \(\varepsilon^a\), i.e.

\[
P_\pi(\tau < i_{k+1} - 1 | \tau \geq i_k) \leq \varepsilon^a \leq P_\pi(\tau < i_{k+1} | \tau \geq i_k) \leq \varepsilon^a
\]

for all \(k \in \mathbb{N}\) respectively

\[
1 - \prod_{j=i_k}^{i_{k+1}-2} \varrho(p_j, \emptyset) < \varepsilon^a \leq 1 - \prod_{j=i_k}^{i_{k+1}-1} \varrho(p_j, \emptyset).
\]

Observe that it may occur that a set \(B_k\) consists of only one stage. In that case

\[
1 - \varrho(p_i, \emptyset) \geq \varepsilon^a
\]

holds.

Since \(\pi_i\) is terminating for all \(i \in \mathbb{N}\) the sets \(B_k\) are finite for all \(k \in \mathbb{N}\).

2. Classification of the blocks in type I and type II

Now for each player the blocks defined in the preceding section are categorized into two types.

**Definition 2.3.10** (Block of type I and type II). Let \(m \in \mathcal{N}\) be an arbitrarily but fixed chosen player. A set \(B_k, k \in \mathbb{N}\), is called block of type I for player \(m\), if

\[
P_\pi(\tau < i_{k+1} | \tau \geq i_k) \geq \varepsilon^b
\]

holds, otherwise \(B_k\) is called block of type II.
2.3. Quitting games in general

Example for a block classification:

\[ \mathbb{N} = \{1, 2, \ldots, i_2 - 1, i_2, i_2 + 1, \ldots, i_3 - 1, i_3, i_3 + 1, \ldots, i_3 + 1 - 1, \ldots \} \]

\[ = B_1, \text{ type I} \quad = B_2, \text{ type I} \quad = B_3, \text{ type II} \]

Remark 2.3.11.

1. Observe that \( \varepsilon^b < \varepsilon^a \) and that the probability measure \( P_{(s-m, c)} \) is used in the definition.

2. Interpretation of Definition 2.3.10:
   - Type I: The termination of the game in this block is not essentially caused by player \( m \). That means, the probability that the action \( \text{quit} \) is played by the other players is at least \( \varepsilon^b \), \( b > a \), \( b \in (0, 1) \).
   - Type II: The termination of the game in this block is essentially due to player \( m \). That means, the probability that at least one player \( n \in \mathbb{N} \setminus \{m\} \) is playing the action \( \text{quit} \) is less than \( \varepsilon^b \), \( b > a \), \( b \in \mathbb{R} \).

Definition 2.3.12 (sparse blocks). 5 Let a player \( m \in \mathcal{N} \) be given. The blocks of type I for player \( m \) are sparse, if there exist at least \( \lceil \frac{1}{\varepsilon^d} \rceil + 1 \) consecutive blocks of type II. Otherwise the blocks of type I are regularly scattered 6.

Blocks of type I are sparse for a player \( m \in \mathcal{N} \):

\[ (B_k)_{k \in \mathbb{N}} = (B_1, B_2, B_3, \ldots, B_j, \ldots, B_{\lceil 1/\varepsilon^d \rceil + j}, \ldots) \]

\[ \text{type I} \quad \text{blocks of type II} \quad \text{type I} \]

In the next step, two different cases are considered:

1. Blocks of type I are sparse for at least one player \( m \in \mathcal{N} \): We assume, that the first \( \lceil 1/\varepsilon^d \rceil + 1 \) blocks are from type II for player \( m \in \mathcal{N} \) and show that

\[ \tilde{\pi} := (\tilde{p}_i)_{i \in \mathbb{N}} \text{ with } \tilde{p}_i^n := \begin{cases} 0 & \text{for } n \in \mathcal{N} \setminus \{m\} \\ \max(\varepsilon, p_i^m) & \text{for } n = m \end{cases} \]

is a stationary \( (2r_{\max}(\varepsilon + \varepsilon^{b-a} + \varepsilon^a + 4\varepsilon^b) + \varepsilon) \)-equilibrium in \( G \).

2. Blocks of type I are regularly scattered: We prove that \( \pi \) is a \( (2\varepsilon^{1-b-d} + 4\varepsilon^{1-b} + 16r_{\max}\varepsilon^a + \varepsilon) \)-equilibrium in \( G \).

---

5 This definition differs from the definition in [36] in the requirement that there must be \( \lceil 1/\varepsilon^d \rceil + 1 \) consecutive blocks of type II. Solan and Vieille defined the sparse property for \( \lceil 1/\varepsilon^d \rceil \) consecutive type II blocks. Later on we will see, why it is necessary to have one more.

6 \([z] := n, n \in \mathbb{Z} \) such that \( n - 1 < z \leq n \)
3. Case 1.: Blocks of type I are sparse

**Assumption 2.3.13.** Assume that the blocks of type I are sparse for at least one player \( m \in \mathcal{N} \) under the given strategy profile \( \pi \). Let the first block of type I has an index \( l \geq 1/\varepsilon^d + 2, l \in \mathbb{N} \).

**Remark 2.3.14.** Let \( \pi \) be a strategy profile which fulfills the requirements of Proposition 2.3.8. If the first block of type I does not have the index \( l \geq 1/\varepsilon^d + 2 \), one has to shift the strategy profile. In detail: Let \( j \) denote the index of the first of the (at least) \([1/\varepsilon^d] + 1\) consecutive blocks of type II. Then consider now the strategy profile \( \tilde{\pi} := \pi_j \). The requirements of Proposition 2.3.8 hold for the strategy profile \( \tilde{\pi} \) as well.

Now all blocks \((B_k)_{1 \leq k < l}\) are of type II and

\[
P_{(\pi^{-n,m})}(\tau < i_{k+1}|\tau \geq i_k) < \varepsilon^b\]

holds for \( 1 \leq k < l \).

We prove that \( \tilde{\pi} := (\tilde{p}_i)_{i \in \mathbb{N}} \) with \( \tilde{p}_i := (c^{-m}, \max(\varepsilon, p_i^m)) \) for all \( i \in \mathbb{N} \), i.e.

\[
\tilde{p}_i^n := \begin{cases} 
0 & \text{for } n \in \mathcal{N} \setminus \{m\}, \\
\max(\varepsilon, p_i^m) & \text{if } n = m
\end{cases}, \quad \forall i \in \mathbb{N},
\]

is a stationary \( \eta_{\varepsilon}\)-equilibrium\(^7\) in \( G \), where \( \eta_{\varepsilon} := \varepsilon + 2r_{\max}(\varepsilon + \varepsilon^b - a + \varepsilon^a + 4 \cdot \varepsilon^b) \) and \( r_{\max} = \max \{ |r_S^n| \mid n \in \mathcal{N}, S \in \mathcal{P}(\mathcal{N}) \} \).

We have to show:

\[
\forall n \in \mathcal{N}, \forall \hat{\pi}^n \in \Pi^n : \quad \gamma^n(\hat{\pi}) \geq \gamma^n((\hat{\pi}^{-n}, \hat{\pi}^n)) - \eta_{\varepsilon}. \tag{2.10}
\]

\(^7\)Solan and Vieille stated, that \( \hat{\pi} \) is an \( \varepsilon^{1/\alpha} \)-equilibrium. But this is coupled with several conditions on how to choose \( a, b \) and \( d \). We only use the conditions given in the first step of the proof (see p. 53).
With (1.37) and (1.38), for the expected payoff under $\bar{\pi}$

$$\gamma(\bar{\pi}) = \gamma_{\bar{\pi}}(\bar{p}_1) = \frac{1}{1 - (1 - \bar{p}_1^m)} \cdot \bar{p}_1^m \cdot r_{\{m\}} = r_{\{m\}}$$

holds.

**Consider player** $m$: For all stationary strategies $\tilde{\pi}^m$ for player $m$,

$$\gamma^m((\bar{\pi}^{-m}, \tilde{\pi}^m)) = \begin{cases} 0 & \text{for } \tilde{\pi}^m = c \\ r_{\{m\}} & \text{otherwise,} \end{cases}$$

and since $r_{\{m\}} \geq -\eta$, player $m$ does not profit more than $\eta$ by deviating. This implies that (2.10) is true for player $m^8$.

**Consider the players** $n \in N \setminus \{m\}$: Because $\bar{\pi}$ is a stationary strategy profile, it is sufficient to consider only stationary strategies as alternative strategy. Furthermore, because of the structure from the profile $\bar{\pi}$ and with Remark 1.4.9, (2.10) can be reduced to

$$\gamma^n(\bar{\pi}) \geq \gamma^n((\bar{\pi}^{-n}, q)) - \eta$$

for the players $n \in N \setminus \{m\}$.

Additionally

$$\gamma^m((\bar{\pi}^{-n}, q)) = \gamma_{\bar{\pi}}((\bar{p}_1^{-n}, 1)) = \gamma_{\bar{\pi}}((\bar{p}_1^{-n}, 1)).$$

This implies that proving that $\bar{\pi}$ is an $\eta$-equilibrium in $G$ for the players $n \in N \setminus \{m\}$ is equivalent to showing that $\bar{p}_1$ is an $\eta$-equilibrium in $\Gamma_{\gamma(\bar{\pi})}$. Based on the fact, that $p_1$ is $\varepsilon$-perfect in $\Gamma_{\gamma(\bar{\pi})}$, which implies that $p_1$ is an $\varepsilon$-equilibrium in $\Gamma_{\gamma(\bar{\pi})}$ as well, the proof that $\bar{p}_1$ is an $\eta$-equilibrium in $\Gamma_{\gamma(\bar{\pi})}$ is arranged in 2 steps:

(i) It will be shown, that $\bar{p}_1$ is an $($ $\varepsilon + 8 \cdot \varepsilon^b \cdot r_{\text{max}}$)-equilibrium in $\Gamma_{\gamma(\bar{\pi})}$ for the players

$\text{ }(n \in N \setminus \{m\})^9$.

---

8Here we need the first requirement of Theorem 2.3.3 mentioned also in Proposition 2.3.8.
9Solan and Vieille stated in [36], that $\bar{p}_1$ is an $(\varepsilon + 2 \cdot r_{\text{max}} \cdot N\varepsilon^b)$-equilibrium in $\Gamma_{\gamma(\bar{\pi})}$, where $N$ is the number of players. Though they make no distinction of the cases $\bar{p}_1^m = p_1^m$ and $\bar{p}_1^m = \varepsilon$. Nevertheless the estimate proven here is better than the one by Solan and Vieille since the number of players is at least four.

---
Based on (i), \( \bar{p}_1 \) is an \((\varepsilon + 2r_{\text{max}}(\varepsilon + \varepsilon^b - a + 4 \cdot \varepsilon^b))\)-equilibrium in \( \Gamma_\gamma(\pi) \) for the players \( n \in N \setminus \{ m \} \) \(^{10}\).

\[ \begin{align*}
\text{Overview:} \\
p_1 &\ \varepsilon\text{-perfect in } \Gamma_\gamma(\pi) \ \overset{T1.2.12}{\implies} \ p_1 \ \varepsilon\text{-equilibrium in } \Gamma_\gamma(\pi) \\
&\implies (i) \ \bar{p}_1 \ (\varepsilon + 8\varepsilon^b r_{\text{max}})\text{-equilibrium in } \Gamma_\gamma(\pi) \\
&\implies (ii) \ \bar{p}_1 \ (\varepsilon + 2r_{\text{max}}(\varepsilon + \varepsilon^b - a + 4 \cdot \varepsilon^b))\text{-equilibrium in } \Gamma_\gamma(\pi) \\
&\implies (2.11) \ \bar{\pi}_1 \ (\varepsilon + 2r_{\text{max}}(\varepsilon + \varepsilon^b - a + 4 \cdot \varepsilon^b))\text{-equilibrium in } \Gamma_\gamma(\pi) \\
\end{align*} \]

(i): In order to prove this, one needs several estimates, which should be stated and proven first.

**Estimate 2.3.15.**

\[ \left\| \gamma_\gamma(\pi_2)(p_1) - \gamma_\gamma(\pi_2)(\bar{p}_1) \right\| \leq \begin{cases} 2 \cdot \varepsilon^b \cdot r_{\text{max}} & \text{for } p_{1m}^m = p_{1m}^m \notag \\ 4 \cdot \varepsilon^b \cdot r_{\text{max}} & \text{for } \bar{p}_{1m}^m = \varepsilon \end{cases} \]

**Proof.** Consider the expected payoff to the players in the one-step game \( \Gamma_\gamma(\pi_2) \) under the strategy profile \( p_1 \). Using that the expected payoff in a one-step game is linear in the strategy of one player, here \( p_{1m}^m \), yields (see additionally [13] or Appendix (A.1), (A.4) and (A.5))

\[ \begin{align*}
\gamma_\gamma(\pi_2)(p_1) \\
= p_{1m}^m \left( \varrho((p_{1m}^m, 1), \{ m \}) r_{\{ m \}} + \sum_{S \in \mathcal{P}(N \setminus \{ m \}) \setminus \{ \varnothing \}} \varrho((p_{1m}^m, 1), S \cup \{ m \}) r_{S \cup \{ m \}} \right) \\
+ (1 - p_{1m}^m) \left( \varrho((p_{1m}^m, 0), \varnothing) \gamma_\gamma(\pi_2) + \sum_{S \in \mathcal{P}(N \setminus \{ m \}) \setminus \{ \varnothing \}} \varrho((p_{1m}^m, 0), S) r_S \right). \tag{2.12} \\
\end{align*} \]

On the other hand, the expected payoff to the players in the one-step game \( \Gamma_\gamma(\pi_2) \) under the strategy profile \( \bar{p}_1 \) is given by

\[ \gamma_\gamma(\pi_2)(\bar{p}_1) = \bar{p}_{1m}^m \cdot r_{\{ m \}} + (1 - \bar{p}_{1m}^m) \cdot \gamma(\pi_2). \tag{2.13} \]

\(^{10}\)In [36], \( \varepsilon + 2r_{\text{max}}(\varepsilon + \varepsilon^b - a + 4 \cdot \varepsilon^b) \) is bounded by \( \varepsilon^{1/6} \), using several assumptions on \( a, b \) and \( d \).
With (2.12) and (2.13), we get
\[ \left\| \gamma_{\gamma(\pi_2)}(p_1) - \gamma_{\gamma(\pi_2)}(\bar{p}_1) \right\| \\
= \left\| p_1^m \left( \psi\left((p_1^{-m}, 1), \{m\}\right) r_{\{m\}} \right) + \sum_{S \in \mathcal{P}(\mathcal{N} \setminus \{m\}) \setminus \{\emptyset\}} \psi\left((p_1^{-m}, 1), S \cup \{m\}\right) r_{S \cup \{m\}} \right\| \\
+ \left( 1 - p_1^m \right) \left( \psi\left((p_1^{-m}, 0), \emptyset\right) \gamma(\pi_2) + \sum_{S \in \mathcal{P}(\mathcal{N} \setminus \{m\}) \setminus \{\emptyset\}} \psi\left((p_1^{-m}, 0), S\right) r_S \right) \\
- \bar{p}_1^m \cdot r_{\{m\}} - \left( 1 - \bar{p}_1^m \right) \cdot \gamma(\pi_2) \right\| \\
\leq \left\| \left( p_1^m \psi\left((p_1^{-m}, 1), \{m\}\right) - \bar{p}_1^m \right) r_{\{m\}} \right\| + \left\| \left( 1 - p_1^m \right) \ \sum_{S \in \mathcal{P}(\mathcal{N} \setminus \{m\}) \setminus \{\emptyset\}} \psi\left((p_1^{-m}, 0), S\right) r_S \right\| \\
+ \left\| \left( 1 - p_1^m \right) \cdot \psi\left((p_1^{-m}, 0), \emptyset\right) - \left( 1 - \bar{p}_1^m \right) \right\| \cdot \gamma(\pi_2) \right\|.
\]

\[ \sum_{S \in \mathcal{P}(\mathcal{N} \setminus \{m\}) \setminus \{\emptyset\}} \psi\left((p_1^{-m}, 1), S \cup \{m\}\right) \] as well as \[ \sum_{S \in \mathcal{P}(\mathcal{N} \setminus \{m\}) \setminus \{\emptyset\}} \psi\left((p_1^{-m}, 0), S\right) \] denote the probability that at least one player \( n \in \mathcal{N} \setminus \{m\} \) plays the action \( \text{quit} \). Since \( \mathcal{B}_1 \) is a block of type II (cf. Assumption 2.3.13),

\[ \sum_{S \in \mathcal{P}(\mathcal{N} \setminus \{m\}) \setminus \{\emptyset\}} \psi\left((p_1^{-m}, 1), S \cup \{m\}\right) = \sum_{S \in \mathcal{P}(\mathcal{N} \setminus \{m\}) \setminus \{\emptyset\}} \psi\left((p_1^{-m}, 0), S\right) \leq \varepsilon^b \quad (2.14) \]

holds.

\[ r_S \leq 1 \cdot r_{\text{max}} \] for all \( S \in \mathcal{P}(\mathcal{N}) \) implies

\[ \left\| \gamma_{\gamma(\pi_2)}(p_1) - \gamma_{\gamma(\pi_2)}(\bar{p}_1) \right\| \]
\[ \leq \left\| \left( p_1^m \cdot \psi\left((p_1^{-m}, 1), \{m\}\right) - \bar{p}_1^m \right) \cdot r_{\{m\}} \right\| + (1 - p_1^m) \cdot \varepsilon^b \cdot r_{\text{max}} + p_1^m \cdot \varepsilon^b \cdot r_{\text{max}} \]
\[ + \left\| \left( 1 - p_1^m \right) \cdot \psi\left((p_1^{-m}, 0), \emptyset\right) - \left( 1 - \bar{p}_1^m \right) \right\| \cdot \gamma(\pi_2) \right\| \\
\leq \left( p_1^m \cdot \psi\left((p_1^{-m}, 1), \{m\}\right) - \bar{p}_1^m \right) \cdot r_{\text{max}} + \varepsilon^b \cdot r_{\text{max}} \]
\[ + \left( 1 - p_1^m \right) \cdot \psi\left((p_1^{-m}, 0), \emptyset\right) - \left( 1 - \bar{p}_1^m \right) \right\| \cdot r_{\text{max}}. \]

Due to the term \((*)\), one has to observe two different cases in order to continue the evaluation. In the first case, \( \bar{p}_1^m = \max(\varepsilon, p_1^m) = p_1^m \), and in the second, \( \bar{p}_1 = \varepsilon = \max(\varepsilon, p_1^m) \).
Chapter 2. Equilibria in Quitting games

a) \( \bar{p}^m_1 = p^m_1 \). We have

\[
\| \gamma_{\pi_2}(p_1) - \gamma_{\pi_2}(\bar{p}_1) \| \\
\leq r_{\max} \left( p^m_1 \left( \phi((p^m_1, 1), \{m\}) - 1 \right) + \varepsilon^b + \left( 1 - p^m_1 \right) \left( \phi((p^m_1, 0), \emptyset) - 1 \right) \right) \\
= r_{\max} \left( p^m_1 \left( 1 - \phi((p^m_1, 1), \{m\}) \right) + \varepsilon^b + (1 - p^m_1) \left( 1 - \phi((p^m_1, 0), \emptyset) \right) \right) \\
< \varepsilon^b \text{ (cf. (2.14))} \\
< 2 \cdot \varepsilon^b \cdot r_{\max}.
\]

b) \( \bar{p}_1 = \varepsilon \) implies

\[
\varepsilon \geq p^m_1 \geq p^m_1 \cdot \phi((p^m_1, 1), \{m\}),
\]

thus

\[
\left| p^m_1 \cdot \phi((p^m_1, 1), \{m\}) - \bar{p}_1^m \right| = \left| p^m_1 \cdot \phi((p^m_1, 1), \{m\}) - \varepsilon \right| \leq \varepsilon < \varepsilon^b.
\]

Observe that \( \varepsilon \in (0, 1) \) and \( \varepsilon < 1 \).

Furthermore

\[
\left| (1 - p^m_1) \cdot \phi((p^m_1, 0), \emptyset) - (1 - \bar{p}_1^m) \right| \\
= \left| (1 - p^m_1) \cdot \phi((p^m_1, 0), \emptyset) - (1 - \varepsilon) \right| \\
= \left| (1 - p^m_1) \cdot \left( 1 - \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{m\})\setminus\{\emptyset\}} \phi((p^m_1, 0), S) \right) - (1 - \varepsilon) \right| \\
= \left| (1 - p^m_1) - (1 - p^m_1) \cdot \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{m\})\setminus\{\emptyset\}} \phi((p^m_1, 0), S) \right| \\
= \left| \varepsilon - p^m_1 - (1 - p^m_1) \cdot \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{m\})\setminus\{\emptyset\}} \phi((p^m_1, 0), S) \right| \\
\leq \varepsilon - p^m_1 \left( 1 - \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{m\})\setminus\{\emptyset\}} \phi((p^m_1, 0), S) \right) + \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{m\})\setminus\{\emptyset\}} \phi((p^m_1, 0), S) \leq \varepsilon + \varepsilon^b < 2 \cdot \varepsilon^b
\]

holds. Finally we have \( \| \gamma_{\pi_2}(\bar{p}_1) - \gamma_{\pi_2}(p_1) \| \leq 4 \cdot \varepsilon^b \cdot r_{\max} \).

Estimate 2.3.16.

\[
\forall n \in \mathcal{N}\setminus\{m\} : \left| \gamma^n_{\pi_2}((p^n_1, 1)) - \gamma^n_{\pi_2}((\bar{p}^n_1, 1)) \right| \leq \begin{cases} 2\varepsilon^b \cdot r_{\max} & \text{for } \bar{p}^m_1 = p^m_1 \\ 4\varepsilon^b \cdot r_{\max} & \text{for } \bar{p}^m_1 = \varepsilon \end{cases}
\]
2.3. Quitting Games in General

Proof. Choose a player \( n \neq m \) arbitrary but fixed. In the profile \((p_1^{-n}, 1)\), this player plays the action *quit* with certainty. Thus for the expected payoff to \( n \) in the one-step game \( \Gamma_{\gamma(\pi_2)} \) under the strategy profile \((p_1^{-n}, 1)\),

\[
\gamma_n^{\gamma(\pi_2)}((p_1^{-n}, 1)) = \sum_{S \in \mathcal{P}(N)} \varrho(((p_1^{-n}, 1), S) \cdot r^n_S
\]

holds. Using the linearity of the expected payoff \( \gamma_n^{\gamma(\pi_2)}((p_1^{-n}, 1)) \) in the strategy \( p_1^m \), one obtains (cf. [13] or Appendix (A.3), (A.4) and (A.5))

\[
\begin{align*}
\gamma_n^{\gamma(\pi_2)}((p_1^{-n}, 1)) &= p_1^m \cdot \left( \varrho\left(((p_1^{-n}, 1)^{-m}, 1), \{m, n\}\right) \cdot r^n_{\{m, n\}} \right. \\
&\quad + \sum_{S \in \mathcal{P}(N \setminus \{m, n\}) \setminus \emptyset} \varrho\left(((p_1^{-n}, 1)^{-m}, 1), S \cup \{m, n\}\right) \cdot r^n_{S \cup \{m, n\}} \\
&\quad + (1 - p_1^m) \cdot \left( \varrho\left(((p_1^{-n}, 1)^{-m}, 1), \emptyset\right) \cdot r^n_{\emptyset} \right) \\
&\quad + \sum_{S \in \mathcal{P}(N \setminus \{m, n\}) \setminus \emptyset} \varrho\left(((p_1^{-n}, 1)^{-m}, 1), S \cup \emptyset\right) \cdot r^n_{S \cup \emptyset} \right). 
\end{align*}
\]

On the other hand, for the expected payoff \( \gamma_n^{\gamma(\pi_2)}((\bar{p}_1^{-n}, 1)) \) to the player \( n \) in the one-step game \( \Gamma_{\gamma(\pi_2)} \) under the strategy profile \((\bar{p}_1^{-n}, 1)\), we have

\[
\gamma_n^{\gamma(\pi_2)}((\bar{p}_1^{-n}, 1)) = \bar{p}_1^m \cdot r^n_{\{m, n\}} + (1 - \bar{p}_1^m) \cdot r^n_{\{n\}}.
\]

Observe, that all players except player \( m \) play the action *continue* in the strategy profile \( \bar{p}_1 \) with certainty.

This yields

\[
\begin{align*}
&|\gamma_n^{\gamma(\pi_2)}((p_1^{-n}, 1)) - \gamma_n^{\gamma(\pi_2)}((\bar{p}_1^{-n}, 1))| \\
&= \left| \left( p_1^m \varrho\left(((p_1^{-n}, 1)^{-m}, 1), \{m, n\}\right) - \bar{p}_1^m \right) \cdot r^n_{\{m, n\}} \right. \\
&\quad + \left( (1 - p_1^m) \varrho\left(((p_1^{-n}, 1)^{-m}, 0), \{n\}\right) - (1 - \bar{p}_1^m) \right) \cdot r^n_{\{n\}} \\
&\quad + p_1^m \sum_{S \in \mathcal{P}(N \setminus \{m, n\}) \setminus \emptyset} \varrho\left(((p_1^{-n}, 1)^{-m}, 1), S \cup \{m, n\}\right) \cdot r^n_{S \cup \{m, n\}} \\
&\quad + (1 - p_1^m) \sum_{S \in \mathcal{P}(N \setminus \{m, n\}) \setminus \emptyset} \varrho\left(((p_1^{-n}, 1)^{-m}, 0), S \cup \emptyset\right) \cdot r^n_{S \cup \emptyset} \left|.
\end{align*}
\]
Chapter 2. Equilibria in Quitting games

Since $r^n_{\{m,n\}}$, $r^n_m$, $r^n_{S \cup \{m,n\}}$ and $r^n_{\overline{S} \cup \{n\}}$ are lower or equal $r_{\text{max}}$, we obtain

$$|\gamma^n_{\gamma_{(\pi_2)}}((p_1^{-n}, 1)) - \gamma^n_{\gamma_{(\pi_2)}}((\bar{p}_1^{-n}, 1))|$$

$$\leq r_{\text{max}} \cdot \left( |p^m_{1} g(((p_1^{-n}, 1)^{-m}, 1), \{m, n\}) - \bar{p}^m_{1}| + |(1 - p^m_{1}) g(((p_1^{-n}, 1)^{-m}, 0), \{n\}) - (1 - \bar{p}^m_{1})| + p^m_{1} \sum_{S \in P(N \setminus \{m,n\}) \setminus \{\emptyset\}} g(((p_1^{-n}, 1)^{-m}, 1), S \cup \{m, n\}) \right)$$

Both sums $\sum_{S \in P(N \setminus \{m,n\}) \setminus \{\emptyset\}} g(((p_1^{-n}, 1)^{-m}, 1), S \cup \{m, n\})$ on the one hand and $\sum_{S \in P(N \setminus \{m,n\}) \setminus \{\emptyset\}} g(((p_1^{-n}, 1)^{-m}, 0), S \cup \{n\})$ on the other describe the probability that at least one player except the players $m$ and $n$ plays the action quit. Since $B_1$ is a block of type II,

$$\sum_{S \in P(N \setminus \{m,n\}) \setminus \{\emptyset\}} g(((p_1^{-n}, 1)^{-m}, 1), S \cup \{m, n\}) < \varepsilon^b$$

(2.15)

respectively

$$\sum_{S \in P(N \setminus \{m,n\}) \setminus \{\emptyset\}} g(((p_1^{-n}, 1)^{-m}, 0), S \cup \{n\}) < \varepsilon^b$$

holds, which implies

$$|\gamma^n_{\gamma_{(\pi_2)}}((p_1^{-n}, 1)) - \gamma^n_{\gamma_{(\pi_2)}}((\bar{p}_1^{-n}, 1))|$$

$$\leq r_{\text{max}} \cdot \left( |p^m_{1} g(((p_1^{-n}, 1)^{-m}, 1), \{m, n\}) - \bar{p}^m_{1}| + |(1 - p^m_{1}) g(((p_1^{-n}, 1)^{-m}, 0), \{n\}) - (1 - \bar{p}^m_{1})| + \varepsilon^b \right).$$

In order to continue the evaluation, we have to distinguish again between the two cases $\bar{p}^m_{1} = p^m_{1}$ and $\bar{p}^m_{1} = \varepsilon$.

a) $\bar{p}^m_{1} = p^m_{1}$. We have

$$|\bar{p}^m_{1} g(((p_1^{-n}, 1)^{-m}, 1), \{m, n\}) - \bar{p}^m_{1}| = |p^m_{1} g(((p_1^{-n}, 1)^{-m}, 1), \{m, n\}) - p^m_{1}|$$

$$= p^m_{1} \cdot |g(((p_1^{-n}, 1)^{-m}, 1), \{m, n\}) - 1|$$

$$< p^m_{1} \cdot \varepsilon^b$$
respectively
\[
(1 - \bar{p}_1^m) \varrho(((p_1^{-n}1)^{m-1},0),\{n\}) - (1 - \bar{p}_1^m)
= (1 - \bar{p}_1^m) \varrho(((p_1^{-n}1)^{m-1},0),\{n\}) - 1
< (1 - \bar{p}_1^m) \cdot \varepsilon^b
\]
and finally
\[
|\gamma_\epsilon^n(\bar{p}_1^{-n},1)) - \gamma_{\epsilon}^n(\bar{p}_1^{-n},1))| < (p_1^m \cdot \varepsilon^b + (1 - p_1^m) \cdot \varepsilon^b \cdot r_{\max}) = 2 \cdot \varepsilon^b \cdot r_{\max}.
\]
b) \(\bar{p}_1^m = \varepsilon\). Thus
\[
|p_1^m \varrho(((p_1^{-n}1)^{m-1},1),\{m,n\}) - \bar{p}_1^m| = |p_1^m \varrho(((p_1^{-n}1)^{m-1},1),\{m,n\}) - \varepsilon|
\]
holds, and since \(\varepsilon \geq \bar{p}_1^m \geq p_1^m \cdot \varrho(((p_1^{-n}1)^{m-1},1),\{m,n\}),
\[
|\bar{p}_1^m \varrho(((p_1^{-n}1)^{m-1},1),\{m,n\}) - \bar{p}_1^m| \leq \varepsilon \leq \varepsilon^b
\]
follows. Furthermore
\[
(1 - \bar{p}_1^m) \varrho(((p_1^{-n}1)^{m-1},0),\{n\}) - (1 - \bar{p}_1^m)
= |(1 - \bar{p}_1^m) \varrho(((p_1^{-n}1)^{m-1},0),\{n\}) - 1 - \varepsilon|
= |(1 - \bar{p}_1^m)(1 - \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{m,n\}) \setminus \emptyset} \varrho(((p_1^{-n}1)^{m-1},0),S \cup \{n\})) - (1 - \varepsilon)|
\]
Expansion and transposing leads to
\[
(1 - \bar{p}_1^m) \varrho(((p_1^{-n}1)^{m-1},0),\{n\}) - (1 - \bar{p}_1^m)
= |\varepsilon - \bar{p}_1^m - \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{m,n\}) \setminus \emptyset} \varrho(((p_1^{-n}1)^{m-1},0),S \cup \{n\})|
\leq |\varepsilon - \bar{p}_1^m(1 - \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{m,n\}) \setminus \emptyset} \varrho(((p_1^{-n}1)^{m-1},0),S \cup \{n\})|
= |\varepsilon - \varrho(((p_1^{-n}1)^{m-1},0),\{n\})|
+ \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{m,n\}) \setminus \emptyset} \varrho(((p_1^{-n}1)^{m-1},0),S \cup \{n\})
< \varepsilon^b \ (\text{cf. (2.15)})
\]
Finally we get
\[
|\gamma_{\gamma_\epsilon}^n(\bar{p}_1^{-n},1)) - \gamma_{\gamma_\epsilon}^n(\bar{p}_1^{-n},1))| \leq 4 \cdot \varepsilon^b \cdot r_{\max}.
\]
Overview: Estimates in Step (i):

Estimate 2.3.15: \[ \| \gamma_{\pi_2}(p_1) - \gamma_{\pi_2}(\bar{p}_1) \| \leq \begin{cases} 2 \cdot \varepsilon^b \cdot r_{\text{max}} & \text{for } \bar{p}_1^m = p_1^m \\ 4 \cdot \varepsilon^b \cdot r_{\text{max}} & \text{for } \bar{p}_1^m = \varepsilon \end{cases} \]

Estimate 2.3.16:

\[ \forall n \in \mathcal{N} \setminus \{m\}: \ |\gamma_n^{\pi_2}((p_1^{-n}, 1)) - \gamma_n^{\pi_2}((\bar{p}_1^{-n}, 1))| \leq \begin{cases} 2 \cdot \varepsilon^b \cdot r_{\text{max}} & \text{for } \bar{p}_1^m = p_1^m \\ 4 \cdot \varepsilon^b \cdot r_{\text{max}} & \text{for } \bar{p}_1^m = \varepsilon \end{cases} \]

With these estimates, we show that (i) holds. Because of Estimate 2.3.15,

\[ \gamma_n^{\pi_2}(\bar{p}_1) \geq \gamma_n^{\pi_2}((p_1^{-n}, 1)) - 4 \cdot \varepsilon^b \cdot r_{\text{max}} \]

for all \( n \in \mathcal{N} \setminus \{m\} \).

Since \( p_1 \) is an \( \varepsilon \)-equilibrium in \( \Gamma_{\pi_2} \), one has

\[ \gamma_n^{\pi_2}(p_1) \geq \gamma_n^{\pi_2}((p_1^{-n}, 1)) - \varepsilon \implies \gamma_n^{\pi_2}(\bar{p}_1) \geq \gamma_n^{\pi_2}((p_1^{-n}, 1)) - \varepsilon - 4 \varepsilon^b \cdot r_{\text{max}} \]

for all \( n \in \mathcal{N} \setminus \{m\} \). Using Estimate 2.3.16, one finally obtains

\[ \gamma_n^{\pi_2}(\bar{p}_1) \geq \gamma_n^{\pi_2}((\bar{p}_1^{-n}, 1)) - 4 \varepsilon^b \cdot r_{\text{max}} - \varepsilon - 4 \varepsilon^b \cdot r_{\text{max}} \]

\[ \geq \gamma_n^{\pi_2}((\bar{p}_1^{-n}, 1)) - 8 \varepsilon^b \cdot r_{\text{max}} - \varepsilon \]

\[ (2.16) \]

for all \( n \in \mathcal{N} \setminus \{m\} \), which finishes the proof of (i).

(ii): Again the needed estimates are stated first.

**Estimate 2.3.17.** For the probability, that the game terminates during the first \( i_l - 1 \) stages only because of player \( m \), the inequality:

\[ \mathbb{P}_\pi \left( \{ Y^m_\tau = 1 \} \cap \{ Y^n_\tau = 0 \ \forall n \neq m \} \cap \{ \tau < i_l \} \right) \geq 1 - \varepsilon^{b-a} - \varepsilon \]

holds.

**Proof.** By Assumption 2.3.13, the first block of type I has an index \( l > 1/\varepsilon^d + 2 \) and all blocks \( (B_k)_{1 \leq k < l} \) are of type II. The definition of this blocks, i.e.

\[ \mathbb{P}_{(\pi-m,e)}(\tau < i_{k+1} | \tau \geq i_k) < \varepsilon^b \]
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for all $1 \leq k < l$, together with the definition of the sets $B_i, i \in \mathbb{N}$, especially (2.9) (see p. 54) yield

$$P_\pi(\exists n \neq m : Y_\tau^n = 1 \mid \tau \in B_k)$$

$$= \frac{P_\pi(\{\exists n \neq m : Y_\tau^n = 1\} \cap \{\tau \in B_k\})}{P_\pi(\tau \in B_k)}$$

$$= \frac{P_\pi(\{\exists n \neq m : Y_\tau^n = 1\} \cap \{\tau < i_{k+1}\} \mid \tau \geq i_k)}{P_\pi(\tau < i_{k+1} \mid \tau \geq i_k)}$$

$$< \frac{\varepsilon^b}{\varepsilon^a} = \varepsilon^{b-a}$$

for all $k \in \{1, \ldots, l - 1\}$ with $P_\pi(\tau \in B_k) \neq 0$. That means the probability that at least one player $n \in \mathcal{N} \setminus \{m\}$ plays the action \textit{quit} under the condition that the game terminates in the block $B_k$, $k \in \{1, \ldots, l - 1\}$, is bounded by $\varepsilon^{b-a}$.

This implies

$$P_\pi(\{Y_m^\tau = 1\} \cap \{Y_n^\tau = 0 \ \forall n \neq m\} \mid \tau \in B_k)$$

$$= 1 - P_\pi(\exists n \neq m : Y_\tau^n = 1 \mid \tau \in B_k) - P_\pi( Y_\tau = 0 \mid \tau \in B_k)$$

$$\geq 1 - \varepsilon^{b-a}.$$  \hfill (2.17)

The second requirement for the choice of $a$ and $d$, i.e. $(1 - \varepsilon)^{1/\varepsilon^d} \leq \varepsilon$ (cf. p. 53), stated at the beginning of the proof of Proposition 2.3.8, gives that the probability that the game does not terminate before stage $i_l$ is reached with respect to the given strategy profile $\pi$ is bounded by $\varepsilon$, since

$$P_\pi(\tau \geq i_l) = \prod_{k=1}^{l-1} (1 - P_\pi(\tau < i_{k+1} \mid \tau \geq i_k)) \leq (1 - \varepsilon^a)^{l-1} \leq (1 - \varepsilon^a)^{1/\varepsilon^d} \leq \varepsilon.$$

Using $\bigcup_{k=1}^{l-1} \{\omega \in \Omega \mid \tau(\omega) \in B_k\} = \{\omega \in \Omega \mid \tau(\omega) < i_l\}$, we obtain for the probability that player $m$ quits alone during the first $l - 1$ blocks:

$$P_\pi(\{Y_m^\tau = 1\} \cap \{Y_n^\tau = 0 \ \forall n \neq m\} \cap \{\tau < i_l\})$$

$$= \sum_{k=1}^{l-1} P_\pi(\{Y_m^\tau = 1\} \cap \{Y_n^\tau = 0 \ \forall n \neq m\} \cap \{\tau \in B_k\})$$

$$= \sum_{k=1}^{l-1} P_\pi(\{Y_m^\tau = 1\} \cap \{Y_n^\tau = 0 \ \forall n \neq m\} \mid \tau \in B_k) \cdot P_\pi(\tau \in B_k).$$
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Inequality (2.17) is leading to

\[
P_\pi\left( \{Y^m_\tau = 1\} \cap \{Y^n_\tau = 0 \ \forall n \neq m\} \cap \{\tau < \hat{u}\} \right)
\geq \sum_{k=1}^{l-1} (1 - \varepsilon^{b-a}) \cdot P_\pi(\tau \in B_k)
\]

\[
= (1 - \varepsilon^{b-a}) \cdot \sum_{k=1}^{l-1} P_\pi(\tau \in B_k)
= \left(1 - \varepsilon^{b-a}\right) \cdot \frac{1}{\text{\textbraceleft P_\pi(\tau < \hat{u}) \text\textbraceright}}
\geq (1 - \varepsilon^{b-a}) \cdot (1 - \varepsilon)
= 1 - \varepsilon^{b-a} - \varepsilon + \varepsilon^{1+b-a}
\geq 1 - \varepsilon^{b-a} - \varepsilon.
\]

\[\Box\]

Remark 2.3.18. Assume, that at least one block \(B_k, k \in \{1, \ldots, l - 1\}\) exists, such that \(P_\pi(\tau \in B_k) = 0\). Because of the definition of the blocks, this case only occurs if a player plays \textit{quit} with certainty in at least one of the previous blocks. But the blocks \(B_k, k \in \{1, \ldots, l - 1\}\), are of type II, which implies, that only player \(m\) may \textit{quit} with certainty. Let \(B_i\) be the first block with \(P_\pi(\tau \in B_i) = 1\). Then

\[
P_\pi\left( \{Y^m_\tau = 1\} \cap \{Y^n_\tau = 0 \ \forall n \neq m\} \cap \{\tau < \hat{u}\} \right)
= \sum_{k=1}^{l-1} P_\pi\left( \{Y^m_\tau = 1\} \cap \{Y^n_\tau = 0 \ \forall n \neq m\} \cap \{\tau \in B_k\} \right)
\geq P_\pi\left( \{Y^m_\tau = 1\} \cap \{Y^n_\tau = 0 \ \forall n \neq m\} \cap \{\tau \in B_i\} \right)
= P_\pi\left( \{Y^m_\tau = 1\} \cap \{Y^n_\tau = 0 \ \forall n \neq m\} \right| \tau \in B_i) \cdot P_\pi(\tau \in B_i)
> 1 - \varepsilon^b
\geq 1 - \varepsilon^{b-a}.
\]

Estimate 2.3.19.

\[\|\gamma(\pi) - r_{(m)}\| \leq 2 \cdot r_{\max}(\varepsilon + \varepsilon^{b-a})\]

Proof. We have

\[
\|\gamma(\pi) - r_{(m)}\| = \left\|E_\pi(r(Y_\tau)1_{\{\tau < +\infty\}}) - r_{(m)}\right\|
= \left\|E_\pi(r(Y_\tau)1_{\{\tau < +\infty\}}) - r_{(m)}\right\|
= \max_{n \in \mathbb{N}} \left|E_\pi(r^n(Y_\tau)1_{\{\tau < +\infty\}}) - r^n_{(m)}\right|.
\]
2.3. Quitting games in general

Since \( r^n \) as well as \( r^n_s \) are bounded by \( r_{\text{max}} \) for all \( n \in \mathcal{N} \) and \( S \in \mathcal{P}(\mathcal{N}) \), the random variables \( r^n(Y_\tau_I \mathbf{1}_{\{\tau < +\infty\}} - r^n_{\{m\}} \) on \( (\Omega, \mathcal{A}, \mathbb{P}_\pi) \) are bounded by \( 2 \cdot r_{\text{max}} \), \( n \in \mathcal{N} \).

Set

\[
D := \{ Y^m_\tau = 1 \} \cap \{ Y^n_\tau = 0 \forall n \neq m \} \cap \{ \tau < i \}.
\]

This is the set of all \( \omega \in \Omega \), for which the game terminates during the first \( i - 1 \) stages and the quitting coalition consists only of player \( m \). Then

\[
\|\gamma(\pi) - r_{\{m\}}\| \leq \max_{n \in \mathcal{N}} \left( 2 \cdot r_{\text{max}} \cdot \mathbb{P}_\pi(D^c) + \sup_{\omega \in D} |r^n(Y_{\tau(\omega)}(\omega))\mathbf{1}_{\{\tau < +\infty\}}(\omega) - r^n_{\{m\}}| \right),
\]

where \( D^c := \Omega \setminus D \) (cf. Lemma A.0.1). \( D \) contains only those \( \omega \in \Omega \), where player \( m \) quits alone until the stage \( i \) is reached, i.e.

\[
r(Y_{\tau(\omega)}(\omega))\mathbf{1}_{\{\tau(\omega) < +\infty\}}(\omega) = r_{\{m\}} \quad \forall \omega \in D.
\]

Application of Estimate 2.3.17 finally leads to

\[
\|\gamma(\pi) - r_{\{m\}}\| \leq \max_{n \in \mathcal{N}} \left( 2 \cdot r_{\text{max}} \cdot (1 - \mathbb{P}_\pi(D)) + \sup_{\omega \in D} |r^n_{\{m\}} - r^n_{\{m\}}| \right)
\leq 2 \cdot r_{\text{max}} \cdot (1 - \varepsilon - \varepsilon^{b-a})
= 2 \cdot r_{\text{max}} (\varepsilon + \varepsilon^{b-a}).
\]

**Estimate 2.3.20.** If \( 1 - \varrho(p_1, \emptyset) < \varepsilon^a \) then \( \|\gamma(\pi) - \gamma(\pi_2)\| \leq 2 \cdot r_{\text{max}} \varepsilon^a \).

**Proof.**

\[
\|\gamma(\pi) - \gamma(\pi_2)\| = \left\| \sum_{S \in \mathcal{P}(\mathcal{N})} \varrho(p_1, S) \cdot r_S + \varrho(p_1, \emptyset) \cdot \gamma(\pi_2) - \gamma(\pi_2) \right\|
\leq \left\| \sum_{S \in \mathcal{P}(\mathcal{N})} \varrho(p_1, S) \cdot r_S \right\| + \left\| (\varrho(p_1, \emptyset) - 1) \cdot \gamma(\pi_2) \right\|
\leq \sum_{S \in \mathcal{P}(\mathcal{N}) \setminus \{\emptyset\}} \varrho(p_1, S) \cdot r_{\text{max}} + \varrho(p_1, \emptyset) - 1 \cdot r_{\text{max}}
= (1 - \varrho(p_1, \emptyset)) r_{\text{max}} + (1 - \varrho(p_1, \emptyset)) r_{\text{max}}
< \varepsilon^a + \varepsilon^a
< 2 \cdot r_{\text{max}} \varepsilon^a.
\]

\[
< 2 \cdot r_{\text{max}} \varepsilon^a.
\]
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Estimate 2.3.21.

\[ \| \gamma(\pi_2) - r_{(m)} \| \leq 2 \cdot r_{max} \cdot (\varepsilon + \varepsilon^{b-a} + \varepsilon^a) \]

Proof. We consider two different cases. In the first case, \((1 - \varrho(p_1, \emptyset)) < \varepsilon^a\) holds, and in the second, \((1 - \varrho(p_1, \emptyset)) \geq \varepsilon^a\). The second case occurs only if \(|B_1| = 1\), i.e. \(B_1 = \{1\}\).

1. \((1 - \varrho(p_1, \emptyset)) < \varepsilon^a\): The Estimates 2.3.19 and 2.3.20 yield

\[
\| \gamma(\pi_2) - r_{(m)} \| = \| \gamma(\pi_2) - \gamma(\pi) + \gamma(\pi) - r_{(m)} \|
\leq \| \gamma(\pi_2) - \gamma(\pi) \| + \| \gamma(\pi) - r_{(m)} \|
\leq 2r_{max}\varepsilon + 2r_{max}(\varepsilon + \varepsilon^{b-a}) = 2r_{max}(\varepsilon^a + \varepsilon + \varepsilon^{b-a}).
\]

2. \((1 - \varrho(p_1, \emptyset)) \geq \varepsilon^a\): We show: \(\| \gamma(\pi_2) - r_{(m)} \| \leq 2r_{max}(\varepsilon + \varepsilon^{b-a})\), which implies Estimate 2.3.21.

The proof is analogous to the proof of Estimate 2.3.17. Observe that because of \((1 - \varrho(p_1, \emptyset)) \geq \varepsilon^a\), the block \(B_1 = \{1\}\), and the block \(B_2\) starts with the index \(i_2 = 2\).

\[
\| \gamma(\pi_2) - r_{(m)} \| = \| \mathbb{E}_{\pi_2} (r(Y_{\tau})I_{\{\tau<+\infty\}}) - r_{(m)} \|
\leq \max_{n \in \mathbb{N}} | \mathbb{E}_{\pi_2} (r^n(Y_{\tau})I_{\{\tau<+\infty\}} - r^n_{(m)}) |
\]

With \(D = \{ Y^m_{\tau} = 1 \} \cap \{ Y^n_{\tau} = 0 \ \forall \ n \neq m \} \cap \{ \tau < i_l \}\) (cf. proof of Estimate 2.3.17),

\[
\| \gamma(\pi_2) - r_{(m)} \|
\leq \max_{n \in \mathbb{N}} \left( 2 \cdot r_{max} \cdot P_{\pi_2}(D^c) + \sup_{\omega \in D} | r^n(Y_{\tau}(\omega))I_{\{\tau<+\infty\}}(\omega) - r^n_{(m)} | \right)
\leq 2 \cdot r_{max} \cdot (1 - P_{\pi_2}(D))
\]

follows (cf. Lemma A.0.1).

Now we calculate respectively estimate the probability of \(D\) under the strategy profile \(\pi_2\). Using that the first block of type I has an index \(l \geq 1/\varepsilon^d + 2\) and with the same arguments as in the proof of Estimate 2.3.17, we obtain\(^{11}\)

\[
P_{\pi_2}(\tau \geq i_l) = \prod_{k=2}^{l-1} \left( 1 - P_{\pi_2}(\tau < i_{k+1}|\tau \geq i_k) \right) \leq (1 - \varepsilon^a)^{l-2} \leq (1 - \varepsilon^a)^{1/\varepsilon^d} \leq \varepsilon
\]

and

\[
P_{\pi_2}(\{ Y^m_{\tau} = 1 \} \cap \{ Y^n_{\tau} = 0 \ \forall \ n \neq m \} \mid \tau \in B_k) > 1 - \varepsilon^{b-a}
\]

\(^{11}\)Observe the requirements to the choice of \(a\) and \(d\) from the beginning (see. 53), i.e. \((1 - \varepsilon^a)^{1/\varepsilon^d} \leq \varepsilon\).

Furthermore we need for this step that we have \([\frac{1}{\pi_2}] + 1\) consecutive blocks of type I.
for all \( k \in \{2, 3, \ldots \} \). This implies
\[
P_{\pi_2}(D) = \sum_{k=2}^{l-1} P_{\pi_2}(\{ Y^m_\tau = 1 \} \cap \{ Y^n_n = 0 \ \forall n \neq m \} \mid \tau \in B_k) \cdot P_{\pi_2}(\tau \in B_k)
\geq (1 - \varepsilon^{b-a}) \cdot (1 - \varepsilon) > 1 - \varepsilon^{b-a} - \varepsilon
\]
and finally
\[
\|\gamma(\pi_2) - r_{\{m\}}\| \leq 2 \cdot r_{\max} \cdot (1 - (1 - \varepsilon^{b-a} - \varepsilon)) = 2 \cdot r_{\max} \cdot (\varepsilon^{b-a} + \varepsilon).
\]

### 2.3. Quitting games in general

Now we are able to prove (ii), i.e. \( \bar{p}_1 \) is an \( \eta \)-equilibrium in \( \Gamma_{\gamma(\bar{\pi})} \) for the players \( n \in \mathcal{N} \setminus \{m\} \). With Estimate 2.3.21,
\[
\|\gamma_{\gamma(\pi_2)}(\bar{p}_1) - \gamma_{\gamma(\bar{\pi})}(\bar{p}_1)\| = \|\bar{p}_1 \cdot r_{\{m\}} + (1 - \bar{p}_1) \cdot \gamma(\pi_2) - \bar{p}_1 \cdot r_{\{m\}} - (1 - \bar{p}_1) \cdot \gamma(\bar{\pi})\|
\geq \|\gamma(\pi_2) - \gamma(\bar{\pi})\| - \|\gamma(\pi_2) - r_{\{m\}}\|\|
\leq 2r_{\max}(\varepsilon + \varepsilon^{b-a} + \varepsilon^a)
\]
follows and hence \( \gamma_{\gamma(\bar{\pi})}(\bar{p}_1) \geq \gamma_{\gamma(\pi_2)}(\bar{p}_1) - 2r_{\max}(\varepsilon + \varepsilon^{b-a} + \varepsilon^a) \cdot \mathbf{1} \). Estimate (2.16) implies
\[
\gamma_{\gamma(\bar{\pi})}^n(\bar{p}_1) \geq \gamma_{\gamma(\pi_2)}^n(\bar{p}_1) - 2r_{\max}(\varepsilon + \varepsilon^{b-a} + \varepsilon^a)
\geq \gamma_{\gamma(\pi_2)}^n((\bar{p}_1^{-n}, 1)) - 8 \cdot r_{\max} \cdot \varepsilon^b - \varepsilon - 2r_{\max}(\varepsilon + \varepsilon^{b-a} + \varepsilon^a)
= \gamma_{\gamma(\bar{\pi})}^n((\bar{p}_1^{-n}, 1)) - \varepsilon - 2r_{\max}(\varepsilon + \varepsilon^{b-a} + \varepsilon^a + 4 \cdot \varepsilon^b)
\]
for all players \( n \in \mathcal{N} \setminus \{m\} \).

With \( \eta = \varepsilon + 2r_{\max}(\varepsilon + \varepsilon^{b-a} + \varepsilon^a + 4 \cdot \varepsilon^b) \), \( \bar{p}_1 \) is an \( \eta \)-equilibrium in \( \Gamma_{\gamma(\bar{\pi})} \) for the players \( n \neq m \). This finishes the part for the players \( n \neq m \), and together with the consideration for player \( m \), we have shown that \( \bar{\pi} \) is an \( \eta \)-equilibrium in \( G \).
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Remark 2.3.22. To be more accurate: The strategy profile $\bar{\pi}$ is an $\eta_\varepsilon$-equilibrium, where

$$\eta_\varepsilon = \begin{cases} 
\varepsilon + 2r_{\max}(\varepsilon + \varepsilon^{b-a} + \varepsilon^a + 2\varepsilon^b) & \text{if } \bar{p}_1^m = p_1^m \\
\varepsilon + 2r_{\max}(\varepsilon + \varepsilon^{b-a} + \varepsilon^a + 4\varepsilon^b) & \text{if } \bar{p}_1^m = \varepsilon
\end{cases}$$

So for a quitting game with at least two players and the assumption that $\bar{p}_m^1 = p_m^1$, our estimate for $\eta_\varepsilon$ is actually better than the one by Solan and Vieille.

4. Case 2.: Blocks of type I are regularly scattered

Assume that the blocks of type I are regularly scattered in the sense of Definition 2.3.12. We show, that the given strategy profile $\pi$ is an $\tilde{\eta}_\varepsilon$-equilibrium in $G$. Therefore we define a sequence of finite quitting games $(G_{B_k})_{k \in \mathbb{N}}$, where

$$G_{B_k} := G_{B_k,v_k} := (G_B,B_k = \{i_k, \ldots, i_{k+1}-1\}, v_k := \gamma(\pi_{i_{k+1}})),$$

and a sequence of strategy profiles $(\varphi_k)_{k \in \mathbb{N}}$ for the finite quitting games, where $\varphi_k := (p_{i_k}, \ldots, p_{i_{k+1}-1})$. First we prove, that the strategy profiles $\varphi_k$ are $\mu_k$-equilibria in the corresponding finite quitting game $G_{B_k}$, with

$$\mu_k := \max \left\{4 \varepsilon^a \cdot r_{\max} + \varepsilon; \max_{n \in \mathbb{N}} \{11r_{\max}\varepsilon^aP_{(\varphi_k^m,0,0)}(\tau_{B_k} < i_{k+1} - 1) + 2\varepsilon\} \right\}$$

for $k \in \mathbb{N}$. Observe that by Lemma 1.3.5 it is sufficient to consider only pure strategies for the players as alternative strategies in a finite quitting game. Secondly we provide a global estimation for the given quitting game and profile $\pi$. That means we act in three steps:

a) We consider the expected payoff in the finite quitting game for a player $n$ under the assumption, that she changes her strategy $\varphi^n$ to the strategy, in which she plays \textit{quit} with certainty during the stages $i_k, \ldots, i_{k+1} - 1$.

b) We consider the expected payoff in the finite quitting game for a player $n$ under the assumption, that she changes her strategy $\varphi^n$ to the strategy, in which she plays \textit{continue} all the time with certainty.

c) With a) and b), we prove with a global estimation for the whole quitting game, that $\pi$ is an $\tilde{\eta}_\varepsilon$-equilibrium.

Remark 2.3.23. For blocks $B_k$, which consist only of one stage, i.e. $B_k = \{i_k\}$, we have

$$g_{B_k,v_k}(\varphi_k) = g(\varphi_k) = g(p_{i_k}) = \gamma_{\Gamma_{\gamma(\pi_{i_{k+1}})}}(p_{i_k}).$$

With the second requirement from Proposition 2.3.8 (see p. 50), i.e. $p_i$ is an $\varepsilon$-equilibrium in $\Gamma_{\gamma(\pi_{i_{k+1}})}$, it immediately follows that

$$g^n(\varphi_k) = \gamma^n_{\Gamma_{\gamma(\pi_{i_{k+1}})}}(p_{i_k}) \geq \gamma^n_{\Gamma_{\gamma(\pi_{i_{k+1}})}}((p_{i_k}^{-},1)) - \varepsilon = g^n((\varphi_k^{-},1)) - \varepsilon$$
and
\[ g^n(\varphi_k) \geq \gamma^n(\pi_{ik+1}) (p_{ik}^- - \varepsilon) = g^n((\varphi_k^- - \varepsilon)) \]
for all \( n \in \mathcal{N} \), thus \( \mu_k = \varepsilon \).

With regard to the remark above, now we consider only blocks with two or more stages.

4.a) Quitting in the finite quitting game

We first introduce the following notation:

**Notation 2.3.24.** Let \( G_{I,v} = (G, I, v) \) be a given finite quitting game. Denote

- \( \tilde{q}_i := (0, \ldots, 0, 1, 0, \ldots, 0) \) the strategy in the finite quitting game \( G_{I,v} \), where a player chooses to play the action \textit{continue} all the time except for stage \( i \in I \), in which the action \textit{quit} is played with certainty,

- \( \tilde{c} := (0, \ldots, 0) \) the strategy in the finite quitting game \( G_{I,v} \), where a player plays the action \textit{continue} with certainty all the time.

The expected payoff in the given finite quitting game \( G_{B_k} \) under the strategy profile \( \varphi_k \) is

\[ g(\varphi_k) = g_{B_k,v_k}(\varphi_k) = E_{v_k}(r(\tilde{Y}_{\tau_{B_k}})1_{\tau_{B_k} < i} + \gamma(\pi_i)1_{\tau_{B_k} \geq i}) \]
\[ = P_{v_k}(\tau_{B_k} < i)E_{v_k}(r(\tilde{Y}_{\tau_{B_k}})1_{\tau_{B_k} < i}) + P_{v_k}(\tau_{B_k} \geq i)\gamma(\pi_i) \]

(see (1.30) p. 33). Therefore with the definition of the sets \( B_k \) (see (2.9) p. 54) and \( i_k \leq i < i_{k+1} - 1 \), we get

\[ |g^n(\varphi_k) - \gamma^n(\pi_i)| \]
\[ = \left| P_{v_k}(\tau_{B_k} < i)E_{v_k}(r^n(\tilde{Y}_{\tau_{B_k}})1_{\tau_{B_k} < i}) + (P_{v_k}(\tau_{B_k} \geq i) - 1)\gamma^n(\pi_i) \right| \]
\[ \leq \left| P_{v_k}(\tau_{B_k} < i)E_{v_k}(r^n(\tilde{Y}_{\tau_{B_k}})1_{\tau_{B_k} < i}) \right| + \left| (P_{v_k}(\tau_{B_k} \geq i) - 1)\gamma^n(\pi_i) \right| \]
\[ = \left| P_{v_k}(\tau_{B_k} < i)E_{v_k}(r^n(\tilde{Y}_{\tau_{B_k}})1_{\tau_{B_k} < i}) \right| + \left| (P_{v_k}(\tau_{B_k} \geq i) - 1)\gamma^n(\pi_i) \right| \]
\[ \leq P_{v_k}(\tau_{B_k} < i) \cdot r_{\max} + (1 - P_{v_k}(\tau_{B_k} \geq i)) \cdot r_{\max} \]
\[ \leq 2 \cdot P_{v_k}(\tau_{B_k} < i) \cdot r_{\max} \]
\[ \leq 2 \cdot \varepsilon^n \cdot r_{\max} \]

(2.19)

and similarly

\[ g((\varphi_k^- - \varepsilon, \tilde{q}_i)) = P_{(\varphi_k^- - \varepsilon, \tilde{q}_i)}(\tau_{B_k} < i) \cdot E_{(\varphi_k^- - \varepsilon, \tilde{q}_i)}(r(\tilde{Y}_{\tau_{B_k}})1_{\tau_{B_k} < i}) \]
\[ + P_{(\varphi_k^- - \varepsilon, \tilde{q}_i)}(\tau_{B_k} \geq i) \cdot \gamma(\pi_{i+1})(p_i, 1) \]
for all $n \in \mathcal{N}$. Again the construction of the family of sets $(B_k)_{k \in \mathbb{N}}$ yields

$$\mathbb{P}_{(p_k - \bar{q}_k)}(\tau_{B_k} < i) \leq \mathbb{P}_{p_k}(\tau_{B_k} < i) < \varepsilon^n$$
and hence

$$\left| g^n((\varphi_k^- - n, \bar{q}_k)) - \gamma^n_{\pi, n+1}((p_i, 1)) \right| \leq 2 \cdot \varepsilon^n \cdot r_{\text{max}}$$

follows. The inequalities (2.19) and (2.20) imply

$$g^n((\varphi_k^- n, \bar{q}_k)) - g^n(\varphi_k)$$

$$= g^n((\varphi_k^- n, \bar{q}_k)) - \gamma^n_{\pi, n+1}((p_i, 1)) + \gamma^n_{\pi, n+1}((p_i, 1)) - \gamma^n(\pi_i) + \gamma^n(\pi_i) - g^n(\varphi_k)$$

$$\leq 2\varepsilon^n \cdot r_{\text{max}} + \gamma^n_{\pi, n+1}((p_i, 1)) - \gamma^n(\pi_i) + 2\varepsilon^n \cdot r_{\text{max}}.$$  

Since $p_i$ is an $\varepsilon$-equilibrium in $\Gamma_{\gamma, n+1}$ (cf. Proposition 2.3.8)

$$\gamma^n(\pi_i) = \gamma^n_{\pi, n+1}(p_i) \geq \gamma^n_{\pi, n+1}((p_i^- n, 1)) - \varepsilon \implies \gamma^n_{\pi, n+1}((p_i^- n, 1)) \leq \gamma^n(\pi_i) + \varepsilon$$

holds for all $n \in \mathcal{N}$, and we obtain

$$g^n((\varphi_k^- n, \bar{q}_k)) - g^n(\varphi_k) \leq 4\varepsilon^n \cdot r_{\text{max}} + \gamma^n(\pi_i) + \varepsilon - \gamma^n(\pi_i)$$

$$= 4\varepsilon^n \cdot r_{\text{max}} + \varepsilon,$$

respectively

$$g^n((\varphi_k^- n, \bar{q}_k)) \leq g^n(\varphi_k) + 4\varepsilon^n \cdot r_{\text{max}} + \varepsilon$$

(2.21)

for all $n \in \mathcal{N}$.

4.b) Continuing in the finite quitting game

Fix a player $n \in \mathcal{N}$. For the expected payoff $g((\varphi_k^- n, \bar{c}))$ under the strategy profile $(\varphi_k^- n, \bar{c})$ in the finite quitting game $G_{B_k}$ we have

$$g((\varphi_k^- n, \bar{c})) = E((\varphi_k^- n, \bar{c}) \in \mathbb{R}(\bar{Y}_{\tau_{B_k}})1_{\tau_{B_k} < \tau_{k+1}} + \gamma^n_{\pi, k+1}((p_{k+1}^- n, 0))1_{\tau_{B_k} \geq \tau_{k+1}})$$

$$= E((\varphi_k^- n, \bar{c}) \in \mathbb{R}(\bar{Y}_{\tau_{B_k}})1_{\tau_{B_k} < \tau_{k+1}} - \gamma^n_{\pi, k+1}((p_{k+1}^- n, 0)))$$

Since $p_{k+1}$ is an $\varepsilon$-equilibrium in $\Gamma_{\gamma, k+1}$,

$$\gamma^n_{\pi, k+1} = \gamma^n_{\pi, k+1}(p_{k+1}) \geq \gamma^n_{\pi, k+1}((p_{k+1}^- n, 0)) - \varepsilon$$

holds.
This implies
\[
g^n((\varphi_k^{-n}, \bar{c})) \\
\leq E_{(\varphi_k^{-n}, \bar{c})}(r^n(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}}) + P_{(\varphi_k^{-n}, \bar{c})}(\tau_{B_k} \geq i_{k+1} - 1)\gamma^n(\pi_{i_{k+1} - 1}) + \varepsilon \\
\leq E_{(\varphi_k^{-n}, \bar{c})}(r^n(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}}) + P_{(\varphi_k^{-n}, \bar{c})}(\tau_{B_k} \geq i_{k+1} - 1)\gamma^n(\pi_{i_{k+1} - 1}) + \varepsilon \\
= E_{(\varphi_k^{-n}, \bar{c})}(r^n(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}}) + \varepsilon + (1 - P_{(\varphi_k^{-n}, \bar{c})}(\tau_{B_k} < i_{k+1} - 1))\gamma^n(\pi_{i_{k+1} - 1}) \\
= E_{(\varphi_k^{-n}, \bar{c})}(r^n(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}}) + \varepsilon + \gamma^n(\pi_{i_{k+1} - 1}) \\
- P_{(\varphi_k^{-n}, \bar{c})}(\tau_{B_k} < i_{k+1} - 1)\gamma^n(\pi_{i_{k+1} - 1}). \tag{2.22}
\]

On the other hand, for the expected payoff in the finite quitting game $G_{B_k}$ under the strategy profile $\varphi_k$,
\[
g(\varphi_k) = E_{\varphi_k}(r(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}} + \gamma(\pi_{i_{k+1} - 1})I_{\{\tau_{B_k} \geq i_{k+1} - 1\}}) \\
= E_{\varphi_k}(r(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}}) + E_{\varphi_k}(\gamma(\pi_{i_{k+1} - 1})I_{\{\tau_{B_k} \geq i_{k+1} - 1\}}) \\
= E_{\varphi_k}(r(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}}) + P_{\varphi_k}(\tau_{B_k} \geq i_{k+1} - 1)\gamma(\pi_{i_{k+1} - 1}) \\
= E_{\varphi_k}(r(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}}) + (1 - P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1))\gamma(\pi_{i_{k+1} - 1})
\]
holds or, equivalent
\[
\gamma(\pi_{i_{k+1} - 1}) = g(\varphi_k) - E_{\varphi_k}(r(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}}) + P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1)\gamma(\pi_{i_{k+1} - 1}).
\]

Insertion into (2.22) leads to
\[
g^n((\varphi_k^{-n}, \bar{c})) \leq E_{(\varphi_k^{-n}, \bar{c})}(r^n(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}}) + \varepsilon + g^n(\varphi_k) \\
- E_{\varphi_k}(r^n(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}}) + P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1)\gamma^n(\pi_{i_{k+1} - 1}) \\
- P_{(\varphi_k^{-n}, \bar{c})}(\tau_{B_k} < i_{k+1} - 1)\gamma^n(\pi_{i_{k+1} - 1}). \tag{2.23}
\]

Furthermore with
\[
P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1) = P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1, \hat{Y}_{\tau_{B_k}} = 0) + P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1, \hat{Y}_{\tau_{B_k}} = 1),
\]
one obtains
\[
g^n((\varphi_k^{-n}, \bar{c})) \\
\leq E_{(\varphi_k^{-n}, \bar{c})}(r^n(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}}) + \varepsilon + g^n(\varphi_k) \\
- E_{\varphi_k}(r^n(\hat{Y}_{\tau_{B_k}})I_{\{\tau_{B_k} < i_{k+1} - 1\}}) + P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1, \hat{Y}_{\tau_{B_k}} = 1) \cdot \gamma^n(\pi_{i_{k+1} - 1}) \\
+ \left(\tilde{P}_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1, \hat{Y}_{\tau_{B_k}} = 0) - P_{(\varphi_k^{-n}, \bar{c})}(\tau_{B_k} < i_{k+1} - 1)\right)\gamma^n(\pi_{i_{k+1} - 1}). \tag{2.24}
\]
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Now the following estimate holds:

**Estimate 2.3.25.**

\[
\left| P_{\phi_k} (\tau_{B_k} < i_{k+1} - 1, \tilde{Y}^n_{B_k} = 0) - P_{(\phi_{k}^{-n},\mathcal{E})} (\tau_{B_k} < i_{k+1} - 1) \right| \\
\leq \varepsilon^a \cdot P_{(\phi_{k}^{-n},\mathcal{E})} (\tau_{B_k} < i_{k+1} - 1)
\]

**Proof.** Because of

\[
P_{\phi_k} (\tau_{B_k} < i_{k+1} - 1, \tilde{Y}^n_{B_k} = 0) \leq P_{(\phi_{k}^{-n},\mathcal{E})} (\tau_{B_k} < i_{k+1} - 1),
\]

it is sufficient to show

\[
P_{(\phi_{k}^{-n},\mathcal{E})} (\tau_{B_k} < i_{k+1} - 1) - P_{\phi_k} (\tau_{B_k} < i_{k+1} - 1, \tilde{Y}^n_{B_k} = 0) \\
\leq \varepsilon^a \cdot P_{(\phi_{k}^{-n},\mathcal{E})} (\tau_{B_k} < i_{k+1} - 1).
\]

Therefore define two sequences \((V_i)_{i \in \{i_k, \ldots, i_{k+1} - 2\}}\) and \((W_i)_{i \in \{i_k, \ldots, i_{k+1} - 2\}}\) of random variables over \((\Omega_{B_k}, \mathcal{A}_{B_k}, P_{\phi_k})\) with values in \(\{0, 1\}\) by

\[
V_i(\omega) := \begin{cases} 
0 & \text{if } \forall m \in \mathcal{N} \setminus \{n\} : \tilde{Y}^m(\omega) = 0 \\
1 & \text{otherwise}
\end{cases}
\]

respectively

\[
W_i(\omega) := \begin{cases} 
0 & \text{if } \tilde{Y}_i^n(\omega) = 0 \\
1 & \text{otherwise}
\end{cases}
\]

for all \(i \in \{i_k, \ldots, i_{k+1} - 2\}, \omega \in \Omega_{B_k}\). Observe that \(V_{i_k}, \ldots, V_{i_{k+1} - 2}, W_{i_k}, \ldots, W_{i_{k+1} - 2}\) are independent random variables. Furthermore set

\[
t_1(\omega) := \inf \{ i \in \{i_k, \ldots, i_{k+1} - 2\} | V_i(\omega) = 1 \}
\]

\[
t_2(\omega) := \inf \{ i \in \{i_k, \ldots, i_{k+1} - 2\} | W_i(\omega) = 1 \}
\]

\[
t(\omega) := \begin{cases} 
t_1 & \text{if } t_1(\omega) < t_2(\omega) \\
+\infty & \text{otherwise}
\end{cases}. \tag{2.25}
\]

Then

\[
\{ \omega \in \Omega_{B_k} | t(\omega) = i \} \subseteq \{ \omega \in \Omega_{B_k} | t_1(\omega) = i \}
\]

and

\[
\{ \omega \in \Omega_{B_k} | t_1(\omega) = i \} \setminus \{ \omega \in \Omega_{B_k} | t(\omega) = i \} = \{ \omega \in \Omega_{B_k} | t_1(\omega) = i \} \cap \{ \omega \in \Omega_{B_k} | t_2(\omega) \leq i \}
\]
hold for all $i \in \{i_k, \ldots, i_{k+1} - 2\}$. This implies
\[
P_{\psi_k}(t_1 = i) - P_{\psi_k}(t = i) = P_{\psi_k}(t_1 = i) \cdot P_{\psi_k}(t_2 \leq i)
\leq P_{\psi_k}(t_1 = i) \cdot P_{\psi_k}(t_2 < i_{k+1} - 1).
\]
for all $i \in \{i_k, \ldots, i_{k+1} - 2\}$. Thus
\[
\sum_{i=i_k}^{i_{k+1}-2} (P_{\psi_k}(t_1 = i) - P_{\psi_k}(t = i)) = P_{\psi_k}(\bigcup_{i=i_k}^{i_{k+1}-2} t_1 = i) - P_{\psi_k}(\bigcup_{i=i_k}^{i_{k+1}-2} t = i)
= P_{\psi_k}(t_1 < i_{k+1} - 1) - P_{\psi_k}(t < i_{k+1} - 1)
\]
on the one hand and
\[
\sum_{i=i_k}^{i_{k+1}-2} (P_{\psi_k}(t_1 = i) - P_{\psi_k}(t = i)) \leq \sum_{i=i_k}^{i_{k+1}-2} P_{\psi_k}(t_1 = i) \cdot P_{\psi_k}(t_2 < i_{k+1} - 1)
= P_{\psi_k}(t_1 < i_{k+1} - 1) \cdot P_{\psi_k}(t_2 < i_{k+1} - 1)
\]
on the other. The definition of $t_1$, $t_2$ and $t$ (cf. (2.25)) gives us
\[
P_{\psi_k}(t_1 < i_{k+1} - 1) = P_{(\psi_k^{-n}, \overline{\epsilon})}(\tau_{B_k} < i_{k+1} - 1),
P_{\psi_k}(t_2 < i_{k+1} - 1) = P_{\psi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 1)
\]
and
\[
P_{\psi_k}(t < i_{k+1} - 1) = P_{\psi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 0),
\]
such that
\[
P_{\psi_k}(t_1 < i_{k+1} - 1) - P_{\psi_k}(t < i_{k+1} - 1)
= P_{(\psi_k^{-n}, \overline{\epsilon})}(\tau_{B_k} < i_{k+1} - 1) - P_{\psi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 0)
\leq P_{\psi_k}(t_1 < i_{k+1} - 1) \cdot P_{\psi_k}(t_2 < i_{k+1} - 1)
= P_{(\psi_k^{-n}, \overline{\epsilon})}(\tau_{B_k} < i_{k+1} - 1) \cdot P_{\psi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 1)
\]
follows. Furthermore
\[
P_{\psi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 1) \leq P_{\psi_k}(\tau_{B_k} < i_{k+1} - 1) < \epsilon^a
\]
holds and we obtain
\[
P_{(\psi_k^{-n}, \overline{\epsilon})}(\tau_{B_k} < i_{k+1} - 1) - P_{\psi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 0)
\leq \epsilon^a \cdot P_{(\psi_k^{-n}, \overline{\epsilon})}(\tau_{B_k} < i_{k+1} - 1).
\]
Chapter 2. Equilibria in Quitting games

This estimate yields for the inequality (2.24) that

\[
  g^n((\tilde{\varphi}_k^n, \tilde{\varphi})) \\
  \leq E_{(\tilde{\varphi}_k^n, \tilde{\varphi})}(r^n(\tilde{Y}_{\tau_k^n})I_{\tau_k^n < i_{k+1} - 1}) + \varepsilon + g^n(\varphi_k) \\
  - E_{\varphi_k}(r^n(\tilde{Y}_{\tau_k^n})I_{\tau_k^n < i_{k+1} - 1}) + P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 1) \cdot \gamma^n(\pi_{i_{k+1} - 1}) \\
  + \varepsilon^n \cdot P_{(\tilde{\varphi}_k^n, \tilde{\varphi})}(\tau_{B_k} < i_{k+1} - 1) \cdot r_{max}.
\]

(2.27)

Furthermore we have

\[
  E_{(\tilde{\varphi}_k^n, \tilde{\varphi})}(r^n(\tilde{Y}_{\tau_k^n})I_{\tau_k^n < i_{k+1} - 1}) \\
  = P_{(\tilde{\varphi}_k^n, \tilde{\varphi})}(\tau_{B_k} < i_{k+1} - 1)E_{(\tilde{\varphi}_k^n, \tilde{\varphi})}(r^n(\tilde{Y}_{\tau_k^n})|\tau_{B_k} < i_{k+1} - 1) \\
  \tag{2.28}
\]

and

\[
  E_{\varphi_k}(r^n(\tilde{Y}_{\tau_k^n})I_{\tau_k^n < i_{k+1} - 1}) \\
  = P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 1)E_{\varphi_k}(r(\tilde{Y}_{\tau_k^n})|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 1) \\
  + P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 0)E_{\varphi_k}(r(\tilde{Y}_{\tau_k^n})|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 0). \\
  \tag{2.29}
\]

Using both (2.28) and (2.29), we get for (2.27)

\[
  \begin{align*}
  &g^n((\tilde{\varphi}_k^n, \tilde{\varphi})) \\
  &\leq P_{(\tilde{\varphi}_k^n, \tilde{\varphi})}(\tau_{B_k} < i_{k+1} - 1)E_{(\tilde{\varphi}_k^n, \tilde{\varphi})}(r^n(\tilde{Y}_{\tau_k^n})|\tau_{B_k} < i_{k+1} - 1) + \varepsilon + g^n(\varphi_k) \\
  &\quad - P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 1)E_{\varphi_k}(r^n(\tilde{Y}_{\tau_k^n})|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 1) \\
  &\quad - P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 0)E_{\varphi_k}(r^n(\tilde{Y}_{\tau_k^n})|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 0) \\
  &\quad + P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 1) \cdot \gamma^n(\pi_{i_{k+1} - 1}) \\
  &\quad + \varepsilon^n \cdot P_{(\tilde{\varphi}_k^n, \tilde{\varphi})}(\tau_{B_k} < i_{k+1} - 1) \cdot r_{max} \\
  &\quad = g^n(\varphi_k) + \varepsilon + \varepsilon^n \cdot P_{(\tilde{\varphi}_k^n, \tilde{\varphi})}(\tau_{B_k} < i_{k+1} - 1) \cdot r_{max} \\
  &\quad + P_{(\tilde{\varphi}_k^n, \tilde{\varphi})}(\tau_{B_k} < i_{k+1} - 1)E_{(\tilde{\varphi}_k^n, \tilde{\varphi})}(r^n(\tilde{Y}_{\tau_k^n})|\tau_{B_k} < i_{k+1} - 1) \\
  &\quad - P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 0)E_{\varphi_k}(r^n(\tilde{Y}_{\tau_k^n})|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 0) \\
  &\quad + P_{\varphi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 1) \left(\gamma^n(\pi_{i_{k+1} - 1})\right) \\
  &\quad - E_{\varphi_k}(r^n(\tilde{Y}_{\tau_k^n})|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 1). \\
  \tag{2.30}
  \end{align*}
\]

In the next steps we provide estimates on

- \[ |E_{(\tilde{\varphi}_k^n, \tilde{\varphi})}(r^n(\tilde{Y}_{\tau_k^n})|\tau_{B_k} < i_{k+1} - 1) - E_{\varphi_k}(r^n(\tilde{Y}_{\tau_k^n})|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_k^n} = 0)| \]

(see Estimate 2.3.26) and
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- $\gamma^n(\pi_{k+1-1}) - E_{\psi_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 1)$

(see Estimate 2.3.27).

**Estimate 2.3.26.** Let $n \in \mathcal{N}$ be given, then

$$|E_{(\psi^{-}_k, \varepsilon)}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} < i_{k+1} - 1) - E_{\psi_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 0)| \leq 2r_{\max} \cdot \varepsilon^n.$$

**Proof.** We have (cf. Appendix (A.6) and (A.8))

$$E_{(\psi^{-}_k, \varepsilon)}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} < i_{k+1} - 1) = \sum_{i=k}^{i_{k+1}-2} P_{(\psi^{-}_k, \varepsilon)}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1) E_{(\psi^{-}_k, \varepsilon)}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} = i)$$

respectively

$$E_{\psi_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 0) = \sum_{i=k}^{i_{k+1}-2} P_{\psi_k}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 0) E_{\psi_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} = i, \tilde{Y}_{\tau_{B_k}} = 0).$$

Furthermore (cf. Appendix (A.10))

$$E_{\psi_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} = i, \tilde{Y}_{\tau_{B_k}} = 0) = E_{(\psi^{-}_k, \varepsilon)}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} = i)$$

holds, thus

$$\left| E_{(\psi^{-}_k, \varepsilon)}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} < i_{k+1} - 1) - E_{\psi_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 0) \right|$$

$$= \left| \sum_{i=k}^{i_{k+1}-2} E_{(\psi^{-}_k, \varepsilon)}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} = i) \left( P_{(\psi^{-}_k, \varepsilon)}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1) - P_{\psi_k}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 0) \right) \right|$$

$$\leq \sum_{i=k}^{i_{k+1}-2} \left| E_{(\psi^{-}_k, \varepsilon)}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} = i) \right| \cdot \left| P_{(\psi^{-}_k, \varepsilon)}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1) - P_{\psi_k}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 0) \right|$$

$$\leq r_{\max} \sum_{i=k}^{i_{k+1}-2} P_{(\psi^{-}_k, \varepsilon)}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1) - P_{\psi_k}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 0)$$
follows. Using $V_k, \ldots, V_{k+1-2}, W_k, \ldots, W_{k+1-2}, t_1, t_2$ and $t$ from the proof of Estimate 2.3.25, one obtains
\[
P_{(p_k^\tau_x)}(\tau_{B_k} = i|\tau_{B_k} < i_{k+1} - 1) = P_{p_k}(t_1 = i|t_1 < i_{k+1} - 1)
\]
and
\[
P_{p_k}(\tau_{B_k} = i|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 0) = P_{p_k}(t_1 = i|t < i_{k+1} - 1).
\]
In order to prove the Estimate 2.3.26, it is left to show that
\[
\sum_{i=1}^{i_{k+1}-2} \left| P_{p_k}(t_1 = i|t_1 < i_{k+1} - 1) - P_{p_k}(t_1 = i|t < i_{k+1} - 1) \right| \leq 2 \cdot e^a. \quad (2.31)
\]
Since
\[
P_{p_k}(t_1 = i) = P_{p_k}(t_1 = i, t_2 \leq i) + P_{p_k}(t_1 = i, t_2 > i)
= P_{p_k}(t_1 = i, t_2 \leq i) + P_{p_k}(t = i)
\]
respectively
\[
P_{p_k}(t_1 = i|t < i_{k+1} - 1)
= P_{p_k}(t_1 = i, t_2 \leq i|t < i_{k+1} - 1) + P_{p_k}(t_1 = i, t_2 > i|t < i_{k+1} - 1)
= 0 + P_{p_k}(t = i|t < i_{k+1} - 1),
\]
we have
\[
\left| P_{p_k}(t_1 = i|t_1 < i_{k+1} - 1) - P_{p_k}(t_1 = i|t < i_{k+1} - 1) \right|
= \left| P_{p_k}(t_1 = i|t_1 < i_{k+1} - 1) - P_{p_k}(t = i|t < i_{k+1} - 1) \right|
= \left| \frac{P_{p_k}(t_1 = i)}{P_{p_k}(t_1 < i_{k+1} - 1)} \right| - \left| \frac{P_{p_k}(t = i)}{P_{p_k}(t < i_{k+1} - 1)} \right|
\leq \left| \frac{P_{p_k}(t_1 = i, t_2 \leq i)}{P_{p_k}(t_1 < i_{k+1} - 1)} \right| + \left| \frac{P_{p_k}(t_1 < i_{k+1} - 1)}{P_{p_k}(t_1 < i_{k+1} - 1)} \right|
\leq \left| \frac{P_{p_k}(t_1 = i, t_2 \leq i)}{P_{p_k}(t_1 < i_{k+1} - 1)} \right| + \left| \frac{P_{p_k}(t_1 < i_{k+1} - 1)}{P_{p_k}(t_1 < i_{k+1} - 1)} \right|.
\] (2.32)
Because $t_1$ and $t_2$ are independent, we get for the term (*)
\[
\frac{P_{p_k}(t_1 = i, t_2 \leq i)}{P_{p_k}(t_1 < i_{k+1} - 1)} = \frac{P_{p_k}(t_1 = i)}{P_{p_k}(t_1 < i_{k+1} - 1)} \cdot \frac{P_{p_k}(t_2 \leq i)}{P_{p_k}(t_2 < i_{k+1} - 1)}
\leq \frac{P_{p_k}(t_1 = i)}{P_{p_k}(t_1 < i_{k+1} - 1)} \cdot P_{p_k}(t_2 < i_{k+1} - 1).
\] (2.33)
Consider now the term (**) in (2.32):

$$
\left| \frac{P_{\psi k}(t = i)}{P_{\psi k}(t < i_{k+1} - 1)} - \frac{P_{\psi k}(t = i)}{P_{\psi k}(t < i_{k+1} - 1)} \right| = P_{\psi k}(t = i) \cdot \frac{1}{P_{\psi k}(t = i)} \cdot \frac{1}{P_{\psi k}(t < i_{k+1} - 1) - P_{\psi k}(t < i_{k+1} - 1)} \cdot \frac{P_{\psi k}(t < i_{k+1} - 1) - P_{\psi k}(t < i_{k+1} - 1)}{P_{\psi k}(t < i_{k+1} - 1) - P_{\psi k}(t < i_{k+1} - 1)}
$$

for all $i \in \{i_k, \ldots, i_{k+1} - 2\}$. Inserting this and (2.33) into (2.32) leads to

$$
\left| P_{\psi k}(t_1 = i|t_1 < i_{k+1} - 1) - P_{\psi k}(t_1 = i|t < i_{k+1} - 1) \right| \leq \left( \frac{P_{\psi k}(t_1 = i)}{P_{\psi k}(t_1 < i_{k+1} - 1)} + \frac{P_{\psi k}(t = i)}{P_{\psi k}(t < i_{k+1} - 1)} \right) \cdot P_{\psi k}(t_2 < i_{k+1} - 1)
$$

and finally

$$
\sum_{i = i_k}^{i_{k+1}-2} \left| P_{\psi k}(t_1 = i|t_1 < i_{k+1} - 1) - P_{\psi k}(t_1 = i|t < i_{k+1} - 1) \right| \leq 2 \cdot P_{\psi k}(t_2 < i_{k+1} - 1)
$$

Since $P_{\psi k}(t_2 < i_{k+1} - 1) < \varepsilon^a$, inequality (2.31) follows and therefore Estimate 2.3.26.

**Estimate 2.3.27.** Let $n \in \mathcal{N}$ be given, then

$$
\gamma^n(\tau_{i_{k+1}-1}) - E_{\psi k}(r^n(\hat{Y}_{\tau_{B_k}})|\tau_{B_k} < i_{k+1} - 1, \hat{Y}_{\tau_{B_k}} = 1) \leq 7r_{\max}P_{(\psi_{k-n}^-,\varepsilon)}(\tau_{B_k} < i_{k+1} - 1 + \varepsilon).
$$

**Proof.** The proof is organized in four steps.

1. We show

$$
\left| E_{\psi k}(r^n(\hat{Y}_{\tau_{B_k}})|\tau_{B_k} < i_{k+1} - 1, \hat{Y}_{\tau_{B_k}} = 1) - r^n_{\{n\}} \right| \leq 2r_{\max}P_{(\psi_{k-n}^-,\varepsilon)}(\tau_{B_k} < i_{k+1} - 1).
$$
2. Let \( \kappa \) be the last time before \( i_{k+1} - 1 \) with \( p_k^n > 0 \). Then
\[
\gamma^n(\pi_{\kappa+1}) - r^n_{\{n\}} \leq 4r_{\max}\mathbb{P}_{(\mathcal{P}_k^n, \mathcal{E})}(\tau_{B_k} < i_{k+1} - 1) + \varepsilon.
\]

3. We prove the inequality\(^{12}\)
\[
|\gamma^n(\pi_{\kappa+1}) - \gamma^n(\pi_{i_{k+1}-1})| \leq 2r_{\max}\mathbb{P}_{(\mathcal{P}_k^n, \mathcal{E})}(\tau_{B_k} < i_{k+1} - 1).
\]

4. Finally we argue that 1., 2. and 3. imply the estimate.

1.: We have (see Appendix (A.7))
\[
E_{\mathcal{P}_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 1)
= \sum_{i=i_k}^{i_{k+1}-2} \mathbb{P}_{\mathcal{P}_k}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 1) E_{\mathcal{P}_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} = i, \tilde{Y}_{\tau_{B_k}}^n = 1)
\]
for all players \( n \in \mathcal{N} \). Since
\[
\sum_{i=i_k}^{i_{k+1}-2} \mathbb{P}_{\mathcal{P}_k}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 1) = 1,
\]
one obtains
\[
\left| E_{\mathcal{P}_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 1) - r^n_{\{n\}} \right|
= \left| \sum_{i=i_k}^{i_{k+1}-2} \mathbb{P}_{\mathcal{P}_k}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 1) E_{\mathcal{P}_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} = i, \tilde{Y}_{\tau_{B_k}}^n = 1)
\right|
\]
\[
= \left| \sum_{i=i_k}^{i_{k+1}-2} \mathbb{P}_{\mathcal{P}_k}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 1) \cdot \left( E_{\mathcal{P}_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} = i, \tilde{Y}_{\tau_{B_k}}^n = 1)
- r^n_{\{n\}} \right) \right|
\]
\[
\leq \sum_{i=i_k}^{i_{k+1}-2} \mathbb{P}_{\mathcal{P}_k}(\tau_{B_k} = i | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 1) \cdot \left| E_{\mathcal{P}_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} = i, \tilde{Y}_{\tau_{B_k}}^n = 1)
- r^n_{\{n\}} \right| \cdot \tag{2.35}
\]

\(^{12}\)If no \( \kappa \in \{i_k, \ldots, i_{k+1} - 2\} \) exists such that \( p_k^n > 0 \), then player \( n \) plays either continue all the time in the block \( B_k \) or she plays quit only in the stage \( i_{k+1} - 1 \) with positive probability. In the first case, \( g((\mathcal{P}_k^n, \mathcal{E})) = g(\mathcal{P}_k) \) holds and nothing is to show for the part “Continuing in the finite quitting game” for player \( n \). In the second case, the Inequality (2.23) immediately reduces to \( g^n((\mathcal{P}_k^n, \mathcal{E})) \leq \varepsilon + g^n(\mathcal{P}_k). \)
For $E_{p_k^n}(r^n(\tilde{Y}_{\tau_{B_k}})|\tau_{B_k} = i, \tilde{Y}_{\tau_{B_k}} = 1)$ the following holds:

$$
\frac{E_{p_k^n}(r^n(\tilde{Y}_{\tau_{B_k}})|\tau_{B_k} = i, \tilde{Y}_{\tau_{B_k}} = 1)}{r^n_{\{i\}}} = \sum_{S \in P(\mathcal{N}\setminus\{n\})} \varrho((p_i^{-n}, 1), S \cup \{n\}) \cdot r^n_{S \cup \{n\}}
$$

which implies

$$
\left| \frac{E_{p_k^n}(r^n(\tilde{Y}_{\tau_{B_k}})|\tau_{B_k} = i, \tilde{Y}_{\tau_{B_k}} = 1)}{r^n_{\{i\}}} - r^n_{\{i\}} \right| 
\leq \left| r^n_{\{i\}} \cdot \left( \varrho((p_i^{-n}, 1), \{n\}) - 1 \right) \right| + \sum_{S \in P(\mathcal{N}\setminus\{n\}) \setminus \emptyset} \varrho((p_i^{-n}, 1), S \cup \{n\}) \cdot r^n_{S \cup \{n\}}
$$

$$
\leq r_{max} \cdot (1 - \varrho((p_i^{-n}, 1), \{n\})) + r_{max} \cdot \sum_{S \in P(\mathcal{N}\setminus\{n\}) \setminus \emptyset} \varrho((p_i^{-n}, 1), S \cup \{n\})
$$

$$
= 2 \cdot r_{max} \cdot \sum_{S \in P(\mathcal{N}\setminus\{n\}) \setminus \emptyset} \varrho((p_i^{-n}, 1), S \cup \{n\}). \tag{2.36}
$$

The probability that at least one player $m \in \mathcal{N}\setminus\{n\}$ plays quit in one stage $i \in \{i_k, \ldots, i_{k+1} - 2\}$ is lower or equal to the probability that at least one player $m \in \mathcal{N}\setminus\{n\}$ plays quit during the stages $i_k$ and $i_{k+1} - 2$, i.e. (cf. (A.2))

$$
\sum_{S \in P(\mathcal{N}\setminus\{n\}) \setminus \emptyset} \varrho((p_i^{-n}, 1), S \cup \{n\}) \leq P_{(p_i^{-n}, \emptyset)}(\tau_{B_k} < i_{k+1} - 1) \tag{2.37}
$$

for all $i \in \{i_k, \ldots, i_{k+1} - 2\}$, where $\emptyset_k = (p_{i_k}, \ldots, p_{i_{k+1} - 1})$. Now (2.34), (2.35), (2.36) and (2.37) imply

$$
\left| E_{p_k^n}(r^n(\tilde{Y}_{\tau_{B_k}})|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 1) - r^n_{\{i\}} \right| \leq 2 r_{max} P_{(p_i^{-n}, \emptyset)}(\tau_{B_k} < i_{k+1} - 1).
$$

2. Let $\kappa \in B_k$ be the last stage before $i_{k+1} - 1$, for which $p_i^n > 0$ holds, hence $p_i^n = 0$ for all $i \in \{\kappa + 1, \ldots, i_{k+1} - 2\}$. Since $p_{\kappa}$ is $\varepsilon$-perfect in $\Gamma_{\gamma(p_{\kappa}^{-n})}$ (cf. requirement two in Proposition 2.3.8) with $p_{\kappa}^n > 0$,

$$
\gamma_{\gamma(p_{\kappa}^{-n})}((p_i^{-n}, 1)) - \gamma_{\gamma(p_{\kappa}^{-n})}((p_{\kappa}^{-n}, 0)) \geq -\varepsilon
$$

holds.
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By decomposition of the first sum (cf. Appendix (A.4) and (A.5)), one obtains

\[-\varepsilon \leq \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{n\})\setminus\{\varnothing\}} \varrho((p_{\kappa}^{-n}, 1), S \cup \{n\}) r_{S\cup\{n\}}^n + \varrho((p_{\kappa}^{-n}, 1), \{n\}) r_{\{n\}}^n - \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{n\})\setminus\{\varnothing\}} \varrho((p_{\kappa}^{-n}, 0), S) r_S^n - \varrho((p_{\kappa}^{-n}, 0), \varnothing) \gamma^n(\pi_{\kappa+1}).\]

Furthermore \(\varrho((p_{\kappa}^{-n}, 1), S \cup \{n\}) = \varrho((p_{\kappa}^{-n}, 0), S)\) for all \(S \subseteq \mathcal{N}\setminus\{n\}\), which implies

\[-\varepsilon \leq \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{n\})\setminus\{\varnothing\}} \varrho((p_{\kappa}^{-n}, 0), S) (r_{S\cup\{n\}}^n - r_S^n) + \varrho((p_{\kappa}^{-n}, 0), \varnothing) (r_{\{n\}}^n - \gamma^n(\pi_{\kappa+1})),\]

or equivalently

\[\varrho((p_{\kappa}^{-n}, 0), \varnothing) (\gamma^n(\pi_{\kappa+1}) - r_{\{n\}}^n) \leq \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{n\})\setminus\{\varnothing\}} \varrho((p_{\kappa}^{-n}, 0), S) (r_{S\cup\{n\}}^n - r_S^n) + \varepsilon.\]

Using \(\varrho((p_{\kappa}^{-n}, 0), \varnothing) = 1 - \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{n\})\setminus\{\varnothing\}} \varrho((p_{\kappa}^{-n}, 0), S)\) we get

\[\gamma^n(\pi_{\kappa+1}) - r_{\{n\}}^n \leq \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{n\})\setminus\{\varnothing\}} \varrho((p_{\kappa}^{-n}, 0), S) (r_{S\cup\{n\}}^n - r_S^n) + \varepsilon + \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{n\})\setminus\{\varnothing\}} \varrho((p_{\kappa}^{-n}, 0), S) (\gamma^n(\pi_{\kappa+1}) - r_{\{n\}}^n)\]

\[= \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{n\})\setminus\{\varnothing\}} \varrho((p_{\kappa}^{-n}, 0), S) (r_{S\cup\{n\}}^n - r_S^n + \gamma^n(\pi_{\kappa+1}) - r_{\{n\}}^n) + \varepsilon\]

\[\leq 4r_{\max} \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{n\})\setminus\{\varnothing\}} \varrho((p_{\kappa}^{-n}, 0), S) + \varepsilon. \tag{2.38}\]

Remark 2.3.28. If we take into account that \(r_{\{n\}}^n \geq 0\), this inequality improves to

\[\gamma^n(\pi_{\kappa+1}) - r_{\{n\}}^n \leq 3r_{\max} \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{n\})\setminus\{\varnothing\}} \varrho((p_{\kappa}^{-n}, 0), S) + \varepsilon.\]

If we assume that \(r_{\{n\}}^n \geq r_S^n\) for all \(S \in \mathcal{P}(\mathcal{N})\), where \(n \in S\) (like in Theorem 2.3.3), we get

\[\gamma^n(\pi_{\kappa+1}) - r_{\{n\}}^n \leq 2r_{\max} \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{n\})\setminus\{\varnothing\}} \varrho((p_{\kappa}^{-n}, 0), S) + \varepsilon.\]

Finally because of

\[\sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{n\})\setminus\{\varnothing\}} \varrho((p_{\kappa}^{-n}, 0), S) \leq P_{(p_{\kappa}^{-n}, \varepsilon)}(\tau_{B_k} < i_{k+1} - 1)\]
holds for all \( i \in \{i_k, \ldots, i_{k+1} - 2\}, \)

\[
\gamma^n(\pi_{k+1}) - \gamma^n_{\{n\}} \leq 4r_{\max}P_{(p_k^n, \epsilon)}(\tau_{B_k} < i_{k+1} - 1) + \epsilon
\]

follows.

3.: The expected payoff \( \gamma^n(\pi_{k+1}) \) can be calculated in the following way (see Appendix formula (A.11)):

\[
\begin{align*}
\gamma^n(\pi_{k+1}) &= E_{p_k}(r^n(\tilde{Y}_{\tau_{B_k}}) \cdot 1_{\{\tau_{B_k} < i_{k+1} - 1\}} + \gamma^n(\pi_{i_{k+1} - 1}) \cdot 1_{\{\tau_{B_k} \geq i_{k+1} - 1\}} | \tau_{B_k} > k) \\
&= E_{p_k}(r^n(\tilde{Y}_{\tau_{B_k}}) \cdot 1_{\{\tau_{B_k} < i_{k+1} - 1\}} | \tau_{B_k} > k) + \gamma^n(\pi_{i_{k+1} - 1})E_{p_k}(1_{\{\tau_{B_k} \geq i_{k+1} - 1\}} | \tau_{B_k} > k) \\
&= P_{p_k}(\tau_{B_k} < i_{k+1} - 1 | \tau_{B_k} > k) \cdot E_{p_k}(r^n(\tilde{Y}_{\tau_{B_k}}) | \kappa < \tau_{B_k} < i_{k+1} - 1) \\
&\quad + P_{p_k}(\tau_{B_k} \geq i_{k+1} - 1 | \tau_{B_k} > k) \gamma^n(\pi_{i_{k+1} - 1}).
\end{align*}
\]

By the definition of \( \kappa \), \( p_i^n = 0 \) for all \( i \in \{\kappa + 1, \ldots, i_{k+1} - 2\} \), which implies

\[
\gamma^n(\pi_{k+1}) = P_{(p_k^n, \epsilon)}(\tau_{B_k} < i_{k+1} - 1 | \tau_{B_k} > k) \cdot E_{(p_k^n, \epsilon)}(r^n(\tilde{Y}_{\tau_{B_k}}) | \kappa < \tau_{B_k} < i_{k+1} - 1) \\
+ P_{(p_k^n, \epsilon)}(\tau_{B_k} \geq i_{k+1} - 1 | \tau_{B_k} > k) \gamma^n(\pi_{i_{k+1} - 1}).
\]

Hence

\[
\begin{align*}
&|\gamma^n(\pi_{k+1}) - \gamma^n(\pi_{i_{k+1} - 1})| \\
&= \left| P_{(p_k^n, \epsilon)}(\tau_{B_k} < i_{k+1} - 1 | \tau_{B_k} > k) \cdot E_{(p_k^n, \epsilon)}(r^n(\tilde{Y}_{\tau_{B_k}}) | \kappa < \tau_{B_k} < i_{k+1} - 1) \\
&\quad + \left( P_{(p_k^n, \epsilon)}(\tau_{B_k} \geq i_{k+1} - 1 | \tau_{B_k} > k) \right) \gamma^n(\pi_{i_{k+1} - 1}) \right|
\end{align*}
\]

follows. Furthermore with

\[
\begin{align*}
P_{(p_k^n, \epsilon)}(\tau_{B_k} < i_{k+1} - 1 | \tau_{B_k} > k) &= 1 - P_{(p_k^n, \epsilon)}(\tau_{B_k} \geq i_{k+1} - 1 | \tau_{B_k} > k) \\
&= 1 - \frac{P_{(p_k^n, \epsilon)}(\tau_{B_k} \geq i_{k+1} - 1, \tau_{B_k} > k)}{P_{(p_k^n, \epsilon)}(\tau_{B_k} > k)} \\
&= 1 - \frac{P_{(p_k^n, \epsilon)}(\tau_{B_k} \geq i_{k+1} - 1)}{P_{(p_k^n, \epsilon)}(\tau_{B_k} > k)} \\
&\leq 1 - P_{(p_k^n, \epsilon)}(\tau_{B_k} \geq i_{k+1} - 1) \\
&= P_{(p_k^n, \epsilon)}(\tau_{B_k} < i_{k+1} - 1),
\end{align*}
\]
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we obtain

\[ |\gamma^n(\pi_{k+1}) - \gamma^n(\pi_{k+1-1})| \leq P_{(\psi^n_k, \xi)}(\tau_{B_k} < i_{k+1} - 1|\tau_{B_k} > \kappa) \cdot |E_{(\psi^n_k, \xi)}(r^n(\tilde{Y}_{\tau_{B_k}})|\kappa < \tau_{B_k} < i_{k+1} - 1)| \]

\[ + \left(1 - P_{(\psi^n_k, \xi)}(\tau_{B_k} \geq i_{k+1} - 1|\tau_{B_k} > \kappa)\right) \gamma^n(\pi_{k+1-1}) \]

\[ \leq P_{(\psi^n_k, \xi)}(\tau_{B_k} < i_{k+1} - 1) \cdot r_{max} \]

\[ + \left(1 - P_{(\psi^n_k, \xi)}(\tau_{B_k} \geq i_{k+1} - 1|\tau_{B_k} > \kappa)\right) \gamma^n(\pi_{k+1-1}) \]

\[ \leq 2 \cdot P_{(\psi^n_k, \xi)}(\tau_{B_k} < i_{k+1} - 1) \cdot r_{max}. \]

4.: 1., 2. and 3. together imply

\[ \gamma^n(\pi_{k+1-1}) - E_{\psi_k}(r^n(\tilde{Y}_{\tau_{B_k}})|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 1) \]

\[ = \gamma^n(\pi_{k+1-1}) - E_{\psi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 1) + r_{n} - r_{n} \]

\[ + \gamma^n(\pi_{k+1-1}) - \gamma^n(\pi_{k+1}) \]

\[ \leq \left| r_{n} - E_{\psi_k}(\tilde{Y}_{\tau_{B_k}})|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 1) \right| + \gamma^n(\pi_{k+1-1}) - r_{n} \]

\[ + \gamma^n(\pi_{k+1-1}) - \gamma^n(\pi_{k+1}) \]

\[ \leq 2r_{max} P_{(\psi^n_k, \xi)}(\tau_{B_k} < i_{k+1} - 1) + 4r_{max} P_{(\psi^n_k, \xi)}(\tau_{B_k} < i_{k+1} - 1) \]

\[ \leq 8r_{max} P_{(\psi^n_k, \xi)}(\tau_{B_k} < i_{k+1} - 1) + \varepsilon. \]

\[ \square \]

Overview: Estimates in Continuing the finite quitting game

Estimate 2.3.25: \[ |P_{(\psi_k, \xi)}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 0) - P_{(\psi^n_k, \xi)}(\tau_{B_k} < i_{k+1} - 1)| \leq \varepsilon^a \cdot P_{(\psi^n_k, \xi)}(\tau_{B_k} < i_{k+1} - 1) \]

Estimate 2.3.26: \[ |E_{(\psi^n_k, \xi)}(\tilde{Y}_{\tau_{B_k}})|\tau_{B_k} < i_{k+1} - 1) - E_{\psi_k}(\tilde{Y}_{\tau_{B_k}})|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 0) | \leq 2r_{max} \varepsilon^a \]

Estimate 2.3.27: \[ \gamma^n(\pi_{k+1-1}) - E_{\psi_k}(\tilde{Y}_{\tau_{B_k}})|\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}} = 1) \leq 8r_{max} P_{(\psi^n_k, \xi)}(\tau_{B_k} < i_{k+1} - 1) + \varepsilon \]

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Recall now estimate (2.30) on page 76

\[
g^n((\varphi_k^{-n}, \bar{c})) \\
\leq g^n(\varphi_k) + \varepsilon + \varepsilon^n \cdot p_{(\varphi_k^{-n}, \bar{c})}(\tau_{B_k} < i_{k+1} - 1) \\
+ P_{(\varphi_k^{-n}, \bar{c})}(\tau_{B_k} < i_{k+1} - 1) E_{(\varphi_k^{-n}, \bar{c})} \left( r^n(\bar{Y}_{\tau_{B_k}}) \mid \tau_{B_k} < i_{k+1} - 1 \right) \\
- P_{p_k}(\tau_{B_k} < i_{k+1} - 1, \bar{Y}^n_{\tau_{B_k}} = 0) E_{p_k} \left( r^n(\bar{Y}_{\tau_{B_k}}) \mid \tau_{B_k} < i_{k+1} - 1, \bar{Y}^n_{\tau_{B_k}} = 0 \right) \\
+ P_{p_k}(\tau_{B_k} < i_{k+1} - 1, \bar{Y}^n_{\tau_{B_k}} = 1) \left( \gamma^n(\pi_{i_{k+1}-1}) \\
- E_{p_k} \left( r^n(\bar{Y}_{\tau_{B_k}}) \mid \tau_{B_k} < i_{k+1} - 1, \bar{Y}^n_{\tau_{B_k}} = 1 \right) \right)
\]

Estimate 2.3.26 implies

\[
E_{(p_k^{-n}, \bar{c})} \left( r^n(\bar{Y}_{\tau_{B_k}}) \mid \tau_{B_k} < i_{k+1} - 1 \right) \leq E_{p_k} \left( r^n(\bar{Y}_{\tau_{B_k}}) \mid \tau_{B_k} < i_{k+1} - 1, \bar{Y}^n_{\tau_{B_k}} = 0 \right) \\
+ 2r_{max} \cdot \varepsilon^a
\]

and hence

\[
g^n((\varphi_k^{-n}, \bar{c})) \\
\leq g^n(\varphi_k) + \varepsilon + r_{max} \varepsilon^a \cdot P_{(p_k^{-n}, \bar{c})}(\tau_{B_k} < i_{k+1} - 1) \\
+ P_{(p_k^{-n}, \bar{c})}(\tau_{B_k} < i_{k+1} - 1) \left( E_{p_k} \left( r^n(\bar{Y}_{\tau_{B_k}}) \mid \tau_{B_k} < i_{k+1} - 1, \bar{Y}^n_{\tau_{B_k}} = 0 \right) + 2r_{max} \cdot \varepsilon^a \right) \\
- P_{p_k}(\tau_{B_k} < i_{k+1} - 1, \bar{Y}^n_{\tau_{B_k}} = 0) E_{p_k} \left( r^n(\bar{Y}_{\tau_{B_k}}) \mid \tau_{B_k} < i_{k+1} - 1, \bar{Y}^n_{\tau_{B_k}} = 0 \right) \\
+ P_{p_k}(\tau_{B_k} < i_{k+1} - 1, \bar{Y}^n_{\tau_{B_k}} = 1) \left( \gamma^n(\pi_{i_{k+1}-1}) \\
- E_{p_k} \left( r^n(\bar{Y}_{\tau_{B_k}}) \mid \tau_{B_k} < i_{k+1} - 1, \bar{Y}^n_{\tau_{B_k}} = 1 \right) \right)
\]

holds and Estimates 2.3.25 and 2.3.27 lead to

\[
g^n((\varphi_k^{-n}, \bar{c})) \leq g^n(\varphi_k) + \varepsilon + 3r_{max} \varepsilon^a \cdot P_{(p_k^{-n}, \bar{c})}(\tau_{B_k} < i_{k+1} - 1) \\
+ r_{max} \cdot \varepsilon^a \cdot P_{(p_k^{-n}, \bar{c})}(\tau_{B_k} < i_{k+1} - 1) \\
+ P_{p_k}(\tau_{B_k} < i_{k+1} - 1, \bar{Y}^n_{\tau_{B_k}} = 1) \left( 8r_{max} P_{(p_k^{-n}, \bar{c})}(\tau_{B_k} < i_{k+1 - 1}) + \varepsilon \right).
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Since
\[ P_{\psi_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_k^n = 1) \leq P_{\psi_k}(\tau_{B_k} < i_{k+1} - 1) < \varepsilon^a, \]
finally
\[ g^n((\varphi_k^{-n}, \tilde{c})) \leq g^n(\varphi_k) + 12r_{max}\varepsilon^a P_{(\psi_k^{-n}, \varepsilon)}(\tau_{B_k} < i_{k+1} - 1) + 2\varepsilon \] (2.39)
follows for all \( n \in \mathcal{N}. \) This finishes part 4.b) “Continuing in the finite quitting game”.

In summary we have
\[ \begin{align*}
&\text{with 4.a): } g^n((\varphi_k^{-n}, \tilde{q}_k)) \leq g^n(\varphi_k) + 4\varepsilon^a \cdot r_{max} + \varepsilon, \quad \text{(cf. (2.21)) and} \\
&\text{with 4.b): } g^n((\varphi_k^{-n}, \tilde{c})) \leq g^n(\varphi_k) + 12\varepsilon^a \cdot r_{max} P_{(\psi_k^{-n}, \varepsilon)}(\tau_{B_k} < i_{k+1} - 1) + 2\varepsilon,
\end{align*} \]
which implies, that the strategy profile \( \varphi_k \) is an \( \mu_k \)-equilibrium in \( G_{B_k} \), where
\[ \mu_k = \max \left\{ 4\varepsilon^a \cdot r_{max} + \varepsilon; \max_{n \in \mathcal{N}} \left\{ 12\varepsilon^a \cdot r_{max} P_{(\psi_k^{-n}, \varepsilon)}(\tau_{B_k} < i_{k+1} - 1) + 2\varepsilon \right\} \right\}. \]

4.c) Global estimate

The aim is to show, that \( \pi \) is an \( \hat{\eta}_k \)-equilibrium in the given quitting game \( G \) under the assumption that the blocks of type I are regularly scattered for all players \( n \in \mathcal{N}. \) According to Definition 2.3.10 (see p.
\[ P_{(\pi^{-n}, \varepsilon)}(\tau < i_{k+1} | \tau \geq i_k) \geq \varepsilon^b \]
for all sets \( B_k = \{i_k, \ldots, i_{k+1} - 1\}, k \in \mathcal{N}, \) which are of type I for player \( m \in \mathcal{N}. \)

Consider the player \( n \in \mathcal{N}: \) Define a sequence of random variables \( (Z^n_k)_{k \in \mathcal{N}} \) over the probability space \((\Omega, \mathcal{A}, P)\) by
\[ Z^n_k(\omega) := \begin{cases} 
\gamma^n(\pi_k) & \text{if } \tau(\omega) \geq i_k \\
\gamma^n(Y_{\tau}(\omega)) & \text{if } \tau(\omega) < i_k
\end{cases} \]
for all \( \omega \in \Omega, k \in \mathcal{N}. \)

Remark 2.3.29. For \( i_1 = 1, \) we have \( Z^n_k(\omega) = \gamma^n(\pi) \) for all \( \omega \in \Omega, \) and hence
\[ \begin{align*}
E_{(\pi^{-n}, \varepsilon)}(Z^n_k) &= E_{(\pi^{-n}, \varepsilon)}(Z^n_k 1_{\{\tau < i_k\}} + Z^n_k 1_{\{\tau \geq i_k\}}) \\
&= E_{(\pi^{-n}, \varepsilon)}(Z^n_k 1_{\{\tau < i_k\}}) + E_{(\pi^{-n}, \varepsilon)}(Z^n_k 1_{\{\tau \geq i_k\}}) \\
&= E_{(\pi^{-n}, \varepsilon)}(r^n(Y_{\tau}) 1_{\{\tau < i_k\}}) + E_{(\pi^{-n}, \varepsilon)}(\gamma^n(\pi_k) 1_{\{\tau \geq i_k\}}) \\
&= E_{(\pi^{-n}, \varepsilon)}(r^n(Y_{\tau}) 1_{\{\tau < i_k\}}) + \gamma^n(\pi_k) \cdot E_{(\pi^{-n}, \varepsilon)}(1_{\{\tau \geq i_k\}}) \\
&= E_{(\pi^{-n}, \varepsilon)}(r^n(Y_{\tau}) 1_{\{\tau < i_k\}}) + \gamma^n(\pi_k) \cdot P_{(\pi^{-n}, \varepsilon)}(\tau \geq i_k)
\end{align*} \]
for all \( k \in \mathcal{N}. \)
Estimate 2.3.30. Let \( n \in \mathcal{N} \) be given, then
\[
\gamma^n((\pi^{n}, \mathfrak{c})) \leq \sup_{k \in \mathbb{N}} E_{(\pi^{n}, \mathfrak{c})}(Z^n_k).
\]

Proof. Observe that the blocks of type I are regularly scattered for all players \( n \in \mathcal{N} \). Thus the alternative strategy profile \((\pi^{n}, \mathfrak{c})\) is terminating and
\[
\gamma^n((\pi^{n}, \mathfrak{c})) = E_{(\pi^{n}, \mathfrak{c})}(r^n(Y_{\tau})I_{\{\tau < \infty\}}) \leq \sup_{k \in \mathbb{N}} E_{(\pi^{n}, \mathfrak{c})}(Z^n_k).
\]

Estimate 2.3.31.
\[
\gamma^n((\pi^{n}, \mathfrak{u}_i)) \leq \sup_{k \in \mathbb{N}} E_{(\pi^{n}, \mathfrak{c})}(Z^n_k) + \varepsilon + 4\varepsilon a \cdot r_{\max} \quad \forall k, i \in \mathbb{N}, \ n \in \mathcal{N}
\]

Proof. Choose \( i \in \mathbb{N} \) arbitrary but fixed and let \( B_k = \{i_k, \ldots, i_{k+1} - 1\} \) denote the set of stages, where \( i \in B_k \). Then for the expected payoff under the alternative strategy profile \((\pi^{n}, \mathfrak{u}_i)\),
\[
\gamma^n((\pi^{n}, \mathfrak{u}_i)) = E_{(\pi^{n}, \mathfrak{c})}(r^n(Y_{\tau})I_{\{\tau < i_k\}}) + E_{(\pi^{n}, \mathfrak{c})}(r^n(Y_{\tau})I_{\{\tau < i_k\}})
\]
\[
= E_{(\pi^{n}, \mathfrak{c})}(Z^n_k I_{\{\tau < i_k\}}) + P_{(\pi^{n}, \mathfrak{u}_i)}(\tau \geq i_k) \cdot E_{(\pi^{n}, \mathfrak{c})}(r^n(Y_{\tau})|\tau \geq i_k)
\]
follows, where \( g^n((\varphi^n_k, \mathfrak{u}_i)) \) is the expected payoff for player \( n \) in the finite quitting game \( G_{B_k} \) under the strategy profile \( \varphi_k = (p_{i_k}, \ldots, p_{i_k+1-1}) \) (see p. 70). The inequalities (2.21) and (1.30) (see p. 33) yield
\[
\gamma^n((\pi^{n}, \mathfrak{u}_i)) \leq E_{(\pi^{n}, \mathfrak{c})}(Z^n_k I_{\{\tau < i_k\}}) + P_{(\pi^{n}, \mathfrak{u}_i)}(\tau \geq i_k) \cdot (g^n(\varphi_k) + 4\varepsilon a \cdot r_{\max} + \varepsilon)
\]
\[
\leq E_{(\pi^{n}, \mathfrak{c})}(Z^n_k I_{\{\tau < i_k\}}) + P_{(\pi^{n}, \mathfrak{u}_i)}(\tau \geq i_k) \cdot g^n(\varphi_k) + 4\varepsilon a \cdot r_{\max} + \varepsilon.
\]
Since \( i \in B_k \), respectively \( i \geq i_k \), we have
\[
P_{(\pi^{n}, \mathfrak{u}_i)}(\tau \geq i_k) = 1 - P_{(\pi^{n}, \mathfrak{u}_i)}(\tau < i_k) = 1 - P_{(\pi^{n}, \mathfrak{c})}(\tau < i_k)
\]
\[
= P_{(\pi^{n}, \mathfrak{c})}(\tau \geq i_k) = E_{(\pi^{n}, \mathfrak{c})}(I_{\{\tau \geq i_k\}}),
\]
which implies
\[
\gamma^n((\pi^{n}, \mathfrak{u}_i)) \leq E_{(\pi^{n}, \mathfrak{c})}(Z^n_k I_{\{\tau < i_k\}}) + E_{(\pi^{n}, \mathfrak{c})}(I_{\{\tau \geq i_k\}}) \gamma^n(\pi_k) + 4\varepsilon a \cdot r_{\max} + \varepsilon
\]
\[
= E_{(\pi^{n}, \mathfrak{c})}(Z^n_k I_{\{\tau < i_k\}}) + E_{(\pi^{n}, \mathfrak{c})}(\gamma^n(\pi_k) I_{\{\tau \geq i_k\}}) + 4\varepsilon a \cdot r_{\max} + \varepsilon
\]
\[
= E_{(\pi^{n}, \mathfrak{c})}(Z^n_k I_{\{\tau < i_k\}}) + E_{(\pi^{n}, \mathfrak{c})}(Z^n_k I_{\{\tau \geq i_k\}}) + 4\varepsilon a \cdot r_{\max} + \varepsilon
\]
\[
= E_{(\pi^{n}, \mathfrak{c})}(Z^n_k)
\]
\[
\leq \sup_{k \in \mathbb{N}} E_{(\pi^{n}, \mathfrak{c})}(Z^n_k) + 4\varepsilon a \cdot r_{\max} + \varepsilon.
\]
Estimate 2.3.32. Let \( n \in \mathcal{N} \) be a given player, then
\[
\sup_{k \in \mathcal{N}} E_{(\pi^n, c)}(Z^n_k) \leq \gamma^n(\pi) + 2\varepsilon^{1-b-d} + 12r_{\max} \varepsilon^a.
\]

Proof. As shown in 4.b) (cf. (2.39)),
\[
g^n((\varphi^n_k, \bar{c})) \leq g^n(\varphi_k) + 12r_{\max} \varepsilon^a P_{(\varphi^n_k, c)}(\tau_{B_k} < i_k + 1) + 2\varepsilon
\]
holds for all \( n \in \mathcal{N} \). Since
\[
g^n((\varphi^n_k, \bar{c})) = E_{(\pi^n, c)}(r^n(Y_\tau)I_{\{\tau < i_k + 1\}} + \gamma^n(\pi_{i_k + 1})I_{\{\tau \geq i_k\}} \mid \tau \geq i_k)
\]
\[
= E_{(\pi^n, c)}(Z^n_{k+1} \mid \tau \geq i_k),
\]
\[
g^n(\varphi_k) = \gamma^n(\pi_{i_k}) = E_{(\pi^n, c)}(\gamma^n(\pi_{i_k}) \mid \tau \geq i_k) = E_{(\pi^n, c)}(Z^n_{k} \mid \tau \geq i_k)
\]
and
\[
P_{(\varphi^n_k, c)}(\tau_{B_k} < i_k + 1) = P_{(\pi^n, c)}(\tau < i_k + 1 \mid \tau \geq i_k),
\]
we get
\[
E_{(\pi^n, c)}(Z^n_{k+1} \mid \tau \geq i_k) \leq E_{(\pi^n, c)}(Z^n_{k} \mid \tau \geq i_k) + 12r_{\max} \varepsilon^a P_{(\pi^n, c)}(\tau < i_k + 1 \mid \tau \geq i_k) + 2\varepsilon.
\]

By multiplication with \( P_{(\pi^n, c)}(\tau \geq i_k) \), one obtains
\[
P_{(\pi^n, c)}(\tau \geq i_k) E_{(\pi^n, c)}(Z^n_{k+1} \mid \tau \geq i_k)
\]
\[
\leq P_{(\pi^n, c)}(\tau \geq i_k) E_{(\pi^n, c)}(Z^n_{k} \mid \tau \geq i_k) + 12r_{\max} \varepsilon^a \cdot P_{(\pi^n, c)}(\tau \leq i_k + 1) + 2\varepsilon P_{(\pi^n, c)}(\tau \geq i_k)
\]
\[
\leq P_{(\pi^n, c)}(\tau \geq i_k) E_{(\pi^n, c)}(Z^n_{k} \mid \tau \geq i_k) + 12r_{\max} \varepsilon^a \cdot P_{(\pi^n, c)}(\tau \in B_k)
\]
\[
+ 2\varepsilon P_{(\pi^n, c)}(\tau \geq i_k),
\]
which is equal to
\[
E_{(\pi^n, c)}(Z^n_{k+1} I_{\{\tau \geq i_k\}})
\]
\[
\leq E_{(\pi^n, c)}(Z^n_{k} I_{\{\tau \geq i_k\}}) + 12r_{\max} \varepsilon^a \cdot P_{(\pi^n, c)}(\tau \in B_k) + 2\varepsilon P_{(\pi^n, c)}(\tau \geq i_k). \tag{2.40}
\]

With \( Z^n_{k+1} = Z^n_{k} \) for \( \tau < i_k \),
\[
E_{(\pi^n, c)}(Z^n_{k} I_{\{\tau < i_k\}}) = E_{(\pi^n, c)}(r^n(Y_\tau)I_{\{\tau < i_k\}}) = E_{(\pi^n, c)}(Z^n_{k+1} I_{\{\tau < i_k\}})
\]
holds. Adding \( E_{(\pi^n, c)}(r^n(Y_\tau)I_{\{\tau < i_k\}}) \) on both sides of (2.40) leads to
\[
E_{(\pi^n, c)}(Z^n_{k+1}) \leq E_{(\pi^n, c)}(Z^n_{k}) + 12r_{\max} \varepsilon^a P_{(\pi^n, c)}(\tau \in B_k) + 2\varepsilon P_{(\pi^n, c)}(\tau \geq i_k)
\]
and summation over \( k = 1 \) to \( M \in \mathbb{N} \) implies
\[
\sum_{k=1}^{M} E_{(\pi^n,c)}(Z^n_k) \\
\leq \sum_{k=1}^{M} \left( E_{(\pi^n,c)}(Z^n_k) + 12r_{\text{max}}\epsilon^a \cdot P_{(\pi^n,c)}(\tau \in B_k) + 2\varepsilon P_{(\pi^n,c)}(\tau \geq i_k) \right) \\
\leq \sum_{k=1}^{M} E_{(\pi^n,c)}(Z^n_k) + 12r_{\text{max}}\epsilon^a \cdot 1 + 2\varepsilon \sum_{k\in\mathbb{N}} P_{(\pi^n,c)}(\tau \geq i_k).
\]
This leads to
\[
E_{(\pi^n,c)}(Z^n_M) \leq E_{(\pi^n,c)}(Z^n_1) + 12r_{\text{max}}\epsilon^a + 2\varepsilon \sum_{k\in\mathbb{N}} P_{(\pi^n,c)}(\tau \geq i_k)
\]
and finally
\[
\sup_{M\in\mathbb{N}} E_{(\pi^n,c)}(Z^n_M) \leq E_{(\pi^n,c)}(Z^n_1) + 12r_{\text{max}}\epsilon^a + 2\varepsilon \sum_{k\in\mathbb{N}} P_{(\pi^n,c)}(\tau \geq i_k) = \left( \gamma^n(\pi) + 12r_{\text{max}}\epsilon^a + 2\varepsilon \sum_{k\in\mathbb{N}} P_{(\pi^n,c)}(\tau \geq i_k) \right) \tag{2.41}
\]
The blocks of type I are regularly scattered, i.e. two blocks of type I have a distance not more than \([1/\varepsilon^d]\). So in the first \( \delta \cdot ([1/\varepsilon^d] + 1) \) blocks, there are at least \( \delta \) blocks of type I for all players \( n \in \mathcal{N} \). If \( B_k \) is a block of type I for a player \( n \in \mathcal{N} \), then
\[
P_{(\pi^n,c)}(\tau < i_{k+1} | \tau \geq i_k) \geq \epsilon^b \implies P_{(\pi^n,c)}(\tau \geq i_{k+1} | \tau \geq i_k) \leq 1 - \epsilon^b \\
\implies P_{(\pi^n,c)}(\tau \geq i_{\delta \cdot ([1/\varepsilon^d] + 1)}) \leq (1 - \epsilon^b)^\delta.
\]
Furthermore one has
\[
P_{(\pi^n,c)}(\tau \geq i_k) \leq P_{(\pi^n,c)}(\tau \geq i_{\delta \cdot ([1/\varepsilon^d] + 1)})
\]
for all \( k \geq \delta \cdot ([1/\varepsilon^d] + 1) \) and therefore
\[
\sum_{k\in\mathbb{N}} P_{(\pi^n,c)}(\tau \geq i_k) \leq \left( \left\lceil \frac{1}{\varepsilon^d} \right\rceil + 1 \right) \cdot \sum_{\delta \in \mathbb{N}_0} P_{(\pi^n,c)}(\tau \geq i_{\delta \cdot ([1/\varepsilon^d] + 1)}) \\
\leq \left( \frac{1}{\varepsilon^d} + 2 \right) \cdot \sum_{\delta \in \mathbb{N}_0} (1 - \epsilon^b)^\delta \\
= \left( \frac{1}{\epsilon^d} + 2 \right) \cdot \frac{1 - \varepsilon^b}{\epsilon^b},
\]
where \( i_0 := 1 = i_1 \). Inserting this in (2.41) yields
\[
\sup_{M\in\mathbb{N}} E_{(\pi^n,c)}(Z^n_M) \leq \gamma^n(\pi) + 2\epsilon^{1-b-d} + 4\epsilon^{1-b} + 12r_{\text{max}}\epsilon^a.
\]
Overview: Estimates in Global estimate

Estimate 2.3.30: Let \( n \in \mathbb{N} \) be given, then
\[
\gamma^n((\pi^n, c)) \leq \sum_{k \in \mathbb{N}} \mathbb{E}_{(\pi^n, c)}(Z^n_k).
\]

Estimate 2.3.31: \( \gamma^n((\pi^n, q_i)) \leq \sup_{k \in \mathbb{N}} \mathbb{E}_{(\pi^n, c)}(Z^n_k) + \varepsilon + 4\varepsilon r_{\max} \quad \forall k, i \in \mathbb{N}, n \in \mathcal{N} \)

Estimate 2.3.32: Let \( n \in \mathcal{N} \) be a given player, then
\[
\sum_{k \in \mathbb{N}} \mathbb{E}_{(\pi^n, c)}(Z^n_k) \leq \gamma^n(\pi) + 2\varepsilon^{1-b-d} + 12r_{\max}\varepsilon^a.
\]

Estimates 2.3.30, 2.3.31 and 2.3.32 finally yield
\[
\gamma^n((\pi^n, c)) \leq \gamma^n(\pi) + 2\varepsilon^{1-b-d} + 4\varepsilon^{1-b} + 12r_{\max}\varepsilon^a
\]
and
\[
\gamma^n((\pi^n, q_i)) \leq \gamma^n(\pi) + 2\varepsilon^{1-b-d} + 4\varepsilon^{1-b} + 15r_{\max}\varepsilon^a + \varepsilon
\]
for all \( n \in \mathcal{N} \) and all \( i \in \mathbb{N} \), which implies that \( \pi \) is an \( \tilde{\eta}_\varepsilon \)-equilibrium in \( G \), where
\[
\tilde{\eta}_\varepsilon = 2\varepsilon^{1-b-d} + 4\varepsilon^{1-b} + 15r_{\max}\varepsilon^a + \varepsilon.
\]

Now we are done with the proof of Proposition 2.3.8.

Remark to \( \varepsilon = 0 \)

The statement of Proposition 2.3.8 is also true for \( \varepsilon = 0 \). In that case, \( r_{\{n\}}^n \geq 0 \) and the requirement that \( G \) is terminating under every subgame strategy profile \( \pi_i \), are not necessary. Furthermore the property that \( p_i \) is perfect in \( \Gamma_{\gamma(\pi_{i+1})} \) can be replaced by the weaker property that \( p_i \) is an equilibrium in \( \Gamma_{\gamma(\pi_{i+1})} \). The next lemma summarizes these facts.

**Corollary 2.3.33.** Let \( G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})}) \) be a quitting game and \( \pi = (p_i)_{i \in \mathbb{N}} \) a strategy profile in \( G \). Assume that \( p_i \) is perfect in \( \Gamma_{\gamma(\pi_{i+1})} \) for every \( i \in \mathbb{N} \). Then \( \pi \) is a subgame equilibrium in \( G \).

**Proof.** Because of \( \varepsilon = 0 \), we obtain for the block-construction from the proof of Proposition 2.3.8, that the blocks \( B_k \) consist only of one stage, that means \( B_k := \{k\} \) for all \( k \in \mathbb{N} \).
2.4. Equivalent conditions for the existence of approximate equilibria in quitting games

Furthermore all blocks are of type I and we are situated in 4. Case 2.: Blocks of type I are regularly scattered. With Remark 2.3.23 we only have to consider the global estimation.

As shown in Estimate 2.3.30

\[ \gamma^n((\pi^{-n}, c)) \leq \sup_{k \in \mathbb{N}} E_{(\pi^{-n}, c)}(Z_k^n). \]

For \( \gamma^n((\pi^{-n}, q_k)) \) we have

\[
\begin{align*}
\gamma^n((\pi^{-n}, q_k)) &= E_{(\pi^{-n}, c)}(Z_k^n \mathbb{1}_{\{\tau < k\}}) + P_{(\pi^{-n}, q_k)}(\tau \geq k) \cdot g^n((\varphi_k^{-n}, q_k)) \\
&= E_{(\pi^{-n}, c)}(Z_k^n \mathbb{1}_{\{\tau < k\}}) + P_{(\pi^{-n}, q_k)}(\tau \geq k) \cdot \gamma^n(\pi_{k+1})(p_k^{-n}, 1).
\end{align*}
\]

Since \( p_k \) is perfect in \( \Gamma_{\gamma(\pi_{k+1})} \)

\[
\begin{align*}
\gamma^n((\pi^{-n}, q_k)) &\leq E_{(\pi^{-n}, c)}(Z_k^n \mathbb{1}_{\{\tau < k\}}) + P_{(\pi^{-n}, q_k)}(\tau \geq k) \cdot \gamma^n(\pi_{k+1})(p_k) \\
&= E_{(\pi^{-n}, c)}(Z_k^n \mathbb{1}_{\{\tau < k\}}) + E_{(\pi^{-n}, c)}(\mathbb{1}_{\{\tau \geq k\}}) \gamma^n(\pi_k) \\
&= E_{(\pi^{-n}, c)}(Z_k^n) + E_{(\pi^{-n}, c)}(Z_k^n \mathbb{1}_{\{\tau \geq k\}}) \\
&\leq \sup_{k \in \mathbb{N}} E_{(\pi^{-n}, c)}(Z_k^n).
\end{align*}
\]

The same arguments like in 4. Global estimate leads to

\[ \sup_{k \in \mathbb{N}} E_{(\pi^{-n}, c)}(Z_k^n) \leq E_{(\pi^{-n}, c)}(Z_1^n) \leq \gamma^n(\pi) \]

for all players \( n \).

2.4. Equivalent conditions for the existence of approximate equilibria in quitting games

Robert Simon published in the paper “The structure of non-zero-sum stochastic games” ([31]) some equivalent formulations for the existence of approximate equilibria in quitting games. We end this chapter by quoting his results.

We start by introducing some new definitions.

**Definition 2.4.1** (min-max value, \((\varepsilon-)\)min-max profile). Let \( G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})}) \) be a given quitting game. For all players \( n \in \mathcal{N} \), the value \( \chi^n \), defined by

\[
\chi^n := \inf_{\pi \in \Pi} \max_{\mathbb{F}^n \in \Pi^n} \gamma^n((\pi^{-n}, \mathbb{F}^{-n})) \tag{2.42}
\]
Chapter 2. Equilibria in Quitting games

is called min-max value for the player $n$ in $G$. Furthermore a strategy profile
\[
\pi^{-n} \in [0,1]^{|N|-1} \times \mathbb{N}
\]
is called $\varepsilon$-min-max profile for player $n$ in $G$, if and only if
\[
\max_{\bar{x}^n \in \Pi^n} \gamma^n((\pi^{-n}, \bar{x}^n)) \leq \chi^n + \varepsilon.
\]

**Definition 2.4.2** (normal player). Let $G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})})$ be a given quitting game. A player $n \in \mathcal{N}$ is called normal player, if and only if
\[
\tau^n \geq \chi^n
\]
holds.

So a normal player always has the incentive to *quit* the game alone.

**Definition 2.4.3** (feasible, $\varepsilon$-rational). Let $G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})})$ be a given quitting game and $\varepsilon \geq 0$. A vector $v \in \mathbb{R}^N$ is called feasible if, and only if
\[
v \in \text{conv}\{r_S \mid S \in \mathcal{P}(\mathcal{N})\} =: \mathcal{I}_\varepsilon.
\]
The vector $v \in \mathbb{R}^N$ is called $\varepsilon$-rational, if and only if
\[
v^n \geq \chi^n - \varepsilon
\]
holds for all $n \in \mathcal{N}$.

**Notation 2.4.4.** Let a quitting game $G$ and an $\varepsilon \geq 0$ be given. Then the correspondences $E_\varepsilon := E^G_\varepsilon \subseteq \mathbb{R}^N \times [0,1]^N$ and $F_\varepsilon := F^G_\varepsilon \subseteq \mathbb{R}^N \times \mathbb{R}^N$ are defined by
\[
E_\varepsilon(v) := E^G_\varepsilon(v) := \{ p \in [0,1]^N \mid p \varepsilon\text{-perfect in } \Gamma_v \}
\]
respectively
\[
F_\varepsilon(v) := F^G_\varepsilon(v) := \{ \gamma_v(p) \mid p \in E_\varepsilon(v) \}
\]
for all $v \in \mathbb{R}^N$.

**Definition 2.4.5** (finite orbit, infinite orbit, extended orbit). Let $G$ be a given quitting game and $\varepsilon \geq 0$. A sequence $(v_1, \ldots, v_l)$ with $l \in \mathbb{N} \cup \{\infty\}$, $v_i \in \mathbb{R}^N$ for all $i \in \{1, \ldots, l\}$, is called
- finite orbit, if $l < \infty$ and $(v_i, v_{i+1}) \in F_\varepsilon$ for every $i = 1, \ldots, l-1$.
- infinite orbit, if $l = \infty$ and $(v_i, v_{i+1}) \in F_\varepsilon$ for every $i \in \mathbb{N}$.

A sequence $(s_1, s_2, \ldots, s_L)$ of sequences $s_j = (v_{j,1}, v_{j,2}, \ldots, v_{j,n_j})$ with $n_j, L \in \mathbb{N} \cup \{\infty\}$, $j = 1, \ldots, L$, is called extended orbit, if $(v_{j,i}, v_{j,i+1}) \in F_\varepsilon$ for every $i = 1, \ldots, n_j - 1$ and all $j = 1, \ldots, L$ and $\lim_{i \to \infty} v_{j,i} = v_{j+1,1}$ if $n_j = \infty$ and $v_{j,n_j} = v_{j+1,1}$; $j = 1, \ldots, L$, otherwise.

The extended orbit has bounded total variation, if
\[
\sum_{j=1}^{L} \sum_{i=1}^{n_j} \| v_{j,i} - v_{j,i-1} \| < \infty,
\]
and unbounded total variation otherwise.
2.4. Equivalent conditions for the existence of approximate equilibria in quitting games

Now to the theorem from Robert Simon (see [31] p. 18):

**Theorem 2.4.6.** Let $G$ be a quitting game with neither stationary approximate equilibria nor instant approximate equilibria under punishment. Then the following propositions are equivalent:

(a) the game has approximate equilibria;

(b) for all $\varepsilon > 0$ there is a cyclic strategy profile $\pi = (p_1, \ldots, p_k, p_1, \ldots)$ with

(i) $\gamma(\pi_i) \in F_\varepsilon(\gamma(\pi_i+1))$ for all $i \in \mathbb{N}$;

(ii) $\gamma(\pi_i)$ $\varepsilon$-rational for all $i \in \mathbb{N}$ and

(iii) $1 - \varrho(p_i, \emptyset) > 0$ for at least one $i$ with $1 \leq i \leq k$;

(c) for all $\varepsilon > 0$ and all $b > 1$ a finite orbit of $F_\varepsilon$ of $\varepsilon$-rational vectors within a distance of $\varepsilon$ of feasible vectors with a total variation of at least $b$ exists;

(d) for all $\varepsilon > 0$ an infinite orbit of $F_\varepsilon$ of $\varepsilon$-rational vectors with unbounded total variation exists;

(e) for all $\varepsilon > 0$ an infinite extended orbit of $F_\varepsilon$ of $\varepsilon$-rational vectors with unbounded total variation exists.

**Remark 2.4.7.** The proof that (d) implies (a) and (b) is based on the proof of Theorem 2.3.3.

This theorem shows that a general algorithm for the detection of an $\varepsilon$-equilibrium for a given quitting game should consists of three parts. The first one is an algorithm detecting an instant $\varepsilon$-equilibrium under punishment. The second one is an algorithm which searches for a stationary $\varepsilon$-equilibrium and the third one is an algorithm which finds a cyclic $\varepsilon$-equilibrium for a given quitting game $G$. Theorem 2.4.6 also implies that in order to show the non-existence of approximate equilibria for a given quitting game with the help of an algorithm, a first step is to exclude stationary approximate equilibria and instant approximate equilibria under punishment.
Chapter 3.

Basic algorithms and software

As already indicated in the introduction of this work, no algorithm or software is known for the analysis of quitting games respectively stochastic games in general. One reason for that may be the complexity of such algorithms. For one-step games and finite games, we have in the case of a two-player zero-sum game that the search for a Nash-equilibrium can be formulated in terms of linear programming (see, e.g. [22]), which can be solved efficiently. The articles “The Complexity of Computing a Nash Equilibrium” ([6]) and “On the complexity of Nash equilibria and other fixed points” ([11]) show that it is more complicated for multi-player respectively non-zero-sum games. Kousha Etessami and Mihalis Yannakakis proved that the calculation of an $\varepsilon$-equilibrium by application of Brouwer’s fixed point theorem is NP-hard$^1$.

We want to cite McLennen (cf. [26]), who pointed out very well, why such programs or algorithms are nevertheless needed: “[Consider] the complexity of the problem of computing the set of all Nash equilibria, as measured by the concepts of theoretical computer science. Standard notions of complexity depend on the rate at which the resources (time and/or memory) required by an algorithm grow as the size of the input increases. The most fundamental division is between algorithms whose running times grow at rates that are bounded by polynomial functions of the size of the input, and those for which the rate of growth is exponential, or perhaps even faster. Generally, algorithms with polynomial time and space requirements are described as ‘practical’, while those with exponential rates of growth are regarded as ‘impractical’. But many interesting games are small, and still hard for people to solve by hand, so in game theory even exponential algorithms have considerable practical utility.”

We take this as motivation to develop algorithms respectively programs for the analysis of quitting games.

This chapter is structured in the following way, which is similar to the usual recipe game theorists would use to analyze a game (see e.g. [37]). In Section 3.1, we consider symmetric

$^1$NP is the set of all problems, which can be solved with the help of nondeterministic algorithms in polynomial time, cf. [5] pp. 160.
Chapter 3. Basic algorithms and software

quitting games and present two algorithms\(^2\). The first one checks whether a given quitting game is symmetric, and the second one determines an equilibrium for a given symmetric quitting game. We implemented these algorithms in Fortran 90. The executable is called `qg_symmetry` and described at the end of the section\(^3\).

In the case that the given quitting game is not symmetric, one starts to test, if a reduction of the game is possible. This case occurs, if at least one player has a dominated strategy. We state an algorithm for the reduction by dominance in Section 3.2. This algorithm was implemented in Fortran 90, as well. The routine `qg_dominant` is explained at the end of the section, the output is – if reduction is possible – a reduced game which can be further analyzed.

Section 3.3 deals with pure \(\varepsilon\)-equilibria. Again an algorithm is stated and the corresponding implemented program called `qg_pure` is described. Playing pure strategies can be seen as restriction of the support set of a game, i.e. the set of actions which are played with a positive probability. In case of pure \(\varepsilon\)-equilibria, one would allow the players only to play one action for sure at each stage. The next step in the analysis of a game would be to allow all but one player to play both actions. The single player has to play `quit` with certainty. As known from the previous chapters, we call such equilibria instant \(\varepsilon\)-equilibria. Now we consider a special kind of those equilibria called instant \(\varepsilon\)-equilibria under punishment. It is called punishment because, if the single player does not play `quit` in the first stage of the game, he is punished by the other players by playing his min-max profile. In Section 3.4, the necessary algorithms are stated and the implemented routine `qg_instant`, which offers the user several options, is documented. This section finishes with a discussion on the presented algorithm.

In Chapter 1 and 2, we have introduced another important kind of equilibria, the so-called stationary equilibria. In Section 3.5, we construct an algorithm to detect those equilibria. The implementation is named `qg_stationary` and explained, as well.

We close this chapter with a discussion on cyclic \(\varepsilon\)-equilibria, where we state an intuitive algorithm and mention the problems and questions, which occur if one wants to implement it.

One remark at the end: We are mainly interested in finding a sample \(\varepsilon\)-equilibrium, but all programs can be used with a slightly modification to find more than one.

\(^2\)All algorithms are presented in a so called pseudocode, which is close to a programming language.
\(^3\)The software is free and available on request directly from the author or from the Institut für Mathematische Stochastik der Technischen Universität Dresden.Hints referring bugs or implementation alternatives are welcome.
3.1. Symmetric quitting games

In this section we consider an algorithm to detect – or better to determine – equilibria in symmetric quitting games. In this context we first provide an algorithm, which checks, whether a given quitting game has a symmetric payoff structure or not.

Test-algorithm for symmetry

Let a quitting game $G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})})$ be given. The algorithm analyzes the family of payoff vectors. If the game is symmetric, the output of the algorithm are two families of numbers $(\alpha_k)_{k \in \mathbb{N}}$ and $(\beta_k)_{k \in \{1, \ldots, N-1\}}$ such that $\alpha_k$ respectively $\beta_k$ is the payoff to the players who play quit respectively continue, where the quitting coalition $S$ has the size $k$ (see Definition 2.1.1, p. 43).

Algorithm 1: Test for symmetry

<table>
<thead>
<tr>
<th>Algorithm 1: Test for symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong> : $\mathcal{N}$, $(r_S)_{S \in \mathcal{P}(\mathcal{N})}$</td>
</tr>
<tr>
<td><strong>Initialization</strong>: sym := true, $S := \emptyset$, $\alpha_N = r_N^1$</td>
</tr>
<tr>
<td>for $n = 1$ to $N - 1$ do</td>
</tr>
<tr>
<td>$S := S \cup {n}$</td>
</tr>
<tr>
<td>$\alpha_n = r_n^S$</td>
</tr>
<tr>
<td>$\beta_n = r_N^S$</td>
</tr>
<tr>
<td>forall the $S \in \mathcal{P}(\mathcal{N}) \setminus {\emptyset}$ do</td>
</tr>
<tr>
<td>$k :=</td>
</tr>
<tr>
<td>for $n = 1$ to $N$ do</td>
</tr>
<tr>
<td>if $(n \in S$ and $\alpha_k \neq r_n^S$) or $(n \in \mathcal{N} \setminus S$ and $\beta_k \neq r_n^S)$ then</td>
</tr>
<tr>
<td>sym := false</td>
</tr>
<tr>
<td>exit loop</td>
</tr>
<tr>
<td>Output: If sym = true then $(\alpha_k)<em>{k \in \mathcal{N}}$ and $(\beta_k)</em>{k \in {1, m, \ldots, N-1}}$ else the quitting game is not symmetric.</td>
</tr>
</tbody>
</table>

Remark 3.1.1. The run-time complexity of this algorithm for the worst-case scenario, which occurs always, if the given game is symmetric, is $\mathcal{O}(N2^N)$ – because all payoffs for the players have to be compared.

Algorithm to determine an equilibrium in a symmetric quitting game

If the given quitting game $G$ is symmetric, the following algorithm shows how one sample equilibrium for $G$ can be determined.
Chapter 3. Basic algorithms and software

Algorithm 2: Determination of an equilibrium for a symmetric quitting game

\begin{verbatim}
Input : \( N \), \((\alpha_k)_{k \in \mathbb{N}}, (\beta_k)_{k \in \{1, \ldots, N-1\}}\)

if \( \alpha_1 \leq 0 \) then
    \( p := c \)
else if \( \alpha_N \geq \beta_{N-1} \) then
    \( p := q \)
else
    \( \beta_0 := 0 \)
    for \( n = 2 \) to \( N-1 \) do
        if \( \alpha_{n+1} \leq \beta_n \) and \( \alpha_n \geq \beta_{n-1} \) then
            \( p^1 := \ldots := p^n := 1 \)
            \( p^{n+1} := \ldots := p^N := 0 \)
            exit loop

Output: \( p \), where \( \pi = (p, p, \ldots) \) is a Nash-equilibrium in the given quitting game.
\end{verbatim}

In Section 2.1, we mentioned that every symmetric quitting game has a pure, stationary equilibrium. As one can see in the proof of this proposition, the algorithm is going to terminate in one of the three steps and has an equilibrium profile as result. The run-time complexity of this algorithm for the worst-case scenario, which occurs, if the algorithm stops in the third case at the end of the loop for \( n = N-1 \), is \( \mathcal{O}(N) \).

\texttt{qg\_symmetry}

This module reads a game from standard input and tests, whether the quitting game is symmetric or not. If the game is symmetric, one Nash-equilibrium will be determined. The output of this routine – if the game is symmetric – consists of two sequences \((\alpha_k)_{k \in \mathbb{N}}\) and \((\beta_k)_{k \in \{1, \ldots, N-1\}}\) together with the probability vector \( p \), where \( \pi = (p, p, \ldots) \) is a Nash-equilibrium in the given game.

Example 4. We consider the following three-player quitting game:

\begin{tabular}{|c|c|c|}
\hline
Player 3 & c & Player 2 \\
\hline
Player 1 & c & q \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|}
\hline
Player 3 & q & Player 2 \\
\hline
Player 1 & c & q \\
\hline
\end{tabular}

(see Example1.txt, Appendix B, for the input file).
Then the output of `qg_symmetry` is the file `results.txt` with the following content:

```
The quitting game is symmetric.

alpha = 1 0 2  
beta = 1 3
One Nash-equilibrium is generated by  
p = 1 0 0
```

### 3.2. Reduction by dominance

As we have seen in Chapter 2 (cf. Section 2.2), it could be useful to test, if a given game has dominant respectively dominated strategies. We focus here on strongly dominant strategies, since we can reduce the considered game (cf. Remark 2.2.4 3.).

In detail: Assume that \( \pi^m = (p^m, p^m, \ldots) \) is a strongly dominant strategy for player \( m \). By definition we consider only stationary strategies, and in Remark 2.2.4 we argued that \( \pi^m \) is pure, so in fact this leads to the following two cases:

1. If \( p^m = 1 \), i.e. player \( m \) plays always `quit`, the given game has at least one Nash equilibrium, which can be determined by the consideration of a reduced game, which is a one-step game.

2. If \( p^m = 0 \), i.e. player \( m \) plays always `continue`, it is sufficient to consider a reduced game, which is again a quitting game.

#### Algorithm

We present an optimized algorithm for the reduction by dominance. If a dominant strategy for a player \( n \in \mathcal{N} \) exists, we immediately consider the reduced game without player \( n \) and the corresponding payoffs for the other players. Therefore the resulting strategies do not have to be strongly dominant strategies in the given quitting game.
Chapter 3. Basic algorithms and software

Algorithm 3: Reduction by dominance

**Input**: \( \mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})} \)

**Initialization**: \( \mathcal{N}_q := \emptyset, \mathcal{M} := \mathcal{N} \)

for \( n = 1 \) to \( N \) do

- if \( r^n_{\{n\}} < 0 \) then \( \text{dom}(n) := 0 \)
- else
  - if \( r^n_{\{n\}} > 0 \) then \( \text{dom}(n) := 1 \)
  - else \( \text{dom}(n) := 2 \)

(Player 1)

if \( \text{dom}(1) \neq 2 \) then

for \( m = 2 \) to \( N \) do

- if \( (r^1_{\{1,m\}} \leq r^1_{\{m\}} \text{ and } \text{dom}(1) = 1) \text{ or } (r^1_{\{1,m\}} \geq r^1_{\{m\}} \text{ and } \text{dom}(1) = 0) \) then
  - \( \text{dom}(1) := 2, \text{ exit loop} \)

if \( \text{dom}(1) \neq 2 \) then

\( \mathcal{M} := \mathcal{M} \setminus \{1\} \)

if \( \text{dom}(1) = 1 \) then \( \mathcal{N}_q := \{1\} \)

(all the other players)

for \( n = 2 \) to \( N \) do

- \( S := \mathcal{N}_q \)
- for all \( m \in \mathcal{M} \) do
  - \( S := S \cup \{m\} \)
  - if \( (r^n_{\{n\}} \cup S \leq r^n_{\{n\}} \text{ and } \text{dom}(n) = 1) \text{ or } (r^n_{\{n\}} \cup S \geq r^n_{\{n\}} \text{ and } \text{dom}(n) = 0) \) then
    - \( \text{dom}(n) := 2, \text{ exit loop} \)

  if \( \text{dom}(n) \neq 2 \) then

  - \( \mathcal{M} := \mathcal{M} \setminus \{n\} \)
  - if \( \text{dom}(n) = 1 \) then \( \mathcal{N}_q := \mathcal{N}_q \cup \{n\} \)

Output: \( \mathcal{N}_q, \mathcal{M} \), where \( \mathcal{N}_q \) contains all players, for which playing \textit{quit} all the time is strongly dominant, and \( \mathcal{N} \setminus (\mathcal{M} \cup \mathcal{N}_q) \) contains all players, for which playing \textit{continue} all the time is strongly dominant.

\textbf{qg\_dominant}

\textbf{qg\_dominant} reads in a game in standard input (see Appendix B). The program determines strongly dominated strategies (in the reduction sense as explained before) and returns a reduced game, if such a strategy exists. Therefore the user has to provide the name of a file, to which the reduced game is stored. All other results such as how many players have strongly dominant strategies and which player has which dominant strategy, are written to the file \textit{results.txt}.
Example 5. Consider the following three-player game (see Example2.txt, Appendix B):

<table>
<thead>
<tr>
<th>Player 1</th>
<th>c</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>(-1,0,0)</td>
<td>(0,1,0)</td>
</tr>
<tr>
<td>q</td>
<td>(1,0,1)</td>
<td>(1,1,0)</td>
</tr>
</tbody>
</table>

The file containing the reduced game reads as:

1 player/s have a dominant strategy, which are:
Player 2 dominant strategy: 1

The content of the file, where the reduced game is stored is:

```
2
0.0000000 0.0000000
1.0000000 0.0000000
0.0000000 -1.0000000
0.0000000 3.0000000
One-step game
```

3.3. Pure $\varepsilon$-equilibria

In this section, we want to construct an algorithm, which finds a sample pure $\varepsilon$-equilibrium for a given quitting game $G$, if it exists. Instead of testing all possible pure strategy profiles $\pi = (p_1, p_2, \ldots)$ with $p_i \in \{0, 1\}^N$, $i \in \mathbb{N}$, we apply Corollary 1.4.7 (see p. 39) which gives us the following: Assume that a pure $\varepsilon$-equilibrium $\pi$ in a given quitting game $G$ exists, then either one of the three cases occurs:

1. All players play continue with certainty all the time, i.e. $\pi = (\pi^1, \ldots, \pi^N)^T$ with $\pi^n = c$ for all $n \in \mathcal{N}$.

2. One player $m$ plays quit in the first stage, while the other players play continue with certainty in the first stage, i.e. $p_1^m = 1$ and $p_1^{-m} = 0$.

3. At least two player play quit in the first stage, i.e. $m, n \in \mathcal{N}$ exist such that $p_1^m = p_1^n = 1$.

or another pure $\varepsilon$-equilibrium in $G$ exists, such that one of the aforementioned cases applies.

We study all three cases in detail, referring to the property that $\pi$ should be an $\varepsilon$-equilibrium.
For the first case we have: Since \( \pi \) is stationary with Lemma 1.4.8 and Remark 1.4.9, we obtain, that \( \pi \) is an \( \epsilon \)-equilibrium, if and only if \( r_n \leq \epsilon \) for all \( n \in \mathcal{N} \).

To the third case: Because the game terminates for sure in the first stage, \( \gamma(\pi) = \gamma_0(p_1) \) holds for the expected payoff in the quitting game under the strategy profile \( \pi \). Because two players play \textit{quit} in the first stage,

\[
\gamma\left(\left(\pi^{-k}, \tilde{x}^k\right)\right) = \gamma_0\left(\left(p_1^{-k}, \tilde{p}_1^k\right)\right)
\]

holds for all \( \tilde{x}^k = (\tilde{p}_1^k, \tilde{p}_2^k, \ldots) \in \Pi^k \) and all \( k \in \mathcal{N} \). Thus \( \pi \) is an \( \epsilon \)-equilibrium in \( G \) if, and only if \( p_1 \) is an \( \epsilon \)-equilibrium in \( \Gamma_0 \).

In the second case, the game terminates in the first stage for sure, as well, and for the expected payoff we have \( \gamma(\pi) = \gamma_0(p_1) = r_{\{m\}} \). All players \( n \in \mathcal{N} \setminus \{m\} \) are situated in a one-step game. \( \pi \) fulfills the \( \epsilon \)-equilibrium condition for these players in the quitting game, if and only if \( p_1 \) fulfills the \( \epsilon \)-equilibrium condition for these players in the one-step game \( \Gamma_0 \).

Consider the player \( m \), who plays \textit{quit} with certainty in the first stage: Since the game is terminating in stage one only because of her choice, we need to know, what the other players play in the next stages. This fact is important and occurs again, if we consider instant \( \epsilon \)-equilibria in general. In the pure case, we make the following two distinctions:

a) The other players play continue for certainty all the time. In that case, \( \pi \) is an \( \epsilon \)-equilibrium in \( G \) only if \( r_{\{m\}} \geq -\epsilon \).

b) Otherwise there exists at least one player \( n \neq m \), who plays \textit{quit} with certainty in a stage \( i, i \geq 2 \). Corollary 1.4.7 shows that it is sufficient to consider only the case, where \( p_2^n = 1 \) – otherwise we have a block of zero vectors in the profile without player \( m \), which can be eliminated here – and \( \pi \) is an \( \epsilon \)-equilibrium in \( G \) only if

\[
\gamma^m(\pi) = r_{\{m\}} \geq \tilde{p}_1^m \cdot r_{\{m\}} + (1 - \tilde{p}_1^m) \gamma^m_0(\left(p_2^{-m}, \tilde{p}_2^m\right)) - \epsilon
\]

for all \( \tilde{p}_1^m, \tilde{p}_2^m \in [0, 1] \).

In summary, it is sufficient to focus only on all possible combinations of the probability vectors \( p_1 \) and \( p_2 \) in order to verify, whether a given quitting game has a pure \( \epsilon \)-equilibrium or not.

**Algorithm**

For programming purposes, we transform the second case from the previous section into an equilibrium test in the context of one-step games. Therefore we define

\[
\chi_\pi^m := \min_{p^{-m} \in \{0, 1\}^{n-1}} \max_{p^m \in \{0, 1\}} \gamma^m_0 \left(\left(p^{-m}, p^m\right)\right)
\]
as lowest upper bound for the payoff that player \( m \) obtains in response to all pure strategy choices of the other players. We call \( \chi^m \) the pure min-max value for player \( m \). This implies for (3.1), that we have to test whether

\[
\gamma^m_0(p_2) = \chi^m_0 \quad \text{and} \quad p_i \in \{0,1\}^N \quad \text{for} \quad 3 \leq i \in \mathbb{N} \ \text{arbitrary, is a pure } \varepsilon\text{-equilibrium in } G.
\]

**Algorithm 4: Pure \( \varepsilon\)-equilibria**

\[
\text{Input : } N, (r_S)_{S \in \mathcal{P}(N)}, \varepsilon \\
\text{Initialization: } \text{pure} := \text{true}, \ S := 0 \\
\text{for } n = 1 \text{ to } N \text{ do} \\
\quad \text{if } \varepsilon < r^m_n \text{ then } \text{pure} := \text{false}, \text{ exit loop} \\
\text{if pure = true then } p_1 := 0 \\
\quad \text{else} \\
\quad \quad \text{for all the } S \subseteq N, \ S \neq \emptyset \text{ do} \\
\quad \quad \quad \text{pure} := \text{true} \\
\quad \quad \quad \text{if } |S| = 1 \text{ then} \\
\quad \quad \quad \quad \text{for } m = 1 \text{ to } N \text{ do} \\
\quad \quad \quad \quad \quad \text{if } m \in S \text{ then} \\
\quad \quad \quad \quad \quad \quad \text{call}\ \text{calculate}_m^0 \text{ (determine } p_2, \text{ such that } \gamma^m_0(p_2) = \chi^m_0) \\
\quad \quad \quad \quad \quad \quad \text{if } r^m_S < \chi^m_0 - \varepsilon \text{ then } \text{pure} := \text{false}, \text{ exit loop} \\
\quad \quad \quad \quad \quad \quad \text{else if } r^m_S < r^m_{S \cup \{m\}} - \varepsilon \text{ then } \text{pure} := \text{false}, \text{ exit loop} \\
\quad \quad \quad \quad \quad \text{else} \\
\quad \quad \quad \quad \quad \quad \text{for } m = 1 \text{ to } N \text{ do} \\
\quad \quad \quad \quad \quad \quad \text{if } (m \in N \setminus S \text{ and } r^m_S < r^m_{S \setminus \{n\}} - \varepsilon) \text{ then} \\
\quad \quad \quad \quad \quad \quad \quad \text{pure} := \text{false}, \text{ exit loop} \\
\quad \quad \quad \quad \quad \quad \text{else if } r^m_S < r^m_{S \setminus \{n\}} - \varepsilon \text{ then } \text{pure} := \text{false}, \text{ exit loop} \\
\quad \quad \quad \quad \quad \quad \text{else} \\
\quad \quad \quad \quad \quad \quad \text{if pure = true then } p^m_1 = \begin{cases} 1 & \text{for } m \in S \\ 0 & \text{for } m \in N \setminus S \end{cases} \text{ exit loop} \\
\quad \quad \text{else if } |S| = 1 \text{ then } p_2 \text{ is } \varepsilon\text{-equilibrium.} \\
\text{if pure = true then } \text{write } S, p_1 \text{ and if } |S| = 1, p_2 \text{ else the quitting game has no pure } \varepsilon\text{-equilibrium.}
\]

The calculation of \( \chi^m_0 \) is done as follows:
Algorithm 5: CALCULATE-$\chi^m_P$

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$, $(r_S)_{S \in P(N)}$, $m \in N$</td>
<td>$\chi^m_P$, $p_2$</td>
</tr>
<tr>
<td>Initialization: $\chi^m_P := 0$, $p_2 := 0$</td>
<td></td>
</tr>
<tr>
<td>if $r^m_{{m}} &gt; 0$ then $\chi^m_P := r^m_{{m}}$, $p_2^m = 1$</td>
<td></td>
</tr>
<tr>
<td>forall the $S \in P(N \setminus {m}) \setminus {\emptyset}$ do</td>
<td></td>
</tr>
<tr>
<td>if $r^m_S &lt; r^m_{S \cup {m}}$ then</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>if $r^m_S &lt; \chi^m_P$ then</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi^m_P := r^m_S$, $p_2^m := 0$ and $p_k^2 := \begin{cases} 1 &amp; \text{if } k \in S \ 0 &amp; \text{if } k \in N \setminus S \end{cases}$, $k \neq \ m$</td>
<td></td>
</tr>
<tr>
<td>else if $r^m_S &lt; \chi^m_P$ then</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi^m_P := r^m_S$, $p_2^m := 0$ and $p_k^2 := \begin{cases} 1 &amp; \text{if } k \in S \ 0 &amp; \text{if } k \in N \setminus S \end{cases}$, $k \neq m$</td>
<td></td>
</tr>
</tbody>
</table>

Remark 3.3.1. The worst case run-time complexity of Algorithm 3.3 is obviously exponential in $N$.

$qg_{\text{pure}}$

$qg_{\text{pure}}$ reads a game from standard input, additionally an $\varepsilon$ has to be given. Then the program tests all possible vectors $p_1 \in \{0,1\}^N$, if they form a pure $\varepsilon$-equilibrium or not (see Algorithm 3.3). The results are stored in the file results.txt. In the case, that a pure $\varepsilon$-equilibrium exists, the output is the vector $p_1$ and, if only one player plays quit according to $p_1$, then $p_2$ is also given.

Example 6. We take the following three player game (see Example3.txt, Appendix B):

<table>
<thead>
<tr>
<th>Pl. 3</th>
<th>c</th>
<th>c</th>
<th>q</th>
<th>Pl. 2</th>
<th>q</th>
<th>Pl. 3</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pl. 1</td>
<td>c</td>
<td>(3, 1, -1)</td>
<td>(2, 2, 2)</td>
<td>Pl. 1</td>
<td>c</td>
<td>(1, -1, 0)</td>
<td>(0, 1, 1)</td>
</tr>
<tr>
<td></td>
<td>q</td>
<td>(3, 1, 3)</td>
<td>(1, 1, 3)</td>
<td></td>
<td>q</td>
<td>(3, 1, 4)</td>
<td>(-1, -1, -1)</td>
</tr>
</tbody>
</table>

The output of the algorithm, stored in results.txt, is:

The profile $p^i = (p_1, p_2, p, p, ...)$ with $p_1 = 0 \ 1 \ 0$ $p_2 = 1 \ 0 \ 0$

and $p$ arbitrarily forms a pure eps-equilibrium in the given game.
3.4. Instant $\varepsilon$-equilibria

An instant $\varepsilon$-equilibrium $\pi = (p_i)_{i \in N}$ for a given quitting game $G = (N, (r_S)_{S \in P(N)})$ was defined as $\varepsilon$-equilibrium, where at least one player $n \in N$ exists, for which $p_1^n = 1$ holds.

Referring to [31], we want to define a special kind of instant $\varepsilon$-equilibria (see [31] p. 16).

**Definition 3.4.1** (instant $\varepsilon$-equilibrium under punishment). Let $G = (N, (r_S)_{S \in P(N)})$ be a given quitting game, $\varepsilon \geq 0$. A strategy profile $\pi = (p_1, p_2)$ for $G$ is called instant $\varepsilon$-equilibrium under punishment, if at least one player $n \in N$ exists with:

- $p_1^n = 1$,
- $\pi_2^n$ is an $\varepsilon$-min-max profile for player $n$ and
- $p_1$ is an $\varepsilon$-equilibrium in $\Gamma_v = (G, v)$, where $v^n = \chi^n + \varepsilon$, $v \in \mathbb{R}^N$.

Recall that the min-max-value for a player $n$ was given by (cf. Definition 2.4.1, p. 91)

$$\chi^n = \inf_{\pi \in \Pi} \max_{\tilde{\pi} \in \Pi^n} \gamma^n((\pi^n, \tilde{\pi}^n)).$$

**Remark 3.4.2.** It is obvious, that a profile $\pi = (p_1, p_2)$, that satisfies the requirements of Definition 3.4.1, is an $\varepsilon$-equilibrium in $G$. For the players $m \neq n$, this follows immediately from the third condition and the fact that $\gamma^m_{\gamma(\pi_2)}((p_1^{-m}, \hat{p}_1^m)) = \gamma^m((\pi^{-m}, \hat{\pi}^m))$ for all $\hat{\pi}^m = (\hat{p}_1^m, \hat{p}_2^m, \ldots) \in \Pi^m$. For player $n$, we have:

$$\gamma^n(\pi) = \gamma^n_{\gamma(\pi_2)}(p_1) = \gamma^n_0(p_1)$$
$$\geq \max_{\hat{p}_1 \in [0,1]} \gamma_0^n((\pi^{-n}, \hat{p}_1^n)) - \varepsilon$$
$$= \max_{\hat{p}_1 \in [0,1]} \left( \gamma^n((\pi^{-n}, \hat{p}_1^n)) + \varrho((\pi^{-n}, \hat{p}_1^n), \emptyset) \cdot (\chi^n + \varepsilon) \right) - \varepsilon$$
$$\geq \max_{\hat{p}_1 \in [0,1]} \left( \gamma^n((\pi^{-n}, \hat{p}_1^n)) + \varrho((\pi^{-n}, \hat{p}_1^n), \emptyset) \cdot \max_{\pi_2 \in \Pi^n} \gamma((\pi^{-n}, \hat{\pi}_2^n)) \right) - \varepsilon$$
$$= \max_{\hat{p}_1 \in [0,1]} \gamma^n((\pi^{-n}, \hat{p}_1^n)) - \varepsilon.$$
3.4.1. Computing the min-max value

Assume that we want to calculate the min-max-value for player one. In every stage of the quitting game the players $m \neq 1$ have to find a quitting probability combination, such that the payoff to player one becomes minimal. Rigorously we have:

$$
\chi^1 = \inf_{\pi \in \Pi} \max_{\tilde{\pi}^1 \in \Pi^1} \sum_{k \in \mathbb{N}} \prod_{i=1}^{k-1} (1 - \tilde{p}_i) \varphi((p^{-1}_i, 0), \emptyset)
\cdot \left( \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{1\})} (r^1_{S\cup\{1\}} \tilde{p}^1_{k} + r^1_S (1 - \tilde{p}^1_{k})) \varphi((p^{-1}_{k}, 0), S) \right)
= \inf_{\pi \in \Pi} \max_{\tilde{\pi}^1 \in \Pi^1} \sum_{k \in \mathbb{N}} \prod_{i=1}^{k-1} \tilde{p}^1_{k} \cdot \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{1\})} (r^1_{S\cup\{1\}} - r^1_S) \varphi((p^{-1}_{k}, 0), S)
+ \sum_{S \in \mathcal{P}(\mathcal{N}\setminus\{1\})} r^1_S \varphi((p^{-1}_{k}, 0), S)\right).
$$

Because player one reacts on what the other players play, their decisions have to be made independently of the (random) choice of player one, hence it is the same for each stage. Therefore it is sufficient to consider only stationary strategies for the players $m \neq 1$ as so called min-max-strategies. Furthermore in the first chapter we pointed out that, given the other players play stationary strategies, the expected payoff maximizing strategy of player one is pure. Consequently this leads to:

$$
\chi^1 = \inf_{p \in [0,1]} \max_{\tilde{p}^1 \in \{0,1\}} \gamma^1\left(\left( (p, p, \ldots)^{-1}, (\tilde{p}^1, \tilde{p}^1, \ldots) \right) \right)
$$

Furthermore we know (cf. [37]), that for all $\varepsilon > 0$ a profile $\pi$ exists, such that

$$
\chi^1 + \varepsilon = \gamma^1(\pi).
$$

for all players $n \in \mathcal{N}$.

In summary, the determination of the min-max-value for a player is an optimization problem, which can be formulated in the form

$$
\minimize_{p^{-n} \in [0,1]^{N-1}} H_{\gamma}(p^{-n}),
$$

where

$$
H_{\gamma}(p^{-n}) := \max_{b \in \{0,1\}} \gamma^n\left(\left( (p, p, \ldots)^{-n}, (b, b, \ldots) \right) \right).
$$

The function $H : [0,1]^{N-1} \rightarrow \mathbb{R}$ has discontinuous first partial derivatives at each point where the expected payoffs for player $n$ are independent of playing *quit* or *continue* with
3.4. Instant $\varepsilon$-equilibria

certainty (cf. [4]). This makes the solution difficult. An introduction to the topic minimax problems is given by the book “Introduction to Minimax” by Dem’yanov and Molozemov ([9]), some solution concepts are presented for example in [4] and [46].

The main problem is, that the expected payoff is not continuous in the point, where all players play _continue_ all the time. Furthermore the expected payoff in the quitting game is not convex, so that there may exist more than one minimum of the function $H$. For our case we need a global minimum, so we decided to implement an algorithm, which approximates the min-max-value by the consideration of a grid over $[0,1]^N$.

<table>
<thead>
<tr>
<th>Algorithm 6: Approximation of min-max-values</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong> : $N$, $(r_S)_{S \in P(N)}$, number of steps $s$</td>
</tr>
<tr>
<td><strong>Initialization</strong> : $\chi := 1000$ (default value)</td>
</tr>
<tr>
<td>forall the $\tilde{p} \in { p \in [0,1]^{N-1}</td>
</tr>
<tr>
<td>for $n = 1$ to $N$ do</td>
</tr>
<tr>
<td>$p^n_c := \tilde{p}$, $p^n_c := 0$</td>
</tr>
<tr>
<td>$p^n_q := \tilde{p}$, $p^n_q := 1$</td>
</tr>
<tr>
<td>if $\max{\gamma^n((p_c,p_c,\ldots)), \gamma^n(p_q)} &lt; \chi^n$ then</td>
</tr>
<tr>
<td>if $\gamma^n((p_c,p_c,\ldots)) &lt; \gamma^n(p_q)$ then</td>
</tr>
<tr>
<td>$\chi^n = \gamma^n(p_c)$</td>
</tr>
<tr>
<td>Minmax$^n = p_q$</td>
</tr>
<tr>
<td>else</td>
</tr>
<tr>
<td>$\chi^n = \gamma^n((p_c,p_c,\ldots))$</td>
</tr>
<tr>
<td>Minmax$^n = p_c$</td>
</tr>
<tr>
<td><strong>Output</strong> : $\chi$, Minmax, where $\chi$ is the vector with the (approximated) min-max-values for the players and Minmax is an $N \times N$ matrix, where each row contains the min-max profile for the corresponding player.</td>
</tr>
</tbody>
</table>

Remark 3.4.3. The run-time complexity of this algorithm is exponential in the number of players and the chosen number of steps.

3.4.2. Computing $\varepsilon$-equilibria in one-step games

We present an algorithm for the detection of (all) $\varepsilon$-equilibria in a one-step game, which we use for the search for an instant $\varepsilon$-equilibrium in a given quitting game. An overview about the existing literature respectively programs which are useful to find Nash-equilibria in one-step games in general is given in Section 3.4.4.

The algorithm is based on a stepwise scanning of the set of all possible strategy profiles $p \in [0,1]^N$ for a given one-step game $\Gamma_v$. Therefore the results are depending on the chosen number of steps, here denoted by $s \in \mathbb{N}$. 
Algorithm 7: Search for $\varepsilon$-equilibria in one-step games

**Input:** $N$, $(r_S)_{S \in P(N)}$, $v$, $\varepsilon \geq 0$, number of steps $s \in \mathbb{N}$

**Initialization:** $j := 0$

forall the $p \in \{ p \in [0, 1]^N \mid p^n = k \cdot \frac{1}{s}, \ k \in \{0, \ldots, s\}, \ n \in \mathcal{N} \}$ do

```
  eq := true
  for $n = 1$ to $N$ do
    if $\gamma^n_v(p) < \gamma^n_v((p^{-n}, 0)) - \varepsilon$ then
      eq := false, exit loop
    else if $\gamma^n_v(p) < \gamma^n_v((p^{-n}, 1)) - \varepsilon$ then
      eq := false, exit loop
    if eq = true
      then
        $j := j + 1$
        $M(j) := p$
```

**Output:** $j$, $M$, where $j$ denotes the number of found $\varepsilon$-equilibria and $M \in [0, 1]^{j \times N}$ is a matrix, where each row contains an $\varepsilon$-equilibrium profile in the given one-step game.

Since every one-step game has an (0-)equilibrium, the algorithm should detect at least one $\varepsilon$-equilibrium, assumed that the step size is small enough respectively the number of steps is high enough. The next section should answer the question for an optimal step size referring to a given $\varepsilon$.

**Optimal step size**

Let $p \in [0, 1]^N$ be a 0-equilibrium in $\Gamma_v$ and $s$ the given number of steps. Define

$$F_s := \{ p \in [0, 1]^N \mid p^n = k \cdot \frac{1}{s}, \ k \in \{0, \ldots, s\}, \ n \in \mathcal{N} \}.$$ 

If $p \in F_s$, the algorithm should detect $p$ for every $\varepsilon \geq 0$.\(^{4}\) If $p \notin F_s$ then denote by $p_s$ the $p_s \in F_s$, where

$$\| p - p_s \| \leq \| p - \tilde{p}_s \| \ \forall \ \tilde{p}_s \in F_s.$$ 

Furthermore $\| p - p_s \| \leq \frac{1}{2} \cdot \frac{1}{s}$ holds. Consider now player one and the player-wise change from $p$ to $p_s$. Therefore define the family $(p_{s,i})_{i \in \mathcal{N}}$ by

$$p_{s,i} := (p_{1}, \ldots, p_{i}, p_{i+1}^{1}, \ldots, p^{N})^T.$$ 

We distinguish between $p_{s, i}^{1} < p^{1}$ and $p_{s, i}^{1} > p^{1}$.

\(^{4}\)Due to the computational accuracy it might be possible that the algorithm does not detect $p$ for $\varepsilon = 0$. 

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$p_1^1 < p_1^2$: Then a $\mu \in [0, 1]$ exists, such that $p_1^1 = (1 - \mu)p^1$, where $\frac{1}{2s} \geq \mu p^1$. Since $p$ is a 0-equilibrium in $\Gamma_v$, with Lemma 1.2.21 we have

$$\gamma_v^1(p_{s,1}) \geq \gamma_v^1(p) - \mu p^1(r_{max} + \delta_v)$$

$$\geq \gamma_v^1((p^{-1}, b)) - \mu p^1(r_{max} + \delta_v)$$

$$= \gamma_v^1((p_{s,1}^{-1}, b)) - \mu p^1(r_{max} + \delta_v)$$

$$\geq \gamma_v^1((p_{s,1}^{-1}, b)) - \frac{1}{2s}(r_{max} + \delta_v), \quad b \in [0, 1].$$

$p_1^1 > p_1^2$: Then a $\lambda \in [0, 1]$ exists, such that $p_1^1 = (1 - \lambda)p^1 + \lambda$, where $\frac{1}{2s} \geq \lambda(1 - p^1)$. With Theorem 1.2.17, we get

$$\gamma_v^1(p_{s,1}) \geq \gamma_v^1(p) - \lambda(1 - p^1)(r_{max} + \delta_v)$$

$$\geq \gamma_v^1((p^{-1}, b)) - \lambda(1 - p^1)(r_{max} + \delta_v)$$

$$= \gamma_v^1((p_{s,1}^{-1}, b)) - \lambda(1 - p^1)(r_{max} + \delta_v)$$

$$\geq \gamma_v^1((p_{s,1}^{-1}, b)) - \frac{1}{2s}(r_{max} + \delta_v), \quad b \in [0, 1].$$

Case 1 and 2 imply that $p_{s,1}$ is an $\varepsilon_1$-equilibrium in $\Gamma_v$ for player one with

$$\varepsilon_1 := \frac{1}{2s}(r_{max} + \delta_v).$$

Consider now the expected payoff for player one under $p_{s,2}$. If $p_2^2 < p_2$, then again a $\mu \in [0, 1]$ exists, such that $p_2^2 = (1 - \mu)p^2$, where $\frac{1}{2s} \geq \mu p^2$ and since $p$ is a 0-equilibrium in $\Gamma_v$, with Lemma 1.2.21 it follows:

$$\gamma_v^1(p_{s,2}) \geq \gamma_v^1(p_{s,1}) - \mu p^2(r_{max} + \delta_v)$$

$$\geq \gamma_v^1((p_{s,1}^{-1}, b)) - \varepsilon_1 - \mu p^2(r_{max} + \delta_v)$$

$$\geq \gamma_v^1((p_{s,2}^{-1}, b)) - \varepsilon_1 - \frac{1}{s}(r_{max} + \delta_v)$$

$$\geq \gamma_v^1((p_{s,2}^{-1}, b)) - \frac{1}{s}(r_{max} + \delta_v), \quad b \in [0, 1].$$

On the other hand, if $p_2^2 > p_1^2$, then a $\lambda \in [0, 1]$ exists, such that $p_2^2 = (1 - \lambda)p^2 + \lambda$, where $\frac{1}{2s} \geq \lambda(1 - p^2)$, and Theorem 1.2.17 yields

$$\gamma_v^1(p_{s,2}) \geq \gamma_v^1(p_{s,1}) - \lambda(1 - p^2)(r_{max} + \delta_v)$$

$$\geq \gamma_v^1((p_{s,1}^{-1}, b)) - \varepsilon_1 - \lambda(1 - p^2)(r_{max} + \delta_v)$$

$$= \gamma_v^1((p_{s,2}^{-1}, b)) - \varepsilon_1 - 2 \cdot \lambda(1 - p^2)(r_{max} + \delta_v)$$

$$\geq \gamma_v^1((p_{s,2}^{-1}, b)) - \varepsilon_1 - \frac{1}{s}(r_{max} + \delta_v)$$

$$\geq \gamma_v^1((p_{s,2}^{-1}, b)) - \frac{1}{s}(r_{max} + \delta_v), \quad b \in [0, 1].$$
This implies that $p_{s,2}$ is an $\varepsilon_2$-equilibrium in $\Gamma_v$ for player one with

$$\varepsilon_2 := 2 \cdot \frac{1}{s} (r_{\text{max}} + \delta_v).$$

The same procedure for the players three to $N$ gives, that $p_s$ is an $N \cdot \frac{1}{s} (r_{\text{max}} + \delta_v)$-equilibrium in $\Gamma_v$ for player one. The other players are similar, hence $p_s$ is an $N \cdot \frac{1}{s} (r_{\text{max}} + \delta_v)$-equilibrium in $\Gamma_v$ in for all players.

**Corollary 3.4.4.** Let $\Gamma_v$ be a given one-step game and $\varepsilon > 0$. Then Algorithm 7 detects an $\varepsilon$-equilibrium in $\Gamma_v$, if the chosen number of steps is at least $\lceil \frac{N}{\varepsilon} (r_{\text{max}} + \delta_v) \rceil$.

**Remark 3.4.5.** If $\varepsilon$ is small, the number of steps is accordingly high. Because every $\varepsilon$-equilibrium for $\Gamma_v$ is an $\tilde{\varepsilon}$-equilibrium for $\Gamma_v$ as well for $\varepsilon \leq \tilde{\varepsilon}$, it is better to do a precalculation in a first step, choosing an $\tilde{\varepsilon} > \varepsilon$ and a number of steps $\tilde{s} \geq \left\lceil \frac{N}{\tilde{\varepsilon}} (r_{\text{max}} + \delta_v) \right\rceil$ to determine an area, where the $\varepsilon$-equilibrium to be found may be located. In the second step this area should be analyzed again with the real parameter $\varepsilon$ and an adapted number of steps $s$.

Observe that we need to determine (if possible) all $\varepsilon$-equilibria in a given one-step game and not only a sample one (see also Section 3.4.4). If one is only interested in one sample $\varepsilon$-equilibrium, there are of course other more efficient algorithms. Some of them are implemented in *Gambit*, a library of game theory software. We give a brief summary about important algorithms implemented there in Section 4.1.

### 3.4.3. Algorithm to detect an instant $\varepsilon$-equilibrium under punishment in quitting games

The algorithm for the detection of an instant $\varepsilon$-equilibrium under punishment is a combination of the previously described algorithms. Here we assume, that a $\delta$-min-max-value for each player is already given. We denote $\chi_{\delta} := (\chi_{\delta}^1, \ldots, \chi_{\delta}^N)$ as such a vector of $\delta$-min-max values for the players, $\delta \geq 0$, i.e. $\chi^i \leq \chi^i_{\delta} \leq \chi^i + \delta$ for all $i \in N$ and assume, that such a vector is provided. We give a sketch of an algorithm\(^5\):

\(^5\)With abuse of notation, we denote the family of payoffs in the reduced one-step game also $r$ and keep in mind, that these are vectors in $\mathbb{R}^{N-1}$, which are the same as in the corresponding quitting game, but missing the component for player $n$.!

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3.4. Instant \( \varepsilon \)-equilibria

**Algorithm 8:** Search for an instant \( \varepsilon \)-equilibrium

<table>
<thead>
<tr>
<th>Input</th>
<th>( N, (r_S)<em>{S \in \mathcal{P}(N)}, \varepsilon \geq 0, \chi</em>\delta ) with ( \varepsilon \geq \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initializaton:</td>
<td>insteq := false for ( n = 1 ) to ( N ) do</td>
</tr>
<tr>
<td></td>
<td>Determine all ( \varepsilon )-equilibria ( \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_k ) of the one-step game</td>
</tr>
<tr>
<td></td>
<td>( \Gamma_v := (\tilde{N} := N \setminus { n }, (r_{S \cup { n }})_{S \in \mathcal{P}(\tilde{N})}, v := 0) )</td>
</tr>
<tr>
<td></td>
<td>(Using Algorithm 7, we get ( k_n := j - 1 ) and ( \tilde{p}_i := M(i) ))</td>
</tr>
<tr>
<td></td>
<td>for ( i = 1 ) to ( k_n ) do</td>
</tr>
<tr>
<td></td>
<td>( \hat{p}_i^{-n} := \tilde{p}_i, \hat{p}_i^n := 1 )</td>
</tr>
<tr>
<td></td>
<td>if ( \gamma_\delta^S(\hat{p}<em>i) \geq \gamma</em>\delta^S((\hat{p}^{-n}, 0)) + g((\hat{p}^{-n}, 0), \emptyset) \cdot \chi_\delta^n - \varepsilon ) then</td>
</tr>
<tr>
<td></td>
<td>insteq := true, exit all loops</td>
</tr>
<tr>
<td>Output:</td>
<td>If insteq = true then an ( \varepsilon )-instant equilibria exits, where player ( n ) plays quit with certainty, and ( p_1 ) is the first entry in the profile ( \pi ), otherwise no ( \varepsilon )-equilibrium could be found.</td>
</tr>
</tbody>
</table>

**Remark 3.4.6.**

1. The run-time complexity of this algorithm is exponential in the number of players and furthermore dependent of the chosen algorithm for the calculation of the \( \varepsilon \)-equilibria in the reduced one-step games.

2. The algorithm terminates after checking all players. The result is either an \( \varepsilon \)-instant equilibrium or the statement that no such equilibrium could be found. If we assume that we are able to determine all \( \varepsilon \)-equilibria in a given one-step game and that \( \chi_\delta \) is given, then we are able to say that the given quitting game has no instant \( \varepsilon \)-equilibrium under punishment and hence no \( \varepsilon \)-instant equilibrium. For a further discussion of this point see Section 3.4.4.

3. In some cases, it is not necessary to know respectively to calculate the min-max-value for a player, i.e. if the reduced one-step game (see the third step of the algorithm) has an \( \varepsilon \)-equilibrium \( \tilde{p}_i \), such that a player \( m \) exists with \( \tilde{p}_i^m = 1 \). Such a precalculation might be useful, if the min-max-value is not known and has to be calculated in the program.

**qg.instant**

The program **qg.instant** reads in a game in standard input. Since it is already known, that two-player quitting games have \( \varepsilon \)-stationary profiles (cf. [37]), the algorithm is conceptualized only for games with more than two players. We have to mention that this program as a whole is only usable for a rough precalculation. On the other hand, the user has several options in order to use only several parts of the program. The main problem is the calculation of the min-max-values, where we cannot a priori recommend an optimal step size, so we use here the same step size as for the search for \( \varepsilon \)-equilibria in one-step games.

After entering the filename, where the game is stored, and an \( \varepsilon \), the user has the following options...
0. Use the whole program, i.e. the min-max-values and the \( \varepsilon \)-equilibria for the reduced one-step games are determined by a stepwise scanning of \([0, 1]^{N-1}\) – step size is given by the user – (see Algorithm 6 and Algorithm 7). Furthermore it will be tested, if the \( \varepsilon \)-equilibria together with the min-max-values generate an instant \( \varepsilon \)-equilibrium for the given quitting game.

1. The user provides the program with min-max-values and uses only the detection of an \( \varepsilon \)-equilibrium for the reduced one-step games and the test for instant \( \varepsilon \)-equilibria.

2. The user has the possibility to enter a filename, where \( \varepsilon \)-equilibria for the reduced one-step games are stored. The program then determines min-max-values for the players by a stepwise scanning of \([0, 1]^{N-1}\) and test, if the given equilibria generate an instant \( \varepsilon \)-equilibrium in the given quitting game. The input file should have the following form:

- the first line gives the number of \( \varepsilon \)-equilibria for each player (we assume that the numbers are at least one),
- the second line starts with player one and has only the player number 1 as entry,
- then the \( \varepsilon \)-equilibria for the reduced one-step game referring to player one, i.e. the one-step game resulting from the quitting game for the player \( n \neq 1 \), if we assume that player one plays \textit{quit} with certainty,
- after stating all equilibria for player one, the next line contains player number 2 followed by the equilibria referring to him, and so on.

For the three player game provided in Example4.txt, we have for example the following input file for Case 2.:

\[
\begin{align*}
2 & \quad 1 & \quad 1 \\
1 & \quad & \\
0.625 & \quad 0.0 & \quad \text{(\( \varepsilon \)-equil. for reduced one-step game referring to player 1)} \\
1 & \quad 0.3 & \quad \text{(\( \varepsilon \)-equil. for reduced one-step game referring to player 1)} \\
2 & \quad & \\
1.0 & \quad 0.0 & \quad \text{(\( \varepsilon \)-equil. for reduced one-step game referring to player 2)} \\
3 & \quad & \\
0.0 & \quad 1.0 & \quad \text{(\( \varepsilon \)-equil. for reduced one-step game referring to player 3)} \\
\end{align*}
\]

3. The user types in a filename, where the min-max-values and the \( \varepsilon \)-equilibrium profiles are stored, which should only be tested by the program. The input file has the same style like in the previous case, with one difference: In the first line, now the min-max-values are stated, followed by the number of \( \varepsilon \)-equilibria referring to the single players and so on. For the three player game provided in Example4.txt, we have for example the following input file:

\[
\begin{align*}
2.0 & \quad 0.0 & \quad 0.0 & \quad \text{(min-max-value for each player)} \\
1 & \quad 1 & \quad 1 & \quad \text{(number of \( \varepsilon \)-equilibria for each player)} \\
1 & \quad & \\
\end{align*}
\]
3.4. Instant $\varepsilon$-equilibria

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.625</td>
<td>0.0</td>
<td>(\varepsilon\text{-equil. for reduced one-step game referring to player 1})</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>(\varepsilon\text{-equil. for reduced one-step game referring to player 2})</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>1.0</td>
<td>(\varepsilon\text{-equil. for reduced one-step game referring to player 3})</td>
</tr>
</tbody>
</table>

Observe, that the equilibria for the players have to be given in the right order, i.e. the equilibria for player $n$ has to be followed by the equilibria for player $n+1$ and so on. The decision for this input style is for reasons of clarity and comprehensibility. Furthermore the number of the equilibria has to be stated first, since we want to allocate only the really needed memory space.

The output of this program is stored again in the file named results.txt. The program stops, if an instant $\varepsilon$-equilibrium is detected – otherwise it terminates with the output, that no instant $\varepsilon$-equilibrium was detected with the chosen step size – in this case the output file contains the equilibrium, the player who plays quit with certainty and if it was executed in the modus 0 or 2. The output contains beside this also the used min-max-values and the min-max-profile of the player, who plays quit.

3.4.4. Discussion

In this section we discuss aspects and problems related to the detection of instant equilibria under punishment. These are mainly

1. The calculation of the min-max-value
   a) as game theoretic problem concerning a team game.
   b) as problem in operations research.
2. The number of Nash-equilibria in one-step games.
3. Do I really need to know all $\varepsilon$-equilibria of the reduced one-step games?

Calculation of the min-max-value

We want to give some equivalent formulations for the calculation of the min-max-value for a player. First we consider a so-called two-player team game.

Assume that we want to calculate the min-max value of player one. In this case, all the other players play against her and try to minimize her expected payoff, regardless of their own expected payoff. Because of this, we can define a two player game in the following way: Player one from the quitting game is also player one from the new game and all the other players play in a team, which is represented by player two of the new game. The new player two has of course more than two possible actions. The actions of this player are vectors, which components constitute the action played by the corresponding player in the old game. We illustrate this with a small example:
Consider the game from Flesh, Thuijsmann and Vrieze, stated in [14]. The quitting game was given by:

\[
\begin{array}{ccc}
\text{Player 3} & c & q \\
\text{Player 2} & c & \bigcirc & (0,1,3) & (1,0,1) \\
\text{Player 3} & q & (1,3,0) & (1,0,1) \\
\text{Player 1} & c & (1,0,1) & (0,1,1) \\
\text{q} & (3,0,1) & (1,1,0) \\
\end{array}
\]

Since we want to calculate the min-max value $\chi_1$ of player one, the players two and three are in a team and play as player two of the new game against player one. The action sets are given by

\[
A_1 := \{0, 1\} \quad \text{and} \quad A_2 := \left\{ \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \right\},
\]

where $A_i$ denotes the action set for player $i$, $i = 1, 2$.

Furthermore we have to define a payoff function for the new players one and two. This function should reflect three important things:

1. All players, who play against player one are – in the context of calculating the min-max value for player one – only interested in minimizing the payoff for player one. Therefore we can assume that the payoff to the new player two is the negative of the payoff from the new player one. For our example, this leads to the following table of payoffs:

\[
\begin{array}{cccc}
\text{Player 1} & \text{Player 2} & \text{Payoffs} \\
0 & \bigcirc & (0, 0) & (0, 0) \\
1 & (1, -1) & (1, -1) & (0, 0) \\
\end{array}
\]

2. The resulting game is again a repeated game with absorbing states. Since it is sufficient to consider only stationary strategies for the players in order to calculate the min-max value, we can substitute the continue payoff, which is represented in this table by $\bigcirc$ with the expected payoff from the quitting game for player one, which can be calculated easily for a given strategy profile.

3. We have to keep in mind, that the strategies of the players are correlated. That means, we can not define the strategy for player two similarly to the definition from the strategy of the quitting game. We come back to this at the end of this section (keyword team games).

We make an excursion to operations research and define a general $N$-player game (see, e.g. [12]) in order to describe the min-max-value calculation with help of this.
Definition 3.4.7 (General $N$-player game). A general $N$-player game is a tuple

$$\mathcal{G} = (\mathcal{N}, \mathbf{p}, (B_i)_{i \in \mathcal{N}}, \theta)$$

where

- $\mathcal{N} = \{1, 2, \ldots, N\} \subseteq \mathbb{N}$ is a finite set of players, $N \in \mathbb{N}$,
- $\mathbf{p} = (p^1, \ldots, p^N)^T \in \mathbb{R}^n$ with $p^i = (p^i_1, \ldots, p^i_{n_i}) \in \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}$, where $p^i$ is called decision variable of player $i$, $i \in \mathcal{N}$ and $n = \sum_{i=1}^{N} n_i$,
- $(B_i)_{i \in \mathcal{N}}$ is the family of feasible sets for the players, where $B_i \subseteq \mathbb{R}^{n_i}$ is the set of feasible decision variables for player $i$, $i \in \mathcal{N}$ and
- $\theta = (\theta^1, \ldots, \theta^N) \in \mathbb{R}^N$ with $\theta^i : \mathbb{R}^n \to \mathbb{R}$, where $\theta^i$ is called the payoff-function for player $i$, $i \in \mathcal{N}$.

Remark 3.4.8. 1. Analogously to the definition in the quitting game or one-step game we denote with $\mathbf{p}^{-i}$ the vector of decision variables without the decision variable corresponding to player $i$ and $(\mathbf{p}^{-i}, \tilde{p}^i)$ as alternative vector of decision variables for player $i$, where $i$ now plays $\tilde{p}^i \in \mathbb{R}^{n_i}$.

2. In the literature, a more general definition can be found, where the feasible sets for the players depend also on the decision variables of the other players (see, e.g. [12]). In the context of Nash-equilibria we speak in that case of generalized Nash-equilibria.

Definition 3.4.9 (Nash-equilibrium). Let $\mathcal{G} = (\mathcal{N}, \mathbf{p}, (B_i)_{i \in \mathcal{N}}, \theta)$ be a given general $N$-player game. A vector of decision variables $\mathbf{p} = (p^1, \ldots, p^N)$ is called Nash-equilibrium for $\mathcal{G}$, if and only if

$$\theta^i(\mathbf{p}) = \max_{\tilde{p}^i \in B_i} \theta^i((\mathbf{p}^{-i}, \tilde{p}^i))$$

for all players $i \in \mathcal{N}$.

In the context of calculating the min-max value for player one of an $N$-person quitting game $G = (\mathcal{N}, \{r_S\}_{S \in \mathcal{P}(\mathcal{N})})$, we consider the following general two-player game:

$$\mathcal{G} = (\mathcal{N} := \{1, 2\}, \mathbf{p}, (B_i)_{i=1,2}, \theta),$$

where

- $\mathbf{p} = (p^1, p^2)$ with $p^1 \in \mathbb{R}$ and $p^2 = (p^2_1, p^2_2, \ldots, p^2_{n_2}) \in \mathbb{R}^{n_2}$ having $n_2 := 2^{N-1}$,
- $B_1 := [0,1]$ and $B_2 := [0,1]^{n_2}$,
\begin{itemize}
  \item \( \theta = (\theta_1, \theta_2) \) with
    \begin{align*}
      \theta_1(p) &= -\theta_2(p) := \begin{cases} 
        0 & \text{for } p = 0 \\
        \frac{1}{1-\bar{\varrho}(p, \emptyset)} \sum_{S \in \mathcal{P}(N)} \bar{\varrho}(p, S) \cdot r^1_S & \text{otherwise}
      \end{cases}
    \end{align*}
  \end{itemize}

and \( \bar{\varrho} \) being the extension of the function \( \varrho \) to the real numbers, i.e. \( \bar{\varrho} : \mathbb{R}^{n_{x+1}} \times \mathcal{P}(N) \to \mathbb{R} \) such that

\begin{align*}
  (p, S) \mapsto \bar{\varrho}(p, S) := \begin{cases} 
    p^1 \prod_{i \in S, i \neq 1} p^2_{i-1} \prod_{j \in N \setminus S} (1 - p^2_j) & \text{for } 1 \in S \\
    (1 - p^1) \prod_{i \in S} p^2_{i-1} \prod_{j \in N \setminus S} (1 - p^2_j) & \text{for } 1 \notin S.
  \end{cases}
\end{align*}

The aim of the new player two is to minimize the payoff to player one, which is similar to the maximization of her own payoff. Observe that player two does not know the strategy of player one. On the other hand, player one tries to maximize her own payoff given a strategy of player two. For both players, the resulting strategy has to be an element of the corresponding feasible set. That means, we are searching for a Nash-equilibrium for the general 2-player game, if there are more than one, we are looking for the one with the lowest payoff for player one.

Let us return to the idea of forming teams. Instead of forming a new game with only two players, we consider a so-called team game.

Assume that we would still like to calculate the min-max-value for player one. So the first team consists of this player and the second team is formed by all the other players. Or in words of Stengel and Koller ([44]), we have one team, which plays against a single adversary. The payoffs for the team players are in our case equal and equivalent to the negative of the payoff from the adversary divided by the number of players in the team. Under these assumptions, we are dealing with a zero-sum game – it is still an \( N \)-player game. One may think that the determination of an equilibrium for a zero-sum team game is equivalent to the determination of an equilibrium in a two-player zero-sum game, where it is known that this game has an equilibrium which is easy to calculate. But unfortunately this is not the case, the explanation therefor is given in [35] by Solan: “One could attempt to analyze a team game as a two-player game by considering each team as a single player. However, such a reduction misses an important feature of team games. Players on the same team cannot necessarily correlate their actions. Therefore, viewing a team as a single player adds new strategies to the game. In particular, an equilibrium in a team game is not necessarily an equilibrium in the same game, where each team is considered as a single player, and vice versa. Moreover, existence of an equilibrium in a team game does not imply and neither is implied by the existence of an equilibrium in the corresponding two-player game.” In [35] it is shown that every absorbing team game has an equilibrium payoff and that there are \( \varepsilon \)-equilibrium profiles.
3.4. Instant $\varepsilon$-equilibria

Number of Nash-equilibria in one-step games

There exist several articles on the number of Nash-equilibria in finite games. Referring to the number of equilibria in one-step games the following papers are of interest:

1. *The maximal generic number of pure Nash equilibria* from McLennan ([25])

McLennan shows how the maximum number of pure Nash-equilibria can be estimated. Trivially the maximum number of possible action-combinations for the $N$ players as upper bound – for the one-step game corresponding to a quitting game this is $2^N$.

**Definition 3.4.10** (thin). Let $\Gamma_v$ be a given one-step game corresponding to a quitting game $G = (N, (r_s)_{s \in P(N)})$. A subset $E \subset \{0, 1\}^N$ is called thin, if for all $p = (p^1, \ldots, p^N)^T, \tilde{p} = (\tilde{p}^1, \ldots, \tilde{p}^N)^T \in E$ with $p \neq \tilde{p}$ there are at least two players $n$ and $m$ such that $p^n \neq \tilde{p}^n$ and $p^m \neq \tilde{p}^m$.

The definition given here is based on the definition for “thin” stated in the paper (cf. [25] p. 409).

**Lemma 3.4.11.** Let $\Gamma_v = (G, v)$ be a given one-step game and $E$ the set of pure Nash-equilibria for $\Gamma_v$. If $r^n_s \neq r^n_{\tilde{s}}$ holds for all players $n \in N$ and all $S, \tilde{S} \in P(N)$, then $E$ is thin. Furthermore the maximum number of element from $E$ is $2^{N-1}$.

This lemma could be useful for the detection of all pure 0-equilibria in a one-step game. Because if one has found one pure Nash-equilibria, one knows that if another equilibrium exists it has to differ in at least two components from the found one. This may lead to a reduction of the computational effort.

2. *The maximal number of regular totally mixed Nash equilibria* from McKelvey ([23])

This article gives an estimate for the maximal number of regular (see [23] for the definition of regular) totally mixed Nash-equilibria, where totally mixed means that every action is chosen by the strategy with positive probability, i.e. in the context of one-step games corresponding to a quitting game $p \in (0, 1)^N$. The following table shows the maximum number of totally mixed Nash-equilibria for $N$-person one-step games:

<table>
<thead>
<tr>
<th>$N$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>number Nash-equilibria</td>
<td>1</td>
<td>2</td>
<td>9</td>
<td>44</td>
<td>265</td>
<td>1854</td>
<td>14833</td>
<td>133496</td>
<td>$1.3 \times 10^6$</td>
</tr>
</tbody>
</table>

3. *The expected number of Nash equilibria of a normal form game* from McLennan ([26])

In this paper, a formula for the calculation of the expected number of Nash equilibria for a so-called random game is presented. A random game is defined as game where,
given the number of players and nonempty finite sets of pure strategies for the players, the payoffs to the players are chosen randomly (see for example [1]).

Do I need all $\varepsilon$-equilibria?

Assume that we want to test, if an instant $\varepsilon$-equilibrium under punishment for player one exists. Let the $\delta$-min-max-value $\chi_1^\delta$ be given. Then we first have to determine all $\varepsilon$-equilibria for the reduced one-step game, given by $\Gamma_v := (\tilde{N} := N \setminus \{1\}, (r_s)_{s \in \mathcal{P}(\tilde{N})}, v := 0)$. If this game has infinitely many $\varepsilon$-equilibria, then we are in the situation, where a player $m$ and two vectors $\tilde{p}_1$ and $\tilde{p}_2$ exist, such that

- $\tilde{p}_1^{-m} = \tilde{p}_2^{-m}$ and $\tilde{p}_1^m \neq \tilde{p}_2^m$, i.e. the vectors $\tilde{p}_1$ and $\tilde{p}_2$ differ only in the component for player $m$

- all vectors $\hat{p} := (1 - \lambda)\tilde{p}_1 + \lambda\tilde{p}_2$ with $\lambda \in [0, 1]$ are $\varepsilon$-equilibria in the given one-step game,

- all other vectors $\hat{p} := (1 - \lambda)\tilde{p}_1 + \lambda\tilde{p}_2$ with $\lambda \in \mathbb{R} \setminus [0, 1]$ are – if they are in $[0, 1]^{N-1}$ – no $\varepsilon$-equilibrium in the given one-step game.

Then it is sufficient to test, whether the vectors $\tilde{p}_1$ and $\tilde{p}_2$ generate an instant $\varepsilon$-equilibrium for player one in the quitting game. This follows immediately from the linearity of the expected payoff in a one-step game in the strategy of one player (see also Theorem 1.2.17 p. 19).

To answer the question: We do not need all $\varepsilon$-equilibria, but we have to determine the boundary profiles and test them.

3.5. Stationary $\varepsilon$-equilibria

A stationary $\varepsilon$-equilibrium $\pi = (p_i)_{i \in \mathbb{N}}$ for a given quitting game $G = (\mathcal{N}, (r_s)_{s \in \mathcal{P}(\mathcal{N})})$ was defined as $\varepsilon$-equilibrium, where $\pi$ is a stationary strategy profile, i.e. $p_i = p_1$ for all $i \in \mathbb{N}$. Resume, that the expected payoff to the players under a stationary strategy profile $\pi = (p, p, \ldots)$ can be calculated by

$$\gamma(\pi) = \gamma_1(p_1)$$

respectively

$$\gamma(\pi) = \begin{cases} 0 & \text{for } p = c \\ \frac{1}{1 - g(p, \emptyset)} \cdot \gamma_0(p) & \text{otherwise} \end{cases}$$ (3.3)
Furthermore the stationary strategy profile $\pi$ is an $\varepsilon$-equilibrium, if and only if

$$\forall n \in N : \quad \gamma^n(\pi) \geq \max_{b \in \{c, q\}} \gamma^n((\pi^n, b)) - \varepsilon. \quad (3.4)$$

This implies in detail:

**Corollary 3.5.1.** Let $G = (N, (r_S)_{S \in P(N)})$ be a given quitting game and $\varepsilon \geq 0$. A stationary profile $\pi = (p, p, \ldots), p \in [0, 1]^N$, is an (stationary) $\varepsilon$-equilibrium in $G$ if

$$\begin{cases} r_{\{n\}}^n \leq \varepsilon & \text{for } p = c \\ r_{\{n\}}^n \geq -\varepsilon & \text{for } p^n = c^n \land p^n \neq 0 \\ \gamma^0_2(p) \geq (1 - g(p, \emptyset)) \left( \max \left\{ \frac{\gamma^0_2((p^n, 0))}{1 - \varphi((p^n, 0), \emptyset), \gamma^0_2((p^n, 1)) \right\} - \varepsilon \right) & \text{for } p^n \neq c^n \end{cases}$$

for all players $n \in N$.

**Proof.** With (3.3) and (3.4), we immediately obtain the last case, and for $p = c$ we have

$$\gamma^n(\pi) = 0 \geq 1 - g((c^n, 1), \emptyset) \cdot \gamma^0_2((c^n, 1)) - \varepsilon = r_{\{n\}}^n - \varepsilon.$$ 

In the second case, i.e. $p^n = c^n \land p^n \neq 0$ formula (3.4) yields

$$\gamma^n(\pi) = \frac{p^n r_{\{n\}}^n}{1 - (1 - p^n)} = r_{\{n\}}^n \geq \max \left\{ \gamma^n((c, c, \ldots)), \gamma^n((\pi^n, q)) \right\} - \varepsilon$$

$$= \max \left\{ 0, r_{\{n\}}^n \right\} - \varepsilon,$$

hence $r_{\{n\}}^n \geq -\varepsilon$ has to hold. \qed

**Algorithm**

The algorithm is based on a stepwise scanning of the set of all possible stationary strategy profiles $\pi = (p, p, \ldots), p \in [0, 1]^N$, for a given quitting game $G$. Therefore the results are depending on the chosen number of steps, here denoted by $s \in \mathbb{N}$. 

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**Algorithm 9:** Search for a stationary $\varepsilon$-equilibrium

**Input** : $N$, $(r_S)_{S \subseteq P(N)}$, $\varepsilon$, $s$

**Initialization:** $p := 0$, $eq := true$

(all players play continue)

for $n = 1$ to $N$

if $0 < r_{\{n\}} < \varepsilon$ then $eq := false$, exit loop

if $eq \neq true$ then

(all the other cases)

forall the $p \in \{ p \in [0,1]^N \mid p^n = k \cdot \frac{1}{s}, k \in \{0,\ldots,s\}, n \in N \}, p \neq 0$ do

$eq := true$

for $n = 1$ to $N$

if $p^{-n} = 0^{-n}$, $p^n \neq 0$ then

if $r_{\{n\}} < -\varepsilon$ then $eq := false$, exit loop

else if $\gamma_0(p) < (1 - \varrho(p,0))\left( \max \left\{ \frac{\gamma_0((p^{-n},0))}{1-\varrho((p^{-n},0),\varnothing)}, \gamma_0((p^{-n},1)) \right\} - \varepsilon \right)$

then $eq := false$, exit loop

if $eq = true$ then exit loop

Output : If $eq = true$ then $\pi := (p, p, \ldots)$ is an stationary $\varepsilon$-equilibrium in $G$, else no stationary $\varepsilon$-equilibria could be found for $G$ with respect to the step size $s$.

**qg_stationary**

**qg_stationary** reads in a game in standard input format and tests with a stepwise search in $[0,1]^N$, whether a stationary $\varepsilon$-equilibrium exists under a given step size or not. The input of the step size is as power of 2. The results of the program are stored in the file results.txt. The program terminates, if one stationary $\varepsilon$-equilibrium was found or after the whole space was searched according to the given step size. In the first case, the vector $p$ is given, in the second, the remark that no stationary $\varepsilon$-equilibrium with the given $\varepsilon$ and step size will be stated.

### 3.6. Cyclic $\varepsilon$-equilibria

In this section, we want to discuss what is important for an algorithm that should detect cyclic $\varepsilon$-equilibria in a given quitting game. First we have a look at the theoretical background. As a basis of such an algorithm we can use for example Proposition 2.3.8 and the following lemma, which is stated and proven here including $\kappa = 1$ (cf. [31] for $\kappa \in (0,1)$).

**Lemma 3.6.1.** Let $G = (N, (r_S)_{S \subseteq P(N)})$ be a given quitting game, $k \in \mathbb{N}$, $\delta > 0$, $\varepsilon \geq 0$ and $(\bar{p}_1, \ldots, \bar{p}_k)$ a sequence with $\bar{p}_i \in [0,1]^N$, $i \in \{1, \ldots, k\}$. Furthermore let $\bar{v}_i \in \mathbb{R}^N$ be given and define iteratively $\bar{v}_{i+1} := \gamma_{\bar{v}_i}(\bar{p}_i)$, $\forall i \in \{1, \ldots, k\}$.
Assume that for the sequences \((\bar{p}_1, \ldots, \bar{p}_k)\) and \((\bar{v}_1, \ldots, \bar{v}_{k+1})\) the properties

- \(\bar{p}_i\) \(\varepsilon\)-perfect in \(\Gamma_{\bar{v}_i} = (G, \bar{v}_i), \forall i \in \{1, \ldots, k\}\),
- \(\kappa := 1 - \prod_{i=1}^{k} g(\bar{p}_i, \emptyset) \in (0, 1)\) and
- \(\|\bar{v}_i - \bar{v}_{k+1}\| \leq \delta\)

hold. Then for the strategy profile \(\pi := (p_i)_{i \in \mathbb{N}} := (\bar{p}_k, \ldots, \bar{p}_1, \bar{p}_k, \ldots, \bar{p}_1, \ldots)\) in \(G\) we have:

1. \(G\) is terminating under every subgame strategy profile \(\pi_i\), and
2. \(p_i\) is \((\varepsilon + \frac{\delta}{\kappa})\)-perfect in \(\Gamma_{\pi_{i+1}}\)

for all \(i \in \mathbb{N}\).

Proof. 1.:

\[
\prod_{i=1}^{k} g(\bar{p}_i, \emptyset) = 1 - \kappa \in [0, 1)
\]

\[
\implies \prod_{i=1}^{k} g(\bar{p}_i, \emptyset) = \prod_{j=1}^{m} (1 - \kappa) = (1 - \kappa)^{m} \xrightarrow{m \to \infty} 0, \quad k = k \cdot m, \quad m \in \mathbb{N}
\]

Since \(\pi\) is cyclic, this implies that the game terminates under every subgame profile \(\pi_i\).

2. Define the sequence \((v_i)_{i \in \mathbb{N}} := (\bar{v}_k, \ldots, \bar{v}_1, \bar{v}_k, \ldots, \bar{v}_1, \ldots)\) and \(v_0 := \bar{v}_{k+1}\). The requirements of the lemma yield

- \(\|v_i - v_k\| \leq \delta\),
- \(\gamma_{v_i}(p_i) = v_{i-1}, \forall i \in \{1, \ldots, k\}\) and
- \(p_i\) \(\varepsilon\)-perfect in \(\Gamma_{v_i}, \forall i \in \{1, \ldots, k\}\).

It is left to show, that \(p_i\) is \((\varepsilon + \frac{\delta}{\kappa})\)-perfect in \(\Gamma_{\gamma(\pi_{i+1})}\), for all \(i \in \mathbb{N}\), i.e. \(\forall i \in \mathbb{N} \forall n \in \mathcal{N}\) :

\[
\begin{cases}
\gamma_n(\pi_{i+1})\left((p_i^{-n}, 1)\right) - \gamma_n(\pi_{i+1})\left((p_i^{-n}, 0)\right) \leq \varepsilon + \frac{\delta}{\kappa} & \text{for } p_i^n = 0 \\
\gamma_n(\pi_{i+1})\left((p_i^{-n}, 1)\right) - \gamma_n(\pi_{i+1})\left((p_i^{-n}, 0)\right) \in [\varepsilon - \frac{\delta}{\kappa}, \varepsilon + \frac{\delta}{\kappa}] & \text{for } p_i^n \in (0, 1) \\
\gamma_n(\pi_{i+1})\left((p_i^{-n}, 1)\right) - \gamma_n(\pi_{i+1})\left((p_i^{-n}, 0)\right) \geq -\varepsilon - \frac{\delta}{\kappa} & \text{for } p_i^n = 1
\end{cases}
\]

Let \(n \in \mathcal{N}\) be a given player. First we prove the following three estimates:

\[
\left|\gamma_n(\pi_{i+1})\left((p_i^{-n}, 0)\right) - \gamma_n\left((p_i^{-n}, 0)\right)\right| \leq \left|\gamma_n(\pi_{i+1}) - v_i^n\right| \leq \left|\gamma_n(\pi_1) - v_k^n\right| \leq \frac{\delta}{\kappa} \quad (3.5)
\]
Estimate (1.) follows from:
\[
|\frac{\gamma_{\pi_0}}{\kappa}((p_{i}^{-n}, 0)) - \frac{\gamma_{\pi_0}}{\kappa}((p_{i}^{-n}, 0))| = \left| \sum_{S \in \mathcal{P}(N)} r_{S}^n \cdot g((p_{i}^{-n}, 0), S) + g((p_{i}^{-n}, 0), \emptyset) \gamma^{n}(\pi_{i+1}) \right| - \left| \sum_{S \in \mathcal{P}(N)} r_{S}^n \cdot g((p_{i}^{-n}, 0), S) - g((p_{i}^{-n}, 0), \emptyset) v_{i}^n \right| = g((p_{i}^{-n}, 0), \emptyset) \cdot |\gamma^{n}(\pi_{i+1}) - v_{i}^n|
\]

If \(g((p_{i}^{-n}, 0), \emptyset) = 0\), this immediately leads to
\[
|\frac{\gamma_{\pi_0}}{\kappa}((p_{i}^{-n}, 0)) - \frac{\gamma_{\pi_0}}{\kappa}((p_{i}^{-n}, 0))| = 0 \leq \frac{\kappa}{\delta},
\]
otherwise we have
\[
|\frac{\gamma_{\pi_0}}{\kappa}((p_{i}^{-n}, 0)) - \frac{\gamma_{\pi_0}}{\kappa}((p_{i}^{-n}, 0))| \leq |\gamma^{n}(\pi_{i+1}) - v_{i}^n|.
\]

Estimate (2): We consider the expected payoff under the subgame profile \(\pi_{i+1}\). Because \(\pi\) is cyclic, we obtain
\[
\gamma(\pi_{i+1}) = \sum_{j=(i+1) \text{mod } k}^{k} \left( \sum_{S \in \mathcal{P}(N)} r_{S} \cdot g(p_{j}, S) \prod_{l=1}^{j-1} g(p_{l}, \emptyset) \right) + \prod_{j=(i+1) \text{mod } k}^{k} g(p_{j}, \emptyset) \gamma(\pi_{k+1})
\]
\[
= \sum_{j=(i+1) \text{mod } k}^{k} \left( \sum_{S \in \mathcal{P}(N)} r_{S} g(p_{j}, S) \prod_{l=1}^{j-1} g(p_{l}, \emptyset) \right) + \prod_{j=(i+1) \text{mod } k}^{k} g(p_{j}, \emptyset) \gamma(\pi_{1}),
\]
for all \(i \in \mathbb{N}\), where \(\text{mod}\) means the modulo operator. Especially for \(i = 1\) and the case \(\kappa < 1\), we get
\[
\gamma(\pi_{1}) = \sum_{j=1}^{k} \left( \sum_{S \in \mathcal{P}(N)} r_{S} \cdot g(p_{j}, S) \prod_{l=1}^{j-1} g(p_{l}, \emptyset) \right) + \prod_{j=1}^{k} g(p_{j}, \emptyset) \gamma(\pi_{1})
\]
\[
= \sum_{j=1}^{k} \left( \sum_{S \in \mathcal{P}(N)} r_{S} \cdot g(p_{j}, S) \prod_{l=1}^{j-1} g(p_{l}, \emptyset) \right) + (1 - \kappa) \gamma(\pi_{1})
\]
\[
= \frac{1}{\kappa} \cdot \prod_{j=1}^{k} \sum_{S \in \mathcal{P}(N)} r_{S} \cdot g(p_{j}, S) \prod_{l=1}^{j-1} g(p_{l}, \emptyset).
\]

For \(\kappa = 1\),
\[
\prod_{j=1}^{k} g(p_{j}, \emptyset) = 0 \iff \exists j \in \{1, \ldots, k\} : g(p_{j}, \emptyset) = 0
\]
which implies
\[
\gamma(\pi_1) = \sum_{j=1}^{k} \left( \sum_{S \in \mathcal{P}(N)} r_S \cdot g(p_j, S) \cdot \prod_{l=1}^{j-1} g(p_l, \emptyset) \right). \tag{3.9}
\]

Furthermore
\[
v_i = \gamma_{n+1}(p_{i+1}) = \sum_{S \in \mathcal{P}(N)} r_S \cdot g(p_{i+1}, S) + g(p_{i+1}, \emptyset) \cdot v_{i+1}
\]
\[
= \sum_{S \in \mathcal{P}(N)} r_S \cdot g(p_{i+1}, S) + g(p_{i+1}, \emptyset) \cdot \gamma_{n+2}(p_{i+2})
\]
\[
= \ldots
\]
\[
= \sum_{j=(i+1) \mod k}^{k} \left( \sum_{S \in \mathcal{P}(N)} r_S \cdot g(p_j, S) \cdot \prod_{l=1}^{j-1} g(p_l, \emptyset) \right) + \prod_{j=(i+1) \mod k} g(p_j, \emptyset) \cdot v_k \tag{3.10}
\]
holds for all \(i \in \mathbb{N}_0\), and especially for \(i = 0\), we have
\[
v_0 = \sum_{j=1}^{k} \left( \sum_{S \in \mathcal{P}(N)} r_S \cdot g(p_j, S) \cdot \prod_{l=1}^{j-1} g(p_l, \emptyset) \right) + \prod_{j=1}^{k} g(p_j, \emptyset) \cdot v_k
\]
\[
= \sum_{j=1}^{k} \left( \sum_{S \in \mathcal{P}(N)} r_S \cdot g(p_j, S) \cdot \prod_{l=1}^{j-1} g(p_l, \emptyset) \right) + (1 - \kappa) \cdot v_k,
\]
which implies
\[
\sum_{j=1}^{k} \left( \sum_{S \in \mathcal{P}(N)} r_S \cdot g(p_j, S) \cdot \prod_{l=1}^{j-1} g(p_l, \emptyset) \right) = v_0 - (1 - \kappa) \cdot v_k. \tag{3.11}
\]

Consider now \(|\gamma^n(\pi_{i+1}) - v^n_i|\) from (3.6). With (3.7) and (3.10), we get
\[
|\gamma^n(\pi_{i+1}) - v^n_i| = \left| \sum_{j=(i+1) \mod k}^{k} \left( \sum_{S \in \mathcal{P}(N)} r^n_S \cdot g(p_j, S) \prod_{l=1}^{j-1} g(p_l, \emptyset) \right) + \prod_{j=(i+1) \mod k} g(p_j, \emptyset) \gamma^n(\pi_1)
\]
\[
- \sum_{j=(i+1) \mod k}^{k} \left( \sum_{S \in \mathcal{P}(N)} r^n_S \cdot g(p_j, S) \prod_{l=1}^{j-1} g(p_l, \emptyset) \right) - \prod_{j=(i+1) \mod k} g(p_j, \emptyset) \cdot v_k^n \right|
\]
\[
= \left| \prod_{j=(i+1) \mod k} g(p_j, \emptyset) \gamma^n(\pi_1) - \prod_{j=(i+1) \mod k} g(p_j, \emptyset) \cdot v_k^n \right|
\]
\[
= \prod_{j=(i+1) \mod k} g(p_j, \emptyset) \cdot |\gamma^n(\pi_1) - v_k^n| \leq |\gamma^n(\pi_1) - v_k^n|.
\]
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Estimate (3.): Using (3.8) respectively (3.9) and (3.11) gives us

\[
|\gamma^n(\pi_1) - v_k^n| = \frac{1}{\kappa} \cdot \sum_{j=1}^{k} \left( \sum_{S \in \mathcal{P}(X)} r_S^n \cdot g(p_j, S) \cdot \prod_{l=1}^{j-1} g(p_l, \emptyset) \right) - v_k^n
\]

\[
= \frac{1}{\kappa} \left( v_0^n - (1 - \kappa) \cdot v_k^n \right) - v_k^n
\]

\[
\leq \frac{\delta}{\kappa}
\]

and consequently (3.5).

Now we are able to show, that \( p_i \) is \( (\varepsilon + \frac{\delta}{\kappa}) \)-perfect in \( \Gamma_{(\pi_{i+1})} \). Consider

\[
\gamma^n_{(\pi_{i+1})}((p_i^{-n}, 1)) - \gamma^n_{(\pi_{i+1})}((p_i^{-n}, 0)).
\]

Since

\[
\gamma^n_{(\pi_{i+1})}((p_i^{-n}, 1)) = \sum_{S \in \mathcal{P}(X)} r_S^n \cdot g((p_i^{-n}, 1), S) = \gamma^n_{v_i}((p_i^{-n}, 1))
\]

for all \( i \in \mathbb{N} \), one obtains

\[
\gamma^n_{(\pi_{i+1})}((p_i^{-n}, 1)) - \gamma^n_{(\pi_{i+1})}((p_i^{-n}, 0))
\]

\[
= \gamma^n_{v_i}((p_i^{-n}, 1)) - \gamma^n_{v_i}((p_i^{-n}, 0)) + \gamma^n_{v_i}((p_i^{-n}, 0)) - \gamma^n_{(\pi_{i+1})}((p_i^{-n}, 0)).
\]

Due to the fact that \( p_i \) is \( \varepsilon \)-perfect in \( \Gamma_{v_i} \) for \( p_i^n = 0 \),

\[
\gamma^n_{(\pi_{i+1})}((p_i^{-n}, 1)) - \gamma^n_{(\pi_{i+1})}((p_i^{-n}, 0)) \leq \varepsilon + \gamma^n_{v_i}((p_i^{-n}, 0)) - \gamma^n_{(\pi_{i+1})}((p_i^{-n}, 0))
\]

holds and with (3.5)

\[
\gamma^n_{(\pi_{i+1})}((p_i^{-n}, 1)) - \gamma^n_{(\pi_{i+1})}((p_i^{-n}, 0)) \leq \varepsilon + \frac{\delta}{\kappa}
\]

follows for all \( i \in \mathbb{N} \). For \( p_i^n \in (0, 1) \), one obtains analogously

\[
\gamma^n_{(\pi_{i+1})}((p_i^{-n}, 1)) - \gamma^n_{(\pi_{i+1})}((p_i^{-n}, 0)) \geq -\varepsilon - \frac{\delta}{\kappa}
\]

and

\[
\gamma^n_{(\pi_{i+1})}((p_i^{-n}, 1)) - \gamma^n_{(\pi_{i+1})}((p_i^{-n}, 0)) \leq \varepsilon + \frac{\delta}{\kappa}
\]

respectively for \( p_i^n = 1 \)

\[
\gamma^n_{(\pi_{i+1})}((p_i^{-n}, 1)) - \gamma^n_{(\pi_{i+1})}((p_i^{-n}, 0)) \geq -\varepsilon - \frac{\delta}{\kappa}.
\]

\[\square\]
Algorithm

According to Lemma 3.6.1, a first algorithm for the detection of cyclic $\varepsilon$-equilibria could be formulated in the following way:

\begin{algorithm}
\begin{tabular}{l}
\textbf{Input} : $N$, $(r_S)_{S \in \mathcal{P}(N)}$, $\varepsilon \in (0,1)$, $\delta \in (0,1)$, initial value $\bar{v}_k \in \text{conv}\{r_S | S \in \mathcal{P}(N)\}$, maximum number of iterations $l \in \mathbb{N}$
\end{tabular}

\begin{tabular}{l}
\textbf{Initialization: } $k := 1$, cycle:= false
\end{tabular}

\begin{algorithmic}
\While {$k \leq l$ and cycle = false}
\State Determine $E_{\varepsilon}(\bar{v}_k) := \{ p \in [0,1]^N | p \ \varepsilon\text{-perfect in } \Gamma_{\bar{v}_k} \text{ and } \varrho(p, \emptyset) < 1 \}$ (for example with Algorithm 7) and $F_{\varepsilon}(\bar{v}_k) := \{ \gamma_{\bar{v}_k}(p) | p \in E_{\varepsilon}(\bar{v}_k) \}$.
\State \If {$E_{\varepsilon}(\bar{v}_k) \neq \emptyset$} \ForAll {$\bar{v} \in F_{\varepsilon}(\bar{v}_k)$} \If {$\| \bar{v}_1 - \bar{v} \| \leq \delta$} \State cycle:= true
\State $\bar{v}_{k+1} := \bar{v}$
\State choose $\bar{p}_k \in \{ p \in E_{\varepsilon}(\bar{v}_k) | \gamma_{\bar{v}_k}(p) = \bar{v}_{k+1} \}$ \EndIf
\EndFor
\EndIf
\If {cycle \neq true} \State choose $\bar{p}_k \in E_{\varepsilon}(\bar{v}_k)$
\State $\bar{v}_{k+1} := \gamma_{\bar{v}_k}(\bar{p}_k)$
\State $k := k + 1$
\EndIf
\EndWhile

\textbf{Output} : If cycle = true then $\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_{k-1}$ define a cyclic for the given quitting game, $\delta$ and $\varepsilon$, else no cycle could be found in $l$ iterations.
\end{algorithmic}
\end{algorithm}

We illustrate the workflow of the algorithm with an example.

\textbf{Example 7.} Consider the example by Flesh, Thuijsmann and Vrieze (cf. [14]), which is given by (see Example4.txt, Appendix B)

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
Player 3 & c & Player 2 \\
\hline
Player 1 & c & (0,1,3) \\
& q & (1,0,1) \\
\hline
\end{tabular}
\hspace{1cm}
\begin{tabular}{|c|c|}
\hline
Player 3 & q \\
\hline
(3,0,1) & (1,1,0) \\
\hline
\end{tabular}
\hspace{1cm}
\begin{tabular}{|c|c|}
\hline
Player 1 & c & Player 2 \\
\hline
& q & (0,1,1) \\
(0,0,0) & (0,0,0) \\
\hline
\end{tabular}
\end{center}

$c$ represents the action \textit{continue} and $q$ the action \textit{quit}.

We choose as input $N = 3$, $(r_S)_{S \in \mathcal{P}(N)}$ as given in the tabular, $\varepsilon = 10^{-6}$, $\delta = 10^{-6}$, $\bar{v}_1 := (1, 2, 1)^T$ and $l = 100$. Assume that we compute the set $E_{\varepsilon}(\bar{v}_k)$ with help of a
Chapter 3. Basic algorithms and software

modified version\(^6\) of Algorithm 7 with step size \(s = 1/8\).

First we set \(k := 1\) and obtain with the mentioned algorithm the following sets

\[
E_\varepsilon(\vec{v}_1) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0.125 \\ 0.25 \\ 0.375 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.25 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0.25 \\ 0 \\ 0.375 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0.375 \\ 0 \\ 0.5 \\ 0 \end{pmatrix} \right\}
\]

and

\[
F_\varepsilon(\vec{v}_1) = \left\{ \begin{pmatrix} 1.25 \\ 1.75 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1.5 \\ 1.5 \\ 1.25 \\ 1 \\ 1.75 \\ 1 \end{pmatrix}, \begin{pmatrix} 1.75 \\ 2 \\ 1 \end{pmatrix} \right\}
\]

Since \(E_\varepsilon(\vec{v}_1) \neq \emptyset\), we test all \(\vec{v} \in F_\varepsilon(\vec{v}_1)\) according to Step 2 of the algorithm and note that no \(\vec{v}\) exists, such that \(\|\vec{v}_1 - \vec{v}\| \leq \delta\). We decide to choose \(\vec{p}_1 := (0, 0, 0.5)^T\), implying that \(\vec{v}_2 := (2, 1, 1)^T\). Furthermore we set \(k := 2\), which is lower than \(l = 100\), hence we go to Step 1 of the algorithm. We get

\[
E_\varepsilon(\vec{v}_2) = \left\{ \begin{pmatrix} 0 \\ 0.125 \\ 0.25 \\ 0.375 \\ 0.5 \end{pmatrix} \right\}
\]

and

\[
F_\varepsilon(\vec{v}_2) = \left\{ \begin{pmatrix} 1.75 \\ 1 \\ 1.25 \\ 1.5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1.5 \\ 1 \\ 1.25 \\ 1.5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}
\]

Again \(E_\varepsilon(\vec{v}_2) \neq \emptyset\) holds and no \(\vec{v} \in F_\varepsilon(\vec{v}_2)\) exists with \(\|\vec{v}_1 - \vec{v}\| \leq \delta\). Therefore we take \(\vec{p}_2 := (0, 0.5, 0)^T\), \(\vec{v}_3 := (1, 1, 2)^T\) and increase \(k\) to 3. Going back to Step 1 yields

\[
E_\varepsilon(\vec{v}_3) = \left\{ \begin{pmatrix} 0.125 \\ 0 \\ 0 \\ 0.25 \\ 0 \\ 0.375 \\ 0 \end{pmatrix} \right\}
\]

and

\[
F_\varepsilon(\vec{v}_3) = \left\{ \begin{pmatrix} 1.25 \\ 1.25 \end{pmatrix}, \begin{pmatrix} 1.25 \\ 1 \end{pmatrix}, \begin{pmatrix} 1.75 \\ 1.25 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \neq \emptyset.
\]

\(^6\)Instead of \(\varepsilon\)-equilibria for one-step games, we are testing for \(\varepsilon\)-perfect profiles in the one-step game. So one has to change the first if-statement of Algorithm 7 and replace it with the requirements for \(\varepsilon\)-perfect profiles.
Now a vector $\vec{v} \in F_\varepsilon(\vec{v}_3)$ exists such that $\|\vec{v}_1 - \vec{v}\| = 0 \leq \delta$. The corresponding $\vec{p}_3$ is given by $(0.5, 0, 0)^T$, and we have $\vec{v}_4 := \vec{v}_1 = (1, 2, 1)^T$. Hence the algorithm terminates here – since we have to set $cycle = true$ in the algorithm. The output is

$$\vec{p}_3 = \begin{pmatrix} 0.5 \\ 0 \\ 0 \end{pmatrix}, \vec{p}_2 = \begin{pmatrix} 0 \\ 0.5 \\ 0 \end{pmatrix}, \vec{p}_1 = \begin{pmatrix} 0 \\ 0 \\ 0.5 \end{pmatrix}.$$ 

With help of Lemma 3.6.1 and Proposition 2.3.8 it turns out, that $\pi := \begin{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0.5 \end{pmatrix}, \ldots \end{pmatrix}$

is at least an $(\max(\hat{\eta}_\varepsilon, \eta_\varepsilon))$-equilibrium, where $\hat{\varepsilon} = 10^{-6} + \frac{\|\vec{v}_1 - \vec{v}_4\|}{0.875 \times 10^6} = 10^{-6}$.

**Discussion**

Although the algorithm works well for our example, several questions respectively problems arise during or after the execution. Before one starts with the implementations of an program referring to that algorithm, one should answer the following questions:

1. How to choose the initial value?
2. How can the sets $E_\varepsilon(\cdot)$ and $F_\varepsilon(\cdot)$ be computed?
3. What happens, if $E_\varepsilon(\cdot)$ is empty? This only occurs, if $p = 0$ is the sole $\varepsilon$-perfect strategy profile in the one-step game $\Gamma_{\vec{v}_k}$. Instead of stopping the algorithm in that case, one may go one (or more) iteration of the algorithm backward and choose another $\vec{v}_k$ respectively another $\vec{p}_k \in E_\varepsilon(\vec{v}_{k-1})$. How to realize this?
4. The cycle does not need to start with the initial value. Is it possible to formulate and implement an alternative test for a cycle?
5. In all cases, where $|E_\varepsilon(\vec{v}_k)| > 1$, we have a selection problem, if we should choose a $\vec{p}_k$ arbitrarily.
6. Suppose that the algorithm could find a cycle and terminates. The output is a sequence of vectors $\vec{p}_{k-1}, \ldots, \vec{p}_1$. What kind of $\hat{\varepsilon}$-equilibrium is generated by this sequence? Respectively how to evaluate the results?

We consider some of these questions in detail and start with the first one.

**Initial value**

Which initial value $\vec{v}_1$ would be the best? Or do there exist any criteria for good initial values?
Chapter 3. Basic algorithms and software

Example 8. Assume that we choose \( \bar{v}_1 := (0, 0, 0)^T \) as initial value, instead of \((1, 2, 1)^T\) in the prior example. One possible sequence of sets \( E_{\varepsilon}(\cdot) \) and \( F_{\varepsilon}(\cdot) \), produced by the algorithm in Step 1, could be:

\[
E_{\varepsilon}(\bar{v}_1) = \left\{ \bar{p}_1 := \begin{pmatrix} 0.292969 \\ 0.292969 \\ 0.292969 \end{pmatrix} \right\} \quad \text{and} \quad F_{\varepsilon}(\bar{v}_1) = \left\{ \bar{v}_2 := \begin{pmatrix} 0.694885 \\ 0.694885 \end{pmatrix} \right\},
\]

\[
E_{\varepsilon}(\bar{v}_2) = \left\{ \bar{p}_2 := \begin{pmatrix} 0.121094 \\ 0.121094 \\ 0.121094 \end{pmatrix} \right\},
\]

and

\[
F_{\varepsilon}(\bar{v}_2) = \left\{ \bar{v}_3 := \begin{pmatrix} 0.871726 \\ 0.871726 \\ 0.871726 \end{pmatrix} \right\},
\]

\[
E_{\varepsilon}(\bar{v}_3) = \left\{ \begin{pmatrix} 0.054688 \\ 0.054688 \\ 0.054688 \end{pmatrix}, \begin{pmatrix} 0.058954 \\ 0.058954 \\ 0.058954 \end{pmatrix}, \begin{pmatrix} 0.0625 \\ 0.0625 \\ 0.0625 \end{pmatrix} \right\},
\]

and

\[
F_{\varepsilon}(\bar{v}_3) = \left\{ \begin{pmatrix} 0.93752 \\ 0.93752 \\ 0.93752 \end{pmatrix}, \begin{pmatrix} 0.941474 \\ 0.941474 \\ 0.941474 \end{pmatrix}, \begin{pmatrix} 0.94533 \\ 0.94533 \\ 0.94533 \end{pmatrix} \right\}, \ldots
\]

It is obvious, that the initial value \((0, 0, 0)^T\) is not as good as the initial value \((1, 2, 1)^T\).

Unfortunately a good a priori recommendation cannot be given yet, but we can get, with a precalculation, a smaller set than the convex hull \( \mathcal{I}_\varepsilon \) (cf. Definition 2.4.3 of feasible, p. 92), from where the vectors \( \bar{v}_1 \) should be taken from. Therefore we observe that

- at least one \( \varepsilon \)-perfect vector \( p \in [0, 1]^N \) should exist in the one-step game \( \Gamma_{\bar{v}_1} \) with \( g(p, \emptyset) < 1 \) and
- for all players \( n \), the continue payoff \( \bar{v}_1^n \) should be bigger than the min-max value \( \chi^n \) or at least bigger than \( \chi^n - \varepsilon \), i.e. \( \bar{v}_1^n \) should be \( \varepsilon \)-rational (cf. Definition 2.4.3).

To the first point: Using Theorem 1.2.24 (see p. 27), we define \( f : [0, 1]^N \to \mathbb{R}^N \), where \( p \mapsto f(p) = (f_1(p), \ldots f_N(p)) \) and

\[
f_n(p) := \frac{\gamma_0^p((p^{-n}, 1)) - \gamma_0^p((p^{-n}, 0))}{g((p^{-n}, 0), \emptyset)}, \quad n \in \mathcal{N}.
\]
This leads to the set
\[ I_\varepsilon := \left\{ v \in \mathbb{R}^N \left| v^n \in \left[ f_n(p) - \frac{\varepsilon}{\varrho((p^{-n}, 0), \emptyset)}, f_n(p) + \frac{\varepsilon}{\varrho((p^{-n}, 0), \emptyset)} \right], n \in \mathcal{N}, \right. \right. \]
\[ p \in [0, 1)^N, \varrho(p, \emptyset) < 1 \} \]

Observe that \( p \in [0, 1)^N \) is used in the definition of \( I_\varepsilon \), i.e. we exclude all probability vectors \( p \), where at least one player plays \textit{quit} with certainty. This is no restriction, if we can exclude instant \( \varepsilon \)-equilibria – or if we are only interested in a cyclic \( \varepsilon \)-equilibrium detected by the algorithm, which is not an instant one.

Suppose that the algorithm detects a cycle with length \( l \) of vectors \((\bar{p}_k)_{k \in \{1, \ldots, l\}}\), where at least one vector exists, such that \( \bar{p}_m^m = 1 \) for at least one player \( m \in \mathcal{N} \). Proposition 2.3.8 gives us, that the profile generated by the sequence of this vectors is a subgame \( \varepsilon \)-equilibrium. Hence the shifted profile starting with the column vector \( \bar{p}_k \) is an \( \varepsilon \)-equilibrium as well. But it is also an instant-equilibrium for \( G \), which can be found for example by the algorithm stated in Section 3.4.3.

The set \( I_\varepsilon \) combined with the requirements, that the initial value should be feasible, and \( \varepsilon \)-rationality leads to a set \( I \) as set of good initial values \( \bar{v}_1 \), where
\[ I := I_\varepsilon \cup \{ v \in \mathbb{R}^N \left| v^n \geq \chi^n, n \in \mathcal{N} \right. \} \]

**Example 9.** We illustrate the achieved restriction with the following two-player one-step game:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2 \text{ continue}</th>
<th>Player 2 \text{ quit}</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{continue} \hspace{1cm} (\bar{v}_1, \bar{v}_2) \hspace{1cm} (1 , -1 )</td>
<td>\text{quit} \hspace{1cm} (-0.5, 1 )</td>
<td>\text{quit} \hspace{1cm} (-1 , -7 )</td>
</tr>
</tbody>
</table>

For simplification choose \( \varepsilon = 0 \). The function \( f \) is given by
\[ f(p) = \left( f_1(p), f_2(p) \right) = \left( \frac{1 - 1.5p^2}{1 - p^2}, \frac{1 - 7p^1}{1 - p^1} \right) \text{ for } p = (p^1, p^2)^T \in [0, 1)^2. \]
Consider the graphs of $f_1$ and $f_2$:

They show that for all vectors in the convex hull $I_c$ a 0-perfect profile exists. Observe that because of $\rho((0,0)^T, \emptyset) = 0$, the vector $(1,1)^T$ is no element of $I_0$. In the next picture, the convex hull $I_c$ together with the set of all feasible vectors $v$ are represented. One can see that the greatest restriction to the set of the initial values $I$ comes from the postulation that the vectors should be feasible (remember that we only regarded 0-feasible vectors).

**Computation of $E_\epsilon(\cdot)$ and $F_\epsilon(\cdot)$**

Since every one-step game has a Nash-equilibrium, every one-step game has a 0-perfect profile as well. That means the sets $E_\epsilon(\bar{v}_k)$ and $F_\epsilon(\bar{v}_k)$ are empty, if and only if $p = 0$
3.6. Cyclic \( \varepsilon \)-equilibria

is the only \( \varepsilon \)-perfect strategy profile in the one-step game \( \Gamma_{\bar{v}} \). If we forget the condition \( g(p, \emptyset) < 1 \) and focus only on Nash-equilibria for the given one-step game, the Step 3. of the algorithm is equal to determining (if possible) all Nash-equilibria of that game. This could be done for example using of Algorithm 7 or Gambit (cf. 4.2). Hints for the use of Gambit are given in the next chapter. For \( \varepsilon \)-perfect profiles in general, one may modify the Algorithm 7 and has to think about the best step size. Furthermore with the same thoughts like in the previous point, we can restrict the set, in which the \( \varepsilon \)-perfect profiles have to be searched. For an illustration we use again Example 9.

**Example 10** (Continuation of Example 9). We consider again the graphs of \( f_1 \) and \( f_2 \), now under the aspect, that the continue-payoffs for the players have to be \( \varepsilon \)-feasible (here \( \varepsilon = 0 \)).

The dotted lines represent the lower bound of the continue-payoff for player one respectively player two, which is their min-max value. The graph of \( f_1 \) implies, that for \( p^2 > 3/4 \) the continue-payoff \( v^1 \) of player one has to be lower than the min-max value of that player, to achieve, that a profile \( p = (p^1, p^2)^T \) with the strategy \( p^2 \) for player two is 0-perfect in \( \Gamma_{f(p)} = (G, f(p)) \). On the other hand, the graph of \( f_2 \), which only depends on \( p^1 \), shows that for \( p^1 > 1/4 \) the continue-payoff for player two has to be smaller than \( -1 \) in order to let \( p = (p^1, p^2)^T \) be 0-perfect in \( \Gamma_{f(p)} \). Payoffs less than \( -0.5 \) for player one respectively less than \( -1 \) for player two are not feasible (in the sense of Definition 2.4.3).

Consequently we can delimit the set in which we search for 0-perfect profiles \( p \) here from \([0, 1]^2\) to \([0, 0.25] \times [0, 0.75] \).
Evaluation of the result

Let $G$ be a given quitting game such that $r_{1(n)}^n \geq 0$ for all $n \in \mathcal{N}$. Assume that the algorithm terminates because a cycle was detected. Denote with $(\bar{v}_1, \ldots, \bar{v}_{k+1})$ the sequence of continue-payoff vectors and with $(\bar{p}_1, \ldots, \bar{p}_k)$, $k \in \mathbb{N}$, the corresponding sequence of profiles for the one-step games obtained with the algorithm. Then

1. $\|\bar{v}_1 - \bar{v}_{k+1}\| \leq \delta$,
2. $\gamma_{\bar{v}_i}(\bar{p}_i) = \bar{v}_{i+1}$ for all $i \in \{1, \ldots, k\}$ and
3. $\bar{p}_i$ $\varepsilon$-perfect in $\Gamma_{\bar{v}_i}$, for all $i \in \{1, \ldots, k\}$.

Using Lemma 3.6.1, we obtain that the strategy profile $\pi = (p_i)_{i \in \mathbb{N}} := (\bar{p}_k, \ldots, \bar{p}_1, \bar{p}_k, \ldots)$ for the given quitting game $G$ has the following properties:

1. $G$ is terminating under every subgame strategy profile $\pi_i$, $i \in \mathbb{N}$.
2. $p_i$ is $(\varepsilon + \frac{\delta}{\kappa})$-perfect in $\Gamma_{\gamma(\pi_{i+1})}$, $\forall i \in \{1, \ldots, k\}$.

For $(\varepsilon + \frac{\delta}{\kappa}) \in [0, 1)$ Proposition 2.3.8 finally implies, that $\pi$ is a subgame $\hat{\eta}_{\varepsilon}$-equilibrium, or a stationary $\eta_{\varepsilon}$-equilibrium exists in $G$, where

$$\hat{\varepsilon} := \varepsilon + \frac{\delta}{\kappa}$$
$$\hat{\eta}_{\varepsilon} := 2\hat{\varepsilon}^{1-b-d} + 4\hat{\varepsilon}^{1-b} + 15r_{\max} \cdot \hat{\varepsilon}^a + \hat{\varepsilon}$$
$$\eta_{\varepsilon} := \hat{\varepsilon} + 2r_{\max}(\hat{\varepsilon}^{b-a} + \hat{\varepsilon}^a + 4 \cdot \hat{\varepsilon}^b),$$

with $a, b, d \in (0, 1)$, $b > a$, $d > a$ and $(1 - \hat{\varepsilon}^a)^{1/d} \leq \hat{\varepsilon}$.

If $\varepsilon + \frac{\delta}{\kappa} \geq 1$, then Proposition 2.3.8 can not be used. Maybe a change of the input parameters of the algorithm – for example a smaller $\delta$ – leads to a more “successful” output.
Chapter 4.

Parallel computation, Gambit and PHCpack

In the previous chapter we are mainly focused on finding one sample $\varepsilon$-equilibrium of a special kind for a given quitting game. For some reasons, especially in economics, it might be useful to find all, or as many as possible, $\varepsilon$-equilibria. The sequential execution of a brute-force algorithm respectively of an algorithm which is NP-hard has its limitations in the time. The only way to analyze games with more than three or four players and a small step size in an acceptable time is to use high performance computing. Therefore we present in the first section of this chapter two parallel programs, namely the parallel implementations of algorithms for the detection of instant, respectively, stationary $\varepsilon$-equilibria.

Furthermore we want to mention two software packages, which are often used in game theory. We already referred to the first one at the end of Section 3.4.2, it is a library of game theory software called Gambit and can be used for the computation of Nash-equilibria in one-step games. In Section 4.1 we give a brief summary about the algorithms which are implemented there. The second software which is often used to determine Nash-equilibria in games is PHCpack. It is even applied by Gambit to find all Nash-equilibria of a given one-step game. In Section 4.3 we show how it can be used to determine stationary Nash-equilibria for a given quitting game.

4.1. Parallel computation

For some applications it might be useful to know not only one, but all or as much as possible $\varepsilon$-equilibria. For this we provide a parallel version of the programs for the detection of instant and stationary $\varepsilon$-equilibria\(^1\). On the other hand, – motivated by Theorem 2.4.6 by Simon – it would be good to have algorithms, which could exclude instant or stationary

\(^1\)Since both algorithms together find also pure $\varepsilon$-equilibria, we waive a parallel version of the corresponding program.
approximate equilibria. We cannot exclude those equilibria with the implemented algorithms, but we can get a feeling if a given game might have such equilibria or not and may prove the non-existence then in a theoretical way.

Both mentioned algorithms are based on a step wise search of the space $[0,1]^{N-1}$ respectively $[0,1]^N$, i.e. we are concerning a grind on that space and test all grid-points if they are an $\varepsilon$-equilibrium or not, with respect to the regarded structure.

For the implementation we use MPI (Message Passing Interface). The number of processors, which should be used for the execution of the program is given by the user – or the system the user works with. Each processor in a message passing system runs a sub-program, which is written here in Fortran 90. The communication between the processors is usually expensive referring to the time, so that we restrict it to a minimum. At the beginning of both programs one process reads in the number of players, the payoff, the step size, the $\varepsilon$ and the maximal number of results which should be stored from each process. After that this process sends the data to the others. The distribution of the work happens over a segmentation of the space concerning the first player. At the end of each program one process is collecting the results from the others and writes them out in the file `results.txt`.

We run the programs on the SGI Altix 4700 from the ZIH at the Technical University in Dresden. The SGI Altix is featured with 1024 dual-core Intel Itanium processors with 1.6 GHz and 1GB memory per core. For more information on the system see `http://tu-dresden.de/die_tu_dresden/zentrale_einrichtungen/zh`.

The performance of these programs is analyzed with Vampir, a graphical analysis framework that provides a large set of different chart representations of event based performance data generated through source code instrumentation (see `www.vampir.eu`).

**Parallel search for instant-$\varepsilon$-equilibria**

The program is based on Algorithm 8, where the calculation of the used min-max-values is done with a parallel version of Algorithm 6. The search for the instant $\varepsilon$-equilibria under the grid points is based on the test used in Algorithm 7, where we use parallel programming as well.

First we use the parallel program for the detection of instant $\varepsilon$-equilibria to determine those equilibria for a three-player quitting game, from which we know that it has instant $\varepsilon$-equilibria. The input file for the example is given in the Appendix in Example5.txt. We chose as step size 12, i.e. $2^{12}$ steps, $\varepsilon = 0.0001$ and want at maximum 100 results per cpu. The performance is analyzed with Vampir, where we are interested in:

- the total time, which is the sum of the time from each cpu,
4.1. Parallel computation

- the time for the approximation of the min-max-values,
- the time which is needed to test all grid points, if they form an instant $\varepsilon$-equilibrium
- the time which is needed for the execution of the MPI commands and
- the time which is needed for the generation of the trace files for the performance analysis.

<table>
<thead>
<tr>
<th>cpu’s</th>
<th>total time</th>
<th>time min-max-value</th>
<th>time eq. search</th>
<th>MPI</th>
<th>VT_API</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>205.19s</td>
<td>113.67s</td>
<td>91.51s</td>
<td>&lt;10ms</td>
<td>–</td>
</tr>
<tr>
<td>4</td>
<td>236.56s</td>
<td>28.53s</td>
<td>39.61s</td>
<td>18.36s</td>
<td>17.94s</td>
</tr>
<tr>
<td>8</td>
<td>240.11s</td>
<td>14.24s</td>
<td>19.81s</td>
<td>&lt;1s</td>
<td>32.24s</td>
</tr>
</tbody>
</table>

We see, that the total time is slightly increasing, which is because of the communication of the cpu’s on the one side and the generation of the trace files on the other. The time for the approximation of the min-max-values as well as the time for the search of equilibria is decreasing with the number of cpu’s, the time for the approximation of the min-max-value nearly proportional. The behavior of the time for the equilibrium test is due to the maximum number of results. Consider the chart below, which is produced with Vampir:

Process 2 is the first that finishes the equilibrium test (after 12.60s), because it reaches the maximum number of results that should be stored. It has to wait for the other processes in order to collect the results from all.

Now consider a four-player game. We take the example studied by Solan and Vieille in [37] (see Example6.txt in Appendix B). We know that this example has no instant $\varepsilon$-equilibrium for $\varepsilon$ sufficiently small. We choose as step size $8$ (i.e. $2^8$ steps), $\varepsilon = 0.001$ and a maximal amount of 100 equilibria per cpu.

\[\text{The time stated in the tables is the longest time for the execution of the approximation part, the time varies from process to process (the same holds for the execution time of the equilibrium test).}\]
Chapter 4. Parallel computation, Gambit and PHCpack

<table>
<thead>
<tr>
<th>cpu's</th>
<th>Example6.txt</th>
<th>Example7.txt</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>cpu's</td>
<td>total time</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>771.90s</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>823.45s</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>839.50s</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td>52722.06s</td>
</tr>
</tbody>
</table>

(now: step size 10 (i.e. $2^{10}$ steps))

**Parallel search for stationary $\varepsilon$-equilibria**

The program for the parallel detection of stationary $\varepsilon$-equilibria is based on Algorithm 9. First we examine a three-player quitting game, where we know that it has stationary $\varepsilon$-equilibria (see Example7.txt, Appendix B). We chose step size 12 (i.e. $2^{12}$ steps), $\varepsilon = 0.0001$ and want at most 100 results per cpu. With Vampir we get:

<table>
<thead>
<tr>
<th>cpu’s</th>
<th>total time</th>
<th>time eq. search</th>
<th>MPI</th>
<th>VT_API</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>338.06s</td>
<td>337.77s</td>
<td>&lt;10ms</td>
<td>–</td>
</tr>
<tr>
<td>4</td>
<td>3481.57s</td>
<td>870.18s</td>
<td>538.78s</td>
<td>564.89s</td>
</tr>
<tr>
<td>8</td>
<td>3586.96s</td>
<td>447.83s</td>
<td>133.9s</td>
<td>595.93s</td>
</tr>
<tr>
<td>16</td>
<td>3618.67s</td>
<td>225.78s</td>
<td>19.00s</td>
<td>377.55s</td>
</tr>
</tbody>
</table>

The chart for 8 cpu’s is the following:

The first four processes are finishing the equilibrium test first because they reach the maximum limit of results they should store, and have to wait for the last four processes in order to collect the results.
At the end of this section we want to test the parallel program for stationary \( \varepsilon \)-equilibria on the same four-player example like in the previous section (see Example6.txt, Appendix B), for which it is known, that no stationary \( \varepsilon \)-equilibrium exists for \( \varepsilon \) sufficiently small. With Vampir we get:

<table>
<thead>
<tr>
<th>cpu's</th>
<th>total time</th>
<th>time eq. search</th>
<th>MPI</th>
<th>VT_API</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30352.58s</td>
<td>30348.66s</td>
<td>&lt;10ms</td>
<td>–</td>
</tr>
<tr>
<td>4</td>
<td>30706.32s</td>
<td>7676.04s</td>
<td>211.35s</td>
<td>254.02s</td>
</tr>
<tr>
<td>8</td>
<td>31569.08s</td>
<td>3947.19s</td>
<td>208.12s</td>
<td>1124.02s</td>
</tr>
<tr>
<td>16</td>
<td>33086.32s</td>
<td>2067.40s</td>
<td>191.93s</td>
<td>2556.74s</td>
</tr>
</tbody>
</table>

In comparison to the chart for Example6.txt we have the following:

No stationary \( \varepsilon \)-equilibrium could be found, so all processes have to scan their whole grid and are finishing their jobs relatively simultaneously.

4.2. Gambit

As mentioned before, Gambit is a library of game theory software. It is Open Source software and provides a graphical user interface, a library of C++ source code for representing games, suitable for use in other applications and a Python API for scripting applications. A historical overview over the development of Gambit can be found at the web page of the Gambit project [24] or in the paper [39] written by Turocy, who is one of the principal developers of Gambit. In this section we give a short overview on the tools which are usable for the calculation of Nash-equilibria in one-step games. Observe that no \( \varepsilon \)-equilibria are considered where \( \varepsilon > 0 \).

We start with the computation of a sample Nash-equilibrium. This could be used for example to get an equilibrium for a quitting game with dominant strategies, where at least
one player has quit as dominant strategy. I.e. if the program `qg_dominant` detects a dominant strategy for a given quitting game, the output is the player with the dominant strategy and the corresponding reduced game, which is a one-step game, if the dominant strategy for the player is quit. For that reduced game we can use the following command-line tools from Gambit.

**Computation of a sample Nash-equilibrium**

**Using a global Newton method – gambit-gnm**

Gambit provides a method for the computation of Nash equilibria using a global Newton method, introduced by Srihari Govindan and Robert Wilson ([16]). The algorithm is a generalization of the Lemke-Howson algorithm to $N$-person games and based on the structure theorem of Kohlberg and Mertens ([21]). In [19], the algorithm is briefly explained as follows: “They indicate that one of the implications of the structure theorem of Kohlberg and Mertens is that, above each generic ray emanating from the game of interest (represented as a point in a Euclidian space), the graph of the equilibrium correspondence is a one-dimensional manifold. Moreover, at sufficient distance from the relevant game there is a unique equilibrium. Therefore, starting from a sufficiently distant game along any generic ray, one can trace the line segment to the relevant game, tracing the one-dimensional manifold of equilibria along the way, to find an equilibrium of the game of interest at the terminus. Govindan and Wilson propose to trace the manifold using a global Newton method.”

**Using iterated polymatrix approximation – gambit-ipa**

The algorithm for the computation of Nash-equilibria using iterated polymatrix approximation is presented in the paper [17] by Govindan and Wilson. It is based on the previous algorithm, but polymatrix approximations are used to increase the speed of the algorithm.

**Using simplicial subdivision – gambit-simpdiv**

We cite from [24]: “This program implements the algorithm of van der Laan, Talman, and van Der Heyden ([41]). The algorithm proceeds by constructing a triangulated grid over the space of mixed strategy profiles, and uses a path-following method to compute an approximate fixed point. This approximate fixed point can then be used as a starting point on a refinement of the grid. The program continues this process with finer and finer grids until locating a mixed strategy profile at which the maximum regret is small.”
4.2. Gambit

Remark 4.2.1. In a test, we used a one-step game corresponding to a quitting game, which is presented in [14]. The game is given by:

\[
\begin{array}{ccc}
\text{Player 3} & c & \text{Player 2} \\
\text{Player 1} & c & (1, 2, 1) \\
& q & (1, 3, 0) \\
\end{array}
\begin{array}{ccc}
\text{Player 3} & q & \\
\text{Player 1} & c & (3, 0, 1) \\
& q & (0, 1, 1) \\
\end{array}
\begin{array}{ccc}
\text{Player 2} & c & (0, 1, 0) \\
& q & (0, 0, 0) \\
\end{array}
\]

The computed Nash-equilibria using the three methods mentioned before are:

- Global Newton method
  \( p = (0, 1.00012 e - 012, 0.5) \)

- Iterated polymatrix approximation
  no result

- Simplicial subdivision
  \( p = (0, 0, 0) \)

Computation of all Nash-equilibria with Gambit

Sometimes it may be not enough to know only one Nash-equilibrium. If we are interested in instant equilibria, for example, we need to know all Nash-equilibria. Gambit uses therefore a method called “Enumeration of supports”.

“Enumeration of supports” – gambit-enumpoly

The method “Enumeration of supports” is described in [39] written by Turocy and implemented in Gambit. It goes back to a computer program published by Dickhaut and Kaplan ([10]). We explain the main steps of the algorithm in the context of one-step games corresponding to quitting games.

We need some definitions first:

Definition 4.2.2. Let \( \Gamma = (\mathcal{N} = (1, \ldots, N), (r_S)_{S \in \mathcal{P}(\mathcal{N})}, v) \) be a given one-step game.

- A pure strategy \( \tilde{p}^n \in \{0, 1\} \) for player \( n \in \mathcal{N} \) is called dominated if:
  \( \forall p = (p^1, \ldots, p^N) \in \{0, 1\}^N, p^n \neq \tilde{p}^n : \gamma_v^n(p) > \gamma_v^n((p^{-n}, \tilde{p}^n)) \).

- The set \( A(p) := \{ a = (a^1, \ldots, a^N) \in \{0, 1\}^N : \rho(p, \{ n \in \mathcal{N}; a^n = 1 \}) > 0 \} \) is called support set of \( \Gamma_v \) with respect to \( p \), i.e. \( A(p) \) contains all action-vectors, which are played with positive probability under the given profile \( p \).
The set \( A^* := \{ a \in A(p) : p \) is an Nash-equilibrium in \( \Gamma_v \} \) is called the equilibrium-support set of \( \Gamma_v \).

Let \( A_i, i \in \{1, \ldots, 3^N\} \) be a possible support set for a given one-step game \( \Gamma_v \). An action (respectively a pure strategy) \( a \in A^n_i \) for player \( n \in \mathcal{N} \) is called admissible, if

\[
\gamma^n_v((p^{-n}, \tilde{p}^n)) \leq \gamma^n_v((p^{-n}, a)) \quad \forall \tilde{p}^n \in \{0, 1\}^N, p \in A_i
\]

holds. Furthermore the set \( A_i \) is admissible, if all sets \( A^n_i, n \in \mathcal{N} \), are admissible.

The algorithm now works as follows:

1. Elimination of dominated strategies.
   In the first step, all dominated strategies are eliminated. This leads to a reduced game, which has the same Nash-equilibria like the original game.

2. Enumeration of all possible equilibrium-support sets.
   We consider a one-step game without dominated strategies and enumerate all possible support sets. For a one-step game with \( N \) players, there exist \( 3^N \) different support sets. Observe that a support set must contain at least one action for each player. All support sets, where the players have only one possible action available, correspond to pure strategies as equilibrium, which is easy to test. The remaining \( 3^N - 2^N \) support sets apart from one define new games with less players than the given game. This reduces the computational effort for the analysis.
   By another precalculation, one excludes support sets, which contain strategies that are:
   a) “dominated from the inside”, i.e. if the support set is not admissible (this case only occurs, if a player has the actions quit and continue in her support) or
   b) “dominated from the outside”, that means, a player \( n \in \mathcal{N} \) and an action respectively a pure strategy \( a \in A^n_i, i \in \{1, \ldots, 3^N\} \), exists, such that \( a \) is dominated by another action \( \tilde{a} \in \{0, 1\} \setminus A^n_i \) for all \( p = a \in A_i \) (this case only occurs, if a player as only one action available in her support).

3. Calculation of equilibria for the possible support set and testing the obtained equilibria in the context of the given one-step game.
   We consider only admissible support sets. Assume that at least one player has both actions available. Then the support sets define reduced one-step games\(^4\). For these games we have to determine all totally mixed Nash-equilibria – observe that we consider 0-equilibria. Therefore one can rewrite the Nash-equilibrium conditions for \( p \) in terms of polynomial equations. This is explained very well by Datta in [8] for games in general. The resulting polynomial equations are now solvable for example with PHCpack a software for solving polynomial systems (see the next Section and [42] or [7]), which is also used by Gambit.

\(^4\) If no player has at least two actions available we are in the case, where we consider a pure strategy.
Remark 4.2.3. The computed Nash-equilibria applying gambit-enumpoly to the previous example (see Remark 4.2.1) are: \( p = (0, 0, 0) \) and \( p = (0, 0, 0.5) \).

We briefly mention some aspects of a parallel implementation of the algorithm.

Parallel Computation

An algorithm for the computation of all Nash-equilibria using polynomial equations and the enumeration of the support set can easily be implemented in parallel, because each support set forms an independent problem. Such an algorithm was realized by Widger and Grosu. The results are stated in [45]. In comparison with the (sequential) algorithm implemented in Gambit, their algorithm does not contain any check for dominated strategies or for the admissibility of the support sets. It is to expect that they can improve the performance by using this precalculations. Furthermore it is recommendable to check first, if the solution of the polynomial equations sums up to one, instead of checking the Nash-equilibrium conditions for the (pure) strategies that are not in the actual support set.

Application of this method in the context of instant equilibria

Determining all Nash-equilibria with help of “Enumeration of the supports” using of polynomial equations may be more efficient than the method used from us in the previous chapter. But it is somehow problematic in the general case, since this method fails if the one-step game is not generic, that means, if there exists a strategy profile, where the payoff to at least on player is independent of the strategy played by her.

Example 11. Consider Example 7. For the one-step game \( \Gamma_{\tilde{v}_1} \) with \( \tilde{v}_1 = (1, 2, 1)^T \), the routine gambit-enumpoly, which should compute all Nash-equilibria for a given one-step game, has only \( (0, 0, 0)^T \) and \( (0, 0, 0.5)^T \) as result. The equilibria \( (0, 0, \lambda)^T, \lambda \in (0,0.5) \), are not listed.

4.3. PHCpack

As mentioned in the previous section, PHCpack is a solver for polynomial systems (see [42]). It is available on the web page of Jan Verschelde (http://hompages.math.uic.edu/~jan/). PHCpack uses homotopy continuation to compute numerically approximations to all isolated complex solutions of a polynomial system.

In the context of quitting games, it is usable for the computation of stationary Nash-equilibria in totally mixed strategies.
Chapter 4. Parallel computation, Gambit and PHCpack

Calculation of stationary Nash-equilibria using PHCpack

In this section we consider a given quitting game \( G \) and want to find out, whether it has a stationary Nash-equilibrium in totally mixed strategies or not. Therefore we can use the approach by Solan and Vieille, stated in [37]: Assume that \( \pi = (p, p, \ldots) \) is a stationary Nash-equilibrium in totally mixed strategies in \( G \), i.e. \( p \in (0, 1)^N \). Then with Lemma 1.4.8 (see p. 40) we have

\[
\gamma^n(\pi) = \gamma^n((\pi^n, c)) = \gamma^n((\pi^n, q))
\]

for all players \( n \in \mathcal{N} \). Since

\[
\gamma^n((\pi^n, q)) = \gamma^n_0((p^n, 1))
\]

and

\[
\gamma^n((\pi^n, c)) = \gamma^n_0((p^n, 0)) = \gamma^n_0((p^n, 0)) + \varrho((p^n, 0), \emptyset) \gamma^n_0((p^n, 1))
\]

we get the following system of polynomial equations:

\[
0 = \gamma^n_0((p^n, 0)) + \varrho((p^n, 0), \emptyset) \gamma^n_0((p^n, 1)) - \gamma^n_0((p^n, 1))
\]

\[
0 = \sum_{ S \in \mathcal{P}(\mathcal{N}\setminus\{n\}) } \varrho((p^n, 0), S) (r^n_S - r^n_{S\cup\{n\}})
\]

\[
+ \sum_{ S \in \mathcal{P}(\mathcal{N}\setminus\{n\}) } \varrho((p^n, 0), \emptyset) \varrho((p^n, 0), S) r^n_{S\cup\{n\}},
\]

where \( \varrho((p^n, 0), S) \) are squarefree monomials and \( \varrho((p^n, 0), \emptyset) \varrho((p^n, 0), S) \) are not squarefree monomials (see e.g. [8]).

The resulting system of equations is the input in PHCpack. How the input file has to look like is explained in the tutorial provided at the webpage by Jan Verschelde. The output of the so called blackbox solver of PHCpack consists of the approximated isolated solutions of the system, the order of the variables is the same order in which they appeared in the input equations. Since the solutions are complex numbers one has to test all of them, in order to identify those, which are real and element of \( (0, 1)^N \). The resulting solutions generate totally mixed stationary Nash-equilibria in the given quitting game.

Example 12. Consider the three player example from Example 7 again. In order to detect all totally mixed stationary Nash-equilibria of this game we have to consider the following system of polynomial equations:

\[
0 = -p^1 + 2p^2 + (p^2)^2 - p^1(p^2)^2
\]

\[
0 = -p^3 + 2p^4 + (p^1)^2 - (p^1)^2p^3
\]

\[
0 = -p^2 + 2p^3 + (p^3)^2 - p^2(p^3)^2
\]

For a discussion on the performance of PHCpack see for example [7], observe that Datta only considers one-step games.
where the first formula comes from player three, the second one from the second player and the third from player one. The output from PHC are 27 solutions, we state here the first 3:

\[
\begin{align*}
\mathbf{a} & : 4.8229559478865E-01 & 8.76008557781811E-01 \\
\mathbf{b} & : 2.95326778355902E-01 & 9.55396302057907E-01 \\
\mathbf{c} & : -9.9437094477199E-01 & 1.05954821423568E-01 \\
\mathbf{a} & : 4.8229559478865E-01 & 8.76008557781811E-01 \\
\mathbf{b} & : 2.95326778355902E-01 & 9.55396302057907E-01 \\
\mathbf{c} & : 4.05425905382346E-01 & -9.14127909673972E-01 \\
\mathbf{a} & : 4.8229559478865E-01 & 8.76008557781811E-01 \\
\mathbf{b} & : 2.95326778355902E-01 & 9.55396302057907E-01 \\
\mathbf{c} & : 5.88945039394853E-01 & 8.08173088250404E-01 \\
\end{align*}
\]

\[a = p^1, \ b = p^2 \text{ and } c = p^3.\] As expected no real solution in \((0,1)^3\) exists, which confirms the assertion by Flesch et al. in [14].

4.3. PHCpack
Appendix A.

Overview: Frequent calculations

This chapter gives a short overview of the used decompositions, which are not proven yet. For completeness we will explain respectively deduce them here.

For one-step games

First we consider the function $\varrho : [0,1]^N \times \mathcal{P}(N) \rightarrow [0,1]$, where

$$(p, S) \mapsto \varrho(p, S) = \prod_{n \in S} p^n \cdot \prod_{m \in N \setminus S} (1 - p^m).$$

Then

$$\sum_{S \in \mathcal{P}(N \setminus \{n\})} \varrho((p^{-n}, 1), S \cup \{n\}) = \sum_{S \in \mathcal{P}(N \setminus \{n\})} \varrho((p^{-n}, 0), S) = 1 \quad (A.1)$$

follows for all $n \in N$.

Let $n \in N$ be a given player. For the probability, that at least one player $m \in N \setminus \{n\}$ plays the action quit under the strategy profile $p$ in the one-step game $\Gamma_v$, we have

$$\sum_{S \in \mathcal{P}(N \setminus \{n\}) \setminus \{\emptyset\}} \varrho((p^{-n}, 1), S \cup \{n\}) = \sum_{S \in \mathcal{P}(N \setminus \{n\}) \setminus \{\emptyset\}} \varrho((p^{-n}, 0), S). \quad (A.2)$$

We used this function mainly for the calculation of the expected payoffs. Observe for example the expected payoff $\gamma_v$ from a one-step game $\Gamma_v$, $v \in \mathbb{R}^N$, under the strategy profile $p$:

$$\gamma_v(p) = p^n \cdot \gamma_v((p^{-n}, 1)) + (1 - p^n) \cdot \gamma_v((p^{-n}, 0)), \quad (A.3)$$
Appendix A. Overview: Frequent calculations

where we can write alternatively

\[
\gamma_v((p^n, 1)) = \sum_{S \in \mathcal{P}(N)} q((p^n, 1), S) \cdot r_S
\]

\[
= \sum_{S \in \mathcal{P}(N) \setminus \{n\}} q((p^n, 1), S \cup \{n\}) \cdot r_{S \cup \{n\}}
\]

\[
= \sum_{S \in \mathcal{P}(N) \setminus \{n\}} q((p^n, 1), S \cup \{n\}) r_{S \cup \{n\}} + q((p^n, 1), \{n\}) r_{\{n\}}
\]

\[
(A.4)
\]

and

\[
\gamma_v((p^n, 0)) = \sum_{S \in \mathcal{P}(N)} q((p^n, 0), S) \cdot r_S + q((p^n, 0), \emptyset) \cdot v
\]

\[
= \sum_{S \in \mathcal{P}(N) \setminus \{n\}} q((p^n, 0), S) \cdot r_S + q((p^n, 0), \emptyset) \cdot v,
\]

\[
(A.5)
\]

for all \( n \in \mathcal{N} \).

For finite quitting games

This section considers different conditional expected payoffs in the finite quitting game \( G_{B_k} = (G, B_k, v_k := \gamma(\pi_{k+1-1})) \), where \( \tau_{B_k} \) is the stopping time for the game \( G_{B_k} \).

We have:

\[
E_{\nu_k}(\hat{r}^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 0)
\]

\[
= \frac{1}{P_{\nu_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 0)} \int_{\{\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 0\}} \hat{r}^n(\tilde{Y}_{\tau_{B_k}}) \ d\mathbf{P}_{\nu_k}
\]

\[
= \frac{1}{P_{\nu_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 0)} \sum_{i=i_k}^{i_{k+1}-2} \int_{\{\tau_{B_k} = i, \tilde{Y}_{\tau_{B_k}}^n = 0\}} \hat{r}^n(\tilde{Y}_i) \ d\mathbf{P}_{\nu_k}
\]

\[
= \sum_{i=i_k}^{i_{k+1}-2} \frac{P_{\nu_k}(\tau_{B_k} = i, \tilde{Y}_i^n = 0)}{P_{\nu_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 0)} \ E_{\nu_k}(\hat{r}^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} = i, \tilde{Y}_i^n = 0)
\]

\[
= \sum_{i=i_k}^{i_{k+1}-2} \frac{P_{\nu_k}(\tau_{B_k} = i)\ E_{\nu_k}(\hat{r}^n(\tilde{Y}_{\tau_{B_k}}) | \tau_{B_k} = i, \tilde{Y}_i^n = 0)}{P_{\nu_k}(\tau_{B_k} < i_{k+1} - 1, \tilde{Y}_{\tau_{B_k}}^n = 0)}
\]

\[
(A.6)
\]
Besides this,

\[
E_{\nu_k}(\tilde{r}^n(\tilde{Y}_{\tau B_k})|\tau B_k < i_k + 1 - 1, \tilde{Y}^n_{\tau B_k} = 1) = \frac{1}{P_{\nu_k}(\tau B_k < i_k + 1 - 1, \tilde{Y}^n_{\tau B_k} = 1)} \int \tilde{r}^n(\tilde{Y}_{\tau B_k}) dP_{(\nu_k^{-n}, \varepsilon)}(\tau B_k < i_k + 1 - 1) \sum_{i = i_k}^{i_k + 1 - 2} E_{\nu_k}(\tilde{r}^n(\tilde{Y}_{\tau B_k})|\tau B_k = i, \tilde{Y}^n_{\tau B_k} = 1)
\]

and analogously we get

\[
E_{(\nu_k^{-n}, \varepsilon)}(\tilde{r}^n(\tilde{Y}_{\tau B_k})|\tau B_k < i_k + 1 - 1) = \frac{1}{P_{(\nu_k^{-n}, \varepsilon)}(\tau B_k < i_k + 1 - 1)} \int \tilde{r}^n(\tilde{Y}_{\tau B_k}) dP_{(\nu_k^{-n}, \varepsilon)}(\tau B_k < i_k + 1 - 1) \sum_{i = i_k}^{i_k + 1 - 2} \int \tilde{r}^n(\tilde{Y}_{\tau B_k}) dP_{(\nu_k^{-n}, \varepsilon)}(\tau B_k = i)
\]

Besides this,

\[
E_{\nu_k}(\tilde{r}^n(\tilde{Y}_{\tau B_k})|\tau B_k = i, \tilde{Y}^n_{\tau B_k} = 0) = \frac{1}{P_{\nu_k}(\tau B_k = i, \tilde{Y}^n_{\tau B_k} = 0)} \int \tilde{r}^n(\tilde{Y}_{\tau B_k}) dP_{\nu_k}(\tau B_k = i, \tilde{Y}^n_{\tau B_k} = 0)
\]
Appendix A. Overview: Frequent calculations

holds, and furthermore we get

\[ E_{\psi_k}(\tilde{r}^n(\tilde{Y}_{\tau_{B_k}})\mid \tau_{B_k} = i, \tilde{Y}_{\tau_{B_k}} = 0) \]

\[ = \frac{\prod_{j=i}^{i-1} \Theta(p_i, \emptyset) \cdot \sum_{S \in P(\mathcal{N} \setminus \{n\}) \setminus \{\emptyset\}} \Theta(p_i, S) \cdot r^n_S}{\prod_{j=1}^{i-1} \Theta(p_i, \emptyset) \cdot \sum_{S \in P(\mathcal{N} \setminus \{n\}) \setminus \{\emptyset\}} \Theta(p_i, S)} \]

\[ = \frac{\sum_{S \in P(\mathcal{N} \setminus \{n\}) \setminus \{\emptyset\}} \Theta(p_i, S) \cdot r^n_S}{\prod_{j=1}^{i-1} \Theta(p_i, \emptyset) \cdot \sum_{S \in P(\mathcal{N} \setminus \{n\}) \setminus \{\emptyset\}} \Theta(p_i, S)} \]

\[ = \frac{(1 - p^n_i) \sum_{S \in P(\mathcal{N} \setminus \{n\}) \setminus \{\emptyset\}} \Theta((p_i^{-n}, 0), S) \cdot r^n_S}{\prod_{j=1}^{i-1} \Theta(p_i, \emptyset) \cdot \sum_{S \in P(\mathcal{N} \setminus \{n\}) \setminus \{\emptyset\}} \Theta(p_i, S)} \]

\[ = \frac{\sum_{S \in P(\mathcal{N} \setminus \{n\}) \setminus \{\emptyset\}} \Theta((p_i^{-n}, 0), S) \cdot r^n_S}{\prod_{j=1}^{i-1} \Theta(p_i, \emptyset) \cdot \sum_{S \in P(\mathcal{N} \setminus \{n\}) \setminus \{\emptyset\}} \Theta(p_i, S)} \]

\[ = E_{(\psi_k^n, \tau)}(\tilde{r}^n(\tilde{Y}_{\tau_{B_k}})\mid \tau_{B_k} = i) \] (A.10)

for all \( i \in B_k \) and \( n \in \mathcal{N} \).

Another useful equation is the following:

\[ E_{\psi_k}(\tilde{r}^n(\tilde{Y}) \cdot 1_{\{\tau_{B_k} < \kappa + 1\} \mid \tau_{B_k} > \kappa}) \]

\[ = \frac{1}{P_{\psi_k}(\tau_{B_k} > \kappa)} \int_{\{\tau_{B_k} < \kappa\}} \tilde{r}^n(\tilde{Y}) \cdot 1_{\{\tau_{B_k} < \kappa + 1\}} d\psi_k \]

\[ = \frac{P_{\psi_k}(\tau_{B_k} < \kappa + 1 - 1, \tau_{B_k} < \kappa)}{P_{\psi_k}(\tau_{B_k} < \kappa + 1 - 1, \tau_{B_k} < \kappa P_{\psi_k}(\tau_{B_k} > \kappa) \int_{\{\tau_{B_k} < \kappa, \tau_{B_k} < \kappa + 1\}} \tilde{r}^n(\tilde{Y}) d\psi_k \]

\[ = P_{\psi_k}(\tau_{B_k} < \kappa + 1 - 1 \mid \tau_{B_k} > \kappa) \cdot E_{\psi_k}(\tilde{r}^n(\tilde{Y}_{\tau_{B_k}})\mid \kappa < \tau_{B_k} < \kappa + 1) \] (A.11)

**Decomposition lemma**

**Lemma A.0.1.** Let \((\hat{\Omega}, \hat{A}, \hat{P})\) be a probability space and \(U : \hat{\Omega} \to \mathbb{R}\) a random variable with \(|U| \leq K, K \in \mathbb{R}_+\), then

\[ |\hat{E}(U)| \leq K \cdot \hat{P}(A^c) + \sup_{\omega \in \hat{A}} |U(\omega)| \]

holds for all \( A \in \hat{A}\), where \( \hat{E} \) denotes the expected value referring to the probability measure \( \hat{P} \) and \( A^c := \hat{\Omega} \setminus A \).
Proof.

\[ |\hat{E}(U)| = |\hat{E}(U \cdot 1_A + U \cdot 1_{A^c})| \]
\[ \leq |\hat{E}(U \cdot 1_A)| + |\hat{E}(U \cdot 1_{A^c})| \]
\[ \leq \hat{E}(|U| \cdot 1_A) + \hat{E}(|U| \cdot 1_{A^c}) \]

Since \( U \) is bounded by \( K \), one obtains

\[ |\hat{E}(U)| \leq \hat{E}(K \cdot 1_{A^c}) + \hat{E}(\sup_{\hat{\omega} \in A} |U(\hat{\omega})| \cdot 1_A) \]
\[ \leq K \cdot \hat{P}(A^c) + \sup_{\hat{\omega} \in A} |U(\hat{\omega})| \cdot \hat{P}(A) \]
\[ \leq K \cdot \hat{P}(A^c) + \sup_{\hat{\omega} \in A} |U(\hat{\omega})|. \]

\( \square \)
Appendix B.

Instructions and examples for the software

Input file

Since a quitting game is given by the set of players and a family of payoff vectors, one has to think about, how to submit this information to the program. We decided to do it via an input file. In order to keep the data small, which should be read, one has to sort the family of payoff vectors sensible. Therefore we encode the quitting coalitions $S \in \mathcal{P}(\mathcal{N})$ with the bijective function $f$ as follows:

$$f : \mathcal{P}(\mathcal{N}) \to \{0, 1, \ldots, 2^{N-1}\}, \quad S \mapsto f(S) := \sum_{i \in S} 2^{i-1}.$$ 

Because $S = \emptyset$ leads by definition to the payoff $r_S = 0$, we can neglect $S = \emptyset$ as input of the data-file. So the file finally consists of the number of players in the first row, followed by the sorted – according to the coding-number – payoff vectors.

For example: Consider the following game (cf. [14]):

<table>
<thead>
<tr>
<th>Player 3</th>
<th>c</th>
<th>Player 2</th>
<th>c</th>
<th>q</th>
<th>Player 3</th>
<th>q</th>
<th>Player 2</th>
<th>c</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>c</td>
<td>(0, 1, 3)</td>
<td>(1, 0, 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Player 1</td>
<td>c</td>
<td>(1, 3, 0)</td>
<td>(1, 0, 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>q</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The data input file looks like:

```
151
```
Appendix B. Instructions and examples for the software

3
1, 3, 0
0, 1, 3
1, 0, 1
3, 0, 1
0, 1, 1
1, 1, 0
0, 0, 0

Another advantage of the encoding is, that it is easy to determine the action vector belonging to the code-number of a quitting coalition in a program.

Remark. Usually, as written in [11], all input data should be rational and represented by the numerator and denominator written in binary. We refrain at this moment from this requirement and allow input data, which is real respectively rational and not represented by numerator and denominator.

Overview: Implemented programs

We provided the following (sequential) programs for the analysis of quitting games:

<table>
<thead>
<tr>
<th>Name</th>
<th>Function</th>
</tr>
</thead>
</table>
| **qg_symmetry** | Tests, if a given quitting game is symmetric.  
Input: Quitting game  
Output: (in `results.txt`)  
– two vectors of payoffs representing the two families of payoffs for a symmetric game  
– one equilibrium for the game |
| **qg_dominant** | Tests, if the players of the game have dominant strategies.  
Input: Quitting game  
Output: If dominant strategies exist, the players and the corresponding strategy as well as the reduced game are stored in `results.txt`. A hint is given at the end, whether the resulting game is a quitting game or a one-step game. |
| **qg_pure** | Tests, if a quitting game has a pure $\varepsilon$-equilibrium.  
Input: Quitting game, $\varepsilon$  
Output: If a pure $\varepsilon$-equilibrium exists, the equilibrium is stored in `results.txt`, otherwise a hint is given that no pure $\varepsilon$-equilibrium exists. |
Searches for instant $\varepsilon$-equilibria. Executable in different modi. Input: Quitting game, $\varepsilon$, execution modus (number from 0 to 3) Case 0-2: additional input: step size as power of two Case 1: additional input: min-max-value for the players Case 2: file with $\varepsilon$-equilibria for the reduced one-step games Case 3: file with the min-max values for the players and the $\varepsilon$-equilibria for the reduced one-step games Output: If found, the first column-vector of the instant $\varepsilon$-equilibrium, the player who plays quit with certainty, and for the Cases 0 and 2 the used min-max values and the min-max profile of the quit player.

Searches for stationary $\varepsilon$-equilibra. Input: Quitting game, $\varepsilon$, step size as power of two Output: If found, the stationary $\varepsilon$-equilibrium for the game is stored in results.txt.

The following parallel programs are available:

<table>
<thead>
<tr>
<th>Name</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>qg_instant</td>
<td>Searches for instant $\varepsilon$-equilibria.</td>
</tr>
<tr>
<td></td>
<td>Input: Quitting game, step size as power of two, $\varepsilon$, maximum number of results per processor</td>
</tr>
<tr>
<td></td>
<td>Output: If found, the instant $\varepsilon$-equilibria.</td>
</tr>
<tr>
<td>qg_stationary</td>
<td>Searches for stationary $\varepsilon$-equilibra.</td>
</tr>
<tr>
<td></td>
<td>Input: Quitting game, step size as power of two, $\varepsilon$, maximum number of results per processor</td>
</tr>
<tr>
<td></td>
<td>Output: If found, the stationary $\varepsilon$-equilibrium for the game is stored in results.txt.</td>
</tr>
</tbody>
</table>
List of examples

For the sequential programs we examined four examples. We give the input files in the following:

<table>
<thead>
<tr>
<th>Example1.txt</th>
<th>Example2.txt</th>
<th>Example3.txt</th>
<th>Example4.txt</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1, 1, 1</td>
<td>-1, 0, 0</td>
<td>3, 1, -1</td>
<td>1, 3, 0</td>
</tr>
<tr>
<td>1, 1, 1</td>
<td>0, 1, 0</td>
<td>2, 2, 2</td>
<td>0, 1, 3</td>
</tr>
<tr>
<td>0, 0, 3</td>
<td>1, 1, 0</td>
<td>1, 1, 3</td>
<td>1, 0, 1</td>
</tr>
<tr>
<td>1, 1, 1</td>
<td>0, 2, -1</td>
<td>1, -1, 0</td>
<td>3, 0, 1</td>
</tr>
<tr>
<td>0, 3, 0</td>
<td>2, 1, 1</td>
<td>3, 1, 4</td>
<td>0, 1, 1</td>
</tr>
<tr>
<td>3, 0, 0</td>
<td>1, 3, -1</td>
<td>0, 1, 1</td>
<td>1, 1, 0</td>
</tr>
<tr>
<td>2, 2, 2</td>
<td>0, 2, 3</td>
<td>-1, -1, -1</td>
<td>0, 0, 0</td>
</tr>
</tbody>
</table>

For the parallel programs we took the following examples:

<table>
<thead>
<tr>
<th>Example5.txt</th>
<th>Example6.txt</th>
<th>Example7.txt</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>3, 1, -1</td>
<td>1, 4, 0, 0</td>
<td>0, 0, 2</td>
</tr>
<tr>
<td>2, 2, 2</td>
<td>4, 1, 0, 0</td>
<td>-2, 0, 1</td>
</tr>
<tr>
<td>1, 1, 3</td>
<td>1, 1, 1, 1</td>
<td>1, -2, 1</td>
</tr>
<tr>
<td>1, -1, 0</td>
<td>0, 0, 1, 4</td>
<td>0, 1, 1</td>
</tr>
<tr>
<td>3, 1, 4</td>
<td>1, 1, 1, 0</td>
<td>1, 2, 3</td>
</tr>
<tr>
<td>0, 1, 1</td>
<td>0, 1, 1, 1</td>
<td>2, 1, 0</td>
</tr>
<tr>
<td>-1, -1, -1</td>
<td>1, 0, 0, 0</td>
<td>-1, -1, -1</td>
</tr>
<tr>
<td>12</td>
<td>0, 0, 4, 1</td>
<td>10</td>
</tr>
<tr>
<td>0.0001</td>
<td>1, 0, 1, 1</td>
<td>0.001</td>
</tr>
<tr>
<td>100</td>
<td>1, 1, 0, 1</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>0, 1, 0, 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1, 1, 1, 1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0, 0, 0, 1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0, 0, 1, 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1, -1, -1, -1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td></td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
Appendix C.

Further reading

We give a short overview about literature which is related to quitting games and might be interesting for a further reading.

**Repeated games with absorbing states**


**Quitting games in general**

Bibliography


Bibliography


# Index of notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>action vector, $a = (a^1, \ldots, a^N)^T \in A$</td>
<td>3</td>
</tr>
<tr>
<td>$A$</td>
<td>$\sigma$-algebra for a quitting game</td>
<td>6</td>
</tr>
<tr>
<td>$\bar{A}$</td>
<td>$\sigma$-algebra for a one-step game</td>
<td>12</td>
</tr>
<tr>
<td>$A_I$</td>
<td>$\sigma$-algebra for a finite quitting game</td>
<td>29</td>
</tr>
<tr>
<td>$A$</td>
<td>action space</td>
<td>3</td>
</tr>
<tr>
<td>$\chi^m$</td>
<td>min-max value for player $m \in \mathcal{N}$</td>
<td>91</td>
</tr>
<tr>
<td>$\chi^m_p$</td>
<td>pure min-max value for player $m \in \mathcal{N}$</td>
<td>102</td>
</tr>
<tr>
<td>$c$</td>
<td>strategy in the quitting game, where the player plays $continue$ with certainty all the time</td>
<td>34</td>
</tr>
<tr>
<td>$\bar{c}$</td>
<td>strategy in the finite quitting game, where the player plays $continue$ with certainty all the time</td>
<td>71</td>
</tr>
<tr>
<td>$c$</td>
<td>all players play $continue$ with certainty in the stage, i.e. $c := \underline{0} \in [0,1]^N$</td>
<td>34</td>
</tr>
<tr>
<td>$\delta_v$</td>
<td>maximum of the absolute value of the payoff for a player in the one-step game $\Gamma_v$</td>
<td>14</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>expected payoff function for the quitting game</td>
<td>8</td>
</tr>
<tr>
<td>$\gamma_v$</td>
<td>expected payoff function for the one-step game</td>
<td>12</td>
</tr>
<tr>
<td>$g_{I,v} = g$</td>
<td>expected payoff function for the finite quitting game $G_{I,v}$</td>
<td>30</td>
</tr>
<tr>
<td>$\Gamma_v$</td>
<td>one-step game corresponding to the quitting game $G$, with continue payoff $v \in \mathbb{R}^N$ and $\Gamma_v = (G, v)$</td>
<td>10</td>
</tr>
<tr>
<td>$G$</td>
<td>quitting game, $G = (\mathcal{N}, (r_S)_{S \in \mathcal{P}(\mathcal{N})})$</td>
<td>3</td>
</tr>
<tr>
<td>$G_{I,v}$</td>
<td>finite quitting game corresponding to the quitting game $(G)$, with continue payoff $v \in \mathbb{R}^N$, with stages $I$, $G_{I,v} = (G, I, v)$</td>
<td>28</td>
</tr>
<tr>
<td>$H_k$</td>
<td>random variable, denotes the random history up to time $k \in \mathbb{N}$</td>
<td>7</td>
</tr>
<tr>
<td>$\bar{H}_k$</td>
<td>random variable, denotes the random history in the finite quitting game $G_{I,v}$</td>
<td>29</td>
</tr>
<tr>
<td>$I$</td>
<td>set of stages, $I \subset \mathbb{N}$</td>
<td>28</td>
</tr>
<tr>
<td>$\mathcal{N}$</td>
<td>set of players</td>
<td>3</td>
</tr>
<tr>
<td>$N$</td>
<td>number of players</td>
<td>3</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>sample space of a quitting game</td>
<td>6</td>
</tr>
<tr>
<td>$\bar{\Omega}$</td>
<td>sample space of a one-step game</td>
<td>12</td>
</tr>
<tr>
<td>$\Omega_I$</td>
<td>sample space of a finite quitting game</td>
<td>29</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>$\pi$</td>
<td>(strategy) profile in the quitting game, $\pi = (\pi^1, \ldots, \pi^N)^T$</td>
<td></td>
</tr>
<tr>
<td>$\pi^n$</td>
<td>strategy for player $n \in \mathcal{N}$, $\pi^n = (p^n_1, p^n_2, \ldots)$</td>
<td></td>
</tr>
<tr>
<td>$\pi_j$</td>
<td>subgame profile induced by $\pi$ in the quitting game staring at time $j \in \mathbb{N}$</td>
<td></td>
</tr>
<tr>
<td>$\pi^n_j$</td>
<td>subgame strategy for player $n \in \mathcal{N}$ induced by $\pi^n$ in the quitting game staring at time $j \in \mathbb{N}$</td>
<td></td>
</tr>
<tr>
<td>$(\pi^{-n}, \tilde{\pi}^n)$</td>
<td>alternative strategy profile for player $n \in \mathcal{N}$ in the quitting game, $\pi \in \Pi$, $\tilde{\pi}^n \in \Pi^n$</td>
<td></td>
</tr>
<tr>
<td>$\varphi_I := \varphi$</td>
<td>(strategy) profile for the finite quitting game, $\varphi = (\varphi^1, \ldots, \varphi^N)^T$</td>
<td></td>
</tr>
<tr>
<td>$\varphi^n_I := \varphi^n$</td>
<td>strategy for player $n \in \mathcal{N}$ for the finite quitting game $G_{I,v}$, $\varphi^n_I = {i, \ldots, j} = (p^n_i, \ldots, p^n_j)$</td>
<td></td>
</tr>
<tr>
<td>$p$</td>
<td>(strategy) profile for a one-step game</td>
<td></td>
</tr>
<tr>
<td>$p^n$</td>
<td>probability that player $n \in \mathcal{N}$ plays $\text{quit}$ in the one-step game</td>
<td></td>
</tr>
<tr>
<td>$p^n_i$</td>
<td>probability that player $n \in \mathcal{N}$ plays $\text{quit}$ at time $i \in \mathbb{N}$</td>
<td></td>
</tr>
<tr>
<td>$\Pi$</td>
<td>set of all strategy profiles of a quitting game</td>
<td></td>
</tr>
<tr>
<td>$\Pi^n$</td>
<td>set of all strategies for player $n \in \mathcal{N}$</td>
<td></td>
</tr>
<tr>
<td>$(p^{-n}, \tilde{p}^n)$</td>
<td>alternative strategy profile for player $n \in \mathcal{N}$ in a one-step game, $p \in [0,1]^N$, $\tilde{p}^n \in [0,1]$</td>
<td></td>
</tr>
<tr>
<td>$P_p$</td>
<td>probability measure for the one-step game (with respect to the given profile $p \in [0,1]^N$)</td>
<td></td>
</tr>
<tr>
<td>$P_\pi$</td>
<td>probability measure for the quitting game (with respect to a given strategy profile $\pi \in \Pi$)</td>
<td></td>
</tr>
<tr>
<td>$q$</td>
<td>strategy in the quitting game, where the player plays $\text{quit}$ with certainty all the time</td>
<td></td>
</tr>
<tr>
<td>$q_i$</td>
<td>strategy in the quitting game, where the player plays $\text{quit}$ at time $i \in \mathbb{N}$ and $\text{continue}$ in all the other stages with certainty</td>
<td></td>
</tr>
<tr>
<td>$\tilde{q}_i$</td>
<td>strategy for the finite quitting game $G_{I,v}$, where the player plays $\text{quit}$ at time $i \in I$ and $\text{continue}$ in all the other stages from $I$ with certainty</td>
<td></td>
</tr>
<tr>
<td>$q$</td>
<td>all players play $\text{quit}$ with certainty in the stage, i.e. $q := 1 = (1, \ldots, 1)^T \in [0,1]^N$</td>
<td></td>
</tr>
<tr>
<td>$\varrho$</td>
<td>(probability) function</td>
<td></td>
</tr>
<tr>
<td>$r$</td>
<td>payoff function of a quitting game</td>
<td></td>
</tr>
<tr>
<td>$r_{\text{max}}$</td>
<td>maximum of the absolute value of the payoffs a player could get in the quitting game</td>
<td></td>
</tr>
<tr>
<td>$(r_S)_{S \in \mathcal{P}({\mathcal{N}})}$</td>
<td>family of payoff vectors, $r_S = (r^1_S, \ldots, r^N_S)^T \in \mathbb{R}^N$</td>
<td></td>
</tr>
<tr>
<td>$\tilde{r}_v$</td>
<td>payoff function of a one-step game</td>
<td></td>
</tr>
<tr>
<td>$S$</td>
<td>quitting coalition</td>
<td></td>
</tr>
<tr>
<td>$\tau$</td>
<td>stopping time for the quitting game</td>
<td></td>
</tr>
<tr>
<td>$\tau_f$</td>
<td>stopping time for the finite quitting game $G_{I,v}$</td>
<td></td>
</tr>
</tbody>
</table>
\tilde{t} \quad \text{transition law} & 3 \\
v \quad \text{continue payoff vector in a one-step game, } v \in \mathbb{R}^N & 10 \\
X_k \quad \text{random variable, random state of the quitting game at time } k \in \mathbb{N} & 7 \\
\tilde{X}_k \quad \text{random variable, random state of the one-step game at time } k \in \{1, 2\} & 12 \\
\hat{X}_k \quad \text{random variable, random state of the finite quitting game at time } k \in I & 29 \\
\tilde{Y} \quad \text{random variable, random action vector taken in the one-step game} & 12 \\
Y_k \quad \text{random variable, random action vector played at time } k \in \mathbb{N}, \text{ in a quitting game} & 7 \\
\hat{Y}_k \quad \text{random variable, random action vector taken in the finite quitting game at time } k \in I & 29 \\
Y^n_k \quad n\text{-th component of the random variable } Y_k, \text{ denotes the random action taken at time } k \in \mathbb{N} \text{ from player } n \in \mathcal{N} & 7 \\
Z \quad \text{state space} & 3
Affirmation

Hereby I affirm that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken directly or indirectly from other sources are indicated as such. This thesis has not been presented to any other examination board in this or a similar form, neither in Germany nor in any other country.

I have written this dissertation at Dresden University of Technology under the scientific supervision of Prof. Dr. René L. Schilling.

There have been no prior attempts to obtain a PhD at any university.

I accept the requirements for obtaining a PhD (Promotionsordnung) of the Faculty of Science of the TU Dresden, issued March 20, 2000 with the changes in effect since April 16, 2003 and October 01, 2008.

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Die vorliegende Dissertation habe ich an der Technischen Universität Dresden unter der wissenschaftlichen Betreuung von Prof. Dr. René L. Schilling angefertigt.

Es wurden zuvor keine Promotionsvorhaben unternommen.


Dresden, 14. März 2013